Fuzzy Optimization and Decision Making (2022) 21:513–530
https://doi.org/10.1007/s10700-021-09379-4

Residual analysis and parameter estimation of uncertain differential equations

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Accepted: 16 December 2021 / Published online: 30 January 2022

Abstract
All existing methods to estimate unknown parameters in uncertain differential equations are based on difference scheme, and do not work when the time intervals between observations are not short enough. In order to overcome this shortage, this paper presents a concept of residual. Afterwards, an algorithm is designed for calculating residuals of uncertain differential equation corresponding to observed data. In addition, this paper presents a method of moments based on residuals to estimate the unknown parameters in uncertain differential equations. Finally, some examples (including Alibaba stock price) are provided to illustrate the parameter estimation method.

Keywords Uncertainty theory · Uncertain differential equation · Residual analysis · Parameter estimation

1 Introduction

For the purpose of rationally handling the belief degree that something will happen, uncertainty theory was founded by Liu (2007) and perfected by Liu (2009). To this day, uncertainty theory has spawned many theoretical branches and has been successfully applied in various fields of science and engineering.

Among the theoretical branches of uncertainty theory, uncertain differential equation was first proposed by Liu (2008) to model time-varying system. For the purpose of investigating the solution of an uncertain differential equation, Chen and Liu (2010) verified the existence and uniqueness theorem of the solution under linear growth condition and Lipschitz condition. Besides, the stability of uncertain differential equation

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was first investigated by Liu (2009) to study the dependence of the solution on the initial value. Following that, some theorems were verified by Yao et al. (2013) to perfect the stability analysis of uncertain differential equations, which has been developed by many scholars such as stability in p-th moment (Sheng and Wang 2014), almost sure stability (Liu et al. 2014) and stability in mean (Yao et al. 2015), etc. Another vital problem in uncertain differential equations is how to solve it. For the sake of dealing with this problem, Yao-Chen formula, as one of the most important contributions to uncertain differential equations, was proved by Yao and Chen (2013) to associate uncertain differential equation with ordinary differential equations. On the basis of Yao-Chen formula, some numerical solution methods for uncertain differential equations were studied by scholars such as Euler method (Yao and Chen 2013), Runge-Kutta method (Yang and Shen 2015), Adams method (Yang and Ralescu 2015) and Milne method (Gao 2016), among others. Up to now, the theory of uncertain differential equations has been well developed.

Assume an uncertain process follows an uncertain differential equation and some realizations of this process are observed. A critical problem in the practical application of uncertain differential equations is how to estimate the unknown parameters based on the observed data. For the purpose of dealing with this problem, several methods have been proposed. For instance, Yao and Liu (2020) presented the method of moments, Sheng et al. (2020) investigated least squares estimation, Yang et al. (2020) discussed minimum cover estimation, Liu and Liu (2020) proposed maximum likelihood estimation, Liu (2021) studied generalized moment estimation, and Lio and Liu (2021) presented initial value estimation. Based on those methods, uncertain differential equations have been applied to handling the real-life problems such as pharmacokinetics (Liu and Yang 2021), chemical reaction (Tang and Yang 2021) and epidemic spread (Lio and Liu 2021; Jia and Chen 2021; Chen et al. 2021).

However, the above parameter estimation methods are all based on difference scheme and do not work when the time intervals between observations are not short enough. In order to overcome this shortage, this paper proposes a concept of residual, and investigates a method of moments based on residuals to estimate unknown parameters in uncertain differential equations. The structure of this paper adopts the form of six parts, including this Introduction section. Sect. 2 begins by introducing the concept of residual, and Sect. 3 begins by designing an algorithm to calculate residuals of uncertain differential equation corresponding to observed data. Afterwards, the method of moments based on residuals is presented to estimate unknown parameters of uncertain differential equations in Sect. 4. As an application, the method of moments based on residuals is used to model Alibaba stock price in Sect. 5. Finally, a concise conclusion is given in Sect. 6.

2 Residual

In order to make a connection between uncertain differential equation and observed data of some uncertain process, this section will introduce a concept of residual. Let us consider an uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

where f and g are known continuous functions and C_t is a Liu process. Assume

$$x_{t_1}, x_{t_2}, \ldots, x_{t_n} \tag{2}$$

are n observations of some uncertain process X_t at the times $t_1, t_2, \ldots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively.

For any given index i with $2 \le i \le n$, let us first solve the updated uncertain differential equation,

$$\begin{cases} \mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases} \tag{3}$$

where $x_{t_{i-1}}$ is the new initial value at the new initial time t_{i-1}. The uncertainty distribution of X_{t_i} is thus obtained and represented by Φ_{t_i}. Then for any x with $0 < x < 1$, we have

$$\mathcal{M}\{\Phi_{t_i}(X_{t_i}) \le x\} = \mathcal{M}\{X_{t_i} \le \Phi_{t_i}^{-1}(x)\} = \Phi_{t_i}(\Phi_{t_i}^{-1}(x)) = x.$$

Thus $\Phi_{t_i}(X_{t_i})$ is always a linear uncertain variable whose uncertainty distribution is

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

denoted by $\mathcal{L}(0, 1)$. Substitute X_{t_i} with the observed value x_{t_i}, and write

$$\varepsilon_i = \Phi_{t_i}(x_{t_i}). \tag{4}$$

Then ε_i can be regarded as a sample of the linear uncertain variable $\Phi_{t_i}(X_{t_i})$. In other words, ε_i is a sample of linear uncertainty distribution $\mathcal{L}(0, 1)$.

Definition 1 For each index i with $2 \le i \le n$, the term ε_i defined by (4) is called the ith residual of uncertain differential Eq. (1) corresponding to the observed data (2).

Example 1 Assume $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ are observed values of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}C_t \tag{5}$$

where μ and σ are constants. For each index i with $2 \le i \le n$, we solve the updated uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = \mu\mathrm{d}t + \sigma\mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the uncertainty distribution of X_{t_i} as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x)}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

It follows from Definition 1 that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x_{t_i})}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

Example 2 Assume $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ are observed values of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t \tag{6}$$

where μ and σ are constants. For each index i with $2 \leq i \leq n$, we solve the updated uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the uncertainty distribution of X_{t_i} as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(t_i - t_{i-1}) - \ln x)}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

It follows from Definition 1 that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(\ln x_{t_{i-1}} + \mu(t_i - t_{i-1}) - \ln x_{t_i})}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}.$$

Example 3 Assume $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ are observed values of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = \mu t X_t \mathrm{d}t + \sigma t X_t \mathrm{d}C_t \tag{7}$$

where μ and σ are constants. For each index i with $2 \leq i \leq n$, we solve the updated uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = \mu t X_t \mathrm{d}t + \sigma t X_t \mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the uncertainty distribution of X_{t_i} as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi(2\ln x_{t_{i-1}} + \mu(t_i^2 - t_{i-1}^2) - 2\ln x)}{\sqrt{3}\sigma(t_i^2 - t_{i-1}^2)}\right)\right)^{-1}.$$

It follows from Definition 1 that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi(2\ln x_{t_{i-1}} + \mu(t_i^2 - t_{i-1}^2) - 2\ln x_{t_i})}{\sqrt{3}\sigma(t_i^2 - t_{i-1}^2)}\right)\right)^{-1}.$$

Example 4 Assume $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ are observed values of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t \tag{8}$$

where m, a and σ are constants. For each index i with $2 \leq i \leq n$, we solve the updated uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the uncertainty distribution of X_{t_i} as follows,

$$\Phi_{t_i}(x) = \left(1 + \exp\left(\frac{\pi\left((ax_{t_{i-1}} - m)\exp(a(t_{i-1} - t_i)) + m - ax\right)}{\sqrt{3}\sigma\,(1 - \exp(a(t_{i-1} - t_i)))}\right)\right)^{-1}.$$

It follows from Definition 1 that the ith residual is

$$\varepsilon_i = \left(1 + \exp\left(\frac{\pi\left((ax_{t_{i-1}} - m)\exp(a(t_{i-1} - t_i)) + m - ax_{t_i}\right)}{\sqrt{3}\sigma\,(1 - \exp(a(t_{i-1} - t_i)))}\right)\right)^{-1}.$$

3 Numerical method for calculating residuals

For general uncertain differential equations, the following algorithm is able to calculate the ith residual ε_i.

Algorithm 1
Step 0: Set $l = 0$, $r = 1$ and a precision $\delta = 0.0001$.
Step 1: Set $\alpha = (l + r)/2$.
Step 2: Compute $X_{t_i}^{\alpha}$ of the uncertain differential Eq. (3) by Euler method.
Step 3: If $X_{t_i}^{\alpha} < x_{t_i}$, then $l = \alpha$. Otherwise, $r = \alpha$.
Step 4: If $|l - r| > \delta$, then go to Step 1.
Step 5: Output $\varepsilon_i = (l + r)/2$.

Table 1 Observed data in Example 5

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
0.04	1.00	1.83	3.74	4.32	7.69	7.65	16.58	11.06	27.19
0.15	1.20	1.96	3.91	4.66	8.36	8.63	19.47	11.79	28.87
0.28	1.39	2.25	4.36	4.66	8.36	8.68	19.62	11.81	28.92
0.69	2.24	2.49	4.43	5.41	10.08	9.47	21.50	11.99	29.36
1.02	2.49	3.25	6.01	6.23	12.85	9.64	21.88	12.81	31.69
1.38	2.85	4.12	7.28	7.18	15.43	10.39	24.04	13.38	33.30

Table 2 Residuals in Example 5

i	ε_i	i	ε_i	i	ε_i	i	ε_i	i	ε_i
2	0.4183	8	0.2217	14	0.4867	20	0.8483	26	0.6334
3	0.2736	9	0.3072	15	0.5000	21	0.8598	27	0.7124
4	0.5331	10	0.0432	16	0.6300	22	0.6657	28	0.6913
5	0.0951	11	0.5357	17	0.9241	23	0.6051	29	0.8215
6	0.1402	12	0.2729	18	0.7855	24	0.8315	30	0.8169
7	0.4899	13	0.5227	19	0.6922	25	0.9926		

Next we provide some examples to illustrate the above algorithm.

Example 5 Table 1 shows 30 observed data of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = \left(\frac{2}{t^2} + \frac{1}{X_t}\right)\mathrm{d}t + \frac{X_t}{t}\mathrm{d}C_t. \tag{9}$$

According to Algorithm 1, the 29 residuals of uncertain differential Eq. (9) corresponding to the observed data can be obtained and are shown in Table 2 and Fig. 1.

Example 6 Table 3 shows 30 observed data of some uncertain process X_t that follows the uncertain differential equation

$$\mathrm{d}X_t = \exp\left(\frac{2}{X_t}\right)\mathrm{d}t + \ln X_t \mathrm{d}C_t. \tag{10}$$

According to Algorithm 1, the 29 residuals of uncertain differential Eq. (10) corresponding to the observed data can be obtained and are shown in Table 4 and Fig. 2.

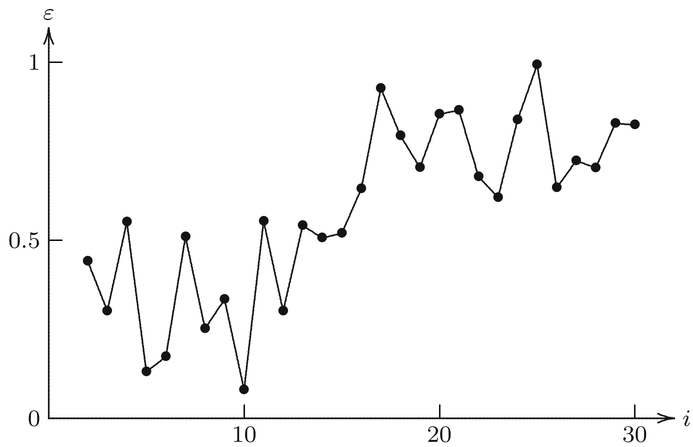

Fig. 1 Residual plot in Example 5

Table 3 Observed data in Example 6

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
0.2	1.00	3.69	10.34	5.98	17.63	8.24	22.02	9.28	23.21
0.32	1.26	4.24	11.94	6.07	17.88	8.44	22.37	9.67	23.16
1.03	3.09	4.44	12.40	6.15	18.08	8.69	22.73	10.22	23.56
1.94	5.56	4.74	13.20	6.81	19.45	8.70	22.75	10.62	23.91
2.12	6.15	5.41	16.19	7.01	19.87	8.83	22.74	11.12	24.97
2.79	7.92	5.49	16.38	7.55	20.65	9.07	22.87	11.62	24.68

Table 4 Residuals in Example 6

i	ε_i	i	ε_i	i	ε_i	i	ε_i	i	ε_i
2	0.5750	8	0.8387	14	0.8039	20	0.3885	26	0.0206
3	0.7403	9	0.6328	15	0.7124	21	0.2659	27	0.0904
4	0.7851	10	0.7702	16	0.5343	22	0.5000	28	0.1150
5	0.9103	11	0.9886	17	0.5452	23	0.0226	29	0.5542
6	0.7621	12	0.6638	18	0.2674	24	0.0663	30	0.0092
7	0.7772	13	0.7309	19	0.4934	25	0.3338		

4 Parameter estimation

Next we will provide a parameter estimation method based on residuals to estimate the unknown parameters in uncertain differential equations. Let us consider the following uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t; \boldsymbol{\theta})\mathrm{d}t + g(t, X_t; \boldsymbol{\theta})\mathrm{d}C_t \tag{11}$$

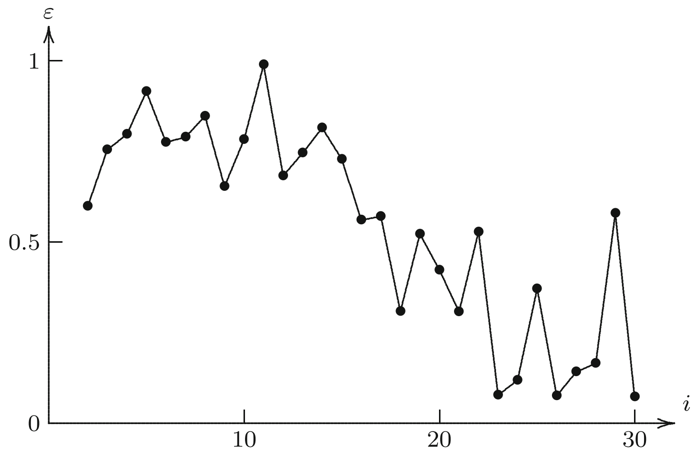

Fig. 2 Residual plot in Example 6

where f and g are known continuous functions but $\boldsymbol{\theta}$ is an unknown vector of parameters. Assume

$$x_{t_1}, x_{t_2}, \ldots, x_{t_n} \tag{12}$$

are observed values of some uncertain process X_t at the times $t_1, t_2, \ldots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively.

For each given $\boldsymbol{\theta}$, we can produce $n-1$ residuals $\varepsilon_2(\boldsymbol{\theta}), \varepsilon_3(\boldsymbol{\theta}), \ldots, \varepsilon_n(\boldsymbol{\theta})$ of the uncertain differential Eq. (11) corresponding to the observed data (12). Note that the $n-1$ residuals $\varepsilon_2(\boldsymbol{\theta}), \varepsilon_3(\boldsymbol{\theta}), \ldots, \varepsilon_n(\boldsymbol{\theta})$ can be regarded as samples of the linear uncertainty distribution $\mathcal{L}(0, 1)$, i.e.,

$$\varepsilon_2(\boldsymbol{\theta}), \varepsilon_3(\boldsymbol{\theta}), \ldots, \varepsilon_n(\boldsymbol{\theta}) \sim \mathcal{L}(0, 1).$$

For each positive integer k, the kth sample moment of the $n-1$ residuals is

$$\frac{1}{n-1}\sum_{i=2}^{n} \varepsilon_i^k(\boldsymbol{\theta}),$$

and the kth population moment of the linear uncertainty distribution $\mathcal{L}(0, 1)$ is

$$\frac{1}{k+1}.$$

The moment estimate $\boldsymbol{\theta}$ is then obtained by equating the first p sample moments to the corresponding first p population moments, where p is the number of unknown

Table 5 Observed data in Example 7

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
0.00	1.00	0.63	1.52	1.23	2.23	2.04	7.06	2.70	20.1
0.12	1.11	0.72	1.64	1.35	2.45	2.16	8.29	2.76	21.6
0.18	1.15	0.87	1.84	1.47	3.04	2.28	11.0	2.91	27.4
0.30	1.22	0.93	1.96	1.59	3.66	2.40	13.5	3.00	31.1
0.39	1.28	1.02	2.09	1.74	4.42	2.49	15.7	3.06	33.6
0.51	1.44	1.08	2.18	1.89	5.36	2.61	18.0	3.12	36.0

parameters. In other words, the moment estimate $\boldsymbol{\theta}$ should solve the system of equations,

$$\frac{1}{n-1}\sum_{i=2}^{n}\varepsilon_i^k(\boldsymbol{\theta}) = \frac{1}{k+1}, \quad k = 1, 2, \ldots, p. \tag{13}$$

The above method to estimate the unknown parameters in uncertain differential equations is called the method of moments based on residuals, and we can solve the system of Eqs. (13) by using MATLAB[1].

Remark 1 Sometimes the system of Eq. (13) has no solution. In this case, we suggest the generalized moment estimation that solves the following minimization problem,

$$\min_{\boldsymbol{\theta}} \sum_{k=1}^{p}\left(\frac{1}{n-1}\sum_{i=2}^{n}\varepsilon_i^k(\boldsymbol{\theta}) - \frac{1}{k+1}\right)^2 \tag{14}$$

where $\varepsilon_2(\boldsymbol{\theta}), \varepsilon_3(\boldsymbol{\theta}), \ldots, \varepsilon_n(\boldsymbol{\theta})$ are the residuals and p is the number of unknown parameters. In particular, we can solve the minimization problem (14) by MATLAB[2].

Example 7 Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t$$

where μ and $\sigma > 0$ are two unknown parameters to be estimated. Suppose that we have 30 observed data as shown in Table 5, and denote the observed data of X_t at the times $t_1, t_2, \ldots, t_{30}$ by $x_{t_1}, x_{t_2}, \ldots, x_{t_{30}}$, respectively. For any given parameters μ and σ, we can obtain the 29 residuals

$$\varepsilon_i(\mu, \sigma) = \left(1 + \exp\left(\frac{\pi(x_{t_{i-1}} + \mu(t_i - t_{i-1}) - x_{t_i})}{\sqrt{3}\sigma(t_i - t_{i-1})}\right)\right)^{-1}, \quad i = 2, 3, \ldots, 30$$

[1] MATLAB R2021a, 9.10.0.1602886, maci64, Optimization Toolbox, “fsolve” function.

[2] MATLAB R2021a, 9.10.0.1602886, maci64, Optimization Toolbox, “fminsearch” function.

Table 6 Observed data in Example 8

t	x_t	t	x_t	t	x_t	t	x_t
0.00	1.00	3.21	16.51	5.65	46.03	7.94	97.9
1.12	4.10	3.93	21.05	6.34	58.7	8.79	116.49
1.97	8.36	4.86	34.94	7.47	86.6	9.83	147.71

according to Example 2. Since the number of unknown parameters is 2 and the first two moments of the linear uncertainty distribution $\mathcal{L}(0,1)$ are $1/2$ and $1/3$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{29}\displaystyle\sum_{i=2}^{30}\left(1+\exp\left(\dfrac{\pi(x_{t_{i-1}}+\mu(t_i-t_{i-1})-x_{t_i})}{\sqrt{3}\sigma(t_i-t_{i-1})}\right)\right)^{-1}=\dfrac{1}{2} \\ \dfrac{1}{29}\displaystyle\sum_{i=2}^{30}\left(1+\exp\left(\dfrac{\pi(x_{t_{i-1}}+\mu(t_i-t_{i-1})-x_{t_i})}{\sqrt{3}\sigma(t_i-t_{i-1})}\right)\right)^{-2}=\dfrac{1}{3}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$\mu^*=1.1201,\ \ \sigma^*=0.5417.$$

Thus we obtain an uncertain differential equation

$$\mathrm{d}X_t = 1.1201X_t\mathrm{d}t + 0.5417X_t\mathrm{d}C_t. \tag{15}$$

Example 8 Consider the uncertain differential equation

$$\mathrm{d}X_t=\mu\sqrt{X_t}\mathrm{d}t+\sigma\sqrt{X_t}\mathrm{d}C_t$$

where μ and $\sigma > 0$ are two unknown parameters to be estimated. Suppose that we have 12 observed data as shown in Table 6. For any given parameters μ and σ, we can produce 11 residuals

$$\varepsilon_2(\mu,\sigma),\varepsilon_3(\mu,\sigma),\ldots,\varepsilon_{12}(\mu,\sigma)$$

by Algorithm 1. Since the number of unknown parameters is 2 and the first two moments of the linear uncertainty distribution $\mathcal{L}(0,1)$ are $1/2$ and $1/3$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{11}\displaystyle\sum_{i=2}^{12}\varepsilon_i(\mu,\sigma)=\dfrac{1}{2} \\ \dfrac{1}{11}\displaystyle\sum_{i=2}^{12}\varepsilon_i^2(\mu,\sigma)=\dfrac{1}{3}. \end{cases}$$

Table 7 Observed data in Example 9

t	x_t	t	x_t	t	x_t	t	x_t	t	x_t
0.00	1.00	0.79	2.12	2.33	2.70	4.54	2.98	6.70	1.35
0.11	1.06	0.98	1.97	2.55	2.99	4.76	2.39	6.86	1.33
0.24	1.31	1.20	1.75	2.97	5.60	4.98	2.06	6.91	1.33
0.29	1.42	1.50	2.37	3.39	4.03	5.40	1.94	7.10	1.15
0.44	1.79	1.80	3.07	3.74	2.23	5.89	1.60	7.36	0.88
0.62	2.10	2.08	3.03	4.19	2.78	6.41	1.08	7.62	0.79

Solving the above system of equations by MATLAB, we can get

$$\mu^* = 2.2759, \ \sigma^* = 0.5166.$$

Thus we obtain an uncertain differential equation

$$\mathrm{d}X_t = 2.2759\sqrt{X_t}\mathrm{d}t + 0.5166\sqrt{X_t}\mathrm{d}C_t. \tag{16}$$

Example 9 Consider the uncertain differential equation

$$\mathrm{d}X_t = \sin(\mu t)X_t\mathrm{d}t + \cos(\sigma t)X_t\mathrm{d}C_t$$

where μ and $\sigma > 0$ are two unknown parameters to be estimated. Suppose that we have 30 observed data as shown in Table 7. For any given parameters μ and σ, we can produce 29 residuals

$$\varepsilon_2(\mu, \sigma), \varepsilon_3(\mu, \sigma), \ldots, \varepsilon_{30}(\mu, \sigma)$$

by Algorithm 1. Since the number of unknown parameters is 2 and the first two moments of the linear uncertainty distribution $\mathcal{L}(0, 1)$ are $1/2$ and $1/3$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{29}\displaystyle\sum_{i=2}^{30} \varepsilon_i(\mu, \sigma) = \dfrac{1}{2} \\ \dfrac{1}{29}\displaystyle\sum_{i=2}^{30} \varepsilon_i^2(\mu, \sigma) = \dfrac{1}{3}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$\mu^* = 4.9662, \ \sigma^* = 1.0786.$$

Table 8 Plasma cimetidine concentration-time data during the constant-rate intravenous infusion in a beagle dog (Yu and Cao (2017))

Time (hr)	Cimetidine (μg/ml)	Time (hr)	Cimetidine (μg/ml)
0.17	1.38	4.0	4.41
0.5	1.88	5.0	4.27
1.0	2.31	6.0	4.42
1.5	2.83	7.0	4.41
2.0	2.96	7.5	4.34
3.0	3.73	8.0	4.06

Thus we obtain an uncertain differential equation

$$\mathrm{d}X_t = \sin(4.9662t)X_t\mathrm{d}t + \cos(1.0786t)X_t\mathrm{d}C_t. \tag{17}$$

Example 10 Pharmacokinetics is the study of the dynamic movement of drug concentrations in the blood of a body. Consider plasma cimetidine concentration in beagle blood during and subsequent to a constant-rate intravenous infusion. Yu and Cao (2017) has provided a collection of data as shown in Table 8. For this experiment of plasma cimetidine concentration, Liu and Yang (2021) derived that the blood plasma cimetidine concentration X_t follows the uncertain pharmacokinetic equation

$$\mathrm{d}X_t = (k_0 - k_1 X_t)\,\mathrm{d}t + \sigma\mathrm{d}C_t$$

where k_0, k_1 and σ are unknown parameters to be estimated. Based on the observed data, the uncertain pharmacokinetic equation was inferred as

$$\mathrm{d}X_t = (5.581 - 1.473X_t)\,\mathrm{d}t + 1.304\mathrm{d}C_t \tag{18}$$

by using the method of moments based on difference scheme.

Let us re-estimate the parameters by using the method of moments based on residuals. Denote the observed data of X_t at the times $t_1, t_2, \ldots, t_{12}$ by $x_{t_1}, x_{t_2}, \ldots, x_{t_{12}}$, respectively. For any fixed parameters k_0, k_1, σ and each index i with $2 \le i \le 12$, we solve the updated uncertain pharmacokinetic equation

$$\begin{cases} \mathrm{d}X_t = (k_0 - k_1 X_t)\,\mathrm{d}t + \sigma\mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the ith residual as follows,

$$\varepsilon_i(k_0, k_1, \sigma) = \left(1 + \exp\left(\frac{\pi\,\left((k_1 x_{t_{i-1}} - k_0)\exp\left(k_1(t_{i-1} - t_i)\right) + k_0 - k_1 x_{t_i}\right)}{\sqrt{3}\sigma\,\left(1 - \exp\left(k_1(t_{i-1} - t_i)\right)\right)}\right)\right)^{-1}.$$

Table 9 N_2O_5 concentration data of the decomposition reaction $2N_2O_5 \longrightarrow 4NO_2 + O_2$ (Henold and Walmsley (1984))

time (sec)	N_2O_5 (mol/L)	time (sec)	N_2O_5 (mol/L)	time (sec)	N_2O_5 (mol/L)
0	0.31	2400	0.151	4800	0.0669
600	0.254	3000	0.116	6000	0.0464
1200	0.208	3600	0.0964		
1800	0.172	4200	0.0812		

Since the number of unknown parameters is 3 and the first three moments of the linear uncertainty distribution $\mathcal{L}(0, 1)$ are $1/2$, $1/3$ and $1/4$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{11}\displaystyle\sum_{i=2}^{12} \varepsilon_i(k_0, k_1, \sigma) = \dfrac{1}{2} \\ \dfrac{1}{11}\displaystyle\sum_{i=2}^{12} \varepsilon_i^2(k_0, k_1, \sigma) = \dfrac{1}{3} \\ \dfrac{1}{11}\displaystyle\sum_{i=2}^{12} \varepsilon_i^3(k_0, k_1, \sigma) = \dfrac{1}{4}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$k_0^* = 1.7429,\ k_1^* = 0.3820,\ \sigma^* = 0.4026.$$

Thus we obtain an uncertain pharmacokinetic equation

$$\mathrm{d}X_t = (1.7429 - 0.3820X_t)\,\mathrm{d}t + 0.4026\mathrm{d}C_t. \tag{19}$$

Example 11 Chemical reaction rate is an important research object in chemical kinetics and is a measure of how fast a chemical reaction goes. Consider the decomposition reaction of N_2O_5 in gas phase. Henold and Walmsley (1984) has provided a collection of experimental data for the concentration of N_2O_5 as shown in Table 9. For this decomposition reaction, Tang and Yang (2021) derived that the N_2O_5 concentration X_t obeys the following uncertain chemical reaction equation

$$\mathrm{d}X_t = -2\mu X_t^2\mathrm{d}t - 2\sigma X_t^2\mathrm{d}C_t$$

according to the law of mass action, where μ and σ are unknown parameters to be estimated. Based on the observed data, the uncertain chemical reaction equation was obtained as

$$\mathrm{d}X_t = -0.0022X_t^2\mathrm{d}t - 0.0010X_t^2\mathrm{d}C_t \tag{20}$$

by using the method of moments based on difference scheme.

Let us employ the method of moments based on residuals to re-estimate the parameters. Denote the observed data of X_t at the times $t_1, t_2, \ldots, t_{10}$ by $x_{t_1}, x_{t_2}, \ldots, x_{t_{10}}$, respectively. For any fixed parameters μ, σ and each index i with $2 \le i \le 10$, we solve the updated uncertain chemical reaction equation

$$\begin{cases} \mathrm{d}X_t = -2\mu X_t^2 \mathrm{d}t - 2\sigma X_t^2 \mathrm{d}C_t \\ X_{t_{i-1}} = x_{t_{i-1}} \end{cases}$$

and obtain the ith residual as follows,

$$\varepsilon_i(\mu, \sigma) = \left(1 + \exp\left(\frac{\pi}{\sqrt{3}\sigma}\left(\frac{x_{t_{i-1}} - x_{t_i}}{2(t_i - t_{i-1})x_{t_{i-1}}x_{t_i}} - \mu\right)\right)\right)^{-1}.$$

Since the number of unknown parameters is 2 and the first two moments of the linear uncertainty distribution $\mathcal{L}(0, 1)$ are 1/2 and 1/3, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{9}\displaystyle\sum_{i=2}^{10} \varepsilon_i(\mu, \sigma) = \dfrac{1}{2} \\ \dfrac{1}{9}\displaystyle\sum_{i=2}^{10} \varepsilon_i^2(\mu, \sigma) = \dfrac{1}{3}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$\mu^* = 0.001345, \ \sigma^* = 0.000835.$$

Thus we obtain an uncertain chemical reaction equation

$$\mathrm{d}X_t = -0.00269 X_t^2 \mathrm{d}t - 0.00167 X_t^2 \mathrm{d}C_t. \tag{21}$$

Example 12 RC circuit is a circuit system composed of resistor and capacitor, and is driven by a voltage source. Consider the simple series RC circuit with a constant potential source (5V). Liu (2021) provided a collection of data for the charge stored on the capacitor as shown in Table 10. For this simple series RC circuit, the charge Q_t stored on the capacitor was derived to follow the uncertain circuit equation

$$\mathrm{d}Q_t = \left(\frac{5}{r} - \frac{Q_t}{rc}\right)\mathrm{d}t + \frac{\sigma}{r}\mathrm{d}C_t$$

based on the fundamental laws of electrical circuits, where r, c and σ are unknown parameters to be estimated. Based on the observed data, the uncertain circuit equation was inferred as

$$\mathrm{d}Q_t = (0.4614 - 0.0921 Q_t)\,\mathrm{d}t + 0.0133 \mathrm{d}C_t \tag{22}$$

by using the method of moments based on difference scheme.

Table 10 Charge data stored on the capacitor of series RC circuit (Liu (2021))

t (s)	q (C)	t (s)	q (C)	t (s)	q (C)	t (s)	q (C)	t (s)	q (C)
0.0025	0.0012	10	3.1355	20	4.3336	34	4.8718	49	4.9663
0.0645	0.0312	13	3.6039	23	4.4971	37	4.9118	50	4.9628
1	0.4624	14	3.7312	26	4.6524	39	4.9495	52	4.9545
4	1.6371	16	3.9386	27	4.6979	42	4.9563	55	4.9785
7	2.5215	18	4.1433	30	4.7587	44	4.9700	56	4.9998
8	2.7731	19	4.2453	31	4.7735	47	4.9857	58	5.0008

Let us re-estimate the parameters by using the method of moments based on residuals. Denote the observed data of Q_t at the times $t_1, t_2, \ldots, t_{30}$ by $q_{t_1}, q_{t_2}, \ldots, q_{t_{30}}$, respectively. For any fixed parameters r, c, σ and each index i with $2 \le i \le 30$, we solve the updated uncertain circuit equation

$$\begin{cases} \mathrm{d}Q_t = \left(\dfrac{5}{r} - \dfrac{Q_t}{rc}\right)\mathrm{d}t + \dfrac{\sigma}{r}\mathrm{d}C_t \\ Q_{t_{i-1}} = q_{t_{i-1}} \end{cases}$$

and obtain the ith residual as follows,

$$\varepsilon_i(r, c, \sigma) = \left(1 + \exp\left(\frac{\pi\left((q_{t_{i-1}} - 5c)\exp\left(\dfrac{t_{i-1} - t_i}{rc}\right) + 5c - q_{t_i}\right)}{\sqrt{3}\sigma c\left(1 - \exp\left(\dfrac{t_{i-1} - t_i}{rc}\right)\right)}\right)\right)^{-1}.$$

Since the number of unknown parameters is 3 and the first three moments of the linear uncertainty distribution $\mathcal{L}(0, 1)$ are $1/2$, $1/3$ and $1/4$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{29}\displaystyle\sum_{i=2}^{30} \varepsilon_i(r, c, \sigma) = \dfrac{1}{2} \\ \dfrac{1}{29}\displaystyle\sum_{i=2}^{30} \varepsilon_i^2(r, c, \sigma) = \dfrac{1}{3} \\ \dfrac{1}{29}\displaystyle\sum_{i=2}^{30} \varepsilon_i^3(r, c, \sigma) = \dfrac{1}{4}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$r^* = 9.7957,\ c^* = 0.9976,\ \sigma^* = 0.1400.$$

Thus we obtain an uncertain circuit equation

$$\mathrm{d}Q_t = (0.5104 - 0.1023Q_t)\,\mathrm{d}t + 0.0143\mathrm{d}C_t. \tag{23}$$

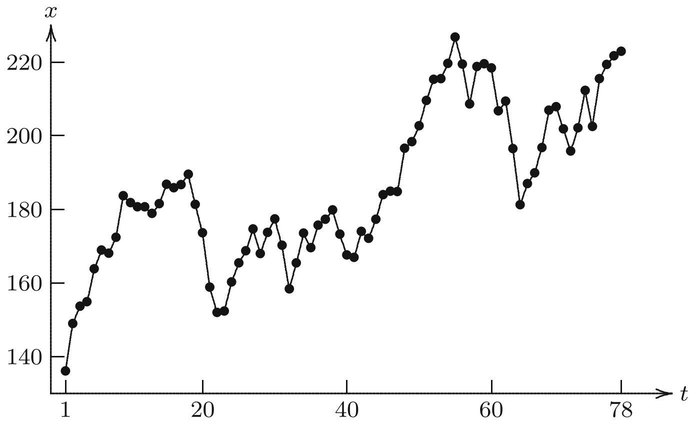

Fig. 3 Alibaba stock prices (weekly average) from January 1, 2019 to June 30, 2020

5 Alibaba stock price

As a famous internet company, Alibaba is closely related to our daily life. The following data

136.03	148.96	153.60	154.81	163.82	168.87	168.02	172.37	183.66	181.75
180.61	180.60	178.81	181.47	186.75	185.84	186.66	189.48	181.26	173.52
158.78	151.91	152.29	160.19	165.34	168.65	174.62	167.96	173.66	177.36
170.18	158.32	165.39	173.44	169.48	175.59	177.25	179.81	173.23	167.59
166.89	173.91	172.04	177.25	183.93	184.89	184.77	196.49	198.28	202.65
209.51	215.24	215.45	219.58	226.68	219.38	208.58	218.73	219.46	218.32
206.71	209.29	196.41	181.17	186.91	189.86	196.70	206.91	207.81	201.74
195.80	202.03	212.23	202.45	215.42	219.26	221.62	222.85		

show Alibaba stock prices (weekly average) in US$ from January 1, 2019 to June 30, 2020 reported by Nasdaq. See Fig. 3.

Let $i = 1, 2, \ldots, 78$ represent the weeks from January 1, 2019 to June 30, 2020, and denote the stock prices by

$$x_1, x_2, \ldots, x_{78}. \tag{24}$$

In order to fit them, we employ the uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\,\mathrm{d}t + \sigma \mathrm{d}C_t$$

where m, a and σ are unknown parameters to be estimated. For any fixed parameters m, a, σ and each index i with $2 \le i \le 78$, we solve the updated uncertain differential equation

$$\begin{cases} \mathrm{d}X_t = (m - aX_t)\,\mathrm{d}t + \sigma\,\mathrm{d}C_t \\ X_{i-1} = x_{i-1} \end{cases}$$

and obtain the ith residual as follows,

$$\varepsilon_i(m, a, \sigma) = \left(1 + \exp\left(\frac{\pi\,((ax_{i-1} - m)\exp(-a) + m - ax_i)}{\sqrt{3}\sigma\,(1 - \exp(-a))}\right)\right)^{-1}.$$

Since the number of unknown parameters is 3 and the first three moments of the linear uncertainty distribution $\mathcal{L}(0, 1)$ are $1/2$, $1/3$ and $1/4$, the system of Eq. (13) becomes

$$\begin{cases} \dfrac{1}{77}\displaystyle\sum_{i=2}^{78} \varepsilon_i(m, a, \sigma) = \dfrac{1}{2} \\ \dfrac{1}{77}\displaystyle\sum_{i=2}^{78} \varepsilon_i^2(m, a, \sigma) = \dfrac{1}{3} \\ \dfrac{1}{77}\displaystyle\sum_{i=2}^{78} \varepsilon_i^3(m, a, \sigma) = \dfrac{1}{4}. \end{cases}$$

Solving the above system of equations by MATLAB, we can get

$$m^* = 45.9292,\ a^* = 0.2404,\ \sigma^* = 8.6308.$$

Thus we obtain an uncertain differential equation

$$\mathrm{d}X_t = (45.9292 - 0.2404X_t)\,\mathrm{d}t + 8.6308\mathrm{d}C_t \tag{25}$$

where X_t is Alibaba stock price.

6 Conclusion

In order to make a connection between uncertain differential equation and observed data of some uncertain process, this paper first introduced the concept of residual, and designed an algorithm to calculate residuals of uncertain differential equation corresponding to observed data. Following that, a method of moments based on residuals was presented to estimate the unknown parameters in uncertain differential equations. Finally, some examples (including Alibaba stock price) were provided to illustrate the method of moments.

Acknowledgements This work was supported by the National Natural Science Foundation of China Grant Nos.61873329 and 12026225.

References

Chen, X., & Liu, B. (2010). Existence and uniqueness theorem for uncertain differential equations. *Fuzzy Optimization and Decision Making, 9*(1), 69–81.

Chen, X., Li, J., Xiao, C., & Yang, P. (2021). Numerical solution and parameter estimation for uncertain SIR model with application to COVID-19. *Fuzzy Optimization and Decision Making, 20*(2), 189–208.

Gao, R. (2016). Milne method for solving uncertain differential equations. *Applied Mathematics and Computation, 274,* 774–785.

Henold, K., & Walmsley, F. (1984). *Chemical Principles, Properties, and Reactions*. Massachusett: Addison Wesley Publishing Company.

Jia, L., & Chen, W. (2021). Uncertain SEIAR model for COVID-19 cases in China. *Fuzzy Optimization and Decision Making, 20*(2), 243–259.

Lio, W., & Liu, B. (2021). Initial value estimation of uncertain differential equations and zero-day of COVID-19 spread in China. *Fuzzy Optimization and Decision Making., 20*(2), 177–188.

Liu, B. (2007). *Uncertainty Theory* (2nd ed.). Berlin: Springer.

Liu, B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain Systems, 2*(1), 3–16.

Liu, B. (2009). Some research problems in uncertainty theory. *Journal of Uncertain Systems, 3*(1), 3–10.

Liu, H., Ke, H., & Fei, W. (2014). Almost sure stability for uncertain differential equation. *Fuzzy Optimization and Decision Making, 13*(4), 463–473.

Liu, Y., & Liu, B. (2020). Estimating unknown parameters in uncertain differential equation by maximum likelihood estimation. Technical Report.

Liu, Y. (2021). Uncertain circuit equation. *Journal of Uncertain Systems, 14*(3), 2150018.

Liu, Z. (2021). Generalized moment estimation for uncertain differential equations. *Applied Mathematics and Computation, 392,* 125724.

Liu, Z., & Yang, X. (2021). A linear uncertain pharmacokinetic model driven by Liu process. *Applied Mathematical Modelling, 89*(2), 1881–1899.

Sheng, Y., & Wang, C. (2014). Stability in the p-th moment for uncertain differential equation. *Journal of Intelligent and Fuzzy Systems, 26*(3), 1263–1271.

Sheng, Y., Yao, K., & Chen, X. (2020). Least squares estimation in uncertain differential equations. *IEEE Transactions on Fuzzy Systems, 28*(10), 2651–2655.

Tang, H., & Yang, X. (2021). Uncertain chemical reaction equation. *Applied Mathematics and Computation, 411,* 126479.

Yang, X., & Ralescu, D. (2015). Adams method for solving uncertain differential equations. *Applied Mathematics and Computation, 270,* 993–1003.

Yang, X., & Shen, Y. (2015). Runge-Kutta method for solving uncertain differential equations. *Journal of Uncertainty Analysis and Applications, 3,* 1–12.

Yang, X., Liu, Y., & Park, G. (2020). Parameter estimation of uncertain differential equation with application to financial market. *Chaos, Solitons and Fractals, 139,* 110026.

Yao, K., & Chen, X. (2013). A numerical method for solving uncertain differential equations. *Journal of Intelligent and Fuzzy Systems, 25*(3), 825–832.

Yao, K., Gao, J., & Gao, Y. (2013). Some stability theorems of uncertain differential equation. *Fuzzy Optimization and Decision Making, 12*(1), 3–13.

Yao, K., Ke, H., & Sheng, Y. (2015). Stability in mean for uncertain differential equation. *Fuzzy Optimization and Decision Making, 14*(3), 365–379.

Yao, K., & Liu, B. (2020). Parameter estimation in uncertain differential equations. *Fuzzy Optimization and Decision Making, 19*(1), 1–12.

Yu, R., & Cao, Y. (2017). A method to determine pharmacokinetic parameters based on andante constant-rate intravenous infusion. *Scientific Reports, 7,* 13279.