The Principle of Supplementarity: A Contextual Probabilistic Viewpoint to Complementarity, the Interference of Probabilities and Incompatibility of Variables in Quantum Mechanics

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We presented a contextual statistical model of the probabilistic description of physical reality. Here contexts (complexes of physical conditions) are considered as basic elements of reality. There is discussed the relation with QM. We propose a realistic analogue of Bohr's principle of complementarity. In the opposite to the Bohr's principle, our principle has no direct relation with mutual exclusivity for observables. To distinguish our principle from the Bohr's principle and to give better characterization, we change the terminology and speak about supplementarity, instead of complementarity. Supplementarity is based on the interference of probabilities. It has quantitative expression trough a coefficient which can be easily calculated from experimental statistical data. We need not appeal to the Hilbert space formalism and noncommutativity of operators representing observables. Moreover, in our model there exists pairs of supplementary observables which cannot be represented in the complex Hilbert space. There are discussed applications of the principle of supplementarity outside quantum physics.

1. INTRODUCTION

Since the creation of statistical mechanics, the probabilistic approach has always played a fundamental role in physics. A crucial step in the development of the statistical approach to physics was made in the process of creation of quantum mechanics. It was however soon realized by

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the founders of this new theory that quantum formalism could not provide a description of physical processes for individual systems. The understanding of this surprising fact induced numerous debates on the possibilities of individual and probabilistic descriptions, and on the relation between them. The debates that followed were characterized by a wide diversity of opinions on the origin of quantum randomness and on its relation to classical randomness, see, e.g., Ref. 1–38 for details and bibliography. Nowadays a rather common opinion is that the classical probabilistic model is incompatible with the quantum one. This opinion is based on a number of "no-go" theorems (von Neumann, Kochen-Specker, Bell, ...). The problem cannot however be considered totally clarified since the *problem of correspondence between quantum and classical probabilistic models* is not a purely mathematical problem, but a physical problem. There are various possibilities to mathematically formalize such a correspondence. Each formalization induces its own model, which in turn usually leads to a new proof of a "no-go" theorem. It should however be stressed that these new "no-go" theorems suffer from the same flaw as any other "no-go" theorem, in the sense that they cannot totally eliminate the possibility of a new formalization of the *quantum-classical probabilistic correspondence.*[2] Of course, the question of the physical adequacy should be treated separately.

In Refs. 29, 30 a new way to establish a correspondence between classical and quantum probabilistic models was proposed. The crucial point of this approach is that physical observables are described by a *contextual probabilistic model*. The main quantum structures are present in this model in a latent form. A quantum representation of the contextual probabilistic model can be constructed on the basis of *two specially chosen observables, a* and *b*. We call them "reference observables" (e.g., the position and momentum); only these observables are considered as representing *objective properties* of physical systems (cf. the views of L. De Broglie, D. Bohm, especially in Ref. 12).

This problem can be considered in a different way, as. Mackey[10] did, by trying to develop a general probabilistic model $\mathcal{M}$ which would contain classical and quantum probabilistic models as special cases (see also

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[2] At my talk at Beckman Institute, University of Urbana-Champain, I was curious: Why did A. Einstein not pay any attention to the von Neumann "no-go" theorem? Leggett paid my attention to the fact that in the original German addition[5] of von Neumann's book[6] there was formulated not a theorem, but *ansatz*. So it seems that A. Einstein considered this chain of mathematical calculations as just some arguments against hidden variables. Moreover, it might be that it was even the position of von Neumann. It might be that both Einstein and von Neumann understood well that it is impossible to prove nonexistence of prequantum deterministic model in the form of a mathematical theorem.

Gudder,[13] L. Ballentine[14]). The origin of the main mathematical structures of quantum mechanics (e.g., the complex Hilbert state space) in such a general probabilistic model should however be clarified.

Before having a closer look at our model, it is perhaps necessary to discuss the meaning of the term *contextuality*, as it can obviously be interpreted in many different ways, see Ref. 31. The most common meaning (especially in the literature on quantum logic[33]) is that the outcome for a measurement of an observable u under a contextual model is calculated using a different (albeit hidden) measure space, depending on whether or not compatible observables $v, w, \ldots$ were also made in the same experiment.[33] We remark that well known "no-go" theorems cannot be applied to such contextual models, see Refs. 33, 37 for details.

This approach to contextuality can be considered as a mathematical formalization, see Ref. 33, of *Bohr's measurement contextuality*. Bohr's interpretation of quantum mechanics is in fact considered as contextual. For N. Bohr the word "context" had the meaning of a "*context of a measurement*", see De Muynck, De Baere, Marten,[38] De Muynck[17] and Plotnitsky.[25−27] For instance, in his answer to the EPR challenge N. Bohr pointed out that position can be determined only in the *context of a position measurement*. The still unsolved and persistent difficulty of any contextual model is that the physical content of such theories appears in the current understanding to be unanchored to what is obtained in any experiment. It seems indeed difficult to explain in any satisfactory way that a measurement on a particle s_1 should or could have both a different meaning and a different associated measure space, depending on whether or not another measurement on a second particle s_2 was performed.

In our approach to the problem of correspondence between classical and quantum probabilistic models, the term contextuality is used in a totally different meaning. Roughly speaking our approach is noncontextual from the conventional viewpoint.[33] Values associated to the reference observables (e.g., position and momentum) are considered as objective properties of physical systems. These observables are therefore not contextual in the sense of Bohr's measurement contextuality.

The basic notion of our approach is the *context*—that is, a complex of physical conditions. Physical systems interact with a context C and in this process a statistical ensemble $\mathcal{S}_C$ is formed (cf. Ref. 14, 34). The notion of context is close to the notion of preparation procedure, see e.g. Ref. 14, 33–36. However, for any preparation procedure $\mathcal{E}$, it is assumed that this procedure could be (at least in principle) realized experimentally. We do not assume this for an arbitrary context C. Contexts are elements of physical reality which exist independently of observers. By using the terminology of Atmanspacher and Primas[32] we can say that context belong

to the ontic level of description of physical reality and preparation procedure to the epistemic level.

Conditional (or better to say contextual) probabilities for reference observables, $\mathbf{P}(a = y/C), \mathbf{P}(b = x/C)$, are used to represent the context C by a complex probability amplitude ψ_C. This amplitude is in fact encoded in a *generalization of the formula of total probability describing the interference of probabilities*. Note that interferences of probabilities can thus be obtained in a classical probabilistic framework (i.e., without the need of the Hilbert space formalism), an observation which was actually the starting point of our considerations. As a result, we found that the quantum probabilistic model can be considered as a Hilbert space projection of the classical contextual probabilistic model. This projection is based on two fixed "reference observables" a and b which play a fundamental role and determine the correspondence between classical prequantum model and quantum model.

Our approach is thus based on two cornerstones:

(a) *contextuality of probabilities*;

(b) *the use of two fixed (incompatible) physical observables in order to represent the classical contextual probabilistic model in the complex Hilbert space.*

We would like to note that the conventional quantum representation is the image of a very special class of contexts $\mathcal{C}^{\mathrm{tr}}$, that is of contexts producing the usual trigonometric interference, while other contexts producing so-called hyperbolic interference[29,30] are also possible. These contexts cannot be represented in a complex Hilbert space but in a so-called hyperbolic Hilbert space[29,30] instead — a module over a two-dimensional Clifford algebra. We will consider however in this paper only contexts having the conventional quantum representation in a complex Hilbert space.

Our contextual statistical approach is realistic. Therefore one may wonder how the *principle of complementarity* should be interpreted or perhaps modified in such an approach. It is precisely the purpose of this paper to study this question.

In a realistic approach one cannot just borrow Bohr's notion of complementarity. The main problem is that of "mutual exclusivity" which was considered by N. Bohr as the main feature of complementarity. We recall the following formulation which was presented in Ref. 1 (vol. 2, p. 40) and was probably Bohr's most refined formulation of what he meant by complementary measurement situations:

> Evidence obtained under different experimental conditions [e.g. those of the position vs. the momentum measurement] cannot be comprehended within a single

picture, but must be regarded as [mutually exclusive and] complementary in the sense that only the totality of the [observable] phenomena exhausts the possible information about the [quantum] objects [themselves].

An extended analysis of Bohr's views to complementarity can be found in works of A. Plotnitsky.[25–27]

Let us consider a realistic model; let a and b be two observables. Then, for any physical system ω, the values $a(\omega)$ and $b(\omega)$ are simultaneously well defined (they "coexist"). In such a situation it is rather unappropriate to talk about the mutual exclusivity of the a-property and the b-property for a system ω. Bohr's principle of complementarity should therefore be modified to retain its part related to completion (in the sense of addition of new information) and to exclude its part related to mutual exclusivity. Hence, in order to distinguish our contextual statistical realistic complementarity from Bohr's complementarity, we propose to change the terminology and use the term *supplementarity*, instead of *complementarity*.[3]

In our approach the principle of complementarity–supplementarity is formulated in mathematical terms. There exist pairs of observables a and b such that b/a-conditioning or a/b-conditioning produces supplementary information. For such observables probabilities interfere producing nontrivial disturbances of the classical formula of total probability. We consider this feature of the contextual statistical realistic model as *the principle of supplementarity*, or to put it in other words, as the *probabilistic principle of complementarity*.

We recall that in the case of two dichotomous random variables $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$ the classical formula of total probability, see e.g. Ref. 39, pp. 25, 77, has the form:

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[3] We understand that the change of terminology, especially for such a fundamental principle of nature, is a very risky choice. It might be better to use a less radical terminology, such as Växjö principle of complementarity for instance. However, since for most physicists complementarity is rigidly associated with mutual exclusivity, we believe we have no other possibility than to introduce a new terminology.

Bohr's principle of complementarity was not formulated in mathematical terms, but as a general philosophic principle. There are various ways to understand (or perhaps misunderstand) this principle. The crucial point is the interpretation of "mutual exlusivity". Here the difference between mathematical *variables* and physical *properties*, as well as their relationship with *observables*, should be taken into account, see Section 7. We emphasize that the direct comparison of the Bohr's principle of complementarity and our principle of supplementarity is really impossible. For N. Bohr mutual exclusivity was related not to objective properties of quantum systems, but to measurement contexts.

$$\mathbf{P}(b=\beta_i) = \mathbf{P}(a=\alpha_1)\mathbf{P}(b=\beta_i/a=\alpha_1) + \mathbf{P}(a=\alpha_2)\mathbf{P}(b=\beta_i/a=\alpha_2).$$

$$\tag{1}$$

As was already emphasized, in the opposition to Bohr's complementarity, supplementarity has no direct relation with mutual exclusivity (however, see Section 7 for more details). Moreover, supplementary observables a and b need not present a complete set of data in a context C. In our approach, representations of reality (e.g., physical reality) based on pairs of supplementary observables a and b are in general not complete. Pairs of supplementary observables produce very rough (statistical) images of the underlying reality (cf. with fuzzy-viewpoint to Quantum Mechanics, see, e.g., Busch et al., Grabowski and Lahti,[34] Gudder, Pulmanova[28]). In particular, in our approach the quantum complex Hilbert space is just a projection of contextual prequantum space — *prespace*.[29,30]

Another difference between our supplementarity and Bohr's complementarity is that we characterize supplementarity by studying interferences of probabilities themselves. This gives the possibility to introduce a simple quantitative measure of supplementarity as a coefficient of interference λ, which we call the *coefficient of supplementarity*. The presence of this quantitative measure shifts discussion from the purely philosophic framework to a physical and mathematical frameworks.[4]

Our model is fundamentally contextual, and supplementarity is consequently also contextual. The coefficient λ depends on a context C under which observables are measured: $\lambda = \lambda_C$. It is therefore meaningless to discuss supplementarity of observables a and b without relation to the concrete complex of physical conditions C under which these observables are measured. Observables can indeed be supplementarity under one context and nonsupplementarity under another one.

Since the Växjö model of reality is a statistical model, it depends on the choice of a probability model. In this paper we use the frequency

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[4] It is noteworthy that our statistical measure of supplementarity λ is not based on the Hilbert space formalism, see Sections 3 and 4. Hence, one need not appeal from the very beginning to the abstract characterization based on *noncommutativity* of operators representing physical observables. It is only under special conditions that supplementary observables can be represented by noncommutative operators in the complex Hilbert space, see Refs. 29, 30 for details. Note also that the coefficient of supplementarity λ can be calculated directly on the basis of experimental statistical data. Since we start directly with statistical data, and since the conventional Hilbert space formalism is considered as a secondary mathematical description, there is no necessity to relate our formalism with waves features or superposition of individual states. This gives the possibility to apply our formalism (and in particular the notion of supplementarity) outside the quantum domain, see, e.g., Refs. 40–42.

probabilistic model (see von Mises,[43] and also Ref. 44 on this model and its applications in quantum physics). This model is essentially more general than the conventional Kolmogorov model.[45] Note that other probabilistic models can be used, e.g., the Cox model with conditional probabilities,[46] see Ballentine[14] on application of this model to Quantum Mechanics.

We remark that our principle of supplementarity is based on rather extended probabilistic considerations developed in Sections 3–5. A reader not too keen on mathematical details is invited to go through those probabilistic sections quickly and pay more attention to Sections 6–8.

There are three appendixes, Section 9 (Bell's inequality) and Sections 10 and 11 on mathematical features of the contextual statistical model (relations between Kolmogorovness, compatibility and complementarity). The latter two sections are essentially mathematical. They may be interesting for people working on mathematical problems of quantum probability and incompatibility.

2. CONTEXTUAL STATISTICAL REALISTIC MODEL

From the very beginning it should be emphasized that the purpose of this section is not to provide a new interpretation of Quantum Mechanics. A general statistical model for observables based on the contextual viewpoint to probability will merely be presented. It will be shown that classical as well as quantum probabilistic models can be obtained as a particular cases of our general contextual model, the *Växjö model*. This model is not reduced to the conventional, classical and quantum models. In particular, it contains a new statistical model: a model with hyperbolic *cosh*-interference (that induces "hyperbolic quantum mechanics").[5]

Realism is one of the main distinguishing features of the Växjö model since it is always possible manipulate objective properties, despite the presence of such essentially quantum effects as, e.g., the interference of probabilities.

As George W. Mackey pointed out in Ref. 10, probabilities cannot be considered as abstract quantities defined outside any reference to a concrete complex of physical conditions C. All probabilities are conditional or better to say contextual.[6] We remark that the same point of view

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[5] It should however be noted that hyperbolic model remains a purely mathematical model, in contrast to those of classical physics and quantum mechanics, as it does not relate to any known physics.

[6] The notion of conditional probability is typically used for events: $\mathbf{P}(A/B)$ is the probability that the event A occurs under the condition that the event B has occurred.

can be found in the works of Kolmogorov[45] and von Mises.[43] However, it seems that Mackey's book was the first thorough presentation of a program of conditional probabilistic description of measurements, both in classical and quantum physics. G. Mackey did a lot to unify classical and quantum probabilistic description and, in particular, demystify quantum probability. One crucial step is however missing in Mackey's work. In his book,[10] Mackey introduced the quantum probabilistic model (based on the complex Hilbert space) by means of a special axiom (Axiom 7, p. 71, in Ref. 10) that looked rather artificial in his general conditional probabilistic framework. The impossibility to derive the quantum probabilistic model from "natural axioms" (which are not based on such a quantum structure as the complex Hilbert space) is clearly the main disadvantage of Mackey's approach.

In our contextual probabilistic approach, structures specifically belonging to the realm of quantum mechanics (e.g., the interference of probabilities, the complex probabilistic amplitudes, Bohr's rule, or the representation of some observables by noncommutative operators) are derived on the basis of two natural axioms (which are, natural from the point of view of classical probabilistic axiomatics).

Mackey's model is based on a system of eight axioms, when our own model requires only two axioms. Let us briefly mention the content of Mackey first axioms. The first four axioms concern conditional structure of probabilities, that is, they can be considered as axioms of a classical probabilistic model. The fifth and sixth axioms are of a logical nature (about questions). We reproduce below Mackey's "quantum axiom", and Mackey's own comments on this axiom (see Ref. 10, pp. 71–72):

Axiom 7. *(G. Mackey) The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subsets of a separable, infinite dimensional Hilbert space.*

"This axiom has rather a different character from Axioms 1 through 4. These all had some degree of physical naturalness and plausibility. Axiom 7 seems entirely *ad. hoc.* Why do we make it? Can we justify making it? What else might we assume? We shall discuss these questions in turn. The first is the easiest to answer. We make it because it "works", that is, it leads to a theory which explains physical phenomena and successfully predicts the results of experiments. It is conceivable that a quite different assumption would do likewise but this is a possibility that no one seems to have explored [But see recent work of Jauch, Stueckelberg, and others at the University of Genève on real and quaternionic Hilbert spaces].

Ideally one would like to have a list of physically plausible assumptions from which one could deduce Axiom 7."

Our activity can be considered as an attempt to find a list of physically plausible assumptions from which the Hilbert space structure can be deduced. We show that this list can consist in two axioms (see our Axioms 1 and 2) and that these axioms can be formulated in the same classical probabilistic manner as Mackey's Axioms 1–4.

Another important difference between the Växjö model and Mackey's model is that our model is not rigidly coupled to the measure theoretic approach to probability. Many probabilistic models can be used to mathematically define contextual probabilities. In this paper we use the von Mises' frequency approach to probability.[43] This approach is essentially more general than the Kolmogorov measure-theoretic one. In general probabilistic data generated by a few collectives, $x, y, z, \ldots, u$, cannot be described by a single Kolmogorov space. There are admittedly some mathematical difficulties in von Mises' approach, like the impossibility to rigorously define randomness on a mathematical level. Nevertheless, in order to define probabilities one need not necessarily to apply von Mises' principle of randomness (which is based on a rather confusing notion of place selection). The frequency probability can be defined by the principle of statistical stabilization of relative frequencies. It means that relative frequencies $\nu_N = \frac{n}{N}$ stabilize when $N \to \infty$, i.e., $|\nu_N - \nu_M| \to 0, N, M \to \infty$. Hence, the limit of ν_N exists and is called the frequency probability. The frequency theory of probability, which is based only on the principle of statistical stabilization is free of contradictions.[44]

Quantum formalism gives good predictions of frequency probabilities, as was verified in an impressive number of experiments, but it does not contain a description of randomness. In the light of the approach described above, the theory of quantum measurements appears to be more about statistical stabilization of relative frequencies than about randomness of data. It seems that in order to create a general frequency probability theory, which would contain classical and quantum probabilities as special cases, one could use the frequency probabilistic model based only on the principle of statistical stabilization.

2.1. Contextual Statistical Model of Observations

A physical *context* C is a complex of physical conditions. Contexts are fundamental elements of any contextual statistical model. Thus construction of any model $\mathcal{M}$ should be started with fixing the collection of contexts of this model; denote the collection of contexts by the symbol $\mathcal{C}$ (so

$\mathcal{C}$ is determined by $\mathcal{M}$). In mathematical formalism $\mathcal{C}$ is an abstract set (of "labels" of contexts). Another fundamental element of any contextual statistical model $\mathcal{M}$ is a set of observables $\mathcal{O}$: any observable $a \in \mathcal{O}$ can be measured under a complex of physical conditions $C \in \mathcal{C}$.[7] For an $a \in \mathcal{O}$, we denote the set of its possible values ("spectrum") by the symbol X_a.

 We do not assume that all these observables can be measured simultaneously; so they need not be compatible. To simplify considerations, we shall consider only discrete observables and, moreover, all concrete investigations will be performed for *dichotomous observables*.

 Axiom 1. *For any observable $a \in \mathcal{O}$, there are defined contexts C_α corresponding to α-filtrations: if we perform a measurement of a under the complex of physical conditions C_α, then we obtain the value $a = \alpha$ with probability 1. It is supposed that the set of contexts $\mathcal{C}$ contains filtration-contexts C_α for all observables $a \in \mathcal{O}$.*

 Axiom 2. *There are defined contextual probabilities $\mathbf{P}(a = \alpha/C)$ for any context $C \in \mathcal{C}$ and any observable $a \in O$.*

 Probabilities $\mathbf{P}(b = \beta/C)$ are interpreted as *contextual probabilities*. Especially important role will be played by probabilities:

$$p^{b/a}(\beta/\alpha) \equiv \mathbf{P}(b = \beta/C_\alpha), \quad a, b \in \mathcal{O}, \quad \alpha \in X_a, \quad \beta \in X_b, \qquad (2)$$

where C_α is the $[a = \alpha]$-filtration context.[8]

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[7] We shall denote observables by Latin letters, $a, b, \dots$, and their values by Greek letters, $\alpha, \beta, \dots$

[8] We prefer to call probabilities with respect to a context $C \in \mathcal{C}$ contextual probabilities. Of course, it would be also possible to call them conditional, but the latter term was already used in other approaches (e.g., Bayes–Kolmogorov). In the opposition of the Bayes-Kolmogorov model, the contextual probability is not probability that an event, say B, occurs under the condition that another event, say C, has been occurred. The contextual probability $\mathbf{P}(b = \beta/C)$ is probability to get the result $b = \beta$ under the complex of physical conditions C. We can say that this is the probability that the event $B_\beta = \{b = \beta\}$ occurs under the complex of physical conditions C. Thus in our approach not event, but context should be considered as a condition; see also Accardi,[15] Ballentine[14] and De Muynck.[17] In particular, the contextual probability $p^{b/a}(\beta/\alpha) \equiv \mathbf{P}(b = \beta/C_\alpha)$, is not probability that the event $B_\beta = \{b = \beta\}$ occurs under the condition that the event $A_\alpha = \{a = \alpha\}$ has been occurred. To find probability $p^{b/a}(\beta/\alpha)$, it is not sufficient to observe the event B_β following the event A_α. It should be first verified that there is really the complex of physical conditions C_α. Then there are performed measurements of the observable b under this context, see also Ballentine[14] and De Muynck.[17]

At the moment we do not fix a definition of probability. Depending on a choice of probability theory we can obtain different models. For any $C \in \mathcal{C}$, there is defined the set of probabilities:

$$E(\mathcal{O}, C) = \{\mathbf{P}(a = \alpha/C) : a \in \mathcal{O}\}. \tag{3}$$

We complete this data by C_α-contextual probabilities: $D(\mathcal{O}, C) = \{\mathbf{P}(a = \alpha/C), \mathbf{P}(b = \beta/C), \ldots, \mathbf{P}(a = \alpha/C_\beta), \mathbf{P}(b = \beta/C_\alpha), \ldots : a, b, \ldots \in \mathcal{O}\}$.

We remark that $D(\mathcal{O}, C)$ does not contain the simultaneous probability distribution of observables $\mathcal{O}$. Data $D(\mathcal{O}, C)$ gives a probabilistic image of the context C through the system of observables $\mathcal{O}$. We denote by the symbol $\mathcal{D}(\mathcal{O}, C)$ the collection of probabilistic data $D(\mathcal{O}, C)$ for all contexts $C \in \mathcal{C}$. There is defined the map:

$$\pi : \mathcal{C} \to \mathcal{D}(\mathcal{O}, C), \quad \pi(C) = D(\mathcal{O}, C). \tag{4}$$

In general this map is not one-to-one. Thus the π-image of contextual reality is very rough: *not all contexts can be distinguished with the aid of probabilistic data produced by the class of observables $\mathcal{O}$.*

Definition 2.1. A contextual statistical model of reality is a triple

$$M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, C)), \tag{5}$$

where $\mathcal{C}$ is a set of contexts and $\mathcal{O}$ is a set of observables which satisfy to axioms 1,2, and $\mathcal{D}(\mathcal{O}, C)$ is probabilistic data about contexts $\mathcal{C}$ obtained with the aid of observables $\mathcal{O}$.

We call observables belonging the set $\mathcal{O} \equiv \mathcal{O}(M)$ *reference of observables*. Inside of a model M observables belonging $\mathcal{O}$ give the only possible references about a context $C \in \mathcal{C}$. Our general model can (but, in principle, need not) be completed by some interpretation of reference observables $a \in \mathcal{O}$. By the Växjö interpretation reference observables are interpreted as *properties of contexts.*

Realistic interpretation of observables: "If an observation of a under a complex of physical conditions $C \in \mathcal{C}$ gives the result $a = \alpha$, then this value is interpreted as the objective property of the context C (at the moment of the observation)."

Remark 2.1 (Number of reference observables). In both most important physical models – in classical and quantum models—the set $\mathcal{O}$ of reference observables consists of *two observables: position and momentum (or*

energy). I think that this number "two" of reference observables plays the crucial role (at least in the quantum model).

2.2. Frequency Description

By taking into account Remark 2.2, we consider a set of reference observables $\mathcal{O} = \{a, b\}$ consisting of two observables a and b. We denotes the sets of values ("spectra") of the reference observables by symbols X_a and X_b, respectively.

Let C be some context. In a series of observations of b (which can be infinite in a mathematical model) we obtain a sequence of values of b:

$$x \equiv x(b/C) = (x_1, x_2, \ldots, x_N, \ldots), \qquad x_j \in X_b. \tag{6}$$

In a series of observations of a we obtain a sequence of values of a:

$$y \equiv y(a/C) = (y_1, y_2, \ldots, y_N, \ldots), \qquad y_j \in X_a. \tag{7}$$

We suppose that the *principle of the statistical stabilization* for relative frequencies[43,44] holds true. This means that the frequency probabilities are well defined:

$$p^b(\beta) \equiv \mathbf{P}_x(b = \beta) = \lim_{N \to \infty} \nu_N(\beta; x), \qquad \beta \in X_b; \tag{8}$$

$$p^a(\alpha) \equiv \mathbf{P}_y(a = \alpha) = \lim_{N \to \infty} \nu_N(\alpha; y), \qquad \alpha \in X_a. \tag{9}$$

Here $\nu_N(\beta; x)$ and $\nu_N(\alpha; y)$ are frequencies of observations of values $b = \beta$ and $a = \alpha$, respectively (under the complex of conditions C).

As was remarked, R. von Mises considered in his theory two principles: (a) *the principle of the statistical stabilization for relative frequencies*; (b) *the principle of randomness*. A sequence of observations for which both principle hold was called a *collective*. An analog of von Mises' theory for sequences of observations which satisfy the principle of statistical stabilization (so relative frequencies converge to limit-probabilities, but these limits need not be invariant with respect to von Mises place selections) was developed in Ref. 44; we call such sequences *S-sequences*. Everywhere in this paper it will be assumed that *sequences of observations are S-sequences*, cf. Ref. 44.

Let $C_\alpha, \alpha \in X_a$, be contexts corresponding to α-filtrations, see Axiom 1. By observation of b under the context C_α we obtain a sequence:

$$x^\alpha \equiv x(b/C_\alpha) = (x_1, x_2, ..., x_N, ...), \qquad x_j \in X_b. \tag{10}$$

It is also assumed that for sequences of observations $x^\alpha, \alpha \in X_a$, the principle of statistical stabilization for relative frequencies holds true and the frequency probabilities are well defined:

$$p^{b/a}(\beta/\alpha) \equiv \mathbf{P}_{x^\alpha}(b = \beta) = \lim_{N \to \infty} \nu_N(\beta; x^\alpha), \qquad \beta \in X_b. \tag{11}$$

Here $\nu_N(\beta; x^\alpha), \alpha \in X_a$, are frequencies of observations of value $b = \beta$ under the complex of conditions C_α. We obtain probability distributions:

$$\mathbf{P}_x(\beta), \qquad \mathbf{P}_y(\alpha), \qquad \mathbf{P}_{x^\alpha}(\beta), \qquad \alpha \in X_a, \ \beta \in X_b. \tag{12}$$

We can repeat all previous considerations by changing b/a-conditioning to a/b-conditioning. We consider contexts $C_\beta, \beta \in X_b$, corresponding to selections with respect to values of the observable b and the corresponding collectives $y^\beta \equiv y(a/C_\beta)$ induced by observations of a in contexts C_β. There can be defined probabilities $p^{a/b}(\alpha/\beta) \equiv \mathbf{P}_{y^\beta}(\alpha)$. Combining these data with data (12) we obtain

$$D(\mathcal{O}, C) = \{p^a(\alpha), p^b(\beta), p^{b/a}(\beta/\alpha), p^{a/b}(\alpha/\beta) : \alpha \in X_a, \beta \in X_b\}. \tag{13}$$

2.3. Systems and Ensemble Representation of Contexts

We now complete the contextual statistical model by considering systems ω (e.g., physical or cognitive, or social,..), (Cf. Ballentine.[14]) In our approach systems as well as contexts are considered as *elements of realty*. In our model a context $C \in \mathcal{C}$ is represented by an ensemble S_C of systems which have been interacted with C. For such systems we shall use notation: $\omega \hookleftarrow C$ The set of all (e.g., physical or cognitive, or social) systems which are used to represent all contexts $C \in \mathcal{C}$ is denoted by the symbol $\Omega \equiv \Omega(\mathcal{C})$. Thus we have a map:

$$C \to S_C = \{\omega \in \Omega : \omega \hookleftarrow C\}. \tag{14}$$

This is the ensemble representation of contexts. We set

$$\mathcal{S} \equiv \mathcal{S}(\mathcal{C}) = \{S : S = S_C, C \in \mathcal{C}\}. \tag{15}$$

This is the collection of all ensembles representing contexts belonging to $\mathcal{C}$. The ensemble representation of contexts is given by the map (14)

$$I : \mathcal{C} \to \mathcal{S}. \tag{16}$$

Reference observables $\mathcal{O}$ are now interpreted as observables on systems $\omega \in \Omega$. In our approach it is not forbidden to interpret the values of the *reference observables* as objective properties of systems. These objective properties coexist in nature and they can be related to individual systems $\omega \in \Omega$. However, the probabilistic description is possible only with respect to a fixed context C. Noncontextual probabilities have no meaning. So values $a(\omega)$ and $b(\omega)$ coexist for a single system $\omega \in \Omega$, but noncontextual ("absolute") probabilities $\mathbf{P}(\omega \in \Omega : a(\omega) = y), \ldots$ are not defined.[9]

Definition 2.2. The ensemble representation of a contextual statistical model $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$ is a triple

$$S(M) = (\mathcal{S}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C})), \tag{17}$$

where $\mathcal{S}$ is a set of ensembles representing contexts $\mathcal{C}$, $\mathcal{O}$ is a set of observables, and $\mathcal{D}(\mathcal{O}, \mathcal{C})$ is probabilistic data about ensembles $\mathcal{S}$ obtained with the aid of observables $\mathcal{O}$.

3. FORMULA OF TOTAL PROBABILITY AND MEASURES OF SUPPLEMENTARITY

Let $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$ be a model in which $\mathcal{O} = \{a, b\}$ and a, b are dichotomous observables. Let $C \in \mathcal{C}$. In general there are no reasons to assume that all probability distributions in $D(a, b, C)$ should be described by a single Kolmogorov probability space (absolute Kolmogorov space) $\mathcal{P} = (\Omega, F, \mathbf{P})$. Thus the classical (Kolmogorovian) formula of total probability:

$$\mathbf{P}(b = \beta) = \sum_{\alpha} \mathbf{P}(a = \alpha)\mathbf{P}(b = \beta/a = \alpha). \tag{18}$$

---

[9] Thus, instead of mutual exclusivity of observables (cf. Bohr's principle of complementarity), we consider contextuality of probabilities and "supplementarity" of the reference observables (in the sense that they give supplementary statistical information about contexts). Oppositely to the very common opinion, such models (with realistic observables) can have nontrivial quantum-like representations (in complex and hyperbolic Hilbert spaces) which are based on the formula of total probability with interference terms.

can be violated. We do not have such a formula in the contextual frequency approach, where the conditional probabilities $\mathbf{P}(b = \beta/a = \alpha)$ are defined as contextual probabilities $\mathbf{P}(b = \beta/C_\alpha)$.[10] In this approach everything is absolutely clear from the very beginning: there are six different S-sequences (or collectives), see Section 2.2:

$$x = x(b/C), \quad y = y(a/C), \quad x^\alpha = x(b/C_\alpha), \quad y^\beta = y(a/C_\beta), \quad (19)$$

where $\alpha = \alpha_1, \alpha_2, \beta = \beta_1, \beta_2$. In the opposition to the Kolmogorov approach, in the contextual frequency approach we have no chance to speculate about a single probability. In the contextual frequency approach in general we have

$$\delta(\beta/a, C) = \mathbf{P}_x(\beta) - \sum_\alpha \mathbf{P}_y(\alpha)\mathbf{P}_{x^\alpha}(\beta) \neq 0. \quad (20)$$

On the other hand, as was mentioned, in the Kolmogorov model we have:

$$\delta(\beta/a, C) = 0. \quad (21)$$

Hence, in the Kolmogorov model by using the Bayesian sum of the conditional probabilities $\mathbf{P}(b = \beta/a = \alpha)$ we find nothing new, but the unconditional probability for $b = \beta$:

$$\mathbf{P}(b = \beta) = \sum_\alpha \mathbf{P}(a = \alpha)\mathbf{P}(b = \beta/a = \alpha). \quad (22)$$

Therefore in these models by obtaining the value $b = \beta$ in a series of observations under the condition $a = \alpha$ we do not obtain new probabilistic information. However, in the contextual approach we obtain new information via conditional observations, see (20). Hence conditional observations give us *supplementary information* which is not contained in statistical data for unconditional observations.

Definition 3.1. The quantity $\delta(\beta/a, C)$ is said to be a probabilistic measure of b/a-supplementarity in the context C.

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[10] We recall that this is the *premeasurement* conditioning. The complex of physical conditions C_α, corresponding to $[a = \alpha]$-selection, is fixed before the b-measurement.

We can rewrite the equality (20) in the form which is similar to the classical formula of total probability:

$$\mathbf{P}_x(\beta) = \sum_\alpha \mathbf{P}_y(\alpha)\mathbf{P}_{x^\alpha}(\beta) + \delta(\beta/a, C), \qquad (23)$$

or by using shorter notations:

$$p^b(\beta) = \sum_\alpha p^a(\alpha)p^{b/a}(\beta/\alpha) + \delta(\beta/a, C). \qquad (24)$$

This formula has the same structure as the quantum formula of total probability:

$$[\text{classical part}] + \text{additional term.}$$

To write the additional term in the same form as in the quantum representation of statistical data, we perform the normalization of the probabilistic measure of supplementarity by the square root of the product of all probabilities:

$$\lambda(\beta/a, C) = \frac{\delta(\beta/a, C)}{2\sqrt{\prod_\alpha p^a(\alpha)p^{b/a}(\beta/\alpha)}}. \qquad (25)$$

The coefficient $\lambda(\beta/a, C)$ also will be called the probabilistic measure of supplementarity.

Of course, it would be better to call λ the coefficient of complementarity, but the latter terminology was already reserved by N. Bohr. By using this coefficient we rewrite (24) in the quantum-like form:

$$p^b(\beta) = \sum_\alpha p^a(\alpha)p^{b/a}(\beta/\alpha) + 2\lambda(\beta/a, C)\sqrt{\prod_\alpha p^a(\alpha)p^{b/a}(\beta/\alpha)}. \quad (26)$$

The coefficient $\lambda(\beta/a, C)$ is well defined only in the case when all probabilities $p^a(\alpha)$, $p^{b/a}(\beta/\alpha)$ are strictly positive. We consider the matrix

$$\mathbf{P}^{b/a} = (p^{b/a}(\beta/\alpha)). \qquad (27)$$

Traditionally this matrix is called the matrix of transition probabilities. In our approach $p^{b/a}(\beta/\alpha) \equiv \mathbf{P}_{x^\alpha}(b = \beta)$ is the probability to obtain the value $b = \beta$ for the S-sequence (collective) x^α. Thus in general we need not speak about states of physical systems and interpret $p^{b/a}(\beta/\alpha)$ as the

probability of the transition from the state α to the state β. We remark that the matrix $\mathbf{P}^{b/a}$ is always *stochastic*:

$$\sum_{\beta} p^{b/a}(\beta/\alpha) = 1 \tag{28}$$

for any $\alpha \in X_a$, because for any S-sequence (or collective) x^α:

$$\sum_{\beta} \mathbf{P}_{x^\alpha}(b = \beta) = 1. \tag{29}$$

We defined a *nondegenerate* S-sequence (or collective) y as such that

$$p^a(\alpha) \equiv \mathbf{P}_y(\alpha) \neq 0 \quad \text{for all} \quad \alpha. \tag{30}$$

A context C is said to be a-nondegenerate (b-nondegenerate) if the corresponding S-sequence (or collective) $y \equiv y(a/C)$ $(x \equiv x(b/C))$ is nondegenerate. We remark that the contexts C_α (S-sequences or collectives) x^α) are b-nondegenerate iff

$$p^{b/a}(\beta/\alpha) \neq 0. \tag{31}$$

The representation (26) is the basis of transition to a (complex or hyperbolic) Hilbert space representation of probabilistic data $D(a, b, C)$. The representation (26) can be used only for nondegenerate contexts C and C_α.

We can repeat all previous considerations by changing b/a-conditioning to a/b-conditioning. We consider contexts C_β corresponding to selections with respect to values of the observable b and the corresponding S-sequences (or collectives) $y^\beta \equiv y(a/C_\beta)$. There can be defined probabilistic measures of supplementarity $\delta(\alpha/b, C)$ and $\lambda(\alpha/b, C), \alpha \in X_a$. We remark that the contexts C_β (S-sequences or collectives y^β) are a-nondegenerate iff

$$p^{a/b}(\alpha/\beta) \neq 0. \tag{32}$$

For nondegenerate contexts C and C_β we have

$$p^a(\alpha) = \sum_{\beta} p^b(\beta) p^{a/b}(\alpha/\beta) + 2\lambda(\alpha/b, C) \sqrt{\prod_{\beta} p^b(\beta) p^{a/b}(\alpha/\beta)}. \tag{33}$$

Definition 3.2. Observables a and b are called (statistically) nondegenerate if (31) and (32) hold.

Theorem 3.1. Let reference observables a and b be nondegenerate and let a context $C \in \mathcal{C}$ be both a and b-nondegenerate. Then quantum-like formulas of total probability (26) and (33) hold true.

In Ref. 1, 22 Theorem 3.1 was proved by using a long series of calculations with relative frequencies; the proof presented in this paper is really straightforward: supplementarity implies the violation of the classical formula of total probability and the perturbation term can be represented in the quantum-like form. As was shown in Refs. 1–3, if $|\lambda| \geqslant 1$, then we get cos-interference (see section 6); if $|\lambda| > 1$, then we get cosh-interference.

4. SUPPLEMENTARY PHYSICAL OBSERVABLES

Definition 4.1. Reference observables a and b are called b/a-supplementary in a context C if

$$\delta(\beta/a, C) \neq 0 \quad \text{for some } \beta \in X_b. \tag{34}$$

Lemma 4.1. For any context $C \in \mathcal{C}$, we have:

$$\sum_{\beta \in X_b} \delta(\beta/a, C) = 0. \tag{35}$$

Proof. We have

$$1 = \sum_{\beta \in X_b} p^b(\beta) = \sum_{\beta \in X_b} \sum_{\alpha \in X_a} p^a(\alpha) p^{b/a}(\beta/\alpha) + \sum_{\beta \in X_b} \delta(\beta/a, C). \tag{36}$$

Since $\mathbf{P}^{b/a}$ is always a stochastic matrix, we have for any $\alpha \in X_a$:

$$\sum_{\beta \in X_b} p^{b/a}(\beta/\alpha) = 1. \tag{37}$$

By using that $\sum_{\alpha \in X_a} p^a(\alpha) = 1$ we obtain (35).

We pay attention to the fact that by Lemma 5.1 the coefficient $\delta(\beta_1/a, C) = 0$ iff $\delta(\beta_2/a, C) = 0$. Thus b/a-supplementarity is equivalent to the condition $\delta(\beta/a, C) \neq 0$ both for β_1 and β_2.

Reference observables a and b are called supplementary in a context C if they are b/a or a/b supplementary:

$$\delta(\beta/a, C) \neq 0 \text{ or } \delta(\alpha/b, C) \neq 0 \quad \text{for some } \beta \in X_b, \alpha \in X_a. \tag{38}$$

By Lemma 4.1 observables are supplementary iff the coefficients $\delta(\beta/a, C) \neq 0$ or $\delta(\alpha/b, C) \neq 0$ for all $\beta \in X_b, \alpha \in X_a$.

Let us consider a contextual model M with the set of contexts $\mathcal{C}$. Observables a and b are said to be supplementary in the model M if there exists $C \in \mathcal{C}$ such that they are supplementary in the context C.

Reference observables a and b are said to be nonsupplementary in the context C if they are neither b/a nor a/b-supplementary:

$$\delta(\beta/a, C) = 0 \quad \text{and} \quad \delta(\alpha/b, C) = 0 \quad \text{for all } \beta \in X_b, \alpha \in X_a. \tag{39}$$

Thus in the case of b/a-supplementarity we have (for $\beta \in X_b$):

$$p^b(\beta) \neq \sum_\alpha p^a(\alpha) p^{b/a}(\beta/\alpha), \tag{40}$$

in the case of a/b-supplementarity we have (for $\alpha \in X_a$):

$$p^a(\alpha) \neq \sum_\beta p^b(\beta) p^{a/b}(\alpha/\beta) \tag{41}$$

in the case of supplementarity we have (40) or (41). In the case of non-supplementarity we have both representations:

$$p^b(\beta) = \sum_\alpha p^a(\alpha) p^{b/a}(\beta/\alpha), \quad \beta \in X_b, \tag{42}$$

$$p^a(\alpha) = \sum_\beta p^b(\beta) p^{a/b}(\alpha/\beta), \quad \alpha \in X_a. \tag{43}$$

5. THE PRINCIPLE OF SUPPLEMENTARITY

Our principle can be formulated in the following way: There exist physical observables, say a and b, such that for some context C they produce supplementary statistical information; in the sense that the contextual probability distribution of, e.g., the observable, b could not be reconstructed on the basis of the probability distribution of a. The classical formula of total probability is violated; supplementarity of the observables a and b under the context C induces interference of probabilities $p_C^b(x)$ and $p_C^a(y)$.

6. THE HILBERT SPACE REPRESENTATION OF CONTEXTS BASED ON CONJUGATE OBSERVABLES

Definition 6.1. Observables a and b are said to be symmetrically conditioned if

$$p^{a/b}(\alpha/\beta) = p^{b/a}(\beta/\alpha). \tag{44}$$

Definition 6.2. Observables a and b are called (statistically) conjugate if they are symmetrically conditioned and nondegenerate:

$$p^{a/b}(\alpha/\beta) = p^{b/a}(\beta/\alpha) > 0 \tag{45}$$

for all α, β.

We shall see that statistically conjugate observables a and b can be represented by noncommutative operators $\hat{a}$ and $\hat{b}$ in the Hilbert space. Everywhere below a and b will be (statistically) conjugate observables. Suppose that, for every $\beta \in X_b$, the coefficient of supplementarity $|\lambda(b = \beta/a, C)| \geqslant 1$. In this case we can introduce new statistical parameters $\theta(b = \beta/a, C) \in [0, 2\pi]$ and represent the coefficients of statistical disturbance in the trigonometric form: $\lambda(b = \beta/a, C) = \cos\theta(b = \beta/a, C)$. Parameters $\theta(b = \beta/a, C)$ are called *probabilistic phases*. We remark that in general there is no geometry behind these phases. By using the trigonometric representation of the coefficients λ we obtain the well known *formula of interference of probabilities* which is typically derived by using the Hilbert space formalism.

If both coefficients λ are larger than one, we can represent them as $\lambda(b = \beta/a, C) = \pm \cosh \theta(b = \beta/a, C)$ and obtain the formula of hyperbolic interference of probabilities; there can also be found models with the mixed hyper-trigonometric behavior, see Refs. 29, 30.

We consider only the complex Hilbert space representation of trigonometric contexts:

$$C^{\mathrm{tr}} = \{C : |\lambda(\beta/a, C)| \geqslant 1, \beta \in X_b\}. \tag{46}$$

Of course, the system C^{tr} depends on the choice of a pair of reference observables, $C^{\mathrm{tr}} \equiv C^{\mathrm{tr}}_{b/a}$. We set $p^a_C(\alpha) = \mathbf{P}(a = \alpha/C)$, $p^b_C(\beta) = \mathbf{P}(b = \beta/C)$, $p(\beta/\alpha) = \mathbf{P}(b = \beta/a = \alpha)$, $\beta \in X_b, \alpha \in X_a$. (In previous sections we considered probabilities with respect to a fixed context C; therefore this index was omitted). Let context $C \in C^{\mathrm{tr}}$. The interference formula of total probability (26) can be written in the following form:

$$p^b_c(\beta) = \sum_{\alpha \in X_a} p^a_C(\alpha) p(\beta/\alpha) + 2 \cos \theta_C(\beta) \sqrt{\Pi_{\alpha \in X_a} p^a_C(\alpha) p(\beta/\alpha)}, \tag{47}$$

where $\theta_C(\beta) = \theta(b = \beta/a, C) = \pm \arccos \lambda(b = \beta/a, C), \beta \in X_b$. By using the elementary formula: $D = A + B + 2\sqrt{AB} \cos \theta = |\sqrt{A} + e^{i\theta}\sqrt{B}|^2$, for $A, B > 0, \theta \in [0, 2\pi]$, we can represent the probability $p^b_C(\beta)$ as the square of the complex amplitude (Born's rule):

$$p^b_C(\beta) = |\psi_C(\beta)|^2, \tag{48}$$

where a complex probability amplitude is defined by

$$\psi(\beta) \equiv \psi_C(\beta) = \sqrt{p^a_C(\alpha_1) p(\beta/\alpha_1)} + e^{i\theta_C(\beta)} \sqrt{p^a_C(\alpha_2) p(\beta/\alpha_2)}. \tag{49}$$

We denote the space of functions: $\psi\colon X_b \to \mathbf{C}$ by the symbol $\Phi = \Phi(X_b, \mathbf{C})$, where $\mathbf{C}$ is the field of complex numbers. Since $X_b = \{\beta_1, \beta_2\}$, the Φ is the two dimensional complex linear space. By using the representation (49) we construct the map $J^{b/a}\colon C^{\mathrm{tr}} \to \Phi(X_b, \mathbf{C})$ which maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation (48) of probability is nothing other than the famous *Born rule*. The complex amplitude $\psi_C(\beta)$ can be called a *wave function* of the complex of physical conditions (context) C or a (pure) *state*. We set $e^b_\beta(\cdot) = \delta(\beta - \cdot)$. The Born's rule for complex amplitudes (48) can be rewritten in the following form:

$$p^b_C(\beta) = |(\psi_C, e^b_\beta)|^2, \tag{50}$$

where the scalar product in the space $\Phi(X_b, \mathbf{C})$ of complex amplitudes is defined by the standard formula:

$$(\psi_1, \psi_2) = \sum_{\beta \in X_b} \psi_1(\beta)\bar{\psi}_2(\beta). \tag{51}$$

The system of functions $\{e_\beta^b\}_{\beta \in X_b}$ is an orthonormal basis in the Hilbert space $H = (\Phi, (\cdot, \cdot))$ Let $X_b \subset \mathbf{R}$, where $\mathbf{R}$ is the field of real numbers. By using the Hilbert space representation of the Born's rule we obtain the Hilbert space representation of the classical conditional expectation of the observable b:

$$
\begin{aligned}
E(b/C) &= \sum_{\beta \in X_b} \beta \ p_C^b(\beta) = \sum_{\beta \in X_b} \beta \ |\psi_C(\beta)|^2 \\
&= \sum_{\beta \in X_b} \beta \ (\psi_C, e_\beta^b)\overline{(\psi_C, e_\beta^b)} = (\hat{b}\psi_C, \psi_C),
\end{aligned} \tag{52}
$$

where the (self-adjoint) operator $\hat{b} \colon H \to H$ is determined by its eigenvectors: $\hat{b}e_\beta^b = \beta e_\beta^b, \beta \in X_b$. This is the multiplication operator in the space of complex functions $\Phi(X_b, \mathbf{C})$: $\hat{b}\psi(\beta) = \beta\psi(\beta)$. It is natural to represent this observable (in the Hilbert space model) by the operator $\hat{b}$. We would like to have Born's rule not only for the b-observable, but also for the a-observable:

$$p_C^a(\alpha) = |(\psi, e_\alpha^a)|^2, \quad \alpha \in X_a. \tag{53}$$

How can we define the basis $\{e_\alpha^a\}$ corresponding to the a-observable? Such a basis can be found starting with interference of probabilities. We set $u_j^a = \sqrt{p_C^a(\alpha_j)}, p_{ij} = p(\beta_j/\alpha_i), u_{ij} = \sqrt{p_{ij}}, \theta_j = \theta_C(\beta_j)$. We have

$$\psi = u_1^a e_1^a + u_2^a e_2^a, \tag{54}$$

where

$$e_1^a = (u_{11}, \ u_{12}), \qquad e_2^a = (e^{i\theta_1}u_{21}, \ e^{i\theta_2}u_{22}). \tag{55}$$

We consider the *matrix of transition probabilities* $\mathbf{P}^{b/a} = (p_{ij})$. It is always a *stochastic matrix*: $p_{i1} + p_{i2} = 1, i = 1, 2$. We remind that a matrix is called *double stochastic* if it is stochastic and, moreover, $p_{1j} + p_{2j} = 1, j = 1, 2$. The system $\{e_i^a\}$ is an orthonormal basis iff the matrix $\mathbf{P}^{b/a}$ is double stochastic and probabilistic phases satisfy the constraint: $\theta_2 - \theta_1 = \pi \bmod 2\pi$, see Refs. 29, 30.

Thus if the matrix $\mathbf{P}^{b/a}$ is double stochastic, then the a-observable is represented by the operator $\hat{a}$ which is diagonal (with eigenvalues α) in the basis $\{e_\alpha^a\}$. The classical conditional average of the observable a coincides with the quantum Hilbert space average:

$$E(a/C) = \sum_{\alpha \in \alpha} \alpha p_C^a(\alpha) = (\hat{a}\psi_C, \psi_C), \ C \in C^{\mathrm{tr}}. \tag{56}$$

7. COMPLEMENTARITY, SUPPLEMENTARITY AND MUTUAL EXCLUSIVITY

I start this section with an extended citation from the report of one of the referees. The problems discussed by those referee are of great importance for clarifying the role of the mutual exclusivity in Bohr's principle of complementarity:

"I would like now to comment on the author's "supplementarity principle". The author juxtaposes this principle to Bohr's complementarity principle, specifically on the account of the mutual exclusivity of whatever *elements* are involved in a given complementary situation. (I shall explain the term "elements" and my emphasis presently.) According to Bohr's interpretation, quantum mechanics is characterized by complementary physical situations, say, those of the position and momentum measurements for a given quantum object. Bohr's terminology refers to situations that are mutually exclusive and hence not applicable at the same time, and yet both defined as possible in order to achieve a comprehensive (complete) physical description. This mutual exclusivity is no longer required by the author's supplementarity principle, and it is replaced by the rule for conditioned probabilities for the corresponding observables, such as position and momentum.

It appears to me that there is some misunderstanding or confusion here as concerns Bohr's complementarity principle, which is not uncommon in treatments of Bohr. Indeed, in the present case it may be a matter of further clarification or, again, a greater lucidity of exposition than misunderstanding. I should add that one's understanding of Bohr's complementarity principle may depend on which Bohr one reads; in particular there are differences between his earlier and his later (e.g., post-EPR) views concerning complementarity. To some degree, the nature of the "elements" involved in complementary situations change in Bohr's later view of the quantum-mechanical situation (hence my emphasis above). In Bohr's later view, to which the author refers, the situation may be seen as follows.

Such variables as position and momentum not only cannot be measured but also cannot be assigned or indeed defined simultaneously, as the author suggests. It is important, however, that this argumentation applies strictly to a given physical situation of measurement, whereby such *variables* as position or momentum (understood in terms of classical physics) and the corresponding measurable physical *properties* now refer only to measuring instruments impacted by quantum objects.

The author seems to me to be insufficiently attentive to these nuances, especially those concerning the differences between mathematical *variables* and physical *properties*, and between each of these and *observables*. Part of the problem is the ambiguity of the term "observable" in quantum mechanics. It can refer to both an observable property and a corresponding variable, say, as an operator in a Hilbert space. The relationships between both, essentially different, types of "observables", Hilbert-space operators and measurable quantities, and specifically the assignment of probabilities to the outcomes of measurements that result, may be seen as defining the essence of quantum mechanics.

In Bohr, an assignment and the definition of certain physical properties (which, again, pertain to the measuring instruments involved and are seen as classical physical variables) would always be mutually exclusive, and the assignment of probabilities would be affected accordingly. The definition of mathematical variables as Hilbert-space operators is, however, a more complex matter. Both variables potentially involved in one or the other complementary situation of measurement possible at any given point may be viewed as being, at that point, definable *simultaneously* (rather than being mutually exclusive) as *formal mathematical entities*. The reason for that is that we consider formally the same Hilbert space in any given situation of measurement, where we would, for example, consider the commutators corresponding to the uncertainty relations for such variables."

I agree with the referee that in this paper there is investigated "mathematical mutual exclusivity" and the physical consequences of our principle of supplementarity should be investigated in more detail. We shall do this in section 8 by replying to the questions of another referee of the paper.

Now we discuss mainly mathematical framework. I agree with the referee that one should distinguish between mathematical variables, physical properties and observables. The essence of quantum mechanics (as physical theory) is in understanding relationships between these mathematical and physical quantities. Denote mathematical variables u and v, physical properties $\mathcal{U}$ and $\mathcal{V}$ and (physical) observables U and V. Bohr's mutual exclusivity is mutual exclusivity of U and V. Thus by mutual exclusivity N. Bohr understood exclusivity of measurement contexts D_U and D_V, cf., with the remark in introduction on the difference between

Bohr's "measurement contextuality" and our "preparation contextuality". Physical observables U and V are represented in quantum formalism by self-adjoint operators $\hat{u}$ and $\hat{v}$. As was rightly pointed out by the referee, $\hat{u}$ and $\hat{v}$ are definable simultaneously as formal mathematical quantities. The situation with physical properties $\mathcal{U}$ and $\mathcal{V}$ is essentially more complicated. Logically mutual exclusivity of observables U and V does not automatically imply that physical properties $\mathcal{U}$ and $\mathcal{V}$ could not coexist. The views of N. Bohr on this problem are not so clear. One could not say that Bohr rejected the existence of physical properties. The result of a measurement was considered by him as a property of the object, see Ref. 38 for the extended discussion. However, this result can be defined only in the context of its measurement. Therefore for mutually exclusive measurement contexts D_U and D_V it is forbidden to consider the properties $\mathcal{U}$ and $\mathcal{V}$ as simultaneously coexisting. It should be pointed out the simultaneous definability of mathematical variables $\hat{u}$ and $\hat{v}$ (self-adjoint operators in the complex Hilbert space) does not contradict to mutual exclusivity of the corresponding physical observables. The quantum formalism does not describe results of individual measurements. Therefore operators $\hat{u}$ and $\hat{v}$ could not be assigned to a single physical system ω.

Moreover, it seems that various "no-go" theorems, e.g., theorems of von Neumann, Kochen-Specker, Bell, confirm Bohr's views to complementarity. These theorems imply that it is even in principle impossible to go deeper than quantum formalism. It is impossible to construct a prequantum ('classical") probabilistic model containing mathematical variables u, v which could be assigned to individual systems.[11]

The Växjö model induces a totally different viewpoint to relationships between mathematical variables, physical properties and observables. In the opposite to the common "quantum opinion" in this model it is possible to define mathematical variables $u = a$ and $v = b$ which are simultaneously defined for a single physical system $\omega : a(\omega)$ and $b(\omega)$ coexist. These functions represent physical observables U and V which were denoted by the same symbols a and b. Here objective properties $\mathcal{U}$ and $\mathcal{V}$ can coexist and they are represented by $a(\omega)$ $b(\omega)$.

As was pointed out, the conventional quantum formalism is a special representation of the Växjö model in which one ignores the knowledge about $a(\omega)$ and $b(\omega)$. Such an ignorance is convenient in the situation in

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[11] I do not know any Bohr's comment on the possibility to justify his principle of complementarity via "no-go" theorems, in particular, von Neumann theorem. It seems that N. Bohr would not be so interested in such results. For him already Heisenberg's uncertainty relations and the impossibility to combine the position measurement with the interference in the two slit experiment were sufficiently strong arguments in favor of complementarity (in the sense of mutual exclusivity).

that the simultaneous measurement of $U = a$ and $V = b$ is impossible. Here impossibility need not be of fundamental nature, it could be of purely technological nature, cf. Section 8.

Quantum model cannot describe a single-particle context C_ω, but the prequantum contextual model can contain such contexts.

As a consequence of this difference of models Bohr's principle of complementarity does not look so natural for the Växjö model and should be changed to a new principle. We propose to consider supplementarity with coexistence instead complementarity (with mutual exclusivity).

8. EXPERIMENTAL CONSEQUENCES OF THE SUPPLEMENTARITY PRINCIPLE

One of the referees of this paper asked about possible experimental consequences of the supplementarity principle. He pointed out that: "According to Bohr, observables $a(\omega)$, $b(\omega)$, when measured on a system ω, may be considered as relevant information about ω only when they are compatible, while in Khrennikov's realistic contextual model the contrary is claimed to be the case. In order to distinguish both approaches, the author introduces the new notions of supplementarity, and supplementary observables. Khrennikov's claim may be criticized, as Bohr's complementarity standpoint and the associated incompatibility of some specific observables fits perfectly actual laboratory practice. However, nobody knows how physics and present day technical possibilities will evolve in the future and, therefore, the possible theoretical alternative considered by Khrennikov may not be excluded a *priori*." Then the referee added that the physical relevance and interest of the principle of supplementarity should be explained somewhat more explicitly. "Relevant questions are e.g. how the principle of supplementarity related to experiment, how may one profit from additional information? Or is it at present only a theoretical possibility, which may become interesting in the future?"

At the moment the principle of supplementarity has purely theoretical value. It says that quantum probabilistic behavior need not imply the impossibility to construct a realistic underlying model. Therefore one cannot exclude the possibility to find (or create) a context C such that Heisenberg's uncertainty relations would be violated in this context for some pair of conjugate variables a and b. Here Heisenberg's uncertainty relations are considered as an inequality for dispersions of a and b for a statistical ensemble S_C corresponding to the context C, see e.g. Ballentine.[14] Thus there might be found a context C such that dispersions of both a and b would be arbitrary small. Of course, such a context

C would be impossible to represent by a complex probability amplitude ψ_C. Moreover, there could exist *dispersion free* contexts. There is no contradiction with Neumann's conclusion on nonexistence of dispersion free contexts (because in our model such contexts are not represented in the complex Hilbert space). In general the set of trigonometric contexts $\mathcal{C}^{\mathrm{tr}}$ (here $|\lambda| \leqslant 1$, see Sec. 2.9) which can be represented in the complex Hilbert space is a proper subset of the family of all possible contexts $\mathcal{C}$ of a contextual statistical model $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$. We recall that there also exist hyperbolic contexts, $\mathcal{C}^{\mathrm{hyp}}$ (here $|\lambda| \geqslant 1$) which can be represented in so called hyperbolic Hilbert space. It is possible to show that dispersion free contexts belong to the class $\mathcal{C} \setminus (\mathcal{C}^{\mathrm{tr}} \cup \mathcal{C}^{\mathrm{hyp}})$.[12]

Another consequence of the principle of supplementarity which can stimulate experimental research is that quantum-like probabilistic effects might be observed for systems and contexts (e.g. physical or biological) for which existence of an underlying realistic model looks very natural. Here an observation of interference of probabilities would not be considered as a contradiction with the realistic model. As was mentioned in Ref. 1, such effects might be observed for ensembles of macroscopic physical systems (related to specially designed contexts) or for ensembles of cognitive systems, see the previous section.[13]

The referee also proposed to discuss in more detail relation between the present contextual statistical realist model and models with hidden

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[12] It is easy to present mathematical models[(1)] in that $\mathcal{C} \setminus (\mathcal{C}^{\mathrm{tr}} \cup \mathcal{C}^{\mathrm{hyp}}) \neq \emptyset$ and there exist dispersion free contexts. But the referee is completely right that until now such contexts have not been found. There are two possibilities: either they do not exists in nature (as it follows from the conventional viewpoint to QM) or there were never performed extended experimental investigations in this direction. I hope that the development of quantum cryptography (which is fundamentally based on Bohr's complementarity principle) could clarify this problem.

[13] We remark that in cognitive science it is commonly believed that there can be created a realistic neurophysiological model of brain functioning, see e.g. Ref. 47. It is natural to suppose that such a realistic model would be of huge complexity. However, it is not so easy to derive some conclusions about individual behavior of a system (brain) depending on billions of (neuronal) parameters. Therefore in such a situation a probabilistic model plays the important role. If we apply Bohr's the complementarity principle, then it seems that the quantum probabilistic model cannot be applied in this case (in the presence of underlying realistic neurophysiological model). However, if we apply the contextual model, then it is very natural to apply quantum-like model to describe the probabilistic behavior of brain (as the whole), see Refs. 40–42. For example, Penrose[(48)] who evidently uses Bohr's principle of complementarity remarked: "It is hard to see how one could usefully consider a quantum superposition of one neuron *firing* and *not firing*." Therefore he was not able to apply quantum formalism on the neurophysiological macroscale and, as a consequence, he should go to the deepest level of matter described by quantum gravity.

variables (HV). Surprisingly our model does not have close relation with HV-models. Despite the realism of reference observables ($a(\omega), b(\omega)$ are well defined for any physical system ω), we do not assume the existence of the simultaneous (frequency) probability distribution (as people do in all HV-models), but the most important is that even for each single variable probability distributions are contextual – determined by complexes of experimental physical conditions. We did not create a HV-model for QM. The only thing that was demonstrated is that the interference of probabilities is compatible with the realistic viewpoint to (reference) observables.

ACKNOWLEDGMENTS

APPENDIX A: BELL'S INEQUALITY

Typically Bell's inequality is considered as a constraint for local realistic models. The question of relation of Bell's inequality with interpretations of QM was discussed in details by Ballentine,[14] De Muynck, De Baere and Marten,[38] De Muynck.[17] Our contextual statistical realistic interpretation does not contradict to the experimental fact of the violation of Bell's inequality. In our model only *two* reference observables are realistic and Bell's theorem is about realism of *three* observables. But it may be that is not the point! Bell's realism is a very special statistical realism, namely, the Kolmogorovian realism. J. Bell claimed from the very beginning that there exist the Kolmogorov probability measure $\rho(d\lambda)$ on the space of hidden variables Λ (see, e.g., Accardi,[15] see also Ref. 49, 50). But in general there are no reasons for the existence of such a unique probability measure on Λ. For example, let us assume that Λ is the infinite dimensional topological linear

space (e.g., Hilbert or Banach). This assumption is not so unnatural—it is clear that the space of hidden variables should be of the greatest complexity.[49] Moreover, De Muynck, De Baere and Marten[38] and De Muynck[17] proposed to consider quantum observables as averages with respect to trajectories, see also Ref. 49 (where that model was investigated). Such spaces are, of course, infinite dimensional. It is well know that many natural distributions on infinite dimensional spaces are not countably additive (for example, some Gaussian measures, see Ref. 49 for analysis). Such distributions are not probability measures; for them Bell's calculations inducing the Bell inequality cannot be performed. Recently it was shown that if, instead of the Bell–Kolmogorov realism, one uses the frequency probabilistic realism (von Mises realism), then in general there is no Bell's inequality[49] (neither GHZ-paradox) and, moreover, it is possible to obtain the EPR-Bohm correlations. These correlations in the general contextual probabilistic framework were obtained in Ref. 25. In fact, the using of von Mises realism is closely related with objections to Bell's arguments presented by De Baere[51] (the hypothesis of nonreproducibility) and Hess and Philipp[52] (taking into account the time-structure of the experiment).

APPENDIX B: SUPPLEMENTARITY AND KOLMOGOROVNESS

Definition B.1. Probabilistic data $D(a, b, C)$ is said to be Kolmogorovian if there exists a Kolmogorov probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ and random variables ξ_a and ξ_b on $\mathcal{P}$ such that

$$p^a(\alpha) = \mathbf{P}(\xi_a = \alpha), \qquad p^b(\beta) = \mathbf{P}(\xi_b = \beta), \tag{B.1}$$

$$p^{b/a}(\beta/\alpha) = \mathbf{P}(\xi_b = \beta/\xi_a = \alpha), \qquad p^{a/b}(\alpha/\beta) = \mathbf{P}(\xi_a = \alpha/\xi_b = \beta). \tag{B.2}$$

If data $D(a, b, C)$ is Kolmogorovian then the observables a and b can represented by Kolmogorovian random variables ξ_a and ξ_b.

Lemma B.1. Data $D(a, b, C)$ is Kolmogorovian if and only if

$$p^a(\alpha) p^{b/a}(\beta/\alpha) = p^b(\beta) p^{a/b}(\alpha/\beta). \tag{B.3}$$

Proof. (a) If data $D(a, b, C)$ is Kolmogorovian then (B.3) is reduced to the equality $\mathbf{P}(O_1 \cap O_2) = \mathbf{P}(O_2 \cap O_1)$ for $O_1, O_2 \in \mathcal{F}$.

(b) Let (B.3) holds true. We set $\Omega = X_a \times X_b$, $X_a = \{\alpha_1, \alpha_2\}$, $X_b = \{\beta_1, \beta_2\}$. We define the probability distribution on Ω by

$$\mathbf{P}(\alpha, \beta) = p^b(\beta) p^{a/b}(\alpha/\beta) = p^a(\alpha) p^{b/a}(\beta/\alpha), \qquad \text{(B.4)}$$

and define the random variables $\xi_a(\omega) = \alpha, \xi_b(\omega) = \beta$ for a system ω on which the outcomes α, β are found when observables a, b are measured. We have $\mathbf{P}(\xi_a(\omega) = \alpha) = \sum_\beta \mathbf{P}(\alpha, \beta) = \sum_\beta p^a(\alpha) p^{b/a}(\beta/\alpha) = p^a(\alpha) \sum_\beta p^{b/a}(\beta/\alpha) = p^a(\alpha)$.

And in the same way $\mathbf{P}(b = \beta) = p^b(\beta)$. Thus

$$\mathbf{P}(a = \alpha/b = \beta) = \frac{\mathbf{P}(a = \alpha, b = \beta)}{\mathbf{P}(b = \beta)} = \frac{p^b(\beta) p^{a/b}(\alpha/\beta)}{p^b(\beta)} = p^{a/b}(\alpha/\beta). \quad \text{(B.5)}$$

And in the same way we prove that $p^{b/a}(\beta/\alpha) = \mathbf{P}(b = \beta/a = \alpha)$.

We now investigate the relation between Kolmogorovness and non-supplementarity. If $D(a, b, C)$ is Kolmogorovian then the formula of total probability holds true and we have (39). Thus observables a and b are nonsupplementary (in the context C). Thus:

Kolmogorovness implies nonsupplementarity
or as we also can say:
Supplementarity implies non-Kolmogorovness.

However, in the general case nonsupplementarity does not imply that probabilistic data $D(a, b, C)$ is Kolmogorovian. Let us investigate in more detail the case when both matrices $\mathbf{P}^{a/b}$ and $\mathbf{P}^{b/a}$ are *double stochastic*. We recall that a matrix $\mathbf{P}^{b/a} = (p^{b/a}(\beta/\alpha))$ is double stochastic if it is stochastic (so (28) holds true) and, moreover,

$$\sum_\alpha p^{b/a}(\beta/\alpha) = 1, \quad \beta = \beta_1, \beta_2. \qquad \text{(B.6)}$$

Remark B.1. (Double stochasticity as the law of statistical balance). As was mentioned, the equality (28) holds true automatically. This is a consequence of additivity and normalization by 1 of the probability distribution of any collective x^α. But the equality (B.6) is an additional condition on the observables a and b. Thus by considering double stochastic matrices we choose a very special pair of reference observables. In Ref. 29 I tried to find the physical meaning of the equality (28). Since $p^{b/a}(\beta/\alpha_2) = 1 - p^{b/a}(\beta/\alpha_1)$, the C_{α_1} and C_{α_2} contexts

compensate each other in "preparation of the property" $b = \beta$. Thus the equation (B.6) could be interpreted the *law of statistical balance* for the property $b = \beta$. If both matrices $\mathbf{P}^{b/a}$ and $\mathbf{P}^{a/b}$ are double stochastic then we have laws of statistical balance for both properties: $a = \alpha$ and $b = \beta$.

Definition B.2. Observables a and b are said to be statistically balanced if both matrices $\mathbf{P}^{b/a}$ and $\mathbf{P}^{a/b}$ are double stochastic.

It is useful to recall the following well known result about double stochasticity for Kolmogorovian random variables:

Lemma B.2. Let ξ_a and ξ_b be random variables on a Kolmogorov space $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$. Then the following conditions are equivalent

(1) The matrices $\mathbf{P}^{a/b} = (\mathbf{P}(\xi_a = \alpha/\xi_b = \beta))$, $\mathbf{P}^{b/a} = (\mathbf{P}(\xi_b = \beta/\xi_a = \alpha))$ are double stochastic.
(2) Random variables are uniformly distributed:

$$\mathbf{P}(\xi_a = \alpha) = \mathbf{P}(\xi_b = \beta) = \frac{1}{2}. \tag{B.7}$$

(3) Random variables are symmetrically conditioned:

$$\mathbf{P}(\xi_a = \alpha/\xi_b = \beta) = \mathbf{P}(\xi_b = \beta/\xi_a = \alpha). \tag{B.8}$$

This result is not true in the contextual frequency approach and this fact is used in the following proposition:

Proposition B.1. A Kolmogorov model for data $D(a, b, C)$ need not exist even in the case of nonsupplementary statistically balanced observables having the uniform probability distribution.

Proof. Let us consider probabilistic data $D(a, b, C)$ such that $p^a(\alpha) = p^b(\beta) = 1/2$ and both matrices $\mathbf{P}^{a/b}$ and $\mathbf{P}^{b/a}$ are double stochastic. But let $p^{a/b}(\alpha/\beta) \neq p^{b/a}(\beta/\alpha)$. Then by Lemma 10.1 data $D(a, b, C)$ is non-Kolmogorovian, but

$$\delta(\alpha/\beta, C) = 1 - \sum_{\beta} p^{a/b}(\alpha/\beta) = 0, \quad \delta(\beta/\alpha, C) = 1 - \sum_{\alpha} p^{b/a}(\beta/\alpha) = 0.$$
$$\tag{B.9}$$

It seems to be that symmetrical conditioning plays the crucial role in these considerations.

Lemma B.3. If observables a and b are symmetrically conditioned, then they are statistically balanced (so the matrices $\mathbf{P}^{a/b}$ and $\mathbf{P}^{b/a}$ are double stochastic).

Proof. We have

$$\sum_{\beta} p^{a/b}(\alpha/\beta) = \sum_{\beta} p^{b/a}(\beta/\alpha) = \sum_{\beta} \mathbf{P}_{x^{\alpha}}(b = \beta) = 1, \qquad \text{(B.10)}$$

$$\sum_{\alpha} p^{b/a}(\beta/\alpha) = \sum_{\alpha} \mathbf{P}_{y^{\beta}}(a = \alpha) = 1. \qquad \text{(B.11)}$$

As we have seen in Proposition B.1, statistically balanced observables need not be symmetrically conditioned.

Proposition B.2. Let observables a and b be symmetrically conditioned. Probabilistic data $D(a, b, C)$ is Kolmogorovian iff the observables a and b are nonsupplementary in the context C.

Proof. Suppose that a and b are nonsupplementary. We set

$$p^{b/a}(1/1) = p^{b/a}(2/2) = p \quad \text{and} \quad p^{b/a}(1/2) = p^{b/a}(2/1) = 1 - p \quad \text{(B.12)}$$

(we recall that by Lemma B.3 the matrix $\mathbf{P}^{b/a}$ is double stochastic). By (42), (43) we have

$$p^{a}(\alpha_i) = \sum_{\beta} p^{b}(\beta) p^{a/b}(\alpha_i/\beta) = \sum_{\beta} \sum_{\alpha} p^{a}(\alpha) p^{b/a}(\beta/\alpha) p^{a/b}(\alpha_i/\beta)$$

$$= \sum_{\alpha} p^{a}(\alpha) \sum_{\beta} p^{b/a}(\beta/\alpha) p^{b/a}(\beta/\alpha_i). \qquad \text{(B.13)}$$

Let us consider the case $i = 1 : p^{a}(\alpha_1) = p^{a}(\alpha_1)(p^2 + (1 - p)^2) + 2p^{a}(\alpha_2)p(1 - p) = p^{a}(\alpha_1)(1 - 4p + 4p^2) + 2p(1 - p)$.

Thus $p^{a}(\alpha_1) = 1/2$. Hence $p^{a}(\alpha_2) = 1/2$. In the same way we get that $p^{b}(\beta_1) = p^{b}(\beta_2) = 1/2$. Thus the condition (B.3) holds true and there exist a Kolmogorov model $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ for probabilistic data $D(a, b, C)$.

Conclusion. In the case of symmetrical conditioning Kolmogorovness is equivalent to nonsupplementarity.

APPENDIX C: INCOMPATIBILITY, SUPPLEMENTARITY AND EXISTENCE OF THE JOINT PROBABILITY DISTRIBUTION

The notions of incompatible and complementary variables are considered as synonymous in the Copenhagen quantum mechanics. Moreover, compatibility (and consequently noncomplementarity) is considered as equivalent to existence of the simultaneous probability distribution $\mathbf{P}(\alpha, \beta) = \mathbf{P}(a = \alpha, b = \beta)$. We now consider similar questions for our Växjö model.

C.1. Existence of the Simultaneous Probability Distribution

By analogy with Definition 5.1 we may propose the following definition:

Definition C.1. Probabilistic data $W(a, b, C) = \{p^a(\alpha), p^b(\beta)\}$, where C is a context, is said to be Kolmogorovian if there exists a Kolmogorov probability space $\mathcal{P} = (\Omega, \mathcal{F}, \mathbf{P})$ and random variables ξ_a and ξ_b on $\mathcal{P}$ such that

$$p^a(\alpha) = \mathbf{P}(\xi_a = \alpha), \quad p^b(\beta) = \mathbf{P}(\xi_b = \beta). \tag{C.1}$$

However, it is evident that data $W(a, b, C)$ is always Kolmogorovian. For example, we can always define the Kolmogorov measure $\mathbf{P}(\alpha, \beta) = p^a(\alpha)p^b(\beta)$. It is evident that the marginal distributions of this probability coincide with $p^a(\alpha)$ and $p^b(\beta)$. Of course, such a probability is not uniquely defined and in general this is a purely mathematical construction which has no physical meaning. In particular, the probability (C.4) always corresponds to independent random variables ξ_a and ξ_b. However, in general observables a and b are not independent. Therefore the problem of Kolmogorovness of data $W(a, b, C)$ can be investigated in physical framework only by using the frequency probability theory.

We emphasize that mathematical Kolmogorovness of data $W(a, b, C)$ does not imply Kolmogorovness of data $D(a, b, C)$. If data $W(a, b, C)$ is Kolmogorovian, then there are well defined Bayes–Kolmogorov conditional probabilities:

$$\mathbf{P}(A_\alpha / B_\beta) = \frac{\mathbf{P}(A_\alpha \cap B_\beta)}{\mathbf{P}(B_\beta)}, \tag{C.2}$$

where $A_\alpha = \{\omega \in \Omega : a(\omega) = \alpha\}$, $B_\beta = \{\omega \in \Omega : b(\omega) = \beta\}$. However, in general these measure-theoretical conditional probabilities do not coincide with experimental conditional probabilities determined in the frequency framework:[14] $p^{a/b}(\alpha/\beta) = \mathbf{P}_{y\beta}(\alpha)$. In particular, Kolmogorovness of data $W(a, b, C)$ does not imply that probabilistic measures of supplementarity $\delta(\alpha/\beta, C)$ and $\delta(\beta/\alpha, C)$ are equal to zero.

Conclusion. Kolmogorovness of data $W(a, b, C)$ does not imply non-supplementarity of observables a and b in the context C.

C.2. Incompatible and supplementary Observables

As usual, observables a and b are called *compatible* in a context C if it is possible to perform a simultaneous observation of them under C. For any instant of time t, there can be observed a pair of values $z(t) = (a(t), b(t))$. There is well defined a sequence of results of observations:

$$z(a, b/C) = (z_1, z_2, ..., z_N, ...), \quad z_j = (y_j, x_j), \tag{C.3}$$

where $y_j = \alpha_1$ or α_2 and $x_j = \beta_1$ or β_2. Observables a and b are incompatible in a context C if it is impossible to perform a simultaneous observation of them under C.

Observables a and b are said to be *statistically compatible* in a context C if they are compatible in C and the $z(a, b/C)$ is an S-sequence (or collective).[15] Thus there exists the frequency simultaneous probability distribution:

$$\mathbf{P}(\alpha, \beta) \equiv \mathbf{P}_z(\alpha, \beta) = \lim_{N \to \infty} \frac{n_N(\alpha, \beta; z)}{N}. \tag{C.4}$$

We remark that in general existence of this frequency probability distribution (77) has nothing to do with existence of a formal Kolmogorov probability distribution—Kolmogorovness of data $W(a, b, C)$. As was mentioned, data $W(a, b, C)$ is always Kolmogorovian, but sequences of results of observations y and x are not always combinable.

If a and b are statistically compatible in a context C then data $D(a, b, C)$ is Kolmogorovian, because $p^{a/b}(\alpha/\beta) = \mathbf{P}_z(a = \alpha/b = \beta)$ and $p^{b/a}(\beta/\alpha) = \mathbf{P}_z(b = \beta/a = \alpha)$.

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[14] But even in the Kolmogorov framework we cannot define conditional probabilities in the unique way, because a Kolmogorov measure for data $W(a, b, C)$ is not uniquely defined.
[15] In the opposite case observables are statistically incompatible.

   Thus we have

Statistical compatibility implies Kolmogorovness of data $D(a, b, C)$
or
Non-Kolmogorovness of data $D(a, b, C)$ implies statistical incompatibility
Thus
Statistical compatibility implies nonsupplementarity
or
Supplementarity implies statistical incompatibility

   However, since in general nonsupplementarity does not imply Kolmogorovness of data $D(a, b, C)$, see Proposition 5.1, we have:

Nonsupplementarity does not imply statistical compatibility

Thus there can exist a context C such that observables a and b do not produce supplementary information,[16] but they do not have the frequency joint probability distribution. The notions of statistical compatibility and nonsupplementarity are not equivalent. Hence, the notions of supplementarity and incompatibility are neither equivalent:

Statistical incompatibility does not imply supplementarity

   Thus there can exist a context C such that observables a and b do not have the frequency joint probability distribution, but at the same time they do not produce supplementary information.
   Let us consider a contextual statistical model $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$. If physical observables a and b are (statistically) compatible for any $C \in \mathcal{C}$ then they are called (statistically) compatible in the model M. They are called (statistically) incompatible in the model M if there exists $C \in \mathcal{C}$ such that they are (statistically) incompatible for C.

C.3. Compatibility does not Imply Statistical Compatibility

   In quantum physics compatibility of observables—the possibility to perform a simultaneous observation—is typically identified with statistical compatibility—existence of the frequency simultaneous probability distribution (77). This is a natural consequence of the Kolmogorovian psychology. In the frequency probability theory we should distinguish compatibility and statistical compatibility. We present an example in which observables are compatible, but the limit (77) does not exist, so observables are not statistically compatible. Of course, this means that $y = y(a/C)$ and $x = x(b/C)$ are not combinable. We need some well known

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[16] Hence all coefficients $\delta(\alpha/b, C)$, $\delta(\beta/a, C)$ are equal to zero.

results about the generalized probability given by the density of natural numbers, see Ref. 44 (we recall that A. N. Kolmogorov considered the density of natural numbers as an example of probability, but it was before he proposed the conventional axiomatics). For a subset $A \subset \mathbf{N}$ the quantity $\mathbf{P}(A) = \lim_{N \to \infty} \frac{|A \cap \{1, \ldots, N\}|}{N}$, is called the *density* of A if the limit exists. Here the symbol $|V|$ is used to denote the number of elements in a finite set V.

Let $\mathcal{G}$ denote the collection of all subsets of $\mathbf{N}$ which admit density. It is evident that each finite $A \subset \mathbf{N}$ belongs to $\mathcal{G}$ and $\mathbf{P}(A) = 0$. It is also evident that each subset $B = \mathbf{N} \setminus A$, where A is finite, belongs to $\mathcal{G}$ and $\mathbf{P}(B) = 1$ (in particular, $\mathbf{P}(\mathbf{N}) = 1$). The reader can easily find examples of sets $A \in \mathcal{G}$ such that $0 < \mathbf{P}(A) < 1$. The "generalized probability" $\mathbf{P}$ has the following additivity-property (cf. Gudder[11]). Let $A_1, A_2 \in \mathcal{G}$ and $A_1 \cap A_2 = \emptyset$. Then $A_1 \cup A_2 \in \mathcal{G}$ and $\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2)$.

It is possible to find sets $A, B \in \mathcal{G}$ such that, for example, $A \cap B \notin \mathcal{G}$. Let A be the set of even numbers. Take any subset $C \subset A$ which has no density. In fact, you can find C such that $\frac{1}{N}|C \cap \{1, 2, \cdots, N\}|$ is oscillating. There happen two cases: $C \cap \{2n\} = \{2n\}$ or $= \emptyset$. Set $B = C \cup \{2n - 1 : C \cap \{2n\} = \emptyset\}$ Then, both A and B have densities one half. But $A \cap B = C$ has no density. Thus $\mathcal{G}$ is not a set algebra.

We now consider a context C which produces natural numbers. We introduce two dichotomous observables: $a(n) = I_A(n)$, $b(n) = I_B(n)$, where $I_O(x)$ is the characteristic function of a set O. We assume that these observables are compatible: we can, e.g., look at a number n and find both values $a(n)$ and $b(n)$.[17] We obtain two S-sequences:

$$y = y(a/C) = (y_1, \ldots, y_N, \ldots),$$
$$x = x(b/C) = (x_1, \ldots, x_N, \ldots), \quad y_j, x_j = 0, 1. \quad \text{(C.5)}$$

The frequency probability distributions are well defined: $p^a(\alpha) \equiv \mathbf{P}_y(\alpha) = 1/2$, $p^b(\beta) \equiv \mathbf{P}_x(\beta) = 1/2$. However, the S-sequences y and x are not combinable. Thus observables a and b are not statistically compatible; for example, the frequency probability $\mathbf{P}(1, 1)$ does not exist.[18]

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[17] If reader like he can consider natural numbers as systems and interpret $a(n)$ and $b(n)$ as values of observables a and b on the system n.

[18] But, of course, there exist various Kolmogorov probability measures $\mathbf{P}_{\text{Kol}}(\alpha, \beta)$ which have marginal distributions $p^a(\alpha)$, $p^b(\beta)$.

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