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Mathematical programming approach to formulate intuitionistic fuzzy regression model based on least absolute deviations

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Abstract
Fuzzy regression models are widely used to investigate the relationship between explanatory and response variables for many decision-making applications in fuzzy environments. To include more fuzzy information in observations, this study uses intuitionistic fuzzy numbers (IFNs) to characterize the explanatory and response variables in formulating intuitionistic fuzzy regression (IFR) models. Different from traditional solution methods, such as the least-squares method, in this study, mathematical programming problems are built up based on the criterion of least absolute deviations to establish IFR models with intuitionistic fuzzy parameters. The proposed approach has the advantages that the model formulation is not limited to the use of symmetric triangular IFNs and the signs of the parameters are determined simultaneously in the model formulation process. The prediction performance of the obtained models is evaluated in terms of similarity and distance measures. Comparison results of the performance measures indicate that the proposed models outperform an existing approach.

Keywords Fuzzy regression model · Intuitionistic fuzzy number · Mathematical programming · Distance criterion

1 Introduction

Regression analysis is a widely used approach for characterizing the relationship between response and explanatory variables. Due to the characteristics of practical observations, shortage of information, or decision-makers' subjective judgment, observations are usually expressed as linguistic terms characterized by membership functions based on fuzzy sets (Zadeh 1965). For fuzzy observations, Tanaka

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et al. (1982) proposed mathematical programming for formulating a fuzzy regression model using numerical explanatory variables and fuzzy responses. A number of fuzzy approaches have been proposed to establish fuzzy regression models with crisp/fuzzy parameters using various types of explanatory and response variables (Celmins 1987; Chang and Lee 1994; D'Urso and Santoro 2006; Chen and Hsueh 2009; Kelkinnama and Taheri 2012; Chen et al. 2017).

Based on the concept of fuzzy sets, Atanassov (1986) proposed intuitionistic fuzzy sets (IFSs), which include both a membership degree and a non-membership degree to express positive and negative information, respectively. IFSs, which contain more information than do fuzzy sets, have been widely studied and applied in various fields (Atannasov 1999). For solving time series problems, IFSs have been applied to neuron network techniques, such as support vector regression (Lin et al. 2016; Hung and Lin 2013), intuitionistic fuzzy inference (Eyoh et al. 2018; Hájek and Olej 2012), and semi-parametric partially logistic regression (Hesamian and Akbari 2017); however, overfitting may occur and the influence power of explanatory variables cannot be known. A few studies have proposed approaches for formulating intuitionistic fuzzy regression (IFR) models (Parvathi et al. 2013; Arefi and Taheri 2015). Parvathi et al. (2013) applied a linear programming problem to determine the symmetric triangular intuitionistic fuzzy number (TIFN) coefficients of an IFR model. In their study, a key task was to determine the upper and lower bounds of observed crisp data using an IFR model in which the intuitionistic fuzziness is minimized by minimizing the support of the determined coefficients. Based on the concept proposed by Tanaka et al. (1982), the approach presented by Parvathi et al. (2013) produces the crisp parameters of explanatory variables for an objective function of the linear programming problem. Arefi and Taheri (2015) proposed an IFR model based on the least-squares method in which the response and explanatory variables are symmetric TIFNs. In their approach, to simplify computation, a multiplication operation of two symmetric TIFNs is used to approximately produce a symmetric TIFN. This operation is used to obtain a general solution formulation to determine symmetric TIFN parameters. In addition, the formulation was derived based on the premise that explanatory variables and the parameters to be determined are all positive; however, a negative parameter was produced in their example.

The present study proposes an approach for formulating IFR models. Acknowledging the developments of fuzzy regression models, some of their advantages are adopted in this study, since IFSs are the extended version of fuzzy sets. For example, as verified in many studies (Kelkinnama and Taheri 2012) fuzzy regression approaches that adopt the least absolute deviation of the distance between the observed and predicted datasets can produce a more robust estimator than that of those based on least-squares deviation (Stahel and Weisberg 2012). In addition, the signs of the determined parameters can greatly affect the performance of the established fuzzy models based on the fuzzy arithmetic operations. However, some approaches presume that all parameters in the model are positive, although negative parameters can be produced (Chen and Hsueh 2009; Arefi and Taheri 2015), which may affect the interpretation of the explanatory variables and result in poor model performance. Particularly, when least-squares approaches are used for fuzzy regression in the formulation process, the signs of the parameters should be predetermined for deriving the solution equations; however, this is impractical for fuzzy regression analyses with multiple explanatory variables.

With the above considerations, the present study proposes an approach for formulating IFR models. Mathematical programming problems with an objective function for minimizing the absolute deviation of distance are built up based on the definitions of intuitionistic fuzzy numbers (IFNs). The signs of the parameters in the IFR model can be determined for the proposed mathematical programming problem to reflect the corresponding IFN operations. In the following section, some basic definitions of IFSs/IFNs and their properties, such as arithmetic operations and a distance measure, are described. In Sect. 3, a general IFR model is formulated, with the signs of the parameters determined in the formulation process. An example is used to demonstrate the proposed approach and for comparison with an existing approach in Sect. 4. Finally, the conclusions are provided in Sect. 5.

2 Background

This section introduces the basic definitions and properties of IFSs (Atanassov 1986), which are a generalization of those for fuzzy sets. IFSs include membership and non-membership degrees.

Definition 1 (Guha and Chakraborty 2010) Let X denote a universe of discourse. An IFS $\tilde{A}$ in X is given by:

$$\tilde{A} = \{(x, \mu_A(x), v_A(x))|x \in R\} \tag{1}$$

where $\mu_A(x), v_A(x) : X \to [0, 1]$ are functions that satisfy $0 \le \mu_A(x)+v_A(x) \le 1$ for all $x \in X$. As shown in Fig. 1, the values of $\mu_A(x)$ and $v_A(x)$ represent membership and non-membership degrees, respectively; then, the hesitancy degree can be defined as $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$.

Definition 2 (Guha and Chakraborty 2010) An IFN is an IFS characterized by:

(1) An IFN is an intuitionistic fuzzy subset defined on the real line.

(2) A unique value $m \in X$ exists, and $\mu_A(m) = 1$ and $v_A(m) = 0$ are met where m is called the mean value of $\tilde{A}$.

Fig. 1 Intuitionistic fuzzy set

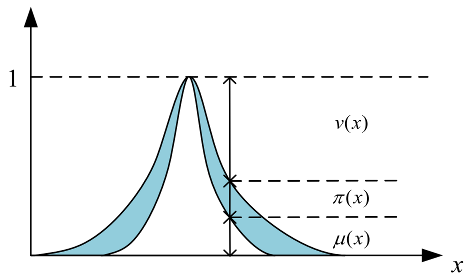
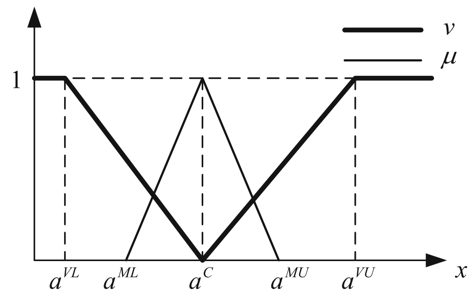


Fig. 2 Triangular intuitionistic fuzzy set



(3) The convexity of the membership function $\mu_A(x)$ is defined as:

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2)), \text{ where } \quad x_1, x_2 \in R \text{ and } \lambda \in [0, 1] \tag{2}$$

(4) The concavity of the non-membership function $\upsilon_A(x)$ is defined as:

$$\upsilon_A(\lambda x_1 + (1-\lambda x_2)) \leq \max(\upsilon_A(x_1), \upsilon_A(x_2)), \text{ where } \quad x_1, x_2 \in R \text{ and } \lambda \in [0, 1] \tag{3}$$

Definition 3 (Mahapatra and Roy 2009) A TIFN $\tilde{A} = (a^{VL}, a^{ML}, a^C, a^{MU}, a^{VU})$ is an IFS in R with the following membership function, $\mu_A(x)$, and non-membership function, $\upsilon_A(x)$, respectively:

$$\mu_A(x) = \begin{cases} \frac{x-a^{ML}}{a^C-a^{ML}}, & a^{ML} \leq x \leq a^C \\ \frac{a^{MU}-x}{a^{MU}-a^C}, & a^C \leq x \leq a^{MU} \\ 0, & \text{otherwise} \end{cases} \text{ and } \upsilon_A(x) = \begin{cases} 1 - \frac{x-a^{VL}}{a^C-a^{VL}}, & a^{VL} \leq x \leq a^C \\ 1 - \frac{a^{VU}-x}{a^{VU}-a^C}, & a^C \leq x \leq a^{VU} \\ 1, & \text{otherwise} \end{cases} \tag{4}$$

For the TIFN shown in Fig. 2, a^C is called the central value; a^{ML} and a^{MU} are the lower and upper bounds of membership, respectively; a^{VL} and a^{VU} are the lower and upper bounds of non-membership, respectively.

Definition 4 (Chakraborty et al. 2014) A TIFN $\tilde{A} = (a^{VL}, a^{ML}, a^C, a^{MU}, a^{VU})$ is called positive, i.e., $\tilde{A} > 0$, if $a^{VL} > 0$; it is called negative, i.e., $\tilde{A} < 0$, if $a^{VU} < 0$. In addition, $\tilde{A} \geq 0$ implies $a^{VL} \geq 0$.

Definition 5 (Guha and Chakraborty 2010) The α-cuts of an IFN $\tilde{A}$ are defined as:

$$\tilde{A}_\alpha = \{\langle x, \mu(x), \upsilon(x)\rangle | \mu(x) \geq \alpha \text{ and } \upsilon(x) \leq 1-\alpha, \ \alpha \in [0, 1]\} \tag{5}$$

The inequality $\upsilon_A(x) \leq 1-\alpha$ is equivalent to $1-\upsilon_A(x) \geq \alpha$ and thus $\tilde{A}_\alpha$ can be expressed as the crisp sets $\tilde{A}_\mu(\alpha) = \{x : \mu_A(x) \geq \alpha\}$ and $\tilde{A}_{1-\upsilon}(\alpha) =$

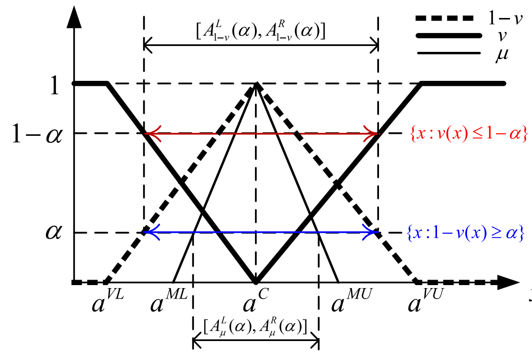


Fig. 3 α-cuts of a TIFN

$\{x : 1 - v_A(x) \geq \alpha\}$. Alternatively, $\tilde{A}_\alpha$ can be represented by the following pair of intervals:

$$\tilde{A}_\alpha = \left\{[\tilde{A}^L_\mu(\alpha), \tilde{A}^R_\mu(\alpha)], [\tilde{A}^L_{1-v}(\alpha), \tilde{A}^R_{1-v}(\alpha)]\right\} \tag{6}$$

Figure 3 shows that the two crisp sets $\{x : v_A(x) \leq 1 - \alpha\}$ and $\tilde{A}_{1-v}(\alpha) = \{x : 1 - v_A(x) \geq \alpha\}$ have the same intervals. For simplicity and compatibility with a previous study (Arefi and Taheri 2015), the notation of $\tilde{A}_{1-v}(\alpha)$ is adopted hereafter.

Based on this definition, the α-cuts of a TIFN $\tilde{A}$ can be formulated in the following general form:

$$\begin{aligned}\tilde{A}_\alpha = \Big\{&\Big[a^{ML} + \alpha(a^C - a^{ML}), a^{MU} - \alpha(a^{MU} - a^C)\Big],\\ &\Big[a^{VL} + \alpha(a^C - a^{VL}), a^{VU} - \alpha(a^{VU} - a^C)\Big]\Big\}\end{aligned} \tag{7}$$

The two extreme cases are $\tilde{A}_{\alpha=0} = \{[a^{ML}, a^{MU}], [a^{VL}, a^{VU}]\}$ and $\tilde{A}_{\alpha=1} = \{[a^C], [a^C]\}$.

Definition 6 (Mahapatra and Roy 2009) Let $\tilde{A} = (a^{VL}, a^{ML}, a^C, a^{MU}, a^{VU})$ and $\tilde{B} = (b^{VL}, b^{ML}, b^C, b^{MU}, b^{VU})$ be two TIFNs. Based on the extension principle, the sum of two TIFNs can be formulated as:

$$\tilde{A} \oplus \tilde{B} = (a^{VL} + b^{VL}, a^{ML} + b^{ML}, a^C + b^C, a^{MU} + b^{MU}, a^{VU} + b^{VU}) \tag{8}$$

The multiplication of a TIFN and a constant k can be expressed as:

$$k\tilde{A} = (ka^{VL}, ka^{ML}, ka^C, ka^{MU}, ka^{VU}), \quad \text{if } k \geq 0 \tag{9}$$

$$k\tilde{A} = (ka^{VU}, ka^{MU}, ka^C, ka^{ML}, ka^{VL}), \quad \text{if } k < 0 \tag{10}$$

The multiplication of two TIFNs can be approximately determined using the following equations based on the signs of two TIFNs:

$$\tilde{A}\otimes\tilde{B}\cong(a^{VL}b^{VL},a^{ML}b^{ML},a^{C}b^{C},a^{MU}b^{MU},a^{VU}b^{VU}),\quad \text{if } \tilde{A}\geq 0 \text{ and } \tilde{B}\geq 0 \tag{11}$$

$$\tilde{A}\otimes\tilde{B}\cong(a^{VU}b^{VL},a^{MU}b^{ML},a^{C}b^{C},a^{ML}b^{MU},a^{VL}b^{VU}),\quad \text{if } \tilde{A}\geq 0 \text{ and } \tilde{B}\leq 0 \tag{12}$$

$$\tilde{A}\otimes\tilde{B}\cong(a^{VU}b^{VU},a^{MU}b^{MU},a^{C}b^{C},a^{ML}b^{ML},a^{VL}b^{VL}),\quad \text{if } \tilde{A}\leq 0 \text{ and } \tilde{B}\leq 0 \tag{13}$$

Property 1 Define a zero TIFN as $\tilde{0}=(0,0,0,0,0)$. Based on the definition of the arithmetic operators of TIFNs, the multiplication of any two TIFNs with different signs, i.e., $\tilde{A}\geq 0$ and $\tilde{B}\leq 0$, will result in zero if and only if $\tilde{A}$ or $\tilde{B}$ is zero.

Proof The case of and $\tilde{B}$ indicates that $0\leq a^{VL}\leq a^{C}\leq a^{MU}\leq a^{VU}$ and $b^{VL}\leq b^{ML}\leq b^{C}\leq b^{MU}\leq b^{VU}\leq 0$ based on the above definition. In addition, the multiplication of $\tilde{A}\otimes\tilde{B}=(a^{VU}b^{VL},a^{MU}b^{ML},a^{C}b^{C},a^{ML}b^{MU},a^{VL}b^{VU})=\{0\}$ implies that $a^{VU}b^{VL}=a^{MU}b^{ML}=a^{C}b^{C}=a^{ML}b^{MU}=a^{VL}b^{VU}=0$. Based on the definition of TIFNs, the inequalities $a^{VU}b^{VL}\leq a^{MU}b^{ML}\leq a^{C}b^{C}\leq a^{ML}b^{MU}\leq a^{VL}b^{VU}$ hold. $\tilde{A}\neq\{0\}$, i.e., $a^{C}>0$, implies that $b^{C}=0$. In addition, the constraint of $0<a^{C}\leq a^{MU}\leq a^{VU}$ makes $b^{VL}=b^{ML}=0$ and then $a^{VU}b^{VL}=a^{MU}b^{ML}=0$ is satisfied. The constraint of $b^{C}\leq b^{MU}\leq b^{VU}\leq 0$ implies that $b^{MU}=b^{VU}=0$ since $b^{C}=0$. Therefore, the TIFN $\tilde{B}$ is a zero TIFN.

Definition 7 (Grzegorzewski 2003) The distance between two IFNs, $\tilde{A}$ and $\tilde{B}$, can be measured by calculating the integral of the average absolute difference of all α-cuts with a parameter p, where $1\leq p\leq\infty$. The distance measure, $D_p(\tilde{A},\tilde{B})$, can be denoted as:

$$D_p(\tilde{A},\tilde{B})=\Bigg(\frac{1}{4}\int_0^1\left|\tilde{A}^L_{\mu}(\alpha)-\tilde{B}^L_{\mu}(\alpha)\right|^p d\alpha+\frac{1}{4}\int_0^1\left|\tilde{A}^R_{\mu}(\alpha)-\tilde{B}^R_{\mu}(\alpha)\right|^p d\alpha$$
$$+\frac{1}{4}\int_0^1\left|\tilde{A}^L_{1-v}(\alpha)-\tilde{B}^L_{1-v}(\alpha)\right|^p d\alpha+\frac{1}{4}\int_0^1\left|\tilde{A}^R_{1-v}(\alpha)-\tilde{B}^R_{1-v}(\alpha)\right|^p d\alpha\Bigg)^{1/p}. \tag{14}$$

Based on Eq. (14), the distance measure is the average of the absolute distance difference between the two-side membership (non-membership) functions of the two IFNs. When the IFNs $\tilde{A}$ and $\tilde{B}$ are triangular, i.e., TIFNs, and $p=1$, the integral of $|\tilde{A}^L_{\mu}(\alpha)-\tilde{B}^L_{\mu}(\alpha)|$ with $0\leq\alpha\leq 1$, i.e., the absolute distance difference between the left-hand-side membership functions of $\tilde{A}$ and $\tilde{B}$, will yield either a trapezoidal area (Fig. 4a) or two triangular areas (Fig. 4b), where the top-side length is $|\tilde{A}^L_{\mu}(1)-\tilde{B}^L_{\mu}(1)|$, the bottom-side length is $|\tilde{A}^L_{\mu}(0)-\tilde{B}^L_{\mu}(0)|$, and the height is 1. Based on basic geometry, if the signs of $(\tilde{A}^L_{\mu}(1)-\tilde{B}^L_{\mu}(1))$ and $(\tilde{A}^L_{\mu}(0)-\tilde{B}^L_{\mu}(0))$ are the same,

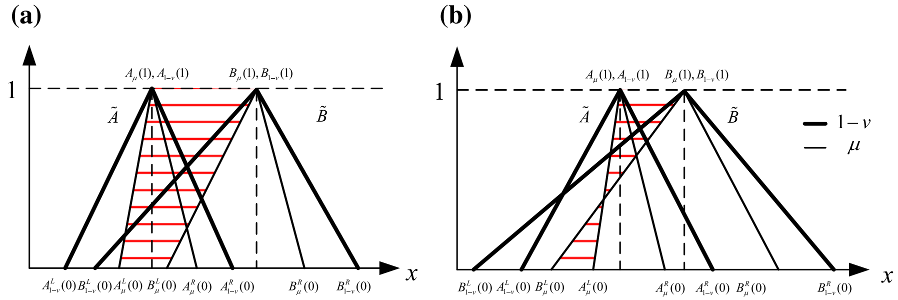


Fig. 4 Integral of the difference of the left spread of membership between two TIFNs

i.e., $\left(\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)\right)\times\left(\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)\right)\geq 0$, a trapezoidal area will be produced; otherwise, two triangular areas are obtained. The area of the former can be expressed as $\frac{1}{2}(|\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)|+|\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)|)$, and that of the latter is $\frac{1}{4}(|\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)| + |\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)|)$. Let $D^{ML}(\tilde{A},\tilde{B})$ denote the integral of $|\tilde{A}_\mu^L(\alpha)-\tilde{B}_\mu^L(\alpha)|$ with $0\leq\alpha\leq 1$; it can be formulated as:

$$
\begin{aligned}
D^{ML}(\tilde{A},\tilde{B}) &= \int_0^1 \left|\tilde{A}_\mu^L(\alpha)-\tilde{B}_\mu^L(\alpha)\right|d\alpha \\
&= \begin{cases} \frac{1}{2}\left(\left|\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)\right|+\left|\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)\right|\right), \\ \text{if } \left(\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)\right)\left(\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)\right)\geq 0 \\ \frac{1}{4}\left(\left|\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)\right|+\left|\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)\right|\right), \\ \text{if } \left(\tilde{A}_\mu^L(1)-\tilde{B}_\mu^L(1)\right)\left(\tilde{A}_\mu^L(0)-\tilde{B}_\mu^L(0)\right)< 0 \end{cases} \\
&= \begin{cases} \frac{1}{2}\left(\left|a^C-b^C\right|+\left|a^{ML}-b^{ML}\right|\right), & \text{if } (a^C-b^C)(a^{ML}-b^{ML})\geq 0 \\ \frac{1}{4}\left(\left|a^C-b^C\right|+\left|a^{ML}-b^{ML}\right|\right), & \text{if } (a^C-b^C)(a^{ML}-b^{ML})< 0 \end{cases}
\end{aligned}
\tag{15}
$$

Similarly, the other three components of the distance measure can be determined as:

$$
D^{VL}(\tilde{A},\tilde{B}) = \begin{cases} \frac{1}{2}\left(\left|a^C-b^C\right|+\left|a^{VL}-b^{VL}\right|\right), & \text{if } (a^C-b^C)(a^{VL}-b^{VL})\geq 0 \\ \frac{1}{4}\left(\left|a^C-b^C\right|+\left|a^{VL}-b^{VL}\right|\right), & \text{if } (a^C-b^C)(a^{VL}-b^{VL})< 0 \end{cases}
\tag{16}
$$

$$D^{MU}(\tilde{A},\tilde{B}) = \begin{cases} \frac{1}{2}\left(\left|a^C - b^C\right| + \left|a^{MU} - b^{MU}\right|\right), & \text{if } (a^C - b^C)(a^{MU} - b^{MU}) \geq 0 \\ \frac{1}{4}\left(\left|a^C - b^C\right| + \left|a^{MU} - b^{MU}\right|\right), & \text{if } (a^C - b^C)(a^{MU} - b^{MU}) < 0 \end{cases} \tag{17}$$

$$D^{VU}(\tilde{A},\tilde{B}) = \begin{cases} \frac{1}{2}\left(\left|a^C - b^C\right| + \left|a^{VU} - b^{VU}\right|\right), & \text{if } (a^C - b^C)(a^{VU} - b^{VU}) \geq 0 \\ \frac{1}{4}\left(\left|a^C - b^C\right| + \left|a^{VU} - b^{VU}\right|\right), & \text{if } (a^C - b^C)(a^{VU} - b^{VU}) < 0 \end{cases} \tag{18}$$

Therefore, the distance measure of two TIFNs can be reformulated as the average of the above four kinds of distance measure as follows:

$$D_{TIFN}(\tilde{A},\tilde{B}) = \frac{1}{4}\left(D^{VL}(\tilde{A},\tilde{B}) + D^{ML}(\tilde{A},\tilde{B}) + D^{MU}(\tilde{A},\tilde{B}) + D^{VU}(\tilde{A},\tilde{B})\right) \tag{19}$$

The above formulation can be considered as a general distance measure for measuring the distance between two TIFNs.

Definition 8 (Arefi and Taheri 2015) A similarity measure of two TIFNs $\tilde{A}$ and $\tilde{B}$ is defined as:

$$S(\tilde{A},\tilde{B}) = 1 - \frac{1}{2}\left[\frac{\int_{-\infty}^{\infty} |\mu_A(x) - \mu_B(x)|dx}{\int_{-\infty}^{\infty} \mu_A(x)dx + \int_{-\infty}^{\infty} \mu_B(x)dx} + \frac{\int_{-\infty}^{\infty} |v_A(x) - v_B(x)|dx}{\int_{-\infty}^{\infty} (1 - v_A(x))dx + \int_{-\infty}^{\infty} (1 - v_B(x))dx}\right] \tag{20}$$

with the value of $0 \leq S(\tilde{A},\tilde{B}) \leq 1$. This index is measured in terms of the area of the average difference between the membership and non-membership functions of two IFNs. However, the degree of difference cannot be determined if the two IFNs have no interactions.

Definition 9 The general formula of a different distance measure based on squared errors between two TIFNs, proposed by Arefi and Taheri (2015), is:

$$\begin{aligned} d^2(\tilde{A},\tilde{B}) = {} & \left(a^C - b^C\right)^2 + \frac{1}{24}\left[\left(s_B^{ML} - s_A^{ML}\right)^2 + \left(s_B^{MR} - s_A^{MR}\right)^2 + \left(s_B^{VL} - s_A^{VL}\right)^2 \right. \\ & \left. + \left(s_B^{VR} - s_A^{VR}\right)^2\right] + \frac{1}{6}(m_A - m_B)\left[\left(s_B^{ML} - s_A^{ML}\right) - \left(s_B^{MR} - s_A^{MR}\right) \right. \\ & \left. + \left(s_B^{VL} - s_A^{VL}\right) - \left(s_B^{VR} - s_A^{VR}\right)\right] \end{aligned} \tag{21}$$

where $s_A^{ML} = a^C - a^{ML}$ and $s_A^{MR} = a^{MU} - a^C$ are called the left and right spreads of the membership function, respectively; similarly, $s_A^{VL} = a^C - a^{VL}$ and $s_A^{VR} = a^{VU} - a^C$ are the left and right spreads of the non-membership function, respectively.

Useful measures are critical for evaluating IFR model performance. Arefi and Taheri (2015) adopted the similarity measure $S(\tilde{A}, \tilde{B})$ and squared error distance $d^2(\tilde{A}, \tilde{B})$ to evaluate the performance of their proposed IFR approach with TIFN explanatory and response datasets. Besides these two measures, the distance measure $D_{TIFN}(\tilde{A}, \tilde{B})$ in terms of the average absolute difference of two TIFNs is used in this study to compare the performance of the proposed approach with that proposed by Arefi and Taheri (2015).

3 Formulations

This study builds up an IFR model based on the criterion of the least absolute difference of distance. With the least absolute deviations criterion, mathematical programming problems are formulated to determine the optimal parameters of TIFNs to minimize the total distance between the observation and prediction variables. To achieve this end, the general distance measure, $D_{TIFN}(\tilde{A}, \tilde{B})$, expressed in Eq. (19), is used as the objective function in the mathematical programming problems. In addition, for comparison with an existing approach (Arefi and Taheri 2015), the observed, predicted, and explanatory variables and parameters are expressed as TIFNs in Definition 3.

Consider the intuitionistic fuzzy observation set $(\tilde{Y}_i, \tilde{X}_{1i}, \tilde{X}_{2i}, \ldots, \tilde{X}_{ji}, \ldots, \tilde{X}_{pi})$, where $\tilde{Y}_i = (y_i^{VL}, y_i^{ML}, y_i^{C}, y_i^{MU}, y_i^{VU})$ is the response variable and $\tilde{X}_{ji} = (x_{ji}^{VL}, x_{ji}^{ML}, x_{ji}^{C}, x_{ji}^{MU}, x_{ji}^{VU})$ represents the jth explanatory variable in the form of TIFNs. The general IFR model can be expressed as:

$$\tilde{Y}_i = \tilde{B}_0 \oplus \tilde{B}_1 \otimes \tilde{X}_{1i} \oplus \tilde{B}_2 \otimes \tilde{X}_{2i} \oplus \cdots \oplus \tilde{B}_p \otimes \tilde{X}_{pi} = \sum_{j=0}^{p} \tilde{B}_j \otimes \tilde{X}_{ji} \tag{22}$$

where $\tilde{B}_j = (b_j^{VL}, b_j^{ML}, b_j^{C}; b_j^{MU}, b_j^{VU})$ is the corresponding intuitionistic fuzzy parameters, and $\tilde{X}_{0i} = (1, 1, 1, 1, 1)$ is specified. Let the predicted fuzzy response variable be denoted as $\hat{\tilde{Y}}_i = (\hat{y}_i^{VL}, \hat{y}_i^{ML}, \hat{y}_i^{C}, \hat{y}_i^{MU}, \hat{y}_i^{VU})$; then, the predicted IFR model can be formulated as:

$$\hat{\tilde{Y}}_i = \hat{\tilde{B}}_0 \oplus (\hat{\tilde{B}}_1 \otimes \tilde{X}_{1i}) \oplus (\hat{\tilde{B}}_2 \otimes \tilde{X}_{2i}) \oplus \cdots \oplus (\hat{\tilde{B}}_p \otimes \tilde{X}_{pi}) = \sum_{j=0}^{p} \hat{\tilde{B}}_j \otimes \tilde{X}_{ji} \tag{23}$$

where $\hat{\tilde{B}}_j = (\hat{b}_j^{VL}, \hat{b}_j^{ML}, \hat{b}_j^{C}; \hat{b}_j^{MU}, \hat{b}_j^{VU})$ is the jth estimated intuitionistic fuzzy parameter of the TIFNs. Consider a model with one explanatory variable, i.e., $\hat{\tilde{Y}}_i = \hat{\tilde{B}}_0 \oplus (\hat{\tilde{B}}_1 \otimes \tilde{X}_i)$. Suppose that the TIFN parameter $\hat{\tilde{B}}_1$ is negative, i.e., $\hat{\tilde{B}}_1 \leq 0$; then, based on the arithmetic operator in Definition 6, the predicted response TIFN $\hat{\tilde{Y}}_i$ is determined as:

$$\begin{cases} \hat{y}_i^{VL} = \hat{b}_0^{VL} + \hat{b}_1^{VL} x_i^{VU} \\ \hat{y}_i^{ML} = \hat{b}_0^{ML} + \hat{b}_1^{ML} x_i^{MU} \\ \hat{y}_i^{C} = \hat{b}_0^{C} + \hat{b}_1^{C} x_i^{C} \\ \hat{y}_i^{MU} = \hat{b}_0^{MU} + \hat{b}_1^{MU} x_i^{ML} \\ \hat{y}_i^{VU} = \hat{b}_0^{VU} + \hat{b}_1^{VU} x_i^{VL} \end{cases} \tag{24}$$

Alternatively, suppose that this TIFN parameter is positive and denoted as $\hat{\tilde{B}}_2$, i.e., $\hat{\tilde{B}}_2 \geq 0$; then, the TIFN $\hat{\tilde{Y}}_i$ can be expressed as:

$$\begin{cases} \hat{y}_i^{VL} = \hat{b}_0^{VL} + \hat{b}_2^{VL} x_i^{VL} \\ \hat{y}_i^{ML} = \hat{b}_0^{ML} + \hat{b}_2^{ML} x_i^{ML} \\ \hat{y}_i^{C} = \hat{b}_0^{C} + \hat{b}_2^{C} x_i^{C} \\ \hat{y}_i^{MU} = \hat{b}_0^{MU} + \hat{b}_2^{MU} x_i^{MU} \\ \hat{y}_i^{VU} = \hat{b}_0^{VU} + \hat{b}_2^{VU} x_i^{VU} \end{cases} \tag{25}$$

The signs of the explanatory TIFN parameters are unknown, which influences the IFR model performance. To overcome this problem, this study sets two dummy TIFNs with different signs, i.e., $\hat{\tilde{B}}_1 \leq 0$ and $\hat{\tilde{B}}_2 \geq 0$. Based on Property 1, if an IFR model has the formulation $\hat{\tilde{Y}}_i = \hat{\tilde{B}}_0 \oplus ((\hat{\tilde{B}}_1 \oplus \hat{\tilde{B}}_2) \otimes \tilde{X}_i)$ subject to $\hat{\tilde{B}}_1 \otimes \hat{\tilde{B}}_2 = 0$, then one dummy TIFN parameter will be determined as the optimal parameter for the explanatory variable and the other one will be zero. Therefore, the predicted TIFN response will have two dummy TIFN parameters for each explanatory TIFN variable in the mathematical programming problems, in which the constraint of their product being zero is added. For the predicted TIFN response with one explanatory variable, the formulations in the mathematical programming problems are:

$$\begin{cases} \hat{y}_i^{VL} = \hat{b}_0^{VL} + \hat{b}_1^{VL} x_i^{VU} + \hat{b}_2^{VL} x_i^{VL} \\ \hat{y}_i^{ML} = \hat{b}_0^{ML} + \hat{b}_1^{ML} x_i^{MU} + \hat{b}_2^{ML} x_i^{ML} \\ \hat{y}_i^{C} = \hat{b}_0^{C} + \hat{b}_1^{C} x_i^{C} + \hat{b}_2^{C} x_i^{C} \\ \hat{y}_i^{MU} = \hat{b}_0^{MU} + \hat{b}_1^{MU} x_i^{ML} + \hat{b}_2^{MU} x_i^{MU} \\ \hat{y}_i^{VU} = \hat{b}_0^{VU} + \hat{b}_1^{VU} x_i^{VL} + \hat{b}_2^{VU} x_i^{VU} \end{cases} \tag{26}$$

With multiple TIFN explanatory variables, the mathematical programming problems are formulated as Eq. (27), in which the objective function $D_{TIFN}(\tilde{A}, \tilde{B})$ in Eq. (19) is adopted to determine the optimal parameters in order to minimize the distance between observed and predicted TIFN responses.

$$\min \sum_{i=1}^{n} D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i) = \frac{1}{4} \sum_{i=1}^{n} \left(D^{VL}(\tilde{Y}_i, \hat{\tilde{Y}}_i) + D^{ML}(\tilde{Y}_i, \hat{\tilde{Y}}_i) + D^{MU}(\tilde{Y}_i, \hat{\tilde{Y}}_i) + D^{VU}(\tilde{Y}_i, \hat{\tilde{Y}}_i) \right)$$

$$
\begin{aligned}
\text{s.t.}\quad & \hat{y}_i^{VL} = \hat{b}_0^{VL} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VL}x_{ji}^{VU} + \hat{b}_{j2}^{VL}x_{ji}^{VL}\right] \\
& \hat{y}_i^{ML} = \hat{b}_0^{ML} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{ML}x_{ji}^{MU} + \hat{b}_{j2}^{ML}x_{ji}^{ML}\right] \\
& \hat{y}_i^{C} = \hat{b}_0^{C} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{C} + \hat{b}_{j2}^{C}\right]x_{ji}^{C} \\
& \hat{y}_i^{MU} = \hat{b}_0^{MU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{MU}x_{ji}^{ML} + \hat{b}_{j2}^{MU}x_{ji}^{MU}\right] \\
& \hat{y}_i^{VU} = \hat{b}_0^{VU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VU}x_{ji}^{VL} + \hat{b}_{j2}^{VU}x_{ji}^{VU}\right] \\
& \hat{b}_{j1}^{C} \le 0;\ \hat{b}_{j2}^{C} \ge 0 \\
& \hat{b}_{jk}^{VL} \le \hat{b}_{jk}^{ML} \le \hat{b}_{jk}^{C} \le \hat{b}_{jk}^{MU} \le \hat{b}_{jk}^{VU} \\
& \hat{b}_{j1}^{VL}\hat{b}_{j2}^{VU} = \hat{b}_{j1}^{ML}\hat{b}_{j2}^{MU} = \hat{b}_{j1}^{C}\hat{b}_{j2}^{C} = \hat{b}_{j1}^{MU}\hat{b}_{j2}^{ML} = \hat{b}_{j1}^{VU}\hat{b}_{j2}^{VL} = 0 \\
& i = 1, \cdots, n,\ j = 1, \ldots, p,\ k = 1, 2
\end{aligned}
\tag{27}
$$

The last three constraints in the above model restrict the two dummy parameters, $\hat{\hat{B}}_{j1}$, and $\hat{\hat{B}}_{j2}$, for each explanatory variable with different signs. The zero restriction, $\hat{\hat{B}}_{j1} \otimes \hat{\hat{B}}_{j2} = 0$, holds for all dummy parameters. The resulting parameters should follow the definition of a TIFN.

In addition, the objective function in Eq. (27) is the average of $D^{VL}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, $D^{ML}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, $D^{MU}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, and $D^{VU}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, in which correct formulations should be decided for each measure in the resolution process based on Eqs. (15)–(18). To deal with this problem, a pair of dummy binary variables, d_{1i}^{VL} and d_{2i}^{VL}, is added in $D^{VL}(\tilde{Y}_i, \hat{\hat{Y}}_i)$; they can be reformulated as:

$$
\begin{aligned}
D^{VL}(\tilde{Y}_i, \hat{\hat{Y}}_i) &= d_{1i}^{VL}\frac{1}{2}\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{VL} - \hat{y}_i^{VL}\right|\right) + d_{2i}^{VL}\frac{1}{4}\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{VL} - \hat{y}_i^{VL}\right|\right) \\
&= \left(\frac{1}{2}d_{1i}^{VL} + \frac{1}{4}d_{2i}^{VL}\right)\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{VL} - \hat{y}_i^{VL}\right|\right)
\end{aligned}
\tag{28}
$$

Additional constraints of Eq. (29) are also added in the mathematical programming problems.

$$
\begin{aligned}
& (d_{1i}^{VL} - d_{2i}^{VL})(y_i^C - \hat{y}_i^C)(y_i^{VL} - \hat{y}_i^{VL}) \ge 0 \\
& d_{1i}^{VL} + d_{2i}^{VL} = 1,\ d_{1i}^{VL}, d_{2i}^{VL} \in \{0, 1\}
\end{aligned}
\tag{29}
$$

The constraints guarantee that when $(y_i^C - \hat{y}_i^C)(y_i^{VL} - \hat{y}_i^{VL}) \ge 0$, then $d_{1i}^{VL} = 1$ and $d_{2i}^{VL} = 0$; otherwise, $d_{1i}^{VL} = 0$ and $d_{2i}^{VL} = 1$, and $D^{VL}(\tilde{Y}_i, \hat{\hat{Y}}_i)$ in Eq. (28) is obtained. Similarly, $D^{ML}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, $D^{MU}(\tilde{Y}_i, \hat{\hat{Y}}_i)$, and $D^{VU}(\tilde{Y}_i, \hat{\hat{Y}}_i)$ can be determined. Therefore, Eq. (27) becomes:

$$
\begin{aligned}
\min \sum_{i=1}^{n} D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i) = \frac{1}{4}\sum_{i=1}^{n} \Bigg\{ & \left(\frac{d_{1i}^{VL}}{2} + \frac{d_{2i}^{VL}}{4}\right)\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{VL} - \hat{y}_i^{VL}\right|\right) \\
& + \left(\frac{d_{1i}^{ML}}{2} + \frac{d_{2i}^{ML}}{4}\right)\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{ML} - \hat{y}_i^{ML}\right|\right) \\
& + \left(\frac{d_{1i}^{MU}}{2} + \frac{d_{2i}^{MU}}{4}\right)\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{MU} - \hat{y}_i^{MU}\right|\right) \\
& + \left(\frac{d_{1i}^{VU}}{2} + \frac{d_{2i}^{VU}}{4}\right)\left(\left|y_i^C - \hat{y}_i^C\right| + \left|y_i^{VU} - \hat{y}_i^{VU}\right|\right) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.}\quad & \hat{y}_i^{VL} = \hat{b}_0^{VL} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VL} x_{ji}^{VU} + \hat{b}_{j2}^{VL} x_{ji}^{VL}\right] \\
& \hat{y}_i^{ML} = \hat{b}_0^{ML} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{ML} x_{ji}^{MU} + \hat{b}_{j2}^{ML} x_{ji}^{ML}\right] \\
& \hat{y}_i^{C} = \hat{b}_0^{C} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{C} + \hat{b}_{j2}^{C}\right] x_{ji}^{C} \\
& \hat{y}_i^{MU} = \hat{b}_0^{MU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{MU} x_{ji}^{ML} + \hat{b}_{j2}^{MU} x_{ji}^{MU}\right] \\
& \hat{y}_i^{VU} = \hat{b}_0^{VU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VU} x_{ji}^{VL} + \hat{b}_{j2}^{VU} x_{ji}^{VU}\right] \\
& (d_{1i}^{VL} - d_{2i}^{VL})(y_i^C - \hat{y}_i^C)(y_i^{VL} - \hat{y}_i^{VL}) \geq 0 \\
& (d_{1i}^{ML} - d_{2i}^{ML})(y_i^C - \hat{y}_i^C)(y_i^{ML} - \hat{y}_i^{ML}) \geq 0 \\
& (d_{1i}^{MU} - d_{2i}^{MU})(y_i^C - \hat{y}_i^C)(y_i^{MU} - \hat{y}_i^{MU}) \geq 0 \\
& (d_{1i}^{VU} - d_{2i}^{VU})(y_i^C - \hat{y}_i^C)(y_i^{VU} - \hat{y}_i^{VU}) \geq 0 \\
& d_{1i}^{VL} + d_{2i}^{VL} = 1,\, d_{1i}^{ML} + d_{2i}^{ML} = 1,\, d_{1i}^{MU} + d_{2i}^{MU} = 1,\, d_{1i}^{VU} + d_{2i}^{VU} = 1 \\
& d_{1i}^{VL}, d_{2i}^{VL}, d_{1i}^{ML}, d_{2i}^{ML}, d_{1i}^{MU}, d_{2i}^{MU}, d_{1i}^{VU}, d_{2i}^{VU} \in \{0, 1\} \\
& \hat{b}_{j1}^{C} \leq 0;\ \hat{b}_{j2}^{C} \geq 0 \\
& \hat{b}_{jk}^{VL} \leq \hat{b}_{jk}^{ML} \leq \hat{b}_{jk}^{C} \leq \hat{b}_{jk}^{MU} \leq \hat{b}_{jk}^{VU} \\
& \hat{b}_{j1}^{VL}\hat{b}_{j2}^{VU} = \hat{b}_{j1}^{ML}\hat{b}_{j2}^{MU} = \hat{b}_{j1}^{C}\hat{b}_{j2}^{C} = \hat{b}_{j1}^{MU}\hat{b}_{j2}^{ML} = \hat{b}_{j1}^{VU}\hat{b}_{j2}^{VL} = 0 \\
& i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, 2
\end{aligned} \tag{30}
$$

Furthermore, considering that the objective function in Eq. (30) is expressed as the sum of the absolute difference between observed and predicted TIFN responses, this will increase computational efforts. To deal with this problem, an efficient approach can be applied to enhance the computational efficiency. For example, let M_i^1 denote

$\max\{y_i^C - \hat{y}_i^C, 0\}$ and M_i^2 be $\max\{\hat{y}_i^C - y_i^C, 0\}$. It is easy to show that $M_i^1 + M_i^2$ is equivalent to $|y_i^C - \hat{y}_i^C|$ and that $M_i^1 - M_i^2$ is equivalent to $y_i^C - \hat{y}_i^C$. Thus, the model can be reformulated as:

$$
\begin{aligned}
\min \sum_{i=1}^{n} D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i) = \frac{1}{4}\sum_{i=1}^{n}\Bigg\{&\left(\frac{d_{1i}^{VL}}{2} + \frac{d_{2i}^{VL}}{4}\right)\left(M_i^1 + M_i^2 + VL_i^1 + VL_i^2\right) \\
&+\left(\frac{d_{1i}^{ML}}{2} + \frac{d_{2i}^{ML}}{4}\right)\left(M_i^1 + M_i^2 + ML_i^1 + ML_i^2\right) \\
&+\left(\frac{d_{1i}^{MU}}{2} + \frac{d_{2i}^{MU}}{4}\right)\left(M_i^1 + M_i^2 + MU_i^1 + MU_i^2\right) \\
&+\left(\frac{d_{1i}^{VU}}{2} + \frac{d_{2i}^{VU}}{4}\right)\left(M_i^1 + M_i^2 + VU_i^1 + VU_i^2\right)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.}\quad & M_i^1 - M_i^2 = y_i^C - \hat{y}_i^C \\
& VL_i^1 - VL_i^2 = y_i^{VL} - \hat{y}_i^{VL} \\
& ML_i^1 - ML_i^2 = y_i^{ML} - \hat{y}_i^{ML} \\
& MU_i^1 - MU_i^2 = y_i^{MU} - \hat{y}_i^{MU} \\
& VU_i^1 - VU_i^2 = y_i^{VU} - \hat{y}_i^{VU} \\
& \hat{y}_i^{VL} = \hat{b}_0^{VL} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VL} x_{ji}^{VU} + \hat{b}_{j2}^{VL} x_{ji}^{VL}\right] \\
& \hat{y}_i^{ML} = \hat{b}_0^{ML} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{ML} x_{ji}^{MU} + \hat{b}_{j2}^{ML} x_{ji}^{ML}\right] \\
& \hat{y}_i^{C} = \hat{b}_0^{C} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{C} + \hat{b}_{j2}^{C}\right] x_{ji}^{C} \\
& \hat{y}_i^{MU} = \hat{b}_0^{MU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{MU} x_{ji}^{ML} + \hat{b}_{j2}^{MU} x_{ji}^{MU}\right] \\
& \hat{y}_i^{VU} = \hat{b}_0^{VU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VU} x_{ji}^{VL} + \hat{b}_{j2}^{VU} x_{ji}^{VU}\right] \\
& (d_{1i}^{VL} - d_{2i}^{VL})(M_i^1 - M_i^2)(VL_i^1 - VL_i^2) \geq 0 \\
& (d_{1i}^{ML} - d_{2i}^{ML})(M_i^1 - M_i^2)(ML_i^1 - ML_i^2) \geq 0 \\
& (d_{1i}^{MU} - d_{2i}^{MU})(M_i^1 - M_i^2)(MU_i^1 - MU_i^2) \geq 0 \\
& (d_{1i}^{VU} - d_{2i}^{VU})(M_i^1 - M_i^2)(VU_i^1 - VU_i^2) \geq 0 \\
& d_{1i}^{VL} + d_{2i}^{VL} = 1, d_{1i}^{ML} + d_{2i}^{ML} = 1, d_{1i}^{MU} + d_{2i}^{MU} = 1, d_{1i}^{VU} + d_{2i}^{VU} = 1 \\
& d_{1i}^{VL}, d_{2i}^{VL}, d_{1i}^{ML}, d_{2i}^{ML}, d_{1i}^{MU}, d_{2i}^{MU}, d_{1i}^{VU}, d_{2i}^{VU} \in \{0, 1\} \\
& \hat{b}_{j1}^{C} \leq 0;\ \hat{b}_{j2}^{C} \geq 0 \\
& \hat{b}_{jk}^{VL} \leq \hat{b}_{jk}^{ML} \leq \hat{b}_{jk}^{C} \leq \hat{b}_{jk}^{MU} \leq \hat{b}_{jk}^{VU} \\
& \hat{b}_{j1}^{VL}\hat{b}_{j2}^{VU} = \hat{b}_{j1}^{ML}\hat{b}_{j2}^{MU} = \hat{b}_{j1}^{C}\hat{b}_{j2}^{C} = \hat{b}_{j1}^{MU}\hat{b}_{j2}^{ML} = \hat{b}_{j1}^{VU}\hat{b}_{j2}^{VL} = 0 \\
& M_i^1, M_i^2, VL_i^1, VL_i^2, ML_i^1, ML_i^2, MU_i^1, MU_i^2, VU_i^1, VU_i^2 \geq 0
\end{aligned}
$$

$$i = 1, \ldots, n,\ j = 1, \ldots, p,\ k = 1, 2 \tag{31}$$

In addition, sometimes an observation dataset contains two types of explanatory variable, i.e., crisp and TIFN explanatory variables. For example, suppose that p explanatory variables are adopted to build up an IFR model, among which $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k$ are TIFNs and $X_{k+1}, X_{k+2}, \ldots, X_p$ are crisp numbers, i.e., X_{ji}^{VL}, X_{ji}^{ML}, X_{ji}^{MU}, and X_{ji}^{VU} are equal to X_{ji}^{C} for $j = k{+}1$ to p. The formulations of the predicted TIFN responses in Eq. (30) become:

$$\begin{aligned}
\hat{y}_i^{VL} &= \hat{b}_0^{VL} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{VL} x_{ji}^{VU} + \hat{b}_{j2}^{VL} x_{ji}^{VL}\right] + \sum_{j=k+1}^{p}\left[\hat{b}_{j1}^{VL} + \hat{b}_{j2}^{VL}\right] x_{ji}^{C} \\
\hat{y}_i^{ML} &= \hat{b}_0^{ML} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{ML} x_{ji}^{MU} + \hat{b}_{j2}^{ML} x_{ji}^{ML}\right] + \sum_{j=k+1}^{p}\left[\hat{b}_{j1}^{ML} + \hat{b}_{j2}^{ML}\right] x_{ji}^{C} \\
\hat{y}_i^{C} &= \hat{b}_0^{C} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{C} + \hat{b}_{j2}^{C}\right] x_{ji}^{C} \\
\hat{y}_i^{MU} &= \hat{b}_0^{MU} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{MU} x_{ji}^{ML} + \hat{b}_{j2}^{MU} x_{ji}^{MU}\right] + \sum_{j=k+1}^{p}\left[\hat{b}_{j1}^{MU} + \hat{b}_{j2}^{MU}\right] x_{ji}^{C} \\
\hat{y}_i^{VU} &= \hat{b}_0^{VU} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{VU} x_{ji}^{VL} + \hat{b}_{j2}^{VU} x_{ji}^{VU}\right] + \sum_{j=k+1}^{p}\left[\hat{b}_{j1}^{VU} + \hat{b}_{j2}^{VU}\right] x_{ji}^{C}
\end{aligned} \tag{32}$$

If all explanatory variables are crisp numbers, the above formulations become:

$$\begin{aligned}
\hat{y}_i^{VL} &= \hat{b}_0^{VL} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VL} + \hat{b}_{j2}^{VL}\right] x_{ji}^{C} \\
\hat{y}_i^{ML} &= \hat{b}_0^{ML} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{ML} + \hat{b}_{j2}^{ML}\right] x_{ji}^{C} \\
\hat{y}_i^{C} &= \hat{b}_0^{C} + \sum_{j=1}^{k}\left[\hat{b}_{j1}^{C} + \hat{b}_{j2}^{C}\right] x_{ji}^{C} \\
\hat{y}_i^{MU} &= \hat{b}_0^{MU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{MU} + \hat{b}_{j2}^{MU}\right] x_{ji}^{C} \\
\hat{y}_i^{VU} &= \hat{b}_0^{VU} + \sum_{j=1}^{p}\left[\hat{b}_{j1}^{VU} + \hat{b}_{j2}^{VU}\right] x_{ji}^{C}
\end{aligned} \tag{33}$$

Based on the above formulations, the proposed approach can deal with explanatory variables of various types, increasing flexibility. The signs of the TIFN parameters are determined in the solution process of a mathematical programming problem based on the criterion of the minimum distance between the observed and predicted TIFN responses. In addition, the mathematical programming model can be easily solved using commercial software, such as LINGO (Anderson et al. 2017).

4 Example and comparison

This study builds up a linear IFR model from mathematical programming problems with the criterion of least absolute deviation between the observed and predicted TIFN responses. Studies on IFR models are very limited. The approach proposed by Parvathi et al. (2013) uses crisp observation and explanatory variables and attempts to determine the IFR model with the least intuitionistic fuzziness, where all the given data can be included. As such, their approach cannot be compared with the approach proposed here. In this section, to demonstrate the proposed approach, the dataset from Arefi and Taheri (2015) is used to formulate an IFR model, and the performance of the model is compared to that of Arefi and Taheri (2015). The performance criteria include the similarity measure and distance measure proposed by Arefi and Taheri (2015). In addition, the distance measure proposed in this study is adopted.

Arefi and Taheri (2015) demonstrated their model using the TIFN dataset (see Table 1) given by Mohammadi and Taheri (2004). They fitted the least-squares regression model $\hat{\tilde{Y}}_{AT}$ as:

$$\hat{\tilde{Y}}_{AT} = (19.9929, 21.0878, 21.9811, 22.8744, 23.9693) \oplus (-0.2339, -0.2338, -0.2221, -0.2104, -0.2103) \otimes \tilde{X}_1 \oplus (2.4701) \otimes \tilde{X}_2 \tag{34}$$

Using the approach proposed here, the TIFN parameters can be solved from the model, as shown in Table 2. The IFR model $\hat{\tilde{Y}}_{CN}$ is expressed as:

$$\hat{\tilde{Y}}_{CN} = (20.7126, 21.0663, 21.0663, 21.1706, 21.7505) \oplus (-0.1969) \otimes \tilde{X}_1 \oplus (2.6922) \otimes \tilde{X}_2 \tag{35}$$

As shown in Table 2, only one dummy TIFN variable was obtained with the corresponding sign. In addition, the parameters of explanatory variables are crisp values to produce the smallest absolute deviation.

Examining the two models, namely Eqs. (34) and (35), the signs of the determined parameters from Arefi and Taheri (2015) and the proposed approach are the same. If traditional regression analysis is applied to build up a regression model using the central value of the TIFN data in the example, the regression estimators of the explanatory will be $\hat{b}_0 = 21.9767$, $\hat{b}_1 = -0.2221$, and $\hat{b}_2 = 2.4727$, indicating that the two approaches can produce the same signs of parameters and approximately equivalent values compared to those obtained using traditional regression analysis. However, the outcomes from Arefi and Taheri's approach (Arefi and Taheri 2015) are questionable since the model formulation is based on least-squares regression analysis under the assumption that the parameters and TIFN explanatory variables are positive. In con-

Table 1 Dataset used in example (Mohammadi 2004)

No.	$\tilde{X}_{1i} = (x_{1i}^{VL}, x_{1i}^{ML}, x_{1i}^{C}, x_{1i}^{MU}, x_{1i}^{VU})$	$\tilde{X}_{2i} = (x_{2i}^{VL}, x_{2i}^{ML}, x_{2i}^{C}, x_{2i}^{MU}, x_{2i}^{VU})$	$\tilde{Y}_{i} = (y_{i}^{VL}, y_{i}^{ML}, y_{i}^{C}, y_{i}^{MU}, y_{i}^{VU})$
1	(29.75, 31.50, 35.00, 38.50, 40.25)	(0.75, 0.79, 0.88, 0.97, 1.01)	(14.85, 15.68, 16.50, 17.33, 18.15)
2	(31.45, 33.30, 37.00, 40.70, 42.55)	(0.96, 1.02, 1.13, 1.24, 1.30)	(16.74, 17.67, 18.60, 19.53, 20.46)
3	(22.95, 24.30, 27.00, 29.70, 31.05)	(1.11, 1.18, 1.31, 1.44, 1.51)	(17.37, 18.34, 19.30, 20.27, 21.23)
4	(24.65, 26.10, 29.00, 31.90, 33.35)	(1.68, 1.78, 1.98, 2.18, 2.28)	(18.27, 19.29, 20.30, 21.32, 22.33)
5	(32.30, 34.20, 38.00, 41.80, 43.70)	(0.87, 0.92, 1.02, 1.12, 1.17)	(15.57, 16.44, 17.30, 18.17, 19.03)
6	(27.20, 28.80, 32.00, 35.20, 36.80)	(1.10, 1.16, 1.29, 1.42, 1.48)	(18.36, 19.38, 20.40, 21.42, 22.44)
7	(24.65, 26.10, 29.00, 31.90, 33.35)	(1.29, 1.37, 1.52, 1.67, 1.75)	(17.37, 18.34, 19.30, 20.27, 21.23)
8	(15.30, 16.20, 18.00, 19.80, 20.70)	(1.13, 1.20, 1.33, 1.46, 1.53)	(19.71, 20.81, 21.90, 23.00, 24.09)
9	(34.00, 36.00, 40.00, 44.00, 46.00)	(1.45, 1.54, 1.71, 1.88, 1.97)	(14.31, 15.11, 15.90, 16.70, 17.49)
10	(23.80, 25.20, 28.00, 30.80, 32.20)	(1.70, 1.80, 2.00, 2.20, 2.30)	(16.47, 17.39, 18.30, 19.22, 20.13)
11	(11.05, 11.70, 13.00, 14.30, 14.95)	(1.43, 1.51, 1.68, 1.85, 1.93)	(20.34, 21.47, 22.60, 23.73, 24.86)
12	(16.15, 17.10, 19.00, 20.90, 21.85)	(1.83, 1.94, 2.15, 2.37, 2.47)	(21.33, 22.52, 23.70, 24.89, 26.07)
13	(26.35, 27.90, 31.00, 34.10, 35.65)	(2.99, 3.17, 3.52, 3.87, 4.05)	(21.96, 23.18, 24.40, 25.62, 26.84)
14	(26.35, 27.90, 31.00, 34.10, 35.65)	(1.98, 2.10, 2.33, 2.56, 2.68)	(19.62, 20.71, 21.80, 22.89, 23.98)
15	(14.45, 15.30, 17.00, 18.70, 19.55)	(1.45, 1.54, 1.71, 1.88, 1.97)	(21.42, 22.61, 23.80, 24.99, 26.18)
16	(11.90, 12.60, 14.00, 15.40, 16.10)	(0.97, 1.03, 1.14, 1.25, 1.31)	(18.72, 19.76, 20.80, 21.84, 22.88)
17	(16.15, 17.10, 19.00, 20.90, 21.85)	(0.84, 0.89, 0.99, 1.09, 1.14)	(15.75, 16.63, 17.50, 18.38, 19.25)
18	(23.80, 25.20, 28.00, 30.80, 32.20)	(0.97, 1.03, 1.14, 1.25, 1.31)	(16.02, 16.91, 17.80, 18.69, 19.58)
19	(22.10, 23.40, 26.00, 28.60, 29.90)	(1.24, 1.31, 1.46, 1.61, 1.68)	(18.18, 19.19, 20.20, 21.21, 22.22)
20	(27.20, 28.80, 32.00, 35.20, 36.80)	(1.54, 1.63, 1.81, 1.99, 2.08)	(18.00, 19.00, 20.00, 21.00, 22.00)
21	(8.50, 9.00, 10.00, 11.00, 11.50)	(1.17, 1.24, 1.38, 1.52, 1.59)	(20.52, 21.66, 22.80, 23.94, 25.08)
22	(32.30, 34.20, 38.00, 41.80, 43.70)	(0.71, 0.76, 0.84, 0.92, 0.97)	(17.19, 18.15, 19.10, 20.06, 21.01)
23	(41.65, 44.10, 49.00, 53.90, 56.35)	(1.26, 1.33, 1.48, 1.63, 1.70)	(10.89, 11.50, 12.10, 12.71, 13.31)
24	(35.70, 37.80, 42.00, 46.20, 48.30)	(0.92, 0.97, 1.08, 1.19, 1.24)	(11.52, 12.16, 12.80, 13.44, 14.08)

Table 2 TIFN parameters for the proposed approach

Intercept		Slope of $\tilde{X}_1$	Slope of $\tilde{X}_2$
$\hat{B}_0^{VL} = 20.7546$	Negative	$\hat{B}_{11}^{VL} = \hat{B}_{11}^{ML} = \hat{B}_{11}^{C}$	$\hat{B}_{21}^{VL} = \hat{B}_{21}^{ML} = \hat{B}_{21}^{C}$
$\hat{B}_0^{ML} = \hat{B}_0^{C} = 21.0663$		$= \hat{B}_{11}^{MU} = \hat{B}_{11}^{VU} = -0.1969$	$= B_{21}^{MU} = \hat{B}_{21}^{VU} = 0$
$\hat{B}_0^{MU} = 21.2173$	Positive	$\hat{B}_{12}^{VL} = \hat{B}_{12}^{ML} = \hat{B}_{12}^{C}$	$\hat{B}_{22}^{VL} = \hat{B}_{2}^{ML} = \hat{B}_{2}^{C}$
$\hat{B}_0^{VU} = 21.7375$		$= \hat{B}_{12}^{MU} = \hat{B}_{12}^{VU} = 0$	$= \hat{B}_{22}^{MU} = \hat{B}_{2}^{VU} = 2.6922$

Table 3 Results of performance comparison

No.	Arefi and Taheri (2015)			Approach		
	$SM(\tilde{Y}_i, \hat{\tilde{Y}}_i)$	$d^2(\tilde{Y}_i, \hat{\tilde{Y}}_i)$	$D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i)$	$SM(\tilde{Y}_i, \hat{\tilde{Y}}_i)$	$d^2(\tilde{Y}_i, \hat{\tilde{Y}}_i)$	$D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i)$
1	0.5608	0.6140	0.7559	0.9212	0.0119	0.1028
2	0.2900	4.7794	1.9495	0.1748	3.5224	1.7459
3	0.6541	0.3787	0.5891	0.9690	0.0132	0.0391
4	0.6336	0.5027	0.6938	0.7283	0.0964	0.4227
5	0.4487	2.2008	1.3543	0.4253	1.1581	0.9396
6	0.2275	5.9011	2.3396	0.1208	5.0699	2.1275
7	0.6333	0.4485	0.6057	0.8697	0.0064	0.1843
8	0.6780	0.5456	0.6610	0.5028	0.7865	0.7543
9	0.3912	2.9830	1.5941	0.1259	3.2841	1.9229
10	0.2066	6.3307	2.4026	0.0463	6.5403	2.6741
11	0.6819	0.5197	0.6589	0.6612	0.1892	0.4655
12	0.6881	0.5945	0.7183	0.6662	0.4474	0.5443
13	0.6193	1.0048	1.0254	0.8918	0.0313	0.2016
14	0.5654	1.4272	1.1305	0.6946	0.4343	0.5301
15	0.4448	2.0048	1.3707	0.2642	2.4260	1.4338
16	0.5656	0.8963	0.8465	0.5286	0.3241	0.6243
17	0.1034	7.5727	2.7066	0.0168	5.8791	2.5328
18	0.5544	1.0316	0.9757	0.4173	0.5482	0.8588
19	0.6550	0.4914	0.7200	0.7947	0.1653	0.2846
20	0.5831	0.9669	0.9890	0.7854	0.2110	0.3276
21	0.7904	0.1894	0.3918	0.7281	0.0942	0.3570
22	0.0848	12.6771	3.4838	0.0201	11.2529	3.2242
23	0.1657	8.5320	2.6535	0.0035	10.3513	3.3259
24	0.1664	7.4196	2.4534	0.0073	7.9225	2.9318
Sum	**11.3919**	70.0124	33.0698	11.3640	**60.7661**	**28.5555**

The bold numbers denote the better performance under the corresponding measure in comparing the two approaches

trast, in the proposed approach, the signs of parameters are determined in the model formulation process.

Furthermore, Arefi and Taheri's approach (2015) was developed based on symmetric TIFNs, and thus the estimated parameters and predicted response are also symmetric TIFNs. However, based on the definitions of the product operator for TIFNs given in Eqs. (11)–(13), the product of two TIFNs does not produce a symmetric TIFN, even if they are symmetric. The proposed predicted TIFN responses are not symmetric, which is more reasonable in theory.

Performance comparisons between Arefi and Taheri's approach (2015) and the proposed approach based on the similarity measure $SM(\tilde{Y}_i, \hat{\tilde{Y}}_i)$ [Eq. (20)], the distance measure $d^2(\tilde{Y}_i, \hat{\tilde{Y}}_i)$ [Eq. (21)], and the absolute distance measure $D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i)$

[Eq. (19)] were conducted item by item. The results are listed in Table 3. Although the similarity measure $SM(\tilde{Y}_i, \hat{\tilde{Y}}_i)$ of the proposed model is 0.2% lower than that of the model obtained using Arefi and Taheri's approach (2015), the proposed approach outperforms Arefi and Taheri's approach (2015) in terms of distance measures $d^2(\tilde{Y}_i, \hat{\tilde{Y}}_i)$ and $D_{TIFN}(\tilde{Y}_i, \hat{\tilde{Y}}_i)$ by 13.2% and 13.7%, respectively.

These results show the feasibility and applicability of the proposed approach. The proposed approach can deal with different TIFN types in a dataset whether they are symmetric or asymmetric. The signs of TIFN parameters are known prior to formulating the model in the proposed approach. In addition, the linear formulation of the proposed mathematical programming problems increases computational performance. In general, the proposed approach is more generalized than existing ones.

5 Conclusion

This study used mathematical programming problems to build up IFR models. The least absolutely deviations between the predicted and observed TIFN responses are considered as the objective function, making the models more robust. The linear formulation of the proposed mathematical programming problems increases computational performance. The formulation of IFR models is derived based on the main components of an IFR model, i.e., the central value and lower and upper bounds of membership and non-membership functions. Unlike existing methods, the proposed approach does not limit observations to be symmetrical TIFNs. More importantly, the signs of parameters can be determined in the resolution process of finding the optimal parameters simultaneously. The proposed approach is general and can be used with TIFNs or crisp numbers. A performance comparison showed that the present IFR model outperforms an existing one in terms of distance measures. In future research, a more robust approach will be developed, and more applications will be used to demonstrate the applicability of the IFR model.

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