Foundations of Physics, Vol. 34, No. 12, December 2004 (© 2004)
DOI: 10.1007/s10701-004-1629-y

The Pioneer Anomaly as Acceleration of the Clocks

Antonio F. Rañada[1]

Received September 30, 2004

This work proposes an explanation of the Pioneer anomaly, the unmodelled and as yet unexplained blueshift detected in the microwave signal of the Pioneer 10 and other spaceships by Anderson et al. in 1998. What they observed is similar to the effect that would have either (i) an anomalous acceleration a_P the ship towards the Sun, or (ii) an acceleration of the clocks $a_t = a_P/c$. The second alternative is investigated here, with a phenomenological model in which the anomaly is an effect of the background gravitational potential $\Psi(t)$ that pervades all the universe and is increasing because of the expansion. It is shown that $2a_t = d\Psi/dt = d^2\tau_{clocks}/dt^2$, evaluated at present time t_0, where t and τ_{clocks} are the coordinate time and the time measured by the atomic clocks, respectively. The result of a simple estimate gives the value $a_t \simeq 1.8\times10^{-18}\,\mathrm{s}^{-1}$, while Anderson et al. suggested $a_t = (2.9\pm0.4)\times10^{-18}\,\mathrm{s}^{-1}$ on the basis of their observations. The calculation are performed near the Newtonian limit but in the frame of general relativity.

KEY WORDS: Pioneer anomaly; Pioneer acceleration; acceleration of the clocks; background gravitational potential.

1. INTRODUCTION

1.1. The Anomaly

Anderson *et al*. reported in 1998 the observation of an unmodelled Doppler blueshift in the microwave signals from the Pioneer 10/11, Galileo and Ulysses spacecrafts that increases linearly in time.(1) They had been observing it since more than 20 years. Obviously, its simplest interpretation is that the ships were not following the predicted orbits, as if our star

[1]Facultad de Física, Universidad Complutense, E-28040 Madrid, Spain; e-mail: afr@fis.ucm.es

pulled a bit too much from them with a force independent of the distance. The corresponding anomalous acceleration, directed towards the Sun and constant, would have the value[2]

$$a_{\rm P} = (8.74 \pm 1.33) \times 10^{-10}\,{\rm m\,s^{-2}}. \tag{1}$$

Intriguingly enough, the effect does not show up in the planets. In their first paper they said: "it is interesting to speculate on the possibility that the origin of the anomalous signal is new physics",[1,2] and later "The veracity of the signal is now undisputed, although the source of the anomaly, some systematic or some not understood physics, is subject to debate".[3] For an interesting argument showing that it may not necessarily be due to systematics (see Ref. 4). The effect is still unexplained.[2]

Anderson *et al.* say that their data show[1] "a steady frequency drift of about $-6 \times 10^{-9}\,{\rm Hz\,s^{-1}}$, or 1.5 Hz over 8 yr. This equates a clock acceleration, $-a_{\rm t}$, of $-2.8 \times 10^{-18}\,{\rm s\,s^{-2}}$", what would mean that the frequencies would drift as

$$\nu = \nu_0[1 + 2a_{\rm t}(t - t_0)], \tag{2}$$

t_0 being here the initial time of the observations (with the best value for $a_{\rm P}$ (1), $a_{\rm t} = (2.9 \pm 0.4) \times 10^{-18}\,{\rm s^{-1}}$). The relation with the Pioneer acceleration is $a_{\rm P} \equiv a_{\rm t}c$. This is important since they say that the drift in the Doppler residuals cannot be removed without either an acceleration of the ship $a_{\rm P}$ or the inclusion of a "clock acceleration" $a_{\rm t}$. Such acceleration $a_{\rm t}$ would imply that all the clocks would be changing with a constant acceleration or, in other words, that there would be a non-uniformity of time. They found that the first alternative leads to problems with the equivalence principle and with the cartography of the solar system. They considered the second by means of several models in which the time is distorted phenomenologically (see Ref. 2, Secs. XI.D and XI.E). The best results were obtained with a model that adds a quadratic term to the definition of the International Atomic Time. However, they found some problems and concluded: "The orbit determination process clearly prefers the constant acceleration model, $a_{\rm P}$, over the quadratic in time model."

1.2. Purpose and Assumption of this Work

This paper considers this second alternative. It shows that, because of the expansion, the background gravitational potential that pervades all the universe produces an acceleration of the cosmological proper time with respect to the coordinate time. In its turn, this implies an acceleration a_t of the atomic clocks. A simple estimation gives a value for a_t that is close to that found by Anderson *et al.* The anomaly is thus in this model an effect of the expansion of the universe. It is assumed, for simplicity, that (i) all the matter and energy of the universe are uniformly distributed, (ii) the space sections $t =$ constant are flat, and (iii) the near Newtonian approximation is adequate and meaningful. The time coordinate t is so chosen as to go in the Newtonian limit to the Newtonian time. The model here presented is the relativistic version of a previous Newtonian one.[5–7] Two other models in which the anomaly is also due to the expansion are proposed in Refs. 8 and 9.

1.3. Two Definitions of the Light Speed

It is important to know precisely which one of the several meanings of "speed of light" is used.[10] Particularly, it must be reminded here that the light speed can be defined in general relativity in two different ways: (i) with respect to the cosmological proper time τ, $c^* = d\ell/d\tau$ (=constant), and (ii) with respect to the coordinate time t, $c = d\ell/dt = c(\mathbf{r}, t)$, where $d\ell$, $d\tau$ and dt are elements of spatial distance, proper time and coordinate time along a null geodesic, respectively (the first is the usual definition). These two speeds will be denoted as c^* and c, respectively, and will be called "*proper speed of light*" and "*ordinary* or *non-proper speed of light*. The derivative of c with respect to t at present time t_0, denoted as $a_\ell = \dot{c}(t_0) = dc(t_0)/dt$, will be called *non-proper acceleration of light* or just acceleration of light if there is no risk of confusion. The first is a universal constant of nature (as it must happen in general relativity), the second is not but, quite on the contrary, it depends generally on space and time $c = c(\mathbf{r}, t)$. The duality between c^* and c reflects the relation between the proper time and the coordinate time.

The element of interval can be written[11–14] (assuming for simplicity that $g_{0i} = 0$)

$$ds^2 = c^{*2}\, d\tau^2 - d\ell^2 \tag{3}$$

with $d\tau = \sqrt{g_{00}}\, dt$ and $d\ell^2 = g_{ij}\, dx^i\, dx^j$, so that c^* is constant and c is equal to

$$c = c(\mathbf{r}, t) = c^* \sqrt{g_{00}}. \tag{4}$$

Near the Newtonian limit, $g_{00} \simeq 1 + 2\Phi/c^2$, at first order, Φ being the gravitational potential, so that $c = c(\mathbf{r}, t) = c^*[1 + \Phi(\mathbf{r}, t)/c^2]$. Taking a non-zero origin for the potential at a reference laboratory R, this can be written, at first order, as

$$c = c(\mathbf{r}, t) = c_0[1 + \Phi(\mathbf{r}, t)/c^2(\mathbf{r}, t) - \Phi_R/c_0^2], \quad (5)$$

where c_0 and Φ_R are the values of c and the potential at that laboratory.

In this paper, we are interested in the effect of the background gravitational potential that pervades all the universe and is due to all the existing matter and energy. Assuming a uniform distribution of matter and energy, it is clear that is time depending but space independent. It will be denoted as $\Phi_{av}(t)$ ("av" stands for average since the mass-energy density is averaged). The same must happen therefore to $c = c(t)$. Instead of (5) one would have then

$$c = c(t) = c_0[1 + \Phi_{av}(t)/c^2(t) - \Phi_{av}(t_0)/c_0^2] \quad (6)$$

with $c_0 = c(t_0)$, t_0 being a reference time that will be the present time or age of the universe, in general. Because of the expansion, it turns out that $\Phi_{av}(t)/c^2(t)$ is an increasing function, as will be shown in Sec. 4 where its derivative with respect to t will be calculated. Consequently, $c(t)$ is also increasing.

If the universe would contain only matter, be it ordinary or dark, the background potential Φ_{av} would be negative so that

$$c(t) < c_0[1 - \Phi_{av}(t_0)/c_0^2]\,. \quad (7)$$

As will be seen in Sec. 3, this implies $c(t) < c^*$, as it could be expected.

However, the effect of the cosmological constant or of dark energy changes dramatically this question. The cosmological model used here is the standard with 27% of matter and 73% of dark energy. For the latter we take either a cosmological constant or the quantum vacuum, but the conclusion would be the same for any kind of dark energy with an equation of state implying repulsion. The potential $\Phi_{av}(t)$ is the addition of the two effects of matter and dark energy.

What is important here is that both the cosmological constant and the quantum vacuum produce a positive potential. Furthermore, it turns out that $\Phi_{av}(t)$ is an increasing positive function after a certain time, in particular now (this will be shown in Sec. 4). Equation (6) implies then that c increases as far as the universe expands and, in particular, it can be larger than the proper speed of light c^*. Although this maybe seem strange and contrary to current wisdom, it must not be a matter of concern since, as it must be emphasized, the proper light speed of light c^* is in fact a universal constant in this model. Moreover, $c < c^*$ if there is only matter, let it be ordinary or dark. It is only the dark energy, which, in addition to accelerate the universe, can make that $c(t)$ could be larger than the proper speed of light c^*. This intriguing result will be considered in Sec. 3. To understand this property, one must keep in mind that almost all our intuitions in general relativity were generated in the study of the gravitation of matter (i.e. without dark energy).

There is moreover a functional relation between the two times $\tau = \tau(t)$, such that τ must accelerate with respect to t. Because of the freedom to choose the coordinates in general relativity, this last statement must be qualified: the time coordinate t is defined here so in the Newtonian limit it goes over the Newtonian time. All this means that

$$c(t) = \frac{d\ell}{d\tau}\frac{d\tau}{dt} = c^*\frac{d\tau}{dt} \quad \text{and} \quad \frac{dc(t)}{dt} = c^*\frac{d^2\tau}{dt^2}\,. \tag{8}$$

Equation (8) states that *the non-proper light speed $c(t)$ must increase if the proper time τ accelerates with respect to the coordinate time t*, its time derivative being equal to the proper light speed c^* times the second derivative of τ with respect to t.

The expression "light speed" means usually the proper speed c^* that, being a universal constant, is of the utmost importance. However, the non-proper light speed c is also used in some important cases. For instance, in the study of the bending of a light ray grazing the Sun surface. Let M and R be the mass and radius of the Sun. The interval around any star is given by the Schwarzschild metric, what implies that $c = c(r) = c_\infty(1 - \eta R/r)$, with $c_\infty = c(\infty)$ and $\eta = GM/c_\infty^2 R \simeq 2.1\times 10^{-6}$. Einstein gave two formulae for this effect. The first (1907) is based only in the equivalence principle and gives $\phi = 2\eta = 0.875''$, just one half of the observed effect. The second (1916), in the frame of general relativity, gives the complete result $\phi = 4\eta = 1.75''$. The first one can be obtained simply by considering the propagation of a wave light with the previous value of the non-proper light speed, in other words as the solution of the variational problem

$$\delta T = \delta \int_1^2 \frac{1 + \eta R/r}{c_\infty}\, d\ell = 0, \tag{9}$$

where $d\ell = dx^2 + dy^2 + dz^2$ is the Euclidean line element, i.e. as a consequence of the application of the Fermat principle to the non-proper light speed c. The complete effect is obtained by taking instead the non-Euclidean spatial line element of the Schwarzschild geometry. In other words, the problem is solved by assuming that the light propagates through space with the non-proper speed $c(r) = c_\infty(1 - \eta R/r)$ (taking into account the Riemannian character of the spatial metric).

These considerations can be summarized as follows: because of the background gravitational potential of all the matter and energy in the universe, (i) the cosmological proper time τ accelerates with respect to the coordinate time, and (ii) the non-proper speed of light $c(t)$ increases so that $a_\ell > 0$.

1.4. Plan of the Paper

In Sec. 2, the Maxwell equations in general relativity are reviewed, with emphasis on the ideas of permittivity and permeability of geometrical origin and on their effect on the non-proper speed of light $c(\mathbf{r}, t)$. It is shown that $2a_\mathrm{t}$ must be equal to the derivative with respect to t of the background potential over $c^2(t)$ and that the corresponding non-proper of light acceleration a_ℓ implies a blueshift. In Sec. 3, the reasons for the acceleration of the atomic clocks are considered, showing that it is equal to $2a_\mathrm{t}$. In Sec. 4, an estimation of a_t is carried out, the agreement being good with the result found by Anderson *et al.* on the basis of their observations.[1] The conclusions will be stated in Sec. 5.

2. THE SPEED OF LIGHT AND THE MAXWELL EQUATIONS

In order to understand the behavior of the non-proper speed of light $c(\mathbf{r}, t)$ (different, don't forget, from the proper speed of light c^*, which is a universal constant), let us consider the effect of a gravitational field on the Maxwell equations, in which the time derivatives are with respect to the coordinate time t. As was seen in Sec 1.3, the proper speed of light c^* and the non-proper speed of light c are equal in absence of potential or, in other words, the reason for their difference is the presence of matter and energy. It is clear that near the special relativity or the Newtonian limits, the electromagnetic fields obey the classical wave equation with the velocity $c(\mathbf{r}, t)$ (remember that flat space sections are assumed). That local value of c must be used at any space–time point in the wave equation. Let us take a reference laboratory where $c = c_0$. The following discussion is based on the well-known textbook *The Classical Theory of Fields* by Landau and Lifshitz (Ref. 15, Sec. 90).

The electromagnetic tensor is defined in general relativity by means of a vector field such that $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The electromagnetic vectors **E**, **D** and antisymmetric tensors B_{ij}, H_{ij} are defined as follows:

$$E_i = F_{0i}, \quad B_{ij} = F_{ij}, \quad D^i = -\sqrt{g_{00}}\, F^{0i}, \quad H^{ij} = \sqrt{g_{00}}\, F^{ij},$$

the vectors **B**, **H** being the dual to the three-tensors B_{ij} and H_{ij}, i.e. $B^i = -e^{ijk}B_{jk}/(2\sqrt{\gamma})$, $H_i = -\sqrt{\gamma}\, e_{ijk}H^{jk}/2$, where $\gamma = \det(\gamma_{ij})$, $\gamma_{ij} = -g_{ij}$ being the three-dimensional metric tensor (assuming for simplicity $g_{0i} = 0$). It follows that(15) (assuming, for simplicity, $\epsilon_0 = 1$, $\mu_0 = 1$ in this argument)

$$\mathbf{D} = \mathbf{E}/\sqrt{g_{00}}, \quad \mathbf{B} = \mathbf{H}/\sqrt{g_{00}}. \tag{10}$$

If the space is empty, i.e. without free charges or currents, the Maxwell equations can be written as

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{\sqrt{\gamma}}\, \partial_t \left(\sqrt{\gamma}\, \mathbf{B}\right), \tag{11}$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{H} = \frac{1}{\sqrt{\gamma}}\, \partial_t \left(\sqrt{\gamma}\, \mathbf{D}\right). \tag{12}$$

In a static situation, these four equations have exactly the same form as in special relativity, since the factors $\sqrt{\gamma}$ cancel. However, Eq. (10) implies that the relative permittivity ϵ_r and permeability μ_r of empty space are different from 1, their common value being $\epsilon_r = \mu_r = (g_{00})^{-1/2}$. This is due to the geometry of space–time. If the potential depends on time and near the Newtonian limit one has $g_{00} = 1 + \Phi(\mathbf{r}, t)/c^2(\mathbf{r}, t) - \Phi_R/c_0^2$, where Φ_R is the potential at a reference laboratory where $c = c_0$, so that the empty space is like an inhomogeneous optical medium with

$$\epsilon_r(\mathbf{r}, t) = \mu_r(\mathbf{r}, t) = 1 - \left[\Phi(\mathbf{r}, t)/c^2(\mathbf{r}, t) - \Phi_R/c_0^2\right]. \tag{13}$$

Since $c = c_0/\sqrt{\epsilon_r \mu_r}$, the non-proper speed of light is given as

$$c(\mathbf{r}, t) = c_0 \left\{1 + \Phi(\mathbf{r}, t)/c^2(\mathbf{r}, t) - \Phi_R/c_0^2\right\}, \tag{14}$$

Φ_R being a reference potential, at present time in a terrestrial laboratory R, where the observed light speed is c_0. As a historical comment, this last equation was first obtained in 1911 by Einstein himself, as a first order approximation in the static case, in a paper entitled "On the influence of gravitation on the propagation of light”[(16)] (in 1907 he had already shown that c depends on Φ as a consequence of the equivalence principle. Note that it is still valid in general relativity, at first order). After a discussion on the synchronization of clocks, he concludes there “if we call the velocity of light at the origin of coordinates c_0, where we take $\Phi = 0$, then the velocity of light at a place with gravitational potential Φ will be given as”

$$c = c_0 \left(1 + \Phi/c^2\right). \tag{15}$$

Einstein had not as yet introduced the general relativistic idea of proper time and used only the coordinate time to define the speed of light.[(17,18)] Note that (14) reduces to (15) in the static case if the potential at the reference laboratory vanishes and that the consideration of the Maxwell equations confirms Eq. (5) from the expression of the interval.

In this work, we are interested in the case of a potential depending only on time. Let $\Phi_{\rm av}(t)$ be the background potential of all the matter and energy and $\Psi(t) = \Phi_{\rm av}(t)/c^2(t)$ its dimensionless expression (the subindex “av” being omitted in Ψ in order to simplify the notation). The permittivity and permeability (13) take then the form near present time t_0 (i.e. the age of the universe)

$$\epsilon_{\rm r}(t) = \mu_{\rm r}(t) = 1 - [\Psi(t) - \Psi(t_0)]. \tag{16}$$

The non-proper speed of light is then

$$c(t) = c_0\,[1 + \Psi(t) - \Psi(t_0)]\,, \tag{17}$$

where $c_0 = c(t_0)$. This can be written as

$$c(t) = c_0[1 + 2a_{\rm t}(t - t_0)] = c_0 + a_\ell(t - t_0), \tag{18}$$

the quantity $a_{\rm t}$ and the non-proper acceleration of light $a_\ell = \dot{c}(t_0)$ being

$$2a_{\rm t} = \dot{\Psi}(t_0), \quad a_\ell = 2a_{\rm t}c_0 = 2\dot{\Psi}(t_0)c_0. \tag{19}$$

As will be seen in the following: (i) a_ℓ is in fact $2a_{\rm P}$, being therefore an adiabatic acceleration, and (ii) $a_{\rm t}$ is what Anderson *et al.* termed acceleration of the clocks.

2.1. The Blueshift

It will be shown now that the non-proper acceleration a_ℓ implies a blue shift of the light with respect to the coordinate time, at first order in a_ℓ. More precisely, it turns out that the frequency ν of a monochromatic light wave with such an adiabatic acceleration a_ℓ increases, its derivative with respect the coordinate time t, $\dot{\nu}$, satisfying

$$\dot{\nu}/\nu_0 = a_\ell/c_0. \tag{20}$$

This means that an adiabatic non-proper acceleration of light has the same radio signature as a blue shift of the emitter, although a peculiar blue shift with no change of the wavelength (i.e. all the increase in velocity is used to increase the frequency).

The derivative with respect to t of the background gravitational potential of all the universe $\Psi(t)$ is positive and of the order of the Hubble parameter $H_0 = 2.3 \times 10^{-18}\,\mathrm{s}^{-1}$, since the galaxies are separating (a calculation will be made in Sec. 4). Equation (16) tells then that the relative permittivity ϵ_r and permeability μ_r of empty space are decreasing, their derivatives with respect to t being negative and also of order H_0, i.e. very small. This can be expressed by saying that the optical density of empty space is decreasing adiabatically. To study the propagation of the light in a medium with time depending permittivity and permeability, we must take the Maxwell equations and deduce the wave equations for the electric field $\mathbf{E}$ and the magnetic intensity $\mathbf{H}$, which are $\nabla^2\mathbf{E} - \partial_t\,(\mu\partial_t(\epsilon\mathbf{E})) = 0$, $\nabla^2\mathbf{H} - \partial_t\,(\epsilon\partial_t(\mu\mathbf{H})) = 0$, or, more explicitly

$$\nabla^2\mathbf{E} - \partial_t^2\mathbf{E}/c^2(t) - (\dot{\mu}/\mu_0 + 2\dot{\epsilon}/\epsilon_0)\,\partial_t\mathbf{E}/c^2(t) - \dot{\epsilon}\dot{\mu}\mathbf{E}/(\epsilon_0\mu_0c^2(t)) = 0, \tag{21}$$

$$\nabla^2\mathbf{H} - \partial_t^2\mathbf{H}/c^2(t) - (2\dot{\mu}/\mu_0 + \dot{\epsilon}/\epsilon_0)\,\partial_t\mathbf{H}/c^2(t) - \dot{\epsilon}\dot{\mu}\mathbf{H}/(\epsilon_0\mu_0c^2(t)) = 0 \tag{22}$$

with $c(t) = c_0 + a_\mathrm{t}(t - t_0)$, since at present time $\epsilon_\mathrm{r} = 1$, $\mu_\mathrm{r} = 1$. Because $\dot{\epsilon}/\epsilon_0$ and $\dot{\mu}/\mu_0$ are of order $H_0 = 2.3 \times 10^{-18}\ \mathrm{s}^{-1}$, the third and the fourth terms in the LHS of (21) and (22) can be neglected for frequencies $\omega \gg H_0$, in other words for all practical purposes. We are left with two classical wave equations with time dependent light velocity $c(t)$.

$$\nabla^2\mathbf{E} - \partial_t^2\mathbf{E}/c^2(t) = 0, \quad \nabla^2\mathbf{H} - \partial_t^2\mathbf{H}/c^2(t) = 0. \tag{23}$$

In order to find the behavior of a monochromatic light beam according to these two wave equations, we take for instance the first one and insert

$\mathbf{E} = \mathbf{E}_0 \exp\{-i[\kappa z - (\omega_0 + \dot{\omega}(t - t_0)/2)(t - t_0)]\}$, where the frequency is the time derivative of the phase of $\mathbf{E}$, i.e. $\omega_0 + \dot{\omega}(t - t_0)$. Neglecting the second time derivatives and working at first order in $\dot{\omega}$ (with $\dot{\omega}(t - t_0) \ll \omega_0$, $\dot{\omega} \ll \omega_0^2$), substitution in (21) gives $\kappa^2 = \left[(\omega_0 + \dot{\omega}(t - t_0))^2 - i\dot{\omega}\right]/c^2(t)$. It follows that $\kappa = k + i\zeta = \pm(\omega_0/c(t))[1 + \dot{\omega}(t - t_0)/\omega_0](\cos\varphi + i \sin\varphi)$, with $\varphi = -\dot{\omega}/2\omega_0^2$, so that $k = \pm(\omega_0/c_0)\,(1 + \dot{\omega}(t - t_0)/\omega_0)/(1 + a_1(t - t_0)/c_0)$ what implies $k = \pm\omega_0/c_0$, $\dot{\omega}/\omega_0 = a_\ell/c_0$, as stated before. Equation (20) has thus been proved. Also, $\zeta = -\dot{\omega}/2\omega_0 c_0 = a_\ell/2c_0^2$. The wave amplitude decreases in the direction of propagation as $e^{-z/\ell}$ with $\ell = 2c_0^2/a_\ell$, but as a_ℓ is of order $H_0 c_0$, ℓ is of order of 5000 Mpc, or, in other words, this attenuation can be neglected. As is easy to show, to take $k + \dot{k}t$ for the wave vector leads to $\dot{k} = 0$. These results are equally valid for the second equation in (23). Taking into account that $a_\ell/c_0 = 2a_\mathrm{t}$ and according to (16)–(19), the frequencies drift as

$$\nu = \nu_0[1 + 2a_\mathrm{t}(t - t_0)], \tag{24}$$

what shows that $2a_\mathrm{t} = \dot{\Psi}(t_0)$ is the acceleration of clocks mentioned by Anderson *et al.* According to these arguments, its value is the derivative with respect to the coordinate time t of the background gravitational potential of all the universe. This will be further studied in next section.

3. THE ACCELERATION OF THE ATOMIC CLOCKS

3.1. The Effect of the Dark Energy: Can $c(t)$ be Larger than c^*?

The observations are made by using atomic clocks, which measure proper time τ not coordinate time t. However, up to now we have used mainly in this paper the coordinate time. This question is addressed in this section. Taking the t derivative of (17) at time t_0, it is found that $\dot{c}(t_0) = c(t_0)\dot{\Psi}(t_0)$. The same argument can be applied to the expression for $c(t)$ near any other fixed time $\tilde{t}$, what implies that $c(t) = c(\tilde{t})\exp[\Psi(t) - \Psi(\tilde{t})]$, an expression valid for $\forall t$. In particular, taking $\tilde{t} = t_0$, one finds

$$c(t) = c_0\, e^{[\Psi(t) - \Psi(t_0)]}. \tag{25}$$

As can be seen (17) is the first order approximation to (25). The shape of the function $c(t)$ defined by (25) does not change if the reference time (t_0 or $\tilde{t}$) is changed, because $c(t_1)e^{-\Psi(t_1)} = c(t_2)e^{-\Psi(t_2)}$. Since c^* and c are equal if $\Psi = 0$, it follows that $c^* = c_0 e^{-\Psi(t_0)}$, what does not depend on the particular value of t_0. In other words, the non-proper speed of light is

$$c(t) = c^* e^{\Psi(t)}. \tag{26}$$

We can precise now some ideas exposed in Sec. 1.3. If there is only matter in the universe (i.e. without dark energy), then $\Psi(t)$ is necessarily negative, so that $c(t) < c^*$. This is the usual situation that everybody has in mind. However, if there is dark energy, $\Psi(t)$ can be positive, and in that case $c(t) > c^*$. We find thus that the non-proper speed of light could be larger than the proper speed and constant of the nature c^*. Even if this is a surprising effect of the dark matter, in addition to accelerate the expansion, it must be emphasized that it is compatible with the constancy of c^* and with general relativity therefore. Indeed, it is an unexpected and new effect of the dark energy.

3.2. On the Time of the Atomic Clocks

The interval can be written then as

$$ds^2 = e^{2[\Psi(t)-\Psi(t_0)]} c_0^2\, dt^2 - d\ell^2 = c^{*\,2}\, d\tau^2 - d\ell^2 \tag{27}$$

so that

$$d\tau = e^{\Psi(t)}\, dt \quad \text{and} \quad \left.\frac{d^2\tau}{dt^2}\right|_{t_0} = \dot{\Psi}(t_0) e^{\Psi(t_0)} = 2a_{\mathrm{t}} e^{\Psi(t_0)} > 0, \tag{28}$$

since $\Psi(t)$ is now an increasing function and $\dot{\Psi}(t_0) > 0$ (the precise calculation is done in Sec. 4). Note that τ is a well defined cosmological proper time and that it accelerates with respect to t, its second derivative being in general non-nil. This means that if, sometime t_{i} in the past, an atomic clock was set to click at the same rate as the coordinate time, it would be advanced now with respect to the coordinate time. In particular, the two time intervals would be different now, since $d\tau = e^{\Psi(t_0)}\, dt$. If there is only matter, be it ordinary or dark, then $\Psi(t_0) < 0$ and $d\tau < dt$. On the other hand, if the dark energy exists, then $\Psi(t_0)$ can be positive and then $d\tau > dt$. This second possibility is the one that actually happens: the background potential not only increases, but is positive also because of the effect of the quantum vacuum, or the dark energy, as is shown in Sec. 4.

However, in the actual measurements these differences between dt and $d\tau$ do not occur, since both times are based now on the international second, defined with reference to the period of a transition of the cesium atom. This means that a small interval of the atomic clock time and of the coordinate time are equal, $d\tau_{\rm clocks} = dt$, at precisely $t = t_0$. Therefore, the time of the atomic clocks has been renormalized, indeed, multiplying the cosmological proper time by the constant scale factor $e^{-\Psi(t_0)}$. This argument shows that the time really measured by the atomic clocks is

$$d\tau_{\rm clocks} = e^{[\Psi(t)-\Psi(t_0)]}\,dt \quad \text{so that} \quad \left.\frac{d^2\tau_{\rm clocks}}{dt^2}\right|_{t_0} = \dot\Psi(t_0) = 2a_{\rm t}. \tag{29}$$

This explains the real meaning of $a_{\rm t}$: *it is one-half the second derivative with respect to the coordinate time t of the time of the atomic clocks, if they have been renormalized by a multiplicative factor to tick now, just at time t_0, at the same rate as the coordinate time* (in order for both to use the same definition of second). Of course, in the future the atomic clocks will advance over t, i.e. $d\tau_{\rm clocks} > dt$ if $t > t_0$.

Taking the first-order approximations near t_0, the intervals of both times are related as $d\tau_{\rm clocks} = [1 + \Psi(t) - \Psi(t_0)]\,dt$. This is equal to $d\tau_{\rm clocks} = [1 + \dot\Psi(t_0)(t - t_0)]\,dt = [1 + 2a_{\rm t}(t - t_0)]\,dt$ (compare with (2)). It follows:

$$\tau_{\rm clocks} - \tau_{\rm clocks}(t_0) = (t - t_0) + a_{\rm t}(t - t_0)^2, \tag{30}$$

what reminds the quadratic in time model tried by Anderson *et al.* to solve the anomaly. Probably, this explains why this was the best among the other models they used with phenomenologically distorted time (see Sec. 1.1 at the end).

In order to refer to the time of the clocks, $a_{\rm t}$ must be multiplied by the factor $dt/d\tau_{\rm clocks}$ what produces the change

$$a_{\rm t} \Rightarrow a_\tau = a_{\rm t}\frac{dt}{d\tau_{\rm clocks}} = a_{\rm t}[1 - 2a_{\rm t}(t - t_0)] = a_{\rm t}\,, \tag{31}$$

neglecting the terms of second order in $a_{\rm t}$. Since $a_{\rm t} = a_\tau$, at first order, Eq. (2) can be written as

$$\nu = \nu_0\{1 + 2a_\tau[\tau_{\rm clocks} - \tau_{\rm clocks}(t_0)]\} \tag{32}$$

at first order, what states that the frequency must drift, according to the measurements made with atomic clocks. This gives a solution to the riddle.

Indeed the two derivatives of the background potential Ψ with respect to the times t and $\tau_{\rm clocks}$ are equal because the intervals of the two times are equal at precisely the time t_0 (or $\tau_{\text{clocks}}(t_0)$). Consequently, the acceleration of the clocks can be calculated as one-half the derivative of Ψ with respect to any one of the two times (at first order). Therefore, this model seems to offer an attractive solution to the anomaly. But in order to test it, its prediction for the inverse time a_t must be calculated and compared with the value proposed by Anderson *et al.* This is done in next section.

4. ESTIMATION OF THE ACCELERATION OF THE CLOCKS

Unfortunately, a rigorous calculation of a_t that would take into account all the eventual effects, is not easy. However, a simple crude estimate of its value can indeed be made, as is done in this section, taking (17) and (19) as starting point. Although involving approximations and simplifications, the result is meaningful since it shows the main ideas of the model, giving a convincing representation of the phenomenon. In order to do that, the shape of the function $\Psi(t)$ near t_0 must be determined first to calculate its time-derivative Eq. (19). The potential of all the universe at the terrestrial laboratory R can be written, with good approximation, as $\Phi_{\text{all}} = \Phi_{\text{loc}}(R) + \Phi_{\text{av}}(t)$. The first term $\Phi_{\text{loc}}(R)$ is the part due to the local inhomogeneities, i.e. the nearby bodies (the Solar System and the Milky Way). It is constant in time since these objects are not expanding. The second $\Phi_{\text{av}}(t)$ is the space averaged potential due to all the mass and energy in the universe (except for the nearby bodies), assuming that they are uniformly distributed. Contrary to the first, it depends on time because of the expansion. The former has a non-vanishing gradient but is small, the latter is space independent, but time dependent and much larger. The value of Φ_{loc}/c_0^2 at the laboratory R is the sum of the effects of the Earth, the Sun and the Milky Way, which are about -7×10^{-10}, -10^{-8} and -6×10^{-7}, respectively, certainly with much smaller absolute values than $\Phi_{\text{av}}(t_0)$, of the order of -10^{-1} as will be seen below.

Since the background gravitational potential of all the universe $\Phi_{\text{av}}(t)$ is increasing because of the expansion (the galaxies are separating and their interaction potential increasing), Eq. (19) imply that a_t and a_ℓ are both positive, as will be seen in the following. In this sense, there is a non-proper acceleration of light $a_\ell = \dot{c}$ (see Sec. 1.3) (as it may be convenient to stress again, all this is compatible with the constancy of the proper speed of light $c^* = d\ell/d\tau$).

Let $\Omega_M = 0.27$, $\Omega_\Lambda = 0.73$ be the corresponding present time relative densities of matter (ordinary plus dark) and dark energy corresponding to the cosmological constant Λ. We take a universe with $k = 0$ and Hubble parameter $H_0 = 71\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 2.3 \times 10^{-18}\,\mathrm{s^{-1}}$. In order to determine the average potential $\Phi_{\mathrm{av}}(t)$, let $\Phi_0(t_0)$ be the gravitational potential produced by the critical density of mass distributed up to the present radius of the visible universe $R_{\mathrm{U}}(t_0) = c_0/H_0 = 4200\,\mathrm{Mpc}$. One has then $\Phi_0(t_0)/c_0^2 = -\int_0^{c_0/H_0} c_0^{-2} G\rho_{\mathrm{cr}} 4\pi r\,dr = 2\pi G\rho_{\mathrm{cr}}/H_0^2 \simeq -0.78$. It must be emphasized that, although this value of the potential might seem to be too large for this approximation to apply, there is no problem in fact since it is space independent and its time derivative is extremely small. It will be absorbed in the redefinition of time, its effect being only to accelerate adiabatically the proper time with respect to the coordinate time, i.e. to accelerate the clocks, precisely what we want to investigate.

Because the radius of the universe is changing, the potential must be multiplied by the factor $[R_{\mathrm{U}}(t)/R_{\mathrm{U}}(t_0)]^2$, with $\dot{R}_{\mathrm{U}}(t_0) = c_0$. It turns out then that $\Phi_0(t)/c_0^2 = 2\pi G\rho_{\mathrm{cr}} R_{\mathrm{U}}^2(t)/c_0^2$.

The present time value of the background potential is then $\Phi_{\mathrm{av}}(t_0) = \Phi_0(t_0)(\Omega_{\mathrm{M}} - 2\Omega_\Lambda) \simeq 0.92 > 0$ (it is positive because of the contribution of the cosmological constant). Because of the expansion of the universe, the gravitational potentials due to matter and dark energy equivalent to the cosmological constant vary in time as the inverse of the scale factor $R(t)$ and as its square $R^2(t)$, respectively (with $R(t) = (\Omega_{\mathrm{M}}/\Omega_\Lambda)^{1/3}\sinh^{2/3}\left[(3\Lambda)^{1/2}t/2\right]$ for this model universe). This implies that the average background gravitational potential can be expressed as

$$\Phi_{\mathrm{av}}(t) = \Phi_0(t_0)\left(\frac{R_{\mathrm{U}}(t)}{R_{\mathrm{U}}(t_0)}\right)^2\left[\frac{\Omega_{\mathrm{M}}}{R(t)} - 2\Omega_\Lambda R^2(t)\right] \tag{33}$$

(remember: $\Psi(t) = \Phi_{\mathrm{av}}(t)/c^2(t)$). Note that, as announced in Sec. 1.3 and since $\Phi_0(t_0) < 0$, the term in Ω_Λ overcomes necessarily the one in Ω_{M}, after a certain time, the potential being positive afterwards, as far as the expansion goes on. The non-proper speed of light $c(t)$ is then larger than c^*. As is easy to see, $\Phi_{\mathrm{av}}(t) \to 0$ at time zero, because $R_{\mathrm{U}}(t) \sim t^2$ in that limit. After a bit of simple algebra, the inverse time $2a_{\mathrm{t}} = \dot{\Psi}(t_0)$ (19) can be expressed as

$$a_{\mathrm{t}} = \frac{H_0}{2}\frac{(1 - 9\Omega_\Lambda)\Phi_0/c_0^2}{1 + 2(1 - 3\Omega_\Lambda)\Phi_0/c_0^2}. \tag{34}$$

Introducing in this equation the values of Ω_Λ and Φ_0, the acceleration of the clocks a_{t} turns out to be

$$a_{\mathrm{t}} \simeq 0.8H_0 \simeq 1.8 \times 10^{-18}\,\mathrm{s^{-1}}. \tag{35}$$

This is to be compared with the value suggested by Anderson *et al.* on the basis of their data $a_t = (2.9 \pm 0.3) \times 10^{-18}\,\text{s}^{-1}$. Taking into account the simplicity of the calculation and the approximations involved, the prediction of this model can be considered to be acceptable. This is encouraging.

4.1. An Intuitive and Phenomenological Understanding of the Phenomenon

Let a photon with frequency $h\nu_0$ travel across the space at time τ_0. Its energy is $h\nu_0$. If the time runs from τ_0 until τ, when the gravitational potential is $\Psi(\tau)$, it will pick up potential energy, its total energy being then $h\nu(\tau) = h\nu_0[1 + \Psi(\tau) - \Psi(\tau_0)]$. It will be seen as having a frequency $\nu(\tau) = \nu_0[1 + \Psi(\tau) - \Psi(\tau_0)]$. The derivative of the frequency with respect to the time of the atomic clocks will be $2a_\tau = d\Psi/d\tau_{\text{clocks}} > 0$. In other words, it must be expected that the frequency of the photons will increase adiabatically, because of the expansion of the universe. This means a small blueshift, just what was observed. Since c^* is constant, the wavelengths much decrease accordingly. That is the Pioneer anomaly.

5. SUMMARY AND CONCLUSIONS

The model here presented suggests an explanation of the Pioneer anomaly that is simple and based in standard physical ideas: *the acceleration a_P is not related to any anomalous or unmodelled motion of the spaceships. Instead, it is an effect of the increasing background gravitational potential that pervades the universe and produces an acceleration $2a_t$ (= $2a_{p/c}$) of the time of the clocks τ_{clocks} with respect to the coordinate time t*, i.e. $2a_t = d^2\tau_{\text{clocks}}/dt^2\big|_{t_0}$, t_0 being the present time (remember that the time coordinate was so chosen as to go to the Newtonian time in the Newtonian limit). This acceleration of the clocks a_t is also equal, in this model, to the time derivative of the background gravitational potential $\Psi(t)$, i.e. $2a_t = d\Psi/dt|_{t_0} = d\Psi/d\tau_{\text{clocks}}|_{t_0}$ (note that $d\tau_{\text{clock}}/dt = 1$, at present time t_0 but not before or after). The anomaly would be thus an interesting case of the dynamics of time.[19,20] A further comment: as it might be worth to point out, this model is similar to the explanation by Mach of the origin of the inertia.

According to this model, the anomaly is a manifestation of the expansion of the universe, which causes the increase of the background potential $\Psi(t)$. This increase, in its turn, accelerates the cosmological time and causes the acceleration $2a_t$. A simple estimation gives a good agreement with the value proposed, on the basis of their observations, by Anderson *et al.* the discoverers of the anomaly (see Eq. (35)).

The model complies with the principles of general relativity, particularly because the proper speed of light (i.e. $c^* = d\ell/d\tau$, τ being the cosmological proper time) is a universal constant. However, what is here called the non-proper speed of light (i.e. $c(t) = d\ell/dt$, t being the coordinate time) is not constant. This is standard. If there is only matter, be it ordinary or dark, then $c(t) < c^*$. On the other hand, it happens that $c(t)$ can accelerate until being larger than the proper light speed c^*, *if and only if there is cosmological constant (or any other form of dark energy that implies repulsion)*. This is unusual and might seem perplexing at first sight, but it does not imply any contradiction as far as c^* is still constant. It would be just another unexpected effect of the dark energy, in addition to accelerate the universe.

The Pioneer anomaly poses a most intriguing riddle for physics and very complex and difficult problems for metrology. Taking this into account, the main conclusion of this paper is that the ideas here presented should be considered by the experts who know the details of the motion of the spacecrafts and of the metrological procedures involved in the observation.

ACKNOWLEDGEMENTS

I am indebted to A. Tiemblo for information on the dynamics of time, to J. L. Sebastián for explanations on microwave detection and to C. Aroca, A. I. Gómez de Castro, E. López and J. L. Rosales for discussions or encouragement. This research was partially supported by Grant number BFM03-5453 of the Spanish Ministry of Science and Technology.

REFERENCES

1. J. D. Anderson, Ph. A. Laing, E. L. Lau, A. S. Liu, M. Martin Nieto, and S. G. Turyshev, *Phys. Rev. Lett.* **81**, 2858 (1998).
2. J. D. Anderson, Ph. A. Laing, E. L. Lau, A. S. Liu, M. Martin Nieto, and S. G. Turyshev, *Phys. Rev. D* **65**, 082004 (2002).
3. J. D. Anderson, E. L. Lau, S. G. Turyshev, Ph. A. Laing, and M. Martin Nieto, *Mod. Phys. Lett. A* **17**, 875 (2002).
4. J. P. Mbelek and M. Michalski, *Int. J. Mod. Phys. D* **13**, 865 (2004).
5. A. F. Rañada, astro-phys/0202224, *Europhys. Lett.* **61**, 174 (2003).
6. A. F. Rañada, gr-qc/0211052, *Europhys. Lett.* **63**, 653 (2003)
7. A. F. Rañada, *Int J. Mod. Phys. D* **12**, 1755 (2003).
8. J. L. Rosales and J. L. Sánchez-Gómez, gr-qc/9810085.
9. J. L. Rosales, gr-qc/0401014.
10. G. F. R. Ellis and J. Ph. Uzan, gr-qc/0305099 v2.

11. C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
12. M. V. Berry, *Principles of Cosmology and Gravitation* (Adam Hilger-Institute of Physics Publishing, Bristol, 1989).
13. C. M. Will, *Theory and Experiment in Gravitational Physics*, Revised edn. (Cambridge University Press, Cambridge, 1993).
14. W. Rindler, *Relativity. Special, General and Cosmological* (Oxford University Press, Oxford, 2001).
15. L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, 4th revised English edn. (Pergamon Press, Oxford, 1975), chapter 10.
16. A. Einstein, *Annalen der Physik* **35**, 898 (1911); reprinted in *The Collected Papers of Albert Einstein*, English translation, Vol. 3 (Princeton University Press, 1993), p. 379.
17. A. Pais, *Subtle is the Lord. The Science and the Life of Albert Einstein* (Oxford University Press, Oxford, 1984).
18. J. M. Sánchez Ron, *El origen y desarrollo de la relatividad* (Alianza Editorial, Madrid, 1985).
19. A. Tiemblo and R. Tresguerres, *Gen. Rel. Grav.* **34**, 31 (2002).
20. J. F. Barbero, A. Tiemblo, and R. Tresguerres, *Phys. Rev. D* **57**, 6104 (1998).