Fuzzy Optimization and Decision Making (2024) 23:199–217
https://doi.org/10.1007/s10700-023-09417-3

# Partial derivatives of uncertain fields and uncertain partial differential equations

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Accepted: 12 October 2023 / Published online: 20 November 2023
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**Abstract**
Multivariate uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain fields based on uncertainty theory. This paper defines partial derivatives of uncertain fields for the first time by putting forward the concept of Liu field. Then the fundamental theorem, chain rule and integration by parts of multivariate uncertain calculus are derived. Finally, this paper presents an uncertain partial differential equation, and gives its integral form.

**Keywords** Uncertainty theory · Uncertain calculus · Uncertain process · Uncertain field · Partial derivative

## 1 Introduction

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes based on uncertainty theory founded by Liu (2007) and perfected by Liu (2009). As the basics of uncertain calculus, Liu (2009) proposed Liu process which is a type of stationary independent increment process whose increments are normal uncertain variables. Following that, the arithmetic and geometric Liu processes were presented. In order to study the integral of uncertain processes with respect to Liu process, Liu integral was invented by Liu (2009). Then some properties of Liu integral, such as linearity, additivity with respect to integration region and integrability of sample-continuous uncertain processes, were proved by Liu (2009). In order to research the differential of uncertain processes, Chen and Ralescu (2013) presented a general Liu process, and defined the differential of the general Liu process. Later, the concept of general Liu process was revised by Ye (2021) via requiring its drift and diffusion to be sample-continuous. Furthermore, Ye (2021) proved that almost all sample

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paths of general Liu process are locally Lipschitz continuous. In order to facilitate the calculation of differential of uncertain processes, Liu (2009) proposed the fundamental theorem of uncertain calculus which was rigorously proved by Ye (2021). On this basis, Liu (2009) investigated the chain rule, the change of variables and the integration by parts. These work laid a theoretical foundation for uncertain differential equations.

Uncertain differential equation is a type of differential equation involving uncertain processes. In order to apply uncertain differential equations in practice, Liu and Liu (2022) proposed the method of moments to estimate the unknown parameters in an uncertain differential equation based on the concept of residual, and Ye and Liu (2023) used uncertain hypothesis test to judge whether the uncertain differential equation fits the observed data. Up to now, uncertain differential equations have many applications such as chemical reaction (Tang and Yang, 2021), electric circuit (Liu, 2021), pharmacokinetics (Liu and Yang, 2021), epidemic spread (Lio and Liu, 2021), software reliability (Liu et al., 2022), finance (Liu and Liu, 2022; Yang and Ke, 2023; Ye and Liu, 2023), birth rate (Ye and Zheng, 2023), and gas futures price (Mehrdoust et al., 2023).

Uncertain partial differential equation is a type of partial differential equation involving uncertain fields. Yang and Yao (2017) proposed the concept of uncertain partial differential equation for the first time when they studied the one-dimensional uncertain heat equation. Following that, the three-dimensional uncertain heat equation (Ye and Yang, 2022) and its application (Ye, 2023) were further studied. In addition, Gao and Ralescu (2019) investigated the uncertain wave equation which is a second-order partial differential equation describing the wave propagation. Furthermore, Yang et al. (2022) deduced the uncertain seepage equation to describe the phenomenon of liquid seepage in fissured porous media. Recently, Yang and Liu (2023) studied the solution method and parameter estimation of uncertain partial differential equation.

This paper aims to study some fundamental theoretical problems of multivariate uncertain calculus, including the concept of partial derivative of uncertain fields, the fundamental theorem, and the integral form of uncertain partial differential equations. The remainder of the paper is organized as follows. Section 2 introduces some basic concepts and theorems of uncertain processes and uncertain fields. Section 3 proposes the concept of Liu field to show the partial derivative and the differential of Liu fields. Section 4 deduces the fundamental theorem of multivariate uncertain calculus from which the techniques of chain rule and integration by parts are derived in Sects. 5 and 6, respectively. On these bases, Sect. 7 presents the uncertain partial differential equation whose integral form is also given. Finally, some conclusions are made in Sect. 8.

## 2 Preliminaries

In this section, we introduce some basic concepts and theorems about uncertain processes and uncertain fields.

**Definition 1** (Liu, 2008) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let $T$ be a totally ordered set. An uncertain process is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set $B$ of real numbers at each time $t$.

We call an uncertain process $X_t$ independent increment process if $X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_k} - X_{t_{k-1}}$ are independent uncertain variables where $t_1$, $t_2, \cdots, t_k$ are any times with $t_1 < t_2 < \cdots < t_k$. An uncertain process $X_t$ is said to have stationary increments if, for any given $t > 0$, the increments $X_{t+s} - X_t$ are identically distributed uncertain variables for all $s > 0$.

**Definition 2** (Liu, 2009) An uncertain process $C_t$ is said to be a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) $C_t$ has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$.

The uncertainty distribution of $C_t$ is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \frac{\sqrt{3}t}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

**Theorem 1** (Liu, 2015) *Let $C_t$ be a Liu process. Then for each time $t > 0$, the ratio $C_t/t$ is a normal uncertain variable with expected value 0 and variabce 1. That is,*

$$\frac{C_t}{t} \sim \mathcal{N}(0, 1)$$

*for any $t > 0$.*

**Definition 3** (Liu, 2009) Let $X_t$ be an uncertain process and let $C_t$ be a Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then Liu integral of $X_t$ with respect to $C_t$ is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process $X_t$ is said to be integrable.

**Definition 4** (Chen and Ralescu, 2013; Ye, 2021) Let $C_t$ be a Liu process, and let $Z_t$ be an uncertain process. If there exist two sample-continuous uncertain processes $\mu_t$ and $\sigma_t$ such that

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}C_s$$

for any $t \geq 0$, then $Z_t$ is called a general Liu process with drift $\mu_t$ and diffusion $\sigma_t$. Furthermore, $Z_t$ has an uncertain differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

**Theorem 2** (*Liu*, 2009) *Let $C_t$ be a Liu process, and let $h(t, c)$ be a continuously differentiable function. Then $Z_t = h(t, C_t)$ has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

**Definition 5** (Liu, 2008) Suppose $f$ and $g$ are continuous functions, and $C_t$ is a Liu process. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, C_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation. A solution is an uncertain process $X_t$ that satisfies (1) identically in $t$.

**Definition 6** (Liu, 2014) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let $T$ be a partially ordered set. An uncertain field is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set $B$ of real numbers at each $t$.

## 3 Partial derivatives

**Definition 7** Let $C_t$ and $D_x$ be Liu processes indexed by temporal variable $t$ and spatial variable $x$ respectively, and let $Z(t, x)$ be an uncertain field. If there exist some sample-continuous uncertain fields $\mu_1(t,x)$, $\sigma_1(t,x)$, $\mu_2(t,x)$ and $\sigma_2(t,x)$ such that

$$Z(t,x) = Z(t_0,x_0) + \int_{t_0}^{t} \mu_1(s,x_0)\mathrm{d}s + \int_{t_0}^{t} \sigma_1(s,x_0)\mathrm{d}C_s + \int_{x_0}^{x} \mu_2(t,y)\mathrm{d}y + \int_{x_0}^{x} \sigma_2(t,y)\mathrm{d}D_y \tag{2}$$

for any temporal variable $t \geq 0$ and any spatial variable $x \geq 0$, then $Z(t, x)$ is called a Liu field with drifts $\mu_1(t,x), \mu_2(t,x)$ and diffusions $\sigma_1(t,x), \sigma_2(t,x)$. Furthermore, $Z(t, x)$ has an uncertain differential

$$\mathrm{d}Z(t,x) = \mu_1(t,x)\mathrm{d}t + \sigma_1(t,x)\mathrm{d}C_t + \mu_2(t,x)\mathrm{d}x + \sigma_2(t,x)\mathrm{d}D_x, \tag{3}$$

and uncertain partial derivatives

$$\frac{\partial Z}{\partial t}(t,x) = \mu_1(t,x) + \sigma_1(t,x)\dot{C}_t, \quad \frac{\partial Z}{\partial x}(t,x) = \mu_2(t,x) + \sigma_2(t,x)\dot{D}_x \tag{4}$$

where $\dot{C}_t$ is the formal derivative $\mathrm{d}C_t/\mathrm{d}t$, and $\dot{D}_x$ is the formal derivative $\mathrm{d}D_x/\mathrm{d}x$. The uncertain differential (3) can be written as

$$\mathrm{d}Z(t,x) = \frac{\partial Z}{\partial t}(t,x)\mathrm{d}t + \frac{\partial Z}{\partial x}(t,x)\mathrm{d}x.$$

***Remark 1*** Based on the defined partial derivatives (4), we can write the Liu field $Z(t, x)$ in (2) as

$$Z(t,x) = Z(t_0,x_0) + \int_{t_0}^{t} \frac{\partial Z}{\partial t}(s,x_0)\mathrm{d}s + \int_{x_0}^{x} \frac{\partial Z}{\partial x}(t,y)\mathrm{d}y.$$

***Example 1*** It follows from

$$C_t + D_x = \int_0^t \mathrm{d}C_s + \int_0^x \mathrm{d}D_y$$

that $Z(t,x) = C_t + D_x$ is a Liu field, and has an uncertain differential

$$\mathrm{d}Z(t,x) = \mathrm{d}C_t + \mathrm{d}D_x$$

and uncertain partial derivatives

$$\frac{\partial Z}{\partial t}(t,x) = \dot{C}_t, \quad \frac{\partial Z}{\partial x}(t,x) = \dot{D}_x.$$

***Example 2*** It follows from

$$tC_t + D_x^2 = \int_0^t C_s\mathrm{d}s + \int_0^t s\mathrm{d}C_s + \int_0^x 2D_y\mathrm{d}D_y$$

that $Z(t,x) = tC_t + D_x^2$ is a Liu field, and has an uncertain differential

$$\mathrm{d}Z(t,x) = C_t\mathrm{d}t + t\mathrm{d}C_t + 2D_x\mathrm{d}D_x$$

and uncertain partial derivatives

$$\frac{\partial Z}{\partial t}(t,x) = C_t + t\dot{C}_t, \quad \frac{\partial Z}{\partial x}(t,x) = 2D_x\dot{D}_x.$$

## 4 Fundamental theorem of multivariate uncertain calculus

**Theorem 3** *Let $C_t$ and $D_x$ be Liu processes indexed by temporal variable $t$ and spatial variable $x$ respectively, and let $h(t, x, c, d)$ be a continuously differentiable function. Then $Z(t,x) = h(t,x,C_t,D_x)$ has uncertain partial derivatives*

$$\frac{\partial Z}{\partial t}(t,x) = \frac{\partial h}{\partial t}(t,x,C_t,D_x) + \frac{\partial h}{\partial c}(t,x,C_t,D_x)\dot{C}_t,$$
$$\frac{\partial Z}{\partial x}(t,x) = \frac{\partial h}{\partial x}(t,x,C_t,D_x) + \frac{\partial h}{\partial d}(t,x,C_t,D_x)\dot{D}_x$$

*and an uncertain differential*

$$\mathrm{d}Z(t,x) = \frac{\partial Z}{\partial t}(t,x)\mathrm{d}t + \frac{\partial Z}{\partial x}(t,x)\mathrm{d}x.$$

***Proof*** By using Theorem 2, we obtain

$$\begin{aligned} Z(t,x) &= Z(t_0,x_0) + Z(t,x_0) - Z(t_0,x_0) + Z(t,x) - Z(t,x_0) \\ &= Z(t_0,x_0) + \int_{t_0}^{t} \frac{\partial h}{\partial s}(s,x_0,C_s,D_{x_0})\mathrm{d}s + \int_{t_0}^{t} \frac{\partial h}{\partial c}(s,x_0,C_s,D_{x_0})\mathrm{d}C_s \\ &\quad + \int_{x_0}^{x} \frac{\partial h}{\partial y}(t,y,C_t,D_y)\mathrm{d}y + \int_{x_0}^{x} \frac{\partial h}{\partial d}(t,y,C_t,D_y)\mathrm{d}D_y. \end{aligned}$$

Thus, it follows from Definition 7 that the theorem is proved immediately. □

***Remark 2*** Let $C_t, D_{x_1}, D_{x_2}, \cdots, D_{x_n}$ be Liu processes, and let $h(t, x_1, \cdots, x_n, c, d_1, \cdots, d_n)$ be a continuously differentiable function. Then it can be proved that

$$Z(t,x_1,\cdots,x_n) = h(t,x_1,\cdots,x_n,C_t,D_{x_1},\cdots,D_{x_n})$$

has uncertain partial derivatives

$$\begin{aligned} \frac{\partial Z}{\partial t}(t,x_1,\cdots,x_n) &= \frac{\partial h}{\partial t}(t,x_1,\cdots,x_n,C_t,D_{x_1},\cdots,D_{x_n}) \\ &\quad + \frac{\partial h}{\partial c}(t,x_1,\cdots,x_n,C_t,D_{x_1},\cdots,D_{x_n})\dot{C}_t, \\ \frac{\partial Z}{\partial x_i}(t,x_1,\cdots,x_n) &= \frac{\partial h}{\partial x_i}(t,x_1,\cdots,x_n,C_t,D_{x_1},\cdots,D_{x_n}) \\ &\quad + \frac{\partial h}{\partial d_i}(t,x_1,\cdots,x_n,C_t,D_{x_1},\cdots,D_{x_n})\dot{D}_{x_i} \end{aligned}$$

for $i = 1, 2, \cdots, n$ and an uncertain differential

$$\mathrm{d}Z(t,x_1,\cdots,x_n)=\frac{\partial Z}{\partial t}(t,x_1,\cdots,x_n)\mathrm{d}t+\sum_{i=1}^{n}\frac{\partial Z}{\partial x_i}(t,x_1,\cdots,x_n)\mathrm{d}x_i.$$

***Remark 3*** Let $Z_1(t,x_1,\cdots,x_n), Z_2(t,x_1,\cdots,x_n),\cdots,Z_m(t,x_1,\cdots,x_n)$ be Liu fields, and let $h(z_1,\cdots,z_m)$ be a continuously differentiable function. Then it can be proved that

$$Z(t,x_1,\cdots,x_n)=h(Z_1,\cdots,Z_n)$$

has uncertain partial derivatives

$$\frac{\partial Z}{\partial t}(t,x_1,\cdots,x_n)=\sum_{i=1}^{n}\frac{\partial h}{\partial z_i}(Z_1,\cdots,Z_n)\cdot\frac{\partial Z_i}{\partial t}(t,x_1,\cdots,x_n)$$
$$\frac{\partial Z}{\partial x_i}(t,x_1,\cdots,x_n)=\sum_{i=1}^{n}\frac{\partial h}{\partial z_i}(Z_1,\cdots,Z_n)\cdot\frac{\partial Z_i}{\partial x_i}(t,x_1,\cdots,x_n)$$

for $i=1,2,\cdots,n$ and an uncertain differential

$$\mathrm{d}Z(t,x_1,\cdots,x_n)=\sum_{i=1}^{n}\frac{\partial h}{\partial z_i}(t,Z_1,\cdots,Z_n)\mathrm{d}Z_i(t,x_1,\cdots,x_n).$$

***Example 3*** Let us calculate uncertain partial derivatives of $xC_t+tD_x$. In this case, we have $h(t,x,c,d)=xc+td$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,x,c,d)=d,\quad \frac{\partial h}{\partial x}(t,x,c,d)=c,\quad \frac{\partial h}{\partial c}(t,x,c,d)=x,\quad \frac{\partial h}{\partial d}(t,x,c,d)=t.$$

It follows from the fundamental theorem that

$$\frac{\partial}{\partial t}(xC_t+tD_x)=D_x+x\dot{C}_t,\quad \frac{\partial}{\partial x}(xC_t+tD_x)=C_t+t\dot{D}_x.$$

***Example 4*** Let us calculate uncertain partial derivatives of $txC_t+txD_x$. In this case, we have $h(t,x,c,d)=txc+txd$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,x,c,d)=xc+xd,\quad \frac{\partial h}{\partial x}(t,x,c,d)=tc+td,$$
$$\frac{\partial h}{\partial c}(t,x,c,d)=tx,\quad \frac{\partial h}{\partial d}(t,x,c,d)=tx.$$

It follows from the fundamental theorem that

$$\frac{\partial}{\partial t}(txC_t+txD_x)=x(C_t+D_x)+tx\dot{C}_t,\quad \frac{\partial}{\partial x}(txC_t+txD_x)=t(C_t+D_x)+tx\dot{D}_x.$$

***Example 5*** Let us calculate uncertain partial derivatives of $A(t,x)=at+bx+\mu C_t+\sigma D_x$. In this case, we have $h(t,x,c,d)=at+bx+\mu c+\sigma d$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t,x,c,d)=a,\quad \frac{\partial h}{\partial x}(t,x,c,d)=b,\quad \frac{\partial h}{\partial c}(t,x,c,d)=\mu,\quad \frac{\partial h}{\partial d}(t,x,c,d)=\sigma.$$

It follows from the fundamental theorem that

$$\frac{\partial A}{\partial t}(t,x)=a+\mu\dot{C}_t,\quad \frac{\partial A}{\partial x}(t,x)=b+\sigma\dot{D}_x.$$

***Example* 6** Let us calculate uncertain partial derivatives of $G(t,x)=\exp(at+bx+\mu C_t+\sigma D_x)$. In this case, we have $h(t,x,c,d)=\exp(at+bx+\mu c+\sigma d)$ whose partial derivatives are

$$\begin{aligned}
&\frac{\partial h}{\partial t}(t,x,c,d)=ah(t,x,c,d), && \frac{\partial h}{\partial x}(t,x,c,d)=bh(t,x,c,d),\\
&\frac{\partial h}{\partial c}(t,x,c,d)=\mu h(t,x,c,d), && \frac{\partial h}{\partial d}(t,x,c,d)=\sigma h(t,x,c,d).
\end{aligned}$$

It follows from the fundamental theorem that

$$\frac{\partial G}{\partial t}(t,x)=aG(t,x)+\mu G(t,x)\dot{C}_t,\quad \frac{\partial G}{\partial x}(t,x)=bG(t,x)+\sigma G(t,x)\dot{D}_x.$$

## 5 Chain rule

**Theorem 4** *Let $f(c,d)$ be a continuously differentiable function. Then $Z(t,x)=f(C_t,D_x)$ has uncertain partial derivatives*

$$\frac{\partial Z}{\partial t}(t,x)=\frac{\partial f}{\partial c}(C_t,D_x)\dot{C}_t,\quad \frac{\partial Z}{\partial x}(t,x)=\frac{\partial f}{\partial d}(C_t,D_x)\dot{D}_x.$$

***Proof*** It follows from Theorem 3 immediately. □

***Example 7*** Let us calculate uncertain partial derivatives of $\sin(C_t+D_x)$. In this case, we have $f(c,d)=\sin(c+d)$ whose partial derivatives are

$$\frac{\partial f}{\partial c}(c,d)=\cos(c+d),\quad \frac{\partial f}{\partial d}(c,d)=\cos(c+d).$$

It follows from the chain rule that

$$\frac{\partial\sin(C_t+D_x)}{\partial t}=\cos(C_t+D_x)\dot{C}_t,\quad \frac{\partial\sin(C_t+D_x)}{\partial x}=\cos(C_t+D_x)\dot{D}_x.$$

***Example 8*** Let us calculate uncertain partial derivatives of $\sin(C_tD_x)$. In this case, we have $f(c,d)=\sin(cd)$ whose partial derivatives are

$$\frac{\partial f}{\partial c}(c,d)=d\cos(cd),\quad \frac{\partial f}{\partial d}(c,d)=c\cos(cd).$$

It follows from the chain rule that

$$\frac{\partial \sin(C_t D_x)}{\partial t} = D_x \cos(C_t D_x)\dot{C}_t, \quad \frac{\partial \sin(C_t D_x)}{\partial x} = C_t \cos(C_t D_x)\dot{D}_x.$$

***Example 9*** Let us calculate uncertain partial derivatives of $(C_t + D_x)^2$. In this case, we have $f(c,d) = (c+d)^2$ whose partial derivatives are

$$\frac{\partial f}{\partial c}(c,d) = 2(c+d), \quad \frac{\partial f}{\partial d}(c,d) = 2(c+d).$$

It follows from the chain rule that

$$\frac{\partial (C_t + D_x)^2}{\partial t} = 2(C_t + D_x)\dot{C}_t, \quad \frac{\partial (C_t + D_x)^2}{\partial x} = 2(C_t + D_x)\dot{D}_x.$$

## 6 Integration by parts

**Theorem 5** *Suppose $Z_1(t,x)$ and $Z_2(t,x)$ are Liu fields. Then we have*

$$\mathrm{d}(Z_1 Z_2) = Z_2 \mathrm{d}Z_1 + Z_1 \mathrm{d}Z_2,$$

*and*

$$\frac{\partial (Z_1 Z_2)}{\partial t}(t,x) = Z_1 \frac{\partial Z_2}{\partial t}(t,x) + Z_2 \frac{\partial Z_1}{\partial t}(t,x),$$
$$\frac{\partial (Z_1 Z_2)}{\partial x}(t,x) = Z_1 \frac{\partial Z_2}{\partial x}(t,x) + Z_2 \frac{\partial Z_1}{\partial x}(t,x).$$

***Proof*** Since $h(z_1, z_2) = z_1 z_2$ is a continuously differentiable function, and

$$\frac{\partial h}{\partial z_1}(z_1, z_2) = z_2, \quad \frac{\partial h}{\partial z_2}(z_1, z_2) = z_1,$$

the theorem follows from Remark 3 immediately. □

***Example 10*** It follows from the integration by parts that

$$\mathrm{d}(C_t D_x) = D_x \mathrm{d}C_t + C_t \mathrm{d}D_x$$

and

$$\frac{\partial (C_t D_x)}{\partial t} = D_x \dot{C}_t, \quad \frac{\partial (C_t D_x)}{\partial x} = C_t \dot{D}_x.$$

***Example 11*** The integration by parts may calculate the uncertain differential and uncertain partial derivatives of

$$Z(t,x) = txC_t D_x.$$

In this case, we define

$$Z_1(t,x) = tC_t, \quad Z_2(t,x) = xD_x.$$

Then

$$\mathrm{d}Z_1(t,x) = C_t\mathrm{d}t + t\mathrm{d}C_t, \quad \mathrm{d}Z_2(t,x) = D_x\mathrm{d}x + x\mathrm{d}D_x.$$

It follows from the integration by parts that

$$\begin{aligned}\mathrm{d}Z(t,x) &= xD_x(C_t\mathrm{d}t + t\mathrm{d}C_t) + tC_t(D_x\mathrm{d}x + x\mathrm{d}D_x)\\ &= xC_tD_x\mathrm{d}t + txD_x\mathrm{d}C_t + tC_tD_x\mathrm{d}x + txC_t\mathrm{d}D_x\end{aligned}$$

and

$$\frac{\partial Z}{\partial t}(t,x) = xC_tD_x + txD_x\dot{C}_t, \quad \frac{\partial Z}{\partial x}(t,x) = tC_tD_x + txC_t\dot{D}_x.$$

***Example 12*** The integration by parts may calculate the uncertain differential and uncertain partial derivatives of

$$Z(t,x) = \left(\int_0^t s\mathrm{d}C_s\right)\cdot\left(\int_0^x \exp(D_y)\mathrm{d}D_y\right).$$

In this case, we define

$$Z_1(t,x) = \int_0^t s\mathrm{d}C_s, \quad Z_2(t,x) = \int_0^x \exp(D_y)\mathrm{d}D_y.$$

Then

$$\mathrm{d}Z_1(t,x) = t\mathrm{d}C_t, \quad \mathrm{d}Z_2(t,x) = \exp(D_x)\mathrm{d}D_x.$$

It follows from the integration by parts that

$$\mathrm{d}Z(t,x) = t\left(\int_0^x \exp(D_y)\mathrm{d}D_y\right)\mathrm{d}C_t + \exp(D_x)\left(\int_0^t s\mathrm{d}C_s\right)\mathrm{d}D_x$$

and

$$\begin{aligned}\frac{\partial Z}{\partial t}(t,x) =& t\left(\int_0^x \exp(D_y)\mathrm{d}D_y\right)\dot{C}_t,\\ \frac{\partial Z}{\partial x}(t,x) =& \exp(D_x)\left(\int_0^t s\mathrm{d}C_s\right)\dot{D}_x.\end{aligned}$$

***Example 13*** Let $f$, $g$, $h$ and $v$ be continuously differentiable functions. It is clear that

$$Z(t,x) = f(t)g(C_t)h(x)v(D_x)$$

is an uncertain field. In order to calculate the uncertain differential and uncertain partial derivatives of $Z(t, x)$, we define

$$Z_1(t,x) = f(t)g(C_t), \quad Z_2(t,x) = h(x)v(D_x).$$

Then

$$\begin{aligned} \mathrm{d}Z_1(t,x) =& f'(t)g(C_t)\mathrm{d}t + f(t)g'(C_t)\mathrm{d}C_t, \\ \mathrm{d}Z_2(t,x) =& h'(x)v(D_x)\mathrm{d}x + h(x)v'(D_x)\mathrm{d}D_x. \end{aligned}$$

It follows from the integration by parts that

$$\begin{aligned} \mathrm{d}Z(t,x) = & h(x)v(D_x)[f'(t)g(C_t)\mathrm{d}t + f(t)g'(C_t)\mathrm{d}C_t] \\ & + f(t)g(C_t)[h'(x)v(D_x)\mathrm{d}x + h(x)v'(D_x)\mathrm{d}D_x] \\ = & f'(t)g(C_t)h(x)v(D_x)\mathrm{d}t + f(t)g'(C_t)h(x)v(D_x)\mathrm{d}C_t \\ & + f(t)g(C_t)h'(x)v(D_x)\mathrm{d}x + f(t)g(C_t)h(x)v'(D_x)\mathrm{d}D_x \end{aligned}$$

and

$$\begin{aligned} \frac{\partial Z}{\partial t}(t,x) =& f'(t)g(C_t)h(x)v(D_x) + f(t)g'(C_t)h(x)v(D_x)\dot{C}_t, \\ \frac{\partial Z}{\partial x}(t,x) =& f(t)g(C_t)h'(x)v(D_x) + f(t)g(C_t)h(x)v'(D_x)\dot{D}_x. \end{aligned}$$

## 7 Uncertain partial differential equation

**Definition 8** Suppose $f_1$, $f_2$, $g_1$ and $g_2$ are continuous functions, and $C_t$ and $D_x$ are Liu processes indexed by temporal variable $t$ and spatial variable $x$, respectively. Then

$$\frac{\partial Z}{\partial t} = f_1(t,x,Z)\frac{\partial Z}{\partial x} + f_2(t,x,Z) + g_1(t,x,Z)\dot{C}_t + g_2(t,x,Z)\dot{D}_x \tag{5}$$

is called an uncertain partial differential equation, where $\dot{C}_t$ is the formal derivative $\mathrm{d}C_t/\mathrm{d}t$, and $\dot{D}_x$ is the formal derivative $\mathrm{d}D_x/\mathrm{d}x$.

The solution of (5) is an uncertain field $Z(t, x)$ satisfying the following uncertain integral

$$\begin{aligned} Z(t,x) = Z(t_0,x_0) + & \int_{t_0}^{t} [f_1(s,x_0,Z)\mu(s,x_0) + f_2(s,x_0,Z)]\mathrm{d}s \\ + & \int_{t_0}^{t} g_1(s,x_0,Z)\mathrm{d}C_s + \int_{x_0}^{x} \mu(t,y)\mathrm{d}y - \int_{x_0}^{x} \frac{g_2(t,y,Z)}{f_1(t,y,Z)}\mathrm{d}D_y \end{aligned} \tag{6}$$

where $\mu(t,x)$ is a sample-continuous uncertain field, and $f_1$ is supposed to be never 0.

***Example 14*** Consider the uncertain partial differential equation

$$\frac{\partial Z}{\partial t} = k\frac{\partial Z}{\partial x} + a\dot{C}_t + b\dot{D}_x \tag{7}$$

where $k$, $a$ and $b$ are real numbers with $k \neq 0$.

First, we deduce a format solution of (7) by the method of characteristics. Indeed, the characteristic equation of (7) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k$$

whose solution is $x = -kt + r$ where $r$ is a constant. Write $x(t) = x = -kt + r$ and $z(t) = Z(t, x(t))$. Then $r = x(t) + kt$, and

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\partial Z}{\partial t} - k\frac{\partial Z}{\partial x} \\ &= a\dot{C}_t + b\dot{D}_x.\end{aligned}$$

Thus we have

$$z(t) = m + aC_t - \frac{b}{k}D_x$$

where $m$ is a constant. Then

$$z(0) = m - \frac{b}{k}D_{x(0)}.$$

Suppose $m = \phi(x(0)) = \phi(r)$ where $\phi$ is an arbitrarily continuously differential function. Thus

$$Z(t, x(t)) = z(t) = \phi(r) + aC_t - \frac{b}{k}D_x = \phi(x(t) + kt) + aC_t - \frac{b}{k}D_x.$$

Substituting $x(t)$ with $x$ obtains

$$Z(t, x) = \phi(x + kt) + aC_t - \frac{b}{k}D_x. \tag{8}$$

Second, we will verify that (8) is the solution of the uncertain partial differential Eq. (7). Write $\mu(t, x) = \phi'(x + kt)$. Since

$$\begin{aligned}&\phi(0) + \int_0^t k\phi'(ks)\mathrm{d}s + \int_0^t a\mathrm{d}C_s + \int_0^x \phi'(y + kt)\mathrm{d}y - \int_0^x \frac{b}{k}\mathrm{d}D_y \\ &= \phi(0) + \phi(kt) - \phi(0) + aC_t + \phi(x + kt) - \phi(kt) - \frac{b}{k}D_x \\ &= \phi(x + kt) + aC_t - \frac{b}{k}D_x \\ &= Z(t, x),\end{aligned}$$

it follows from (6) that

$$Z(t,x) = \phi(x+kt) + aC_t - \frac{b}{k}D_x$$

is indeed the solution of (7).

In addition, we also can use the method of computing partial derivatives to verify that (8) is the solution of the uncertain partial differential Eq. (7). It follows from the fundamental theorem that the uncertain partial derivatives of

$$Z(t,x) = \phi(x+kt) + aC_t - \frac{b}{k}D_x$$

are

$$\frac{\partial Z}{\partial t} = k\phi'(x+kt) + a\dot{C}_t, \quad \frac{\partial Z}{\partial x} = \phi'(x+kt) - \frac{b}{k}\dot{D}_x.$$

Thus

$$\begin{aligned} &\frac{\partial Z}{\partial t} - \left[k\frac{\partial Z}{\partial x} + a\dot{C}_t + b\dot{D}_x\right] \\ &= k\phi'(x+kt) + a\dot{C}_t - \left[k\left(\phi'(x+kt) - \frac{b}{k}\dot{D}_x\right) + a\dot{C}_t + b\dot{D}_x\right] \\ &= 0 \end{aligned}$$

which means the uncertain partial differential Eq. (7) i.e.,

$$\frac{\partial Z}{\partial t} = k\frac{\partial Z}{\partial x} + a\dot{C}_t + b\dot{D}_x$$

holds. Thus the solution of the uncertain partial differential Eq. (7) is

$$Z(t,x) = \phi(x+kt) + aC_t - \frac{b}{k}D_x$$

where $\phi$ is an arbitrarily continuously differential function. For example, when $k = a = b = 1$ and $\phi(x) = x^2$, the solution of the uncertain partial differential Eq. (7) becomes

$$Z(t,x) = (x+t)^2 + C_t - D_x.$$

***Example 15*** Consider the uncertain partial differential equation

$$\frac{\partial Z}{\partial t} = k\frac{\partial Z}{\partial x} + f(t,x) + a\dot{C}_t + b\dot{D}_x \tag{9}$$

where $f(t, x)$ is a a continuously differential function, and $k$, $a$ and $b$ are real numbers with $k \neq 0$.

The characteristic equation of (9) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k$$

whose solution is $x = -kt + r$ where $r$ is a constant. Write $x(t) = x = -kt + r$ and $z(t) = Z(t, x(t))$. Then $r = x(t) + kt$ and

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\partial Z}{\partial t} - k\frac{\partial Z}{\partial x} \\ &= f(t,x) + a\dot{C}_t + b\dot{D}_x.\end{aligned}$$

Thus we have

$$z(t) = m + \int_0^t f(s, x(s))\mathrm{d}s + aC_t - \frac{b}{k}D_x$$

where $m$ is a constant. Then

$$z(0) = m + \int_0^t f(s, x(s))\mathrm{d}s - \frac{b}{k}D_{x(0)}.$$

Suppose $m = \phi(x(0)) = \phi(r)$ where $\phi$ is an arbitrarily continuously differential function. Thus,

$$\begin{aligned}Z(t, x(t)) = z(t) = z(0) &+ \int_0^t f(s, x(s))\mathrm{d}s + aC_t - \frac{b}{k}D_x \\ &= \phi(r) + \int_0^t f(s, -ks + r)\mathrm{d}s + aC_t - \frac{b}{k}D_x \\ &= \phi(x(t) + kt) + \int_0^t f(s, -ks + x(t) + kt)\mathrm{d}s + aC_t - \frac{b}{k}D_x \\ &= \phi(x(t) + kt) + \int_0^t f(s, x(t) + k(t - s))\mathrm{d}s + aC_t - \frac{b}{k}D_x.\end{aligned}$$

Substituting $x(t)$ with $x$ obtains

$$Z(t, x) = \phi(x + kt) + \int_0^t f(s, x + k(t - s))\mathrm{d}s + aC_t - \frac{b}{k}D_x. \tag{10}$$

Second, we will verify that (10) is the solution of the uncertain partial differential equation (9). Write

$$\mu(t, x) = \phi'(x + kt) + \int_0^t \frac{\partial f}{\partial x}(s, x + k(t - s))\mathrm{d}s$$

where $\partial f/\partial x$ is the partial derivative of the function $f(t, x)$ with respect to the second variable $x$. Since

$$\begin{aligned}
&\phi(0)+\int_0^t\left[k\phi'(ks)+k\int_0^s\frac{\partial f}{\partial x}(\tau,k(s-\tau))\mathrm{d}\tau+f(0,s)\right]\mathrm{d}s+\int_0^t a\mathrm{d}C_s\\
&\quad+\int_0^x\left[\phi'(y+kt)+\int_0^t\frac{\partial f}{\partial x}(s,y+k(t-s))\mathrm{d}s\right]\mathrm{d}y-\int_0^x\frac{b}{k}\mathrm{d}D_y\\
&=\phi(0)+\left[\phi(kt)+\int_0^t f(s,k(t-s))\mathrm{d}s-\phi(0)\right]+aC_t-\frac{b}{k}D_x\\
&\quad+\left[\phi(x+kt)+\int_0^t f(s,x+k(t-s))\mathrm{d}s-\phi(kt)-\int_0^t f(s,k(t-s))\mathrm{d}s\right]\\
&=\phi(x+kt)+\int_0^t f(s,x+k(t-s))\mathrm{d}s+aC_t-\frac{b}{k}D_x\\
&=Z(t,x),
\end{aligned}$$

it follows from (6) that

$$Z(t,x)=\phi(x+kt)+\int_0^t f(s,x+k(t-s))\mathrm{d}s+aC_t-\frac{b}{k}D_x$$

is indeed the solution of (7).

In addition, we also can use the method of computing partial derivatives to verify that (10) is the solution of the uncertain partial differential equation (9). It follows from the fundamental theorem that the uncertain partial derivatives of

$$Z(t,x)=\phi(x+kt)+\int_0^t f(s,x+k(t-s))\mathrm{d}s+aC_t-\frac{b}{k}D_x$$

are

$$\begin{aligned}
\frac{\partial Z}{\partial t}&=k\phi'(x+kt)+k\int_0^t\frac{\partial f}{\partial x}(s,x+k(t-s))\mathrm{d}s+f(t,x)+a\dot{C}_t,\\
\frac{\partial Z}{\partial x}&=\phi'(x+kt)+\int_0^t\frac{\partial f}{\partial x}(s,x+k(t-s))\mathrm{d}s-\frac{b}{k}\dot{D}_x.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\partial Z}{\partial t}-\left[k\frac{\partial Z}{\partial x}+f(t,x)+a\dot{C}_t+b\dot{D}_x\right]\\
&=k\phi'(x+kt)+k\int_0^t\frac{\partial f}{\partial x}(s,x+k(t-s))\mathrm{d}s+f(t,x)+a\dot{C}_t\\
&\quad-k\left(\phi'(x+kt)+\int_0^t\frac{\partial f}{\partial x}(s,x+k(t-s))\mathrm{d}s-\frac{b}{k}\dot{D}_x\right)-f(t,x)-a\dot{C}_t-b\dot{D}_x\\
&=0
\end{aligned}$$

which means the uncertain partial differential equation (9), i.e.,

$$\frac{\partial Z}{\partial t} = k\frac{\partial Z}{\partial x} + f(t,x) + a\dot{C}_t + b\dot{D}_x$$

holds. Thus the solution of the uncertain partial differential equation (9) is

$$Z(t,x) = \phi(x+kt) + \int_0^t f(s, x+k(t-s))\mathrm{d}s + aC_t - \frac{b}{k}D_x$$

where $\phi$ is an arbitrarily continuously differential function. For example, when $k = a = b = 1$, $f(t,x) = t + x$ and $\phi(x) = x^2$, the solution of the uncertain partial differential equation (9) becomes

$$\begin{aligned} Z(t,x) &= (x+t)^2 + \int_0^t (x+t)\mathrm{d}s + C_t - D_x \\ &= x^2 + 3xt + 2t^2 + C_t - D_x. \end{aligned}$$

***Example 16*** Consider the uncertain partial differential equation

$$\frac{\partial Z}{\partial t} = \frac{\partial Z}{\partial x} + t + x + Z + \dot{C}_t + \dot{D}_x. \tag{11}$$

**Step 1.** We decompose solving uncertain partial differential equation (11) into solving the following two uncertain partial differential equations

$$\frac{\partial Z_1}{\partial t} = \frac{\partial Z_1}{\partial x} + t + x + Z_1 + \dot{C}_t \tag{12}$$

and

$$\frac{\partial Z_2}{\partial t} = \frac{\partial Z_2}{\partial x} + Z_2 + \dot{D}_x. \tag{13}$$

Then it follows from the fundamental theorem that

$$\frac{\partial Z}{\partial t} = \frac{\partial Z_1}{\partial t} + \frac{\partial Z_2}{\partial t}, \quad \frac{\partial Z}{\partial x} = \frac{\partial Z_1}{\partial x} + \frac{\partial Z_2}{\partial x}.$$

Thus the solution of (11) is

$$Z = Z_1 + Z_2$$

since

$$
\begin{aligned}
&\frac{\partial Z}{\partial t} - \left[\frac{\partial Z}{\partial x} + t + x + Z + \dot{C}_t + \dot{D}_x\right] \\
&\quad = \frac{\partial Z_1}{\partial t} + \frac{\partial Z_2}{\partial t} - \left[\frac{\partial Z_1}{\partial x} + \frac{\partial Z_2}{\partial x} + t + x + Z_1 + Z_2 + \dot{C}_t + \dot{D}_x\right] \\
&\quad = \left[\frac{\partial Z_1}{\partial t} - \left(\frac{\partial Z_1}{\partial x} + t + x + Z_1 + \dot{C}_t\right)\right] + \left[\frac{\partial Z_2}{\partial t} - \left(\frac{\partial Z_2}{\partial x} + Z_2 + \dot{D}_x\right)\right] \\
&\quad = 0.
\end{aligned}
$$

**Step 2.** It follows from the method of characteristics provided by Examples 14 and 15 that the solution of (12) is

$$Z_1(t,x) = e^t \phi_1(x+t) + (x+t)(e^t - 1) + \int_0^t e^{t-s} \mathrm{d}C_s$$

and the solution of (13) is

$$Z_2(t,x) = e^{-x} \phi_2(x+t) - \int_0^x e^{y-x} \mathrm{d}D_y$$

where $\phi_1$ and $\phi_2$ are arbitrarily continuously differential functions. Thus we have

$$
\begin{aligned}
Z(t,x) &= Z_1(t,x) + Z_2(t,x) \\
&= e^t \phi_1(x+t) + e^{-x} \phi_2(x+t) + (x+t)(e^t - 1) + \int_0^t e^{t-s} \mathrm{d}C_s - \int_0^x e^{y-x} \mathrm{d}D_y.
\end{aligned}
$$

In addition, since $\phi_1$ and $\phi_2$ are arbitrarily continuously differential functions and

$$e^t \phi_1(x+t) + e^{-x} \phi_2(x+t) = e^t[\phi_1(x+t) + e^{-(x+t)} \phi_2(x+t)],$$

we can write

$$\phi(x+t) = \phi_1(x+t) + e^{-(x+t)} \phi_2(x+t),$$

and the solution of the uncertain partial differential Eq. (11) is

$$Z(t,x) = e^t \phi(x+t) + (x+t)(e^t - 1) + \int_0^t e^{t-s} \mathrm{d}C_s - \int_0^x e^{y-x} \mathrm{d}D_y,$$

where $\phi$ is an arbitrarily continuously differential function. For example, when $\phi(x) = x$, the solution becomes

$$Z(t,x) = (x+t)(2e^t - 1) + \int_0^t e^{t-s} \mathrm{d}C_s - \int_0^x e^{y-x} \mathrm{d}D_y.$$

## 8 Conclusion

Partial derivative is an important concept and cornerstone in multivariable calculus, as well as in multivariate uncertain calculus. For that matter, this paper first defined the partial derivative of uncertain fields by putting forward the concept of Liu field. In order to calculate the partial derivative of uncertain fields, the fundamental theorem, chain rule and integration by parts of multivariate uncertain calculus were derived. On these bases, this paper proposed a type of uncertain partial differential equation, and gave its integral form. Finally, some examples were documented to illustrate how to solve uncertain partial differential equations.

**Acknowledgements** This work was supported by the National Natural Science Foundation of China (Grant No. 62073009), the China Postdoctoral Science Foundation (Grant No. 2022M710322), and the National Natural Science Foundation of China (Grant No. 61873329).

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