Fuzzy Optimization and Decision Making (2020) 19:135–152
https://doi.org/10.1007/s10700-020-09313-0

A scalarization method for fuzzy set optimization problems

Masamichi Kon[1]

Published online: 10 February 2020

Abstract
In the present paper, we consider fuzzy optimization problems which involve fuzzy sets only in the objective mappings, and give two concepts of optimal solutions which are non-dominated solutions and weak non-dominated solutions based on orderings of fuzzy sets. First, by using level sets of fuzzy sets, the fuzzy optimization problems treated in this paper are reduced to set optimization problems, and relationships between (weak) non-dominated solutions of the fuzzy optimization problems and the reduced set optimization problems are derived. Next, the set optimization problems are reduced to scalar optimization problems which can be regarded as scalarization of the fuzzy optimization problems. Then, relationships between non-dominated solutions of the fuzzy optimization problems and optimal solutions of the reduced scalar optimization problems are derived.

Keywords Fuzzy set optimization · Scalarization · Fuzzy max order · Order preserving property · Set optimization

1 Introduction

The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness in Zadeh (1965). Then, fuzzy set theory has been applied in various areas of decision making theory including economics and optimization, etc., widely. Since the seminal work on fuzzy optimization problems by Bellman and Zadeh (1970), much attention has been focused on fuzzy optimization problems with various formulations such as linear cases, non-linear cases, single-objective, multi-objective, etc. (Gupta and Dangar 2014; Inuiguchi 2005, 2007; Inuiguchi and Ramík 2000; Jamison and Lodwick 1999; Maeda 2008; Ramík and Římánek 1985; Wu 2006, 2007, 2008). The fuzzy optimization problems are mathematical programming problems involving fuzzy parameters. Applying the classical mathematical programming

✉ Masamichi Kon
masakon@hirosaki-u.ac.jp

1 Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan

problems to real-world problems, they have several parameters. Since the parameters usually contain uncertainty or vagueness, the fuzzy optimization model is a more adequate representation of the reality than the classical one.

In the fuzzy optimization problems, representations or treatments of constraints are not discussed, and the case that fuzzy sets are involved only in the objective mappings is considered in this paper. Considering treatments of only objective mappings, one of main approaches is *possibilistic programming approach*. The possibilistic programming approach, which contains many various approaches, transforms the fuzzy mathematical programming problem to the associated conventional mathematical programming problem based on possibility and/or necessity measures (Inuiguchi 2005, 2007; Inuiguchi and Ramík 2000 and references therein). One of well-known other approaches is to consider (weak) non-dominated solutions and so on based on some ordering of fuzzy sets, and it is based on neither possibility nor necessity measures. We refer to this approach as *ordering-based efficiency approach*, and the ordering-based efficiency approach based on the fuzzy max order is adopted in this paper. The fuzzy max order was first proposed in Ramik and Řimánek 1985. Since there is no universal concept of optimal solutions to be accepted widely, it is important to define the concepts of optimal solutions. In the ordering-based efficiency approach, fuzzy number-valued objective mappings are considered in most of papers. Fuzzy optimization problems with fuzzy number-valued objective mappings are considered in Gupta and Dangar (2014), and Wu (2006, 2007, 2008), and fuzzy optimization problems with fuzzy vector-valued objective mappings are considered in Maeda (2008). Within our knowledge, no literature deals with *fuzzy set-valued* objective mappings in fuzzy optimization problems. The fuzzy number-valued and fuzzy vector-valued objective mappings are special cases of the fuzzy set-valued objective mappings. We refer to the fuzzy optimization problem with a fuzzy set-valued objective mapping as *a fuzzy set optimization problem*. We consider fuzzy set optimization problems and propose two concepts of optimal solutions of them. They are non-dominated solutions and weak non-dominated solutions, which are extensions of well-known (weak) non-dominated solutions of fuzzy optimization problems with fuzzy number-valued and fuzzy vector-valued objective mappings in the ordering-based efficiency approach. Another aspect of view, the fuzzy set optimization problems are an extension of set optimization problems. Thus, a lot of literature on the set optimization problems published so far also prove the importance of the fuzzy set optimization problems. Then, it can be expected that the fuzzy set optimization problems enable us to cover more situations of real-world problems in the sense that the fuzzy number-valued, fuzzy vector-valued and set-valued objective mappings are special cases of the fuzzy set-valued objective mappings in the ordering-based efficiency approach. It does not mean that the ordering-based efficiency approach for fuzzy set optimization problems is an extension of the possibilistic programming approach.

In Wu (2006), a scalarization method is proposed by embedding fuzzy numbers into a normed space. It is difficult to apply it's scalarization method to fuzzy set-valued objective mappings. In Maeda (2008), set optimization problems associated with fuzzy optimization problems are derived by using the order preserving property for fuzzy vectors, and it enables us to solve the fuzzy optimization problems by solving the set optimization problems. On the other hand, a scalarization method for set optimization problems is proposed in Hernández and Rodríguez-Marín (2007) and Maeda (2012) by using scalarizations of sets, and it enables us to solve the set optimization problems by solving scalar optimization problems associated with the set optimization problems.

In the present paper, we consider fuzzy set optimization problems. First, the order preserving property for fuzzy vectors is extended to that for fuzzy sets. Then, the fuzzy set optimization problems are reduced to set optimization problems by using the order preserving property for fuzzy sets. Next, the set optimization problems associated with the fuzzy set optimization problems are reduced to scalar optimization problems. It enables us to solve the fuzzy set optimization problems by solving the scalar optimization problems.

The remainder of the present paper is organized as follows. In Sect. 2, orderings and scalarizations of sets are discussed. In Sect. 3, properties of fuzzy sets are investigated, and orderings of fuzzy sets are discussed. In Sect. 4, the fuzzy set optimization problems are considered, and the properties are investigated. In Sect. 5, a scalarization method for the fuzzy set optimization problems is proposed. Finally, conclusions are presented in Sect. 6.

2 Orderings and scalarizations of sets

In this section, orderings and scalarizations of sets are discussed.

For $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. For convenience, we define $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$, $\min \emptyset = \infty$, and $\min \mathbb{R} = -\infty$.

For $A \subset \mathbb{R}^n$, let $\mathrm{int}(A)$ and $\mathrm{cl}(A)$ be the interior and the closure of A, respectively. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all compact subsets of $\mathbb{R}^n$, and let $\mathcal{C}_0(\mathbb{R}^n)$ be the set of all non-empty compact subsets of $\mathbb{R}^n$. We set $\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geq \mathbf{0}\}$ and $\mathbb{R}^n_- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \leq \mathbf{0}\}$. For $A, B \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we set $A + B = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\}$ and $\lambda A = \{\lambda \boldsymbol{x} : \boldsymbol{x} \in A\}$.

Now, we introduce some orderings of sets.

Definition 1 (Jahn and Ha 2011; Kon 2014; Kurano et al. 2000; Kuroiwa et al. 1997; Maeda 2008, 2012) Let $A, B \subset \mathbb{R}^n$.

(i) $A \leq_L B \overset{\text{def}}{\Leftrightarrow} B \subset A + \mathbb{R}^n_+$;

(ii) $A \leq_U B \overset{\text{def}}{\Leftrightarrow} A \subset B + \mathbb{R}^n_-$;

(iii) $A \leq B \overset{\text{def}}{\Leftrightarrow} A \leq_L B$ and $A \leq_U B$;

(iv) $A <_L B \overset{\text{def}}{\Leftrightarrow} B \subset A + \mathrm{int}(\mathbb{R}^n_+)$;

(v) $A <_U B \overset{\text{def}}{\Leftrightarrow} A \subset B + \mathrm{int}(\mathbb{R}^n_-)$;

(vi) $A < B \overset{\text{def}}{\Leftrightarrow} A <_L B$ and $A <_U B$.

The following lemmas provide fundamental properties of orderings given in Definition 1, and Lemma 1 can be shown easily.

Lemma 1 *For $A, B, C, D \subset \mathbb{R}^n$, the following statements hold.*

(i) $A \leq A$;

(ii) $A \leq_L B, B \leq_L C \Rightarrow A \leq_L C$;
(iii) $A \leq_U B, B \leq_U C \Rightarrow A \leq_U C$;
(iv) $A \leq B, B \leq C \Rightarrow A \leq C$;
(v) $A <_L B \Rightarrow A \leq_L B$;
(vi) $A <_U B \Rightarrow A \leq_U B$;
(vii) $A < B \Rightarrow A \leq B$;
(viii) $A \leq_L B, C \leq_L D \Rightarrow A + C \leq_L B + D$;
(ix) $A \leq_U B, C \leq_U D \Rightarrow A + C \leq_U B + D$;
(x) $A \leq B, C \leq D \Rightarrow A + C \leq B + D$;
(xi) $A = \emptyset, B \neq \emptyset \Rightarrow A \not\leq_L B, B \leq_L A, A \leq_U B, B \not\leq_U A, A \not<_L B, B <_L A, A <_U B, B \not<_U A$;
(xii) $\emptyset \leq \emptyset, \emptyset < \emptyset, \mathbb{R}^n \leq \mathbb{R}^n, \mathbb{R}^n < \mathbb{R}^n$;
(xiii) $\lambda \geq 0, A \leq_L B \Rightarrow \lambda A \leq_L \lambda B$;
(xiv) $\lambda \geq 0, A \leq_U B \Rightarrow \lambda A \leq_U \lambda B$;
(xv) $\lambda \geq 0, A \leq B \Rightarrow \lambda A \leq \lambda B$.

Lemma 2 (Kon 2014, Proposition 3.3) *Let $A, B \subset \mathbb{R}^n$. Assume that $A \in \mathcal{C}_0(\mathbb{R}^n)$ or $B \in \mathcal{C}_0(\mathbb{R}^n)$. If $A < B$, then $A \not\geq B$.*

Let $\boldsymbol{k} \in \mathrm{int}(\mathbb{R}^n_+)$, and we set

$$s_L(A; \boldsymbol{k}) = \inf\{t \in \mathbb{R} : A \leq_L \{t\boldsymbol{k}\}\} = \inf\{t \in \mathbb{R} : \{t\boldsymbol{k}\} \subset A + \mathbb{R}^n_+\}, \tag{1}$$
$$s_U(A; \boldsymbol{k}) = \inf\{t \in \mathbb{R} : A \leq_U \{t\boldsymbol{k}\}\} = \inf\{t \in \mathbb{R} : A \subset t\boldsymbol{k} + \mathbb{R}^n_-\}, \tag{2}$$
$$I_A(\boldsymbol{k}) = [s_L(A; \boldsymbol{k}), s_U(A; \boldsymbol{k})] \tag{3}$$

for each $A \subset \mathbb{R}^n$. $s_L(A; \boldsymbol{k})$ and $s_U(A; \boldsymbol{k})$ are scalarizations of A, which may be $-\infty$ or ∞ (Hernández and Rodríguez-Marín 2007; Maeda 2012). $I_A(\boldsymbol{k})$ is the interval associated with A. It follows that

$$s_L(A; \boldsymbol{k}) = \inf\{t \in \mathbb{R} : A \cap (t\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset\}.$$

If $A = \emptyset$, then $s_L(A; \boldsymbol{k}) = \inf \emptyset = \infty$ and $s_U(A; \boldsymbol{k}) = \inf \mathbb{R} = -\infty$, and then $I_A(\boldsymbol{k}) = \emptyset$.

The following theorem provides properties of s_L and s_U.

Theorem 1 *Let $\boldsymbol{k} \in int(\mathbb{R}^n_+)$. In (i) and (ii), let $A, B \subset \mathbb{R}^n$. In (iii)–(viii), let $A, B \in \mathcal{C}(\mathbb{R}^n)$.*

(i) $A \neq \emptyset \Leftrightarrow s_L(A; \boldsymbol{k}) \leq s_U(A; \boldsymbol{k})$;
(ii) $A \subset B \Rightarrow s_L(A; \boldsymbol{k}) \geq s_L(B; \boldsymbol{k}), s_U(A; \boldsymbol{k}) \leq s_U(B; \boldsymbol{k})$;
(iii) $s_L(A; \boldsymbol{k}) = \min\{t \in \mathbb{R} : A \leq_L \{t\boldsymbol{k}\}\} = \min\{t \in \mathbb{R} : \{t\boldsymbol{k}\} \subset A + \mathbb{R}^n_+\}$;
(iv) $s_U(A; \boldsymbol{k}) = \min\{t \in \mathbb{R} : A \leq_U \{t\boldsymbol{k}\}\} = \min\{t \in \mathbb{R} : A \subset t\boldsymbol{k} + \mathbb{R}^n_-\}$;
(v) $A \leq_L B \Rightarrow s_L(A; \boldsymbol{k}) \leq s_L(B; \boldsymbol{k}) \Leftrightarrow I_A(\boldsymbol{k}) \leq_L I_B(\boldsymbol{k})$;
(vi) $A \leq_U B \Rightarrow s_U(A; \boldsymbol{k}) \leq s_U(B; \boldsymbol{k}) \Leftrightarrow I_A(\boldsymbol{k}) \leq_U I_B(\boldsymbol{k})$;
(vii) *If $A \neq \emptyset$ or $B \neq \emptyset$, then*

$$A <_L B \Rightarrow s_L(A; \boldsymbol{k}) < s_L(B; \boldsymbol{k}) \Leftrightarrow I_A(\boldsymbol{k}) <_L I_B(\boldsymbol{k});$$

(viii) If $A \neq \emptyset$ or $B \neq \emptyset$, then

$$A <_U B \Rightarrow s_U(A; \boldsymbol{k}) < s_U(B; \boldsymbol{k}) \Leftrightarrow I_A(\boldsymbol{k}) <_U I_B(\boldsymbol{k}).$$

Proof (i) and (ii) are trivial. (iii) and (iv) are Theorem 3.1 in Maeda (2012), and then (v) and (vi) are Theorem 3.2 in Maeda (2012). In addition, (vii) and (viii) are Theorem 3.3 in Maeda (2012). □

The following lemmas provide further properties of s_L and s_U.

Lemma 3 *Let $\boldsymbol{k} \in int(\mathbb{R}^n_+)$, and let Λ be any index set. In addition, let $A_\lambda \subset \mathbb{R}^n$ for each $\lambda \in \Lambda$.*

(i) $s_L\left(\bigcup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \inf_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k})$;

(ii) $s_U\left(\bigcup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \sup_{\lambda \in \Lambda} s_U(A_\lambda; \boldsymbol{k})$;

(iii) $s_L\left(\bigcap_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) \geq \sup_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k})$;

(iv) $s_U\left(\bigcap_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) \leq \inf_{\lambda \in \Lambda} s_U(A_\lambda; \boldsymbol{k})$.

Proof (i) First, suppose that $\cup_{\lambda \in \Lambda} A_\lambda = \emptyset$. Then for any $\lambda \in \Lambda$, it follows that $A_\lambda = \emptyset$, and that $s_L(A_\lambda; \boldsymbol{k}) = \infty$. Therefore, we have $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \inf_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k}) = \infty$. Next, suppose that $\cup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then, it follows that $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) < \infty$. Suppose that $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = -\infty$, and fix any $t \in \mathbb{R}$. Since $\cup_{\lambda \in \Lambda}(A_\lambda \cap (t\boldsymbol{k} + \mathbb{R}^n_-)) = (\cup_{\lambda \in \Lambda} A_\lambda) \cap (t\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset$, there exists $\lambda_0 \in \Lambda$ such that $A_{\lambda_0} \cap (t\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset$, and it follows that $\inf_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k}) \leq s_L(A_{\lambda_0}; \boldsymbol{k}) \leq t$. By the arbitrariness of $t \in \mathbb{R}$, we have $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \inf_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k}) = -\infty$. Suppose that $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) > -\infty$. Then for any $\lambda \in \Lambda$, since $A_\lambda \subset \cup_{\mu \in \Lambda} A_\mu$, it follows that $s_L(A_\lambda; \boldsymbol{k}) \geq s_L\left(\cup_{\mu \in \Lambda} A_\mu; \boldsymbol{k}\right)$ from Theorem 1 (ii). Now, fix any $\varepsilon > 0$. Then, there exists $t_0 \in [s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right), s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) + \varepsilon[$ such that $(\cup_{\lambda \in \Lambda} A_\lambda) \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset$. Since $\cup_{\lambda \in \Lambda}(A_\lambda \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-)) = (\cup_{\lambda \in \Lambda} A_\lambda) \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset$, there exists $\lambda_1 \in \Lambda$ such that $A_{\lambda_1} \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \neq \emptyset$, and it follows that $s_L(A_{\lambda_1}; \boldsymbol{k}) \leq t_0 < s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) + \varepsilon$. Therefore, we have $s_L\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \inf_{\lambda \in \Lambda} s_L(A_\lambda; \boldsymbol{k})$.

(ii) First, suppose that $\cup_{\lambda \in \Lambda} A_\lambda = \emptyset$. Then for any $\lambda \in \Lambda$, it follows that $A_\lambda = \emptyset$, and that $s_U(A_\lambda; \boldsymbol{k}) = -\infty$. Therefore, we have $s_U\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \sup_{\lambda \in \Lambda} s_U(A_\lambda; \boldsymbol{k}) = -\infty$. Next, suppose that $\cup_{\lambda \in \Lambda} A_\lambda \neq \emptyset$. Then, it follows that $s_U\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) > -\infty$. Suppose that $s_U\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \infty$, and fix any $t \in \mathbb{R}$. Since $\cup_{\lambda \in \Lambda} A_\lambda \not\subset t\boldsymbol{k} + \mathbb{R}^n_-$, there exists $\lambda_0 \in \Lambda$ such that $A_{\lambda_0} \not\subset t\boldsymbol{k} + \mathbb{R}^n_-$, and then $A_{\lambda_0} \not\subset t'\boldsymbol{k} + \mathbb{R}^n_-$ for any $t' \in]-\infty, t]$. Thus, it follows that $\sup_{\lambda \in \Lambda} s_U(A_\lambda; \boldsymbol{k}) \geq s_U(A_{\lambda_0}; \boldsymbol{k}) \geq t$. By the arbitrariness of $t \in \mathbb{R}$, we have $s_U\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) = \sup_{\lambda \in \Lambda} s_U(A_\lambda; \boldsymbol{k}) = \infty$. Suppose that $s_U\left(\cup_{\lambda \in \Lambda} A_\lambda; \boldsymbol{k}\right) < \infty$. Then for any $\lambda \in \Lambda$, since $A_\lambda \subset \cup_{\mu \in \Lambda} A_\mu$, it follows that $s_U(A_\lambda; \boldsymbol{k}) \leq s_U\left(\cup_{\mu \in \Lambda} A_\mu; \boldsymbol{k}\right)$ from

Theorem 1 (ii). Now, fix any $\varepsilon > 0$. Then for any $t \in]-\infty, s_U(\cup_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})[$, since $\cup_{\lambda\in\Lambda} A_\lambda \not\subset t\boldsymbol{k} + \mathbb{R}^n_-$, there exists $\lambda(t) \in \Lambda$ such that $A_{\lambda(t)} \not\subset t\boldsymbol{k} + \mathbb{R}^n_-$, it follows that $A_{\lambda(t)} \not\subset t'\boldsymbol{k} + \mathbb{R}^n_-$ for any $t' \in]-\infty, t]$, and that $s_U(A_{\lambda(t)}; \boldsymbol{k}) \geq t$. Choose any $t_0 \in]s_U(\cup_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) - \varepsilon, s_U(\cup_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})[$. Then, it follows that $\lambda(t_0) \in \Lambda$, and that $s_U(A_{\lambda(t_0)}; \boldsymbol{k}) \geq t_0 > s_U(\cup_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) - \varepsilon$. Therefore, we have $s_U(\cup_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) = \sup_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})$.

(iii) For any $\lambda \in \Lambda$, since $\cap_{\mu\in\Lambda} A_\mu \subset A_\lambda$, it follows that $s_L(A_\lambda; \boldsymbol{k}) \leq s_L(\cap_{\mu\in\Lambda} A_\mu; \boldsymbol{k})$ from Theorem 1 (ii). Therefore, we have $\sup_{\lambda\in\Lambda} s_L(A_\lambda; \boldsymbol{k}) \leq s_L(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})$.

(iv) For any $\lambda \in \Lambda$, since $\cap_{\mu\in\Lambda} A_\mu \subset A_\lambda$, it follows that $s_U(\cap_{\mu\in\Lambda} A_\mu; \boldsymbol{k}) \leq s_U(A_\lambda; \boldsymbol{k})$ from Theorem 1 (ii). Therefore, we have $s_U(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) \leq \inf_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})$. □

The following example shows that (iii) and (iv) in Lemma 3 do not hold with equality.

Example 1 In $\mathbb{R}^2$, let $\boldsymbol{k} = (1, 1) \in \mathrm{int}(\mathbb{R}^2_+)$, and set $A = \{(x, y) \in \mathbb{R}^2 : |x + 1| + |y - 3| \leq 2\}$ and $B = \{(x, y) \in \mathbb{R}^2 : |x - 1| + |y - 3| \leq 2\}$. Then, it follows that $s_L(A; \boldsymbol{k}) = s_L(B; \boldsymbol{k}) = 1$, $s_U(A; \boldsymbol{k}) = s_U(B; \boldsymbol{k}) = 5$ and $s_L(A \cap B; \boldsymbol{k}) = 2$, $s_U(A \cap B; \boldsymbol{k}) = 4$. Thus, we have $s_L(A \cap B; \boldsymbol{k}) = 2 > 1 = \sup\{s_L(A; \boldsymbol{k}), s_L(B; \boldsymbol{k})\}$ and $s_U(A \cap B; \boldsymbol{k}) = 4 < 5 = \inf\{s_U(A; \boldsymbol{k}), s_U(B; \boldsymbol{k})\}$. □

Lemma 4 *Let $\boldsymbol{k} \in int(\mathbb{R}^n_+)$, and let Λ be any index set. In addition, let $A_\lambda \in \mathcal{C}(\mathbb{R}^n)$ for each $\lambda \in \Lambda$. Assume that $A_\lambda \subset A_\mu$ or $A_\lambda \supset A_\mu$ for any $\lambda, \mu \in \Lambda$.*

(i) $s_L\left(\bigcap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}\right) = \sup_{\lambda\in\Lambda} s_L(A_\lambda; \boldsymbol{k})$;

(ii) $s_U\left(\bigcap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}\right) = \inf_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})$.

Proof (i) It follows that $s_L(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) \geq \sup_{\lambda\in\Lambda} s_L(A_\lambda; \boldsymbol{k})$ from Lemma 3 (iii). Suppose that $s_L(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) > \sup_{\lambda\in\Lambda} s_L(A_\lambda; \boldsymbol{k})$, and fix any $t_0 \in]\sup_{\lambda\in\Lambda} s_L(A_\lambda; \boldsymbol{k}), s_L(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})[$. Then for any $\lambda \in \Lambda$, since $s_L(A_\lambda; \boldsymbol{k}) \leq \sup_{\mu\in\Lambda} s_L(A_\mu; \boldsymbol{k}) < t_0$, it follows that $A_\lambda \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \in \mathcal{C}_0(\mathbb{R}^n)$. Since $A_\lambda \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \subset A_\mu \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-)$ or $A_\lambda \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) \supset A_\mu \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-)$ for any $\lambda, \mu \in \Lambda$, it follows that $(\cap_{\lambda\in\Lambda} A_\lambda) \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) = \cap_{\lambda\in\Lambda}(A_\lambda \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-)) \in \mathcal{C}_0(\mathbb{R}^n)$. It contradicts that $(\cap_{\lambda\in\Lambda} A_\lambda) \cap (t_0\boldsymbol{k} + \mathbb{R}^n_-) = \emptyset$ since $t_0 < s_L(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})$.

(ii) It follows that $s_U(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) \leq \inf_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})$ from Lemma 3 (iv). Suppose that $s_U(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}) < \inf_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})$. Fix any $t_0, t_1 \in]s_U(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k}), \inf_{\lambda\in\Lambda} s_U(A_\lambda; \boldsymbol{k})[$ with $t_1 < t_0$. Then for any $\lambda \in \Lambda$, since $t_0 < \inf_{\mu\in\Lambda} s_U(A_\mu; \boldsymbol{k}) \leq s_U(A_\lambda; \boldsymbol{k})$, it follows that $A_\lambda \not\subset t_0\boldsymbol{k} + \mathbb{R}^n_-$, and that $A_\lambda \not\subset t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-)$, and that $A_\lambda \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c \in \mathcal{C}_0(\mathbb{R}^n)$. Since $A_\lambda \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c \subset A_\mu \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c$ or $A_\lambda \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c \supset A_\mu \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c$ for any $\lambda, \mu \in \Lambda$, it follows that $(\cap_{\lambda\in\Lambda} A_\lambda) \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c = \cap_{\lambda\in\Lambda}(A_\lambda \cap (t_0\boldsymbol{k} + \mathrm{int}(\mathbb{R}^n_-))^c) \in \mathcal{C}_0(\mathbb{R}^n)$. Since $t_1 < t_0$, it follows that $(\cap_{\lambda\in\Lambda} A_\lambda) \cap (t_1\boldsymbol{k} + \mathbb{R}^n_-)^c \neq \emptyset$, and that $\cap_{\lambda\in\Lambda} A_\lambda \not\subset t_1\boldsymbol{k} + \mathbb{R}^n_-$. It contradicts that $\cap_{\lambda\in\Lambda} A_\lambda \subset t_1\boldsymbol{k} + \mathbb{R}^n_-$ since $t_1 > s_U(\cap_{\lambda\in\Lambda} A_\lambda; \boldsymbol{k})$. □

Lemma 5 *Let $\boldsymbol{k} \in int(\mathbb{R}^n_+)$, and let $A, B \in \mathcal{C}(\mathbb{R}^n)$.*

(i) $s_L(A + B; \boldsymbol{k}) \leq s_L(A; \boldsymbol{k}) + s_L(B; \boldsymbol{k})$;
(ii) $s_U(A + B; \boldsymbol{k}) \leq s_U(A; \boldsymbol{k}) + s_U(B; \boldsymbol{k})$;
(iii) $\lambda > 0 \Rightarrow s_L(\lambda A; \boldsymbol{k}) = \lambda s_L(A; \boldsymbol{k})$;
(iv) $\lambda > 0 \Rightarrow s_U(\lambda A; \boldsymbol{k}) = \lambda s_U(A; \boldsymbol{k})$.

Proof If $A = \emptyset$ or $B = \emptyset$, then (i) and (ii) are trivial. Suppose that $A \neq \emptyset$ and $B \neq \emptyset$ in (i) and (ii).

(i) Since $A \leq_L \{s_L(A; \boldsymbol{k})\boldsymbol{k}\}$ and $B \leq_L \{s_L(B; \boldsymbol{k})\boldsymbol{k}\}$ from Theorem 1 (iii), it follows that $A + B \leq_L \{(s_L(A; \boldsymbol{k}) + s_L(B; \boldsymbol{k}))\boldsymbol{k}\}$ from Lemma 1 (viii). Therefore, we have $s_L(A + B; \boldsymbol{k}) \leq s_L(A; \boldsymbol{k}) + s_L(B; \boldsymbol{k})$ from the definition of $s_L(A + B; \boldsymbol{k})$.
(ii) Since $A \leq_U \{s_U(A; \boldsymbol{k})\boldsymbol{k}\}$ and $B \leq_U \{s_U(B; \boldsymbol{k})\boldsymbol{k}\}$ from Theorem 1 (iv), it follows that $A + B \leq_U \{(s_U(A; \boldsymbol{k}) + s_U(B; \boldsymbol{k}))\,\boldsymbol{k}\}$ from Lemma 1 (ix). Therefore, we have $s_U(A + B; \boldsymbol{k}) \leq s_U(A; \boldsymbol{k}) + s_U(B; \boldsymbol{k})$ from the definition of $s_U(A + B; \boldsymbol{k})$.
(iii) We have $s_L(\lambda A; \boldsymbol{k}) = \inf\{t \in \mathbb{R} : \lambda A \leq_L \{t\boldsymbol{k}\}\} = \lambda \inf\left\{\frac{t}{\lambda} : t \in \mathbb{R},\ A \leq_L \left\{\frac{t}{\lambda}\boldsymbol{k}\right\}\right\} = \lambda \inf\{t' \in \mathbb{R} : A \leq_L \{t'\boldsymbol{k}\}\} = \lambda s_L(A; \boldsymbol{k})$ from Lemma 1 (xiii).
(iv) We have $s_U(\lambda A; \boldsymbol{k}) = \inf\{t \in \mathbb{R} : \lambda A \leq_U \{t\boldsymbol{k}\}\} = \lambda \inf\left\{\frac{t}{\lambda} : t \in \mathbb{R},\ A \leq_U \left\{\frac{t}{\lambda}\boldsymbol{k}\right\}\right\} = \lambda \inf\{t' \in \mathbb{R} : A \leq_U \{t'\boldsymbol{k}\}\} = \lambda s_U(A; \boldsymbol{k})$ from Lemma 1 (xiv). □

A mapping F such that $F(\boldsymbol{x}) \subset \mathbb{R}^m$ for each $\boldsymbol{x} \in \mathbb{R}^n$ is called a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, and we denote it by $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$.

We define the convexity of set-valued mappings.

Definition 2 Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$.

(i) F is called *a convex mapping* if

$$F(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \leq \lambda F(\boldsymbol{x}) + (1 - \lambda)F(\boldsymbol{y}) \tag{4}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and any $\lambda \in]0, 1[$;
(ii) F is called *a strictly convex mapping* if

$$F(\lambda \boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) < \lambda F(\boldsymbol{x}) + (1 - \lambda)F(\boldsymbol{y}) \tag{5}$$

for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \boldsymbol{y}$ and any $\lambda \in]0, 1[$.

Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$. The set-valued mapping F is said to be *convex-valued* if $F(\boldsymbol{x})$ is a convex set for any $\boldsymbol{x} \in \mathbb{R}^n$. When F is convex-valued, if F is a strictly convex mapping, then F is a convex mapping (Kon 2014, Proposition 3.5).

The following theorem provides properties of composite functions of s_L, s_U and set-valued mappings. Although Theorem 2 (i) is shown in Theorem 3.5 in Maeda (2012), we present another proof.

Theorem 2 *Let $F : \mathbb{R}^n \to \mathcal{C}_0(\mathbb{R}^m)$, and let $\boldsymbol{k} \in int(\mathbb{R}^m_+)$.*

(i) If F is a convex mapping, then $s_L(F(\cdot); \boldsymbol{k}), s_U(F(\cdot); \boldsymbol{k}) : \mathbb{R}^n \to \mathbb{R}$ are convex functions.

(ii) If F is a strictly convex mapping, then $s_L(F(\cdot); \boldsymbol{k}), s_U(F(\cdot); \boldsymbol{k}) : \mathbb{R}^n \to \mathbb{R}$ are strictly convex functions.

Proof (i) The convexity of $s_L(F(\cdot); \boldsymbol{k})$ follows from Theorem 1 (v) and Lemma 5 (i), (iii). The convexity of $s_U(F(\cdot); \boldsymbol{k})$ follows from Theorem 1 (vi) and Lemma 5 (ii), (iv).

(ii) The strict convexity of $s_L(F(\cdot); \boldsymbol{k})$ follows from Theorem 1 (vii) and Lemma 5 (i), (iii). The strict convexity of $s_U(F(\cdot); \boldsymbol{k})$ follows from Theorem 1 (viii) and Lemma 5 (ii), (iv).

□

3 Fuzzy sets

In this section, properties of fuzzy sets are investigated, and orderings of fuzzy sets are discussed.

A function $\widetilde{a} : \mathbb{R}^n \to [0, 1]$ is called *a fuzzy set on* $\mathbb{R}^n$. Let $\mathcal{F}(\mathbb{R}^n)$ be the set of all fuzzy sets on $\mathbb{R}^n$.

Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$. For each $\alpha \in]0, 1]$, $[\widetilde{a}]_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) \geq \alpha\}$ is called *the* α*-level set of* $\widetilde{a}$. The set $\mathrm{supp}(\widetilde{a}) = \{\boldsymbol{x} \in \mathbb{R}^n : \widetilde{a}(\boldsymbol{x}) > 0\}$ is called *the support of* $\widetilde{a}$, and $[\widetilde{a}]_0 = \mathrm{cl}(\mathrm{supp}(\widetilde{a}))$ is called *the* 0*-level set of* $\widetilde{a}$. The fuzzy set $\widetilde{a}$ is said to be *support bounded* if $\mathrm{supp}(\widetilde{a})$ is bounded, and $\widetilde{a}$ is said to be *normal* if there exists $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $\widetilde{a}(\boldsymbol{x}_0) = 1$. The fuzzy set $\widetilde{a}$ is called *a compact fuzzy set* if $[\widetilde{a}]_\alpha \in \mathcal{C}(\mathbb{R}^n)$ for any $\alpha \in]0, 1]$, and $\widetilde{a}$ is called *a convex fuzzy set* if $[\widetilde{a}]_\alpha$ is a convex set for any $\alpha \in]0, 1]$. Let $\mathcal{FC}(\mathbb{R}^n)$ be the set of all compact fuzzy sets on $\mathbb{R}^n$, and let $\mathcal{FC}_0(\mathbb{R}^n)$ be the set of all compact fuzzy sets on $\mathbb{R}^n$ which are support bounded and normal. The fuzzy set $\widetilde{a}$ is called *a fuzzy vector on* $\mathbb{R}^n$ if $\widetilde{a} \in \mathcal{FC}_0(\mathbb{R}^n)$, and $\widetilde{a} \in \mathcal{FC}_0(\mathbb{R})$ is called *a fuzzy number* if $\widetilde{a}$ is convex (Jamison and Lodwick 1999). Let $\mathcal{FN}$ be the set of all fuzzy numbers. Then, it follows that $\mathcal{FN} \subset \mathcal{FC}_0(\mathbb{R}) \subset \mathcal{FC}(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$ and $\mathcal{FC}_0(\mathbb{R}^n) \subset \mathcal{FC}(\mathbb{R}^n) \subset \mathcal{F}(\mathbb{R}^n)$ from the definitions.

It is well-known as the decomposition theorem that $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$ can be represented as

$$\widetilde{a} = \sup_{\alpha \in]0,1]} \alpha c_{[\widetilde{a}]_\alpha}, \tag{6}$$

where for $A \subset \mathbb{R}^n$, $c_A : \mathbb{R}^n \to \{0, 1\}$ is the indicator function defined as $c_A(\boldsymbol{x}) = 1$ if $\boldsymbol{x} \in A$, and $c_A(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin A$ for each $\boldsymbol{x} \in \mathbb{R}^n$ (Dubois et al. 2000; Zadeh 1975).

We set

$$\begin{aligned} \mathcal{S}(\mathbb{R}^n) = \{\{S_\alpha\}_{\alpha \in]0,1]} : S_\alpha \subset \mathbb{R}^n, \alpha \in]0, 1], \text{ and} \\ S_\beta \supset S_\gamma \text{ for } \beta, \gamma \in]0, 1] \text{ with } \beta < \gamma\}, \end{aligned} \tag{7}$$

and set

$$M_{\mathbb{R}^n}(\{S_\alpha\}_{\alpha \in]0,1]}) = \sup_{\alpha \in]0,1]} \alpha c_{S_\alpha} \in \mathcal{F}(\mathbb{R}^n) \tag{8}$$

for each $\{S_\alpha\}_{\alpha \in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$. For simplicity, $M_{\mathbb{R}^n}$ is also written as M.

The following lemma provides a property of level sets of fuzzy sets defined in Eq. (8).

Lemma 6 (Kon 2013, Proposition 4) *Let* $\{S_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$*, and let* $\widetilde{a} = M(\{S_\alpha\}_{\alpha\in]0,1]})$*. Then,* $[\widetilde{a}]_\alpha = \cap_{\beta\in]0,\alpha[} S_\beta$ *for any* $\alpha \in]0, 1]$.

Let $\widetilde{a} \in \mathcal{F}(\mathbb{R}^n)$, and let $\boldsymbol{k} \in \mathrm{int}(\mathbb{R}^n_+)$. Since $\{[\widetilde{a}]_\alpha\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R}^n)$, it follows that $\{I_{[\widetilde{a}]_\alpha}(\boldsymbol{k})\}_{\alpha\in]0,1]} \in \mathcal{S}(\mathbb{R})$ from Theorem 1 (ii). We set

$$\widetilde{a}_I = M(\{I_{[\widetilde{a}]_\alpha}(\boldsymbol{k})\}_{\alpha\in]0,1]}) \in \mathcal{F}(\mathbb{R}). \tag{9}$$

The following theorem provides a property of level sets of fuzzy sets defined in Eq. (9).

Theorem 3 *Let* $\widetilde{a} \in \mathcal{FC}(\mathbb{R}^n)$*, and let* $\boldsymbol{k} \in int(\mathbb{R}^n_+)$*. Then,* $[\widetilde{a}_I]_\alpha = I_{[\widetilde{a}]_\alpha}(\boldsymbol{k})$ *for any* $\alpha \in]0, 1]$.

Proof Fix any $\alpha \in]0, 1]$. Since

$$s_L([\widetilde{a}]_\alpha; \boldsymbol{k}) = s_L\left(\bigcap_{\beta\in]0,\alpha[} [\widetilde{a}]_\beta; \boldsymbol{k}\right) = \sup_{\beta\in]0,\alpha[} s_L([\widetilde{a}]_\beta; \boldsymbol{k}),$$
$$s_U([\widetilde{a}]_\alpha; \boldsymbol{k}) = s_U\left(\bigcap_{\beta\in]0,\alpha[} [\widetilde{a}]_\beta; \boldsymbol{k}\right) = \inf_{\beta\in]0,\alpha[} s_U([\widetilde{a}]_\beta; \boldsymbol{k})$$

from Lemmas 4 and 6, we have

$$\begin{aligned} I_{[\widetilde{a}]_\alpha}(\boldsymbol{k}) &= [s_L([\widetilde{a}]_\alpha; \boldsymbol{k}), s_U([\widetilde{a}]_\alpha; \boldsymbol{k})] \\ &= \left[\sup_{\beta\in]0,\alpha[} s_L([\widetilde{a}]_\beta; \boldsymbol{k}), \inf_{\beta\in]0,\alpha[} s_U([\widetilde{a}]_\beta; \boldsymbol{k})\right] \\ &= \bigcap_{\beta\in]0,\alpha[} [s_L([\widetilde{a}]_\beta; \boldsymbol{k}), s_U([\widetilde{a}]_\beta; \boldsymbol{k})] \\ &= \bigcap_{\beta\in]0,\alpha[} I_{[\widetilde{a}]_\beta}(\boldsymbol{k}) \\ &= [\widetilde{a}_I]_\alpha \end{aligned}$$

from Lemma 6. □

Next, we introduce some orderings on $\mathcal{F}(\mathbb{R}^n)$.

Definition 3 (Kon 2014, Definition 5.1) Let $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$.

(i) $\widetilde{a} \preceq \widetilde{b} \overset{\text{def}}{\Leftrightarrow} [\widetilde{a}]_\alpha \leq [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$;
(ii) $\widetilde{a} \prec \widetilde{b} \overset{\text{def}}{\Leftrightarrow} [\widetilde{a}]_\alpha < [\widetilde{b}]_\alpha$ for any $\alpha \in]0, 1]$.

The order $\preceq$ in Definition 3 is an extension of the fuzzy max order for fuzzy numbers. The fuzzy max order for fuzzy numbers has been primarily defined in Ramík and Řimánek (1985). Then, the fuzzy max order for fuzzy numbers has been extended for fuzzy vectors in Maeda (2008), and for fuzzy sets which are closed, convex, normal, and support bounded in Kurano et al. (2000). Thus, the orders $\preceq$ and $\prec$ in Definition 3, which are further extensions of them, are called *the fuzzy max order* and *the strict fuzzy max order on* $\mathcal{F}(\mathbb{R}^n)$, respectively, in Kon (2014).

The following lemmas provide fundamental properties of the (strict) fuzzy max order.

Lemma 7 (Kon 2014, Proposition 5.1) *Let* $\widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathcal{F}(\mathbb{R}^n)$.

(i) $\widetilde{a} \preceq \widetilde{a}$;
(ii) $\widetilde{a} \preceq \widetilde{b}, \widetilde{b} \preceq \widetilde{c} \Rightarrow \widetilde{a} \preceq \widetilde{c}$;
(iii) $\widetilde{a} \prec \widetilde{b} \Rightarrow \widetilde{a} \preceq \widetilde{b}$.

Lemma 8 (Kon 2014, Proposition 5.2) *Let* $\widetilde{a}, \widetilde{b} \in \mathcal{F}(\mathbb{R}^n)$. *Assume that there exists* $\alpha \in]0, 1]$ *such that* $[\widetilde{a}]_\alpha \in \mathcal{C}_0(\mathbb{R}^n)$ *or* $[\widetilde{b}]_\alpha \in \mathcal{C}_0(\mathbb{R}^n)$. *If* $\widetilde{a} \prec \widetilde{b}$, *then* $\widetilde{a} \not\succeq \widetilde{b}$.

Next, we introduce order preserving properties for fuzzy set-valued mappings.

Definition 4 Let $\widetilde{F} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m)$, and let $X \subset \mathbb{R}^n$.

(i) $\widetilde{F}$ is said to be *order preserving on* X if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, $[\widetilde{F}(\boldsymbol{x})]_0 \leq [\widetilde{F}(\boldsymbol{y})]_0$ and $[\widetilde{F}(\boldsymbol{x})]_1 \leq [\widetilde{F}(\boldsymbol{y})]_1$ imply $\widetilde{F}(\boldsymbol{x}) \preceq \widetilde{F}(\boldsymbol{y})$;
(ii) $\widetilde{F}$ is said to be *strictly order preserving on* X if for any $\boldsymbol{x}, \boldsymbol{y} \in X$, $[\widetilde{F}(\boldsymbol{x})]_0 < [\widetilde{F}(\boldsymbol{y})]_0$ and $[\widetilde{F}(\boldsymbol{x})]_1 < [\widetilde{F}(\boldsymbol{y})]_1$ imply $\widetilde{F}(\boldsymbol{x}) \prec \widetilde{F}(\boldsymbol{y})$.

Let $\widetilde{F} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m)$. The fuzzy set-valued mapping $\widetilde{F}$ is said to be *compact-valued* if $\widetilde{F}(\boldsymbol{x}) \in \mathcal{FC}(\mathbb{R}^m)$ for any $\boldsymbol{x} \in \mathbb{R}^n$. For each $\alpha \in [0, 1]$, we define $F_\alpha : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ as

$$F_\alpha(\boldsymbol{x}) = [\widetilde{F}(\boldsymbol{x})]_\alpha \tag{10}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$.

4 Fuzzy set optimization

In this section, fuzzy set optimization problems are considered, and the properties are investigated.

For $X \subset \mathbb{R}^n$, $X \neq \emptyset$ and $\widetilde{F} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m)$, a problem

$$\left| \begin{array}{ll} \min & \widetilde{F}(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in X \end{array} \right. \tag{FOP}$$

is called *a fuzzy set optimization problem*, and $\widetilde{F}$ is called *a fuzzy set-valued objective mapping*. In (FOP), $\widetilde{F}$ is called *a fuzzy number-valued objective mapping* if $\widetilde{F} : \mathbb{R}^n \to \mathcal{FN}$, and $\widetilde{F}$ is called *a fuzzy vector-valued objective mapping* if $\widetilde{F} : \mathbb{R}^n \to \mathcal{FC}_0(\mathbb{R}^m)$. Our main problem is the fuzzy set optimization problem (FOP).

Next, we introduce some solution concepts for (FOP).

Definition 5 (i) A point $\boldsymbol{x}^* \in X$ is called *a non-dominated solution of* (FOP) if for any $\boldsymbol{x} \in X$, $\widetilde{F}(\boldsymbol{x}) \preceq \widetilde{F}(\boldsymbol{x}^*)$ implies $\widetilde{F}(\boldsymbol{x}^*) \preceq \widetilde{F}(\boldsymbol{x})$;

(ii) A point $\boldsymbol{x}^* \in X$ is called *a weak non-dominated solution of* (FOP) if there is no $\boldsymbol{x} \in X$ such that $\widetilde{F}(\boldsymbol{x}) \prec \widetilde{F}(\boldsymbol{x}^*)$.

The following theorem provides a relationship between non-dominated and weak non-dominated solutions of (FOP).

Theorem 4 *In (FOP), assume that for any $\boldsymbol{x} \in X$, there exists $\alpha \in]0, 1]$ such that $[\widetilde{F}(\boldsymbol{x})]_\alpha \in \mathcal{C}_0(\mathbb{R}^m)$. If $\boldsymbol{x}^* \in X$ is a non-dominated solution of (FOP), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (FOP).*

Proof Suppose that $\boldsymbol{x}^* \in X$ is not a weak non-dominated solution of (FOP). Then, there exists $\boldsymbol{x}_0 \in X$ such that $\widetilde{F}(\boldsymbol{x}_0) \prec \widetilde{F}(\boldsymbol{x}^*)$. Since $\widetilde{F}(\boldsymbol{x}_0) \prec \widetilde{F}(\boldsymbol{x}^*)$, it follows that $\widetilde{F}(\boldsymbol{x}_0) \preceq \widetilde{F}(\boldsymbol{x}^*)$ from Lemma 7 (iii), and that $\widetilde{F}(\boldsymbol{x}^*) \not\preceq \widetilde{F}(\boldsymbol{x}_0)$ from Lemma 8. Therefore, $\boldsymbol{x}^*$ is not a non-dominated solution of (FOP). □

The following example shows that the assumption in Theorem 4 cannot be eliminated.

Example 2 In (FOP), let $n = 1$, $m = 2$, and $X = \mathbb{R}$. Set $A = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : y \geq 2^x\}$, and $D = \{(1, 1)\}$.

(i) Define $\widetilde{F} : \mathbb{R} \to \mathcal{F}(\mathbb{R}^2)$ as

$$\widetilde{F}(x) = \begin{cases} c_A & \text{if } x = 0, \\ c_D & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$, and consider the following fuzzy set optimization problem:

$$\left| \begin{array}{ll} \min & \widetilde{F}(x) \\ \text{s.t.} & x \in \mathbb{R}. \end{array} \right. \tag{FOP1}$$

In this case, $[\widetilde{F}(0)]_\alpha \notin \mathcal{C}_0(\mathbb{R}^2)$ for any $\alpha \in]0, 1]$ since $[\widetilde{F}(0)]_\alpha = A$ is bounded but not closed for each $\alpha \in]0, 1]$. Then, it can be verified that 0 is a non-dominated solution of (FOP1). However, 0 is not a weak non-dominated solution of (FOP1) since $\widetilde{F}(0) \prec \widetilde{F}(0)$.

(ii) Define $\widetilde{F} : \mathbb{R} \to \mathcal{F}(\mathbb{R}^2)$ as

$$\widetilde{F}(x) = \begin{cases} c_B & \text{if } x = 0, \\ c_D & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$, and consider the following fuzzy set optimization problem:

$$\left| \begin{array}{ll} \min & \widetilde{F}(x) \\ \text{s.t.} & x \in \mathbb{R}. \end{array} \right. \tag{FOP2}$$

In this case, $[\widetilde{F}(0)]_\alpha \notin \mathcal{C}_0(\mathbb{R}^2)$ for any $\alpha \in]0, 1]$ since $[\widetilde{F}(0)]_\alpha = B$ is closed but not bounded for each $\alpha \in]0, 1]$. Then, it can be verified that 0 is a non-dominated solution of (FOP2). However, 0 is not a weak non-dominated solution of (FOP2) since $\widetilde{F}(0) \prec \widetilde{F}(0)$.

□

Let $\boldsymbol{k} \in \text{int}(\mathbb{R}^m_+)$. For $\widetilde{F} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m)$ in (FOP), we define $\widetilde{G} : \mathbb{R}^n \to \mathcal{F}(\mathbb{R})$ as

$$\widetilde{G}(\boldsymbol{x}) = \widetilde{F}(\boldsymbol{x})_I \tag{11}$$

for each $\boldsymbol{x} \in \mathbb{R}^n$. Note that $\widetilde{G}$ depends on $\boldsymbol{k}$. Then, we consider the following problem:

$$\left| \begin{array}{ll} \min & \widetilde{G}(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in X. \end{array} \right. \tag{FOP'}$$

The following theorem provides a relationship between non-dominated and weak non-dominated solutions of (FOP').

Theorem 5 *Let $\boldsymbol{k} \in int(\mathbb{R}^m_+)$. In (FOP), assume that $\widetilde{F}$ is compact-valued, and that for any $\boldsymbol{x} \in X$, there exists $\alpha \in]0, 1]$ such that $[\widetilde{F}(\boldsymbol{x})]_\alpha \in \mathcal{C}_0(\mathbb{R}^m)$. If $\boldsymbol{x}^* \in X$ is a non-dominated solution of (FOP'), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (FOP').*

Proof Fix any $\boldsymbol{x} \in X$. Then, there exists $\alpha \in]0, 1]$ such that $[\widetilde{F}(\boldsymbol{x})]_\alpha \in \mathcal{C}_0(\mathbb{R}^m)$. It follows that $[\widetilde{G}(\boldsymbol{x})]_\alpha = [\widetilde{F}(\boldsymbol{x})_I]_\alpha = I_{[\widetilde{F}(\boldsymbol{x})]_\alpha}(\boldsymbol{k}) \in \mathcal{C}_0(\mathbb{R})$ from Theorem 3. Therefore, if $\boldsymbol{x}^* \in X$ is a non-dominated solution of (FOP'), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (FOP') from Theorem 4. □

The following theorem provides a relationship between weak non-dominated solutions of (FOP) and (FOP').

Theorem 6 *Let $\boldsymbol{k} \in int(\mathbb{R}^m_+)$. In (FOP), assume that $\widetilde{F}$ is compact-valued. If $\boldsymbol{x}^* \in X$ is a weak non-dominated solution of (FOP'), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (FOP).*

Proof Suppose that $\boldsymbol{x}^* \in X$ is not a weak non-dominated solution of (FOP). Then, there exists $\boldsymbol{x}_0 \in X$ such that $\widetilde{F}(\boldsymbol{x}_0) \prec \widetilde{F}(\boldsymbol{x}^*)$. Fix any $\alpha \in]0, 1]$. Since $[\widetilde{F}(\boldsymbol{x}_0)]_\alpha < [\widetilde{F}(\boldsymbol{x}^*)]_\alpha$, it follows that $I_{[\widetilde{F}(\boldsymbol{x}_0)]_\alpha}(\boldsymbol{k}) < I_{[\widetilde{F}(\boldsymbol{x}^*)]_\alpha}(\boldsymbol{k})$ from Theorem 1 (vii), (viii). Then, it follows that $[\widetilde{G}(\boldsymbol{x}_0)]_\alpha = [\widetilde{F}(\boldsymbol{x}_0)_I]_\alpha = I_{[\widetilde{F}(\boldsymbol{x}_0)]_\alpha}(\boldsymbol{k}) < I_{[\widetilde{F}(\boldsymbol{x}^*)]_\alpha}(\boldsymbol{k}) = [\widetilde{F}(\boldsymbol{x}^*)_I]_\alpha = [\widetilde{G}(\boldsymbol{x}^*)]_\alpha$ from Theorem 3. Since $\widetilde{G}(\boldsymbol{x}_0) \prec \widetilde{G}(\boldsymbol{x}^*)$ by the arbitrariness of $\alpha \in]0, 1]$, x^* is not a weak non-dominated solution of (FOP'). □

The following example shows that we cannot prove the same result as Theorem 6 for non-dominated solutions.

Example 3 In (FOP), let $n = 1, m = 2$, and $X = \mathbb{R}$. In addition, let $\boldsymbol{k} = (1, 1) \in \text{int}(\mathbb{R}^2_+)$. Set $A = [0, 1] \times [0, 1]$, $B = \{0\} \times [0, 1]$, and $D = [0, 1]$. Define $\widetilde{F} : \mathbb{R} \to \mathcal{F}(\mathbb{R}^2)$ as

$$\widetilde{F}(x) = \begin{cases} c_A & \text{if } x = 0, \\ c_B & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$, and consider the following fuzzy set optimization problem:

$$\begin{vmatrix} \min & \widetilde{F}(x) \\ \text{s.t.} & x \in \mathbb{R}. \end{vmatrix} \tag{FOP3}$$

It follows that $I_{[\widetilde{F}(x)]_\alpha}(\boldsymbol{k}) = [0, 1]$ for any $x \in \mathbb{R}$ and any $\alpha \in]0, 1]$, and that $\widetilde{G}(x) = c_D$ for any $x \in \mathbb{R}$, where $\widetilde{G} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ is defined by Eq. (11). Now, consider the following problem:

$$\begin{vmatrix} \min & \widetilde{G}(x) \\ \text{s.t.} & x \in \mathbb{R}. \end{vmatrix} \tag{FOP3'}$$

Then, it can be verified that each $x \in \mathbb{R}$ is a non-dominated solution of (FOP3'). However, 0 is not a non-dominated solution of (FOP3) since $\widetilde{F}(x) \preceq \widetilde{F}(0)$ and $\widetilde{F}(0) \not\preceq \widetilde{F}(x)$ for any $x \in \mathbb{R}\backslash\{0\}$. □

5 Scalarization method

In this section, a scalarization method for the fuzzy set optimization problems is proposed.

Throughout this section, assume that $\widetilde{F}(\boldsymbol{x}) \in \mathcal{FC}_0(\mathbb{R}^m)$ for any $\boldsymbol{x} \in X$ in (FOP). For the fuzzy set optimization problem (FOP), we consider the following set optimization problem:

$$\begin{vmatrix} \min & F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in X \end{vmatrix} \tag{SOP}$$

where F_0 and F_1 are defined by Eq. (10).

Next, we introduce some solution concepts for (SOP).

Definition 6 (Maeda 2012, Definitions 4.2 and 4.4) (i) A point $\boldsymbol{x}^* \in X$ is called *a non-dominated solution of* (SOP) if for any $\boldsymbol{x} \in X$, $F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x}) \leq F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$ implies $F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*) \leq F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x})$;

(ii) A point $\boldsymbol{x}^* \in X$ is called *a weak non-dominated solution of* (SOP) if there is no $\boldsymbol{x} \in X$ such that $F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x}) < F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$.

The following theorem provides a relationship between non-dominated and weak non-dominated solutions of (SOP).

Theorem 7 *If $\boldsymbol{x}^* \in X$ is a non-dominated solution of (SOP), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (SOP).*

Proof Suppose that $\boldsymbol{x}^* \in X$ is not a weak non-dominated solution of (SOP). Then, there exists $\boldsymbol{x}_0 \in X$ such that $F_0(\boldsymbol{x}_0) \times F_1(\boldsymbol{x}_0) < F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$. Since $F_0(\boldsymbol{x}_0) \times F_1(\boldsymbol{x}_0) < F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$, it follows that $F_0(\boldsymbol{x}_0) \times F_1(\boldsymbol{x}_0) \leq F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$ from Lemma 1 (vii), and that $F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*) \not\leq F_0(\boldsymbol{x}_0) \times F_1(\boldsymbol{x}_0)$ from Lemma 2. Therefore, $\boldsymbol{x}^*$ is not a non-dominated solution of (SOP). □

The following theorem provides relationships between (weak) non-dominated solutions of (FOP) with the (strict) order preserving property and those of (SOP).

Theorem 8 *(i) In (FOP), assume that $\widetilde{F}$ is order preserving on X. Then, $\boldsymbol{x}^* \in X$ is a non-dominated solution of (FOP) if and only if $\boldsymbol{x}^*$ is a non-dominated solution of (SOP);*

(ii) In (FOP), assume that $\widetilde{F}$ is strictly order preserving on X. If $\boldsymbol{x}^* \in X$ is a weak non-dominated solution of (FOP), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (SOP).

Proof (i) First, suppose that $\boldsymbol{x}^* \in X$ is a non-dominated solution of (FOP). Let $\overline{\boldsymbol{x}} \in X$, and suppose that $F_0(\overline{\boldsymbol{x}}) \times F_1(\overline{\boldsymbol{x}}) \leq F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$. It follows that $F_0(\overline{\boldsymbol{x}}) \leq F_0(\boldsymbol{x}^*)$ and $F_1(\overline{\boldsymbol{x}}) \leq F_1(\boldsymbol{x}^*)$. Since $\widetilde{F}$ is order preserving on X, if follows that $\widetilde{F}(\overline{\boldsymbol{x}}) \preceq \widetilde{F}(\boldsymbol{x}^*)$. Since $\boldsymbol{x}^*$ is a non-dominated solution of (FOP), it follows that $\widetilde{F}(\boldsymbol{x}^*) \preceq \widetilde{F}(\overline{\boldsymbol{x}})$. Then, it follows that $F_\alpha(\boldsymbol{x}^*) \leq F_\alpha(\overline{\boldsymbol{x}})$ for any $\alpha \in]0, 1]$, and that

$$\operatorname{supp}(\widetilde{F}(\boldsymbol{x}^*)) = \bigcup_{\alpha \in]0,1]} F_\alpha(\boldsymbol{x}^*) \leq \bigcup_{\alpha \in]0,1]} F_\alpha(\overline{\boldsymbol{x}}) = \operatorname{supp}(\widetilde{F}(\overline{\boldsymbol{x}})).$$

Thus, it follows that

$$F_0(\boldsymbol{x}^*) = \operatorname{cl}(\operatorname{supp}(\widetilde{F}(\boldsymbol{x}^*))) \leq \operatorname{cl}(\operatorname{supp}(\widetilde{F}(\overline{\boldsymbol{x}}))) = F_0(\overline{\boldsymbol{x}}),$$

and we have $F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*) \leq F_0(\overline{\boldsymbol{x}}) \times F_1(\overline{\boldsymbol{x}})$. Therefore, $\boldsymbol{x}^*$ is a non-dominated solution of (SOP).

Next, suppose that $\boldsymbol{x}^* \in X$ is a non-dominated solution of (SOP). Let $\overline{\boldsymbol{x}} \in X$, and suppose that $\widetilde{F}(\overline{\boldsymbol{x}}) \preceq \widetilde{F}(\boldsymbol{x}^*)$. Since $\widetilde{F}(\overline{\boldsymbol{x}}) \preceq \widetilde{F}(\boldsymbol{x}^*)$, it follows that $F_\alpha(\overline{\boldsymbol{x}}) \leq F_\alpha(\boldsymbol{x}^*)$ for any $\alpha \in]0, 1]$, and that $F_0(\overline{\boldsymbol{x}}) \leq F_0(\boldsymbol{x}^*)$ by the same arguments as in the first part. Thus, it follows that $F_0(\overline{\boldsymbol{x}}) \times F_1(\overline{\boldsymbol{x}}) \leq F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$. Since $\boldsymbol{x}^*$ is a non-dominated solution of (SOP), it follows that $F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*) \leq F_0(\overline{\boldsymbol{x}}) \times F_1(\overline{\boldsymbol{x}})$, and that $F_0(\boldsymbol{x}^*) \leq F_0(\overline{\boldsymbol{x}})$ and $F_1(\boldsymbol{x}^*) \leq F_1(\overline{\boldsymbol{x}})$. Since $\widetilde{F}$ is order preserving on X, we have $\widetilde{F}(\boldsymbol{x}^*) \preceq \widetilde{F}(\overline{\boldsymbol{x}})$. Therefore, $\boldsymbol{x}^*$ is a non-dominated solution of (FOP).

(ii) Suppose that $\boldsymbol{x}^* \in X$ is not a weak non-dominated solution of (SOP). Then, there exists $\overline{\boldsymbol{x}} \in X$ such that $F_0(\overline{\boldsymbol{x}}) \times F_1(\overline{\boldsymbol{x}}) < F_0(\boldsymbol{x}^*) \times F_1(\boldsymbol{x}^*)$. It follows that $F_0(\overline{\boldsymbol{x}}) < F_0(\boldsymbol{x}^*)$ and $F_1(\overline{\boldsymbol{x}}) < F_1(\boldsymbol{x}^*)$. Since $\widetilde{F}$ is strictly order preserving on X, we have $\widetilde{F}(\overline{\boldsymbol{x}}) \prec \widetilde{F}(\boldsymbol{x}^*)$. Therefore, $\boldsymbol{x}^*$ is not a weak non-dominated solution of (FOP).

□

The following example shows the necessity of strict order preserving for weak non-dominated solutions in Theorem 8 (ii).

Example 4 In (FOP), let $n = m = 1$ and $X = \mathbb{R}$. Define $\widetilde{F} : \mathbb{R} \to \mathcal{F}(\mathbb{R})$ as

$$\widetilde{F}(0)(y) = \begin{cases} \frac{1}{4}(y-1) & \text{if } y \in [1,5], \\ -\frac{1}{2}(y-7) & \text{if } y \in]5,6], \\ -\frac{1}{8}(y-10) & \text{if } y \in]6,10], \\ 0 & \text{otherwise} \end{cases}$$

for each $y \in \mathbb{R}$, and

$$\widetilde{F}(x)(y) = \max\left\{1 - \frac{|y-4|}{4}, 0\right\}$$

for each $x \in \mathbb{R} \setminus \{0\}$ and each $y \in \mathbb{R}$. Then, it follows that $\widetilde{F}$ is order preserving on $\mathbb{R}$ but not strictly order preserving on $\mathbb{R}$. We consider the following fuzzy set optimization problem:

$$\left|\begin{array}{ll} \min & \widetilde{F}(x) \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right. \tag{FOP4}$$

and the following set optimization problem:

$$\left|\begin{array}{ll} \min & F_0(x) \times F_1(x) \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right. \tag{SOP4}$$

where

$$F_0(x) \times F_1(x) = \begin{cases} [1, 10] \times \{5\} & \text{if } x = 0, \\ [0, 8] \times \{4\} & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}$. It can be verified that each $x \in \mathbb{R}$ is a weak non-dominated solution of (FOP4). However, 0 is not a weak non-dominated solution of (SOP4) since $F_0(x) \times F_1(x) < F_0(0) \times F_1(0)$ for any $x \in \mathbb{R} \setminus \{0\}$. □

We consider the following scalarization problem of the set optimization problem (SOP):

$$\left|\begin{array}{ll} \min & \lambda s_L(F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x}); \boldsymbol{k}) + (1-\lambda) s_U(F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x}); \boldsymbol{k}) \\ \text{s.t.} & \boldsymbol{x} \in X, \end{array}\right. \tag{P}$$

where $\boldsymbol{k} \in \operatorname{int}(\mathbb{R}^{2m}_+)$ and $\lambda \in [0, 1]$ are parameters. The problem (P) is also the scalarization problem of the fuzzy set optimization problem (FOP).

The following theorem provides relationships between (weak) non-dominated solutions of (SOP) and optimal solutions of (P).

Theorem 9 (Maeda 2012, Theorem 4.2) *(i) If $\boldsymbol{x}^* \in X$ is a unique optimal solution of (P), then $\boldsymbol{x}^*$ is a non-dominated solution of (SOP);*

(ii) If $\boldsymbol{x}^* \in X$ is an optimal solution of (P), then $\boldsymbol{x}^*$ is a weak non-dominated solution of (SOP).

The following theorem provides a relationship between non-dominated solutions of (FOP) and optimal solutions of (P).

Theorem 10 *In (FOP), assume that X is a convex set, and that $\widetilde{F}$ is order preserving on X. In addition, assume that F_0 and F_1 are strictly convex mappings. If $\boldsymbol{x}^* \in X$ is an optimal solution of (P), then $\boldsymbol{x}^*$ is a non-dominated solution of (FOP).*

Proof We show that $F_0 \times F_1 : \mathbb{R}^n \to \mathcal{C}_0(\mathbb{R}^{2m})$ is a strictly convex mapping, where $(F_0 \times F_1)(\boldsymbol{x}) = F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^n$. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \boldsymbol{y}$, and let $\lambda \in]0, 1[$. Since F_0 and F_1 are strictly convex mappings, it follows that $F_0(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) < \lambda F_0(\boldsymbol{x}) + (1 - \lambda)F_0(\boldsymbol{y})$ and $F_1(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) < \lambda F_1(\boldsymbol{x}) + (1 - \lambda)F_1(\boldsymbol{y})$. Thus, we have

$$\begin{aligned}
&(F_0 \times F_1)(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \\
&\quad = F_0(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \times F_1(\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y}) \\
&\quad < (\lambda F_0(\boldsymbol{x}) + (1 - \lambda)F_0(\boldsymbol{y})) \times (\lambda F_1(\boldsymbol{x}) + (1 - \lambda)F_1(\boldsymbol{y})) \\
&\quad = \lambda F_0(\boldsymbol{x}) \times \lambda F_1(\boldsymbol{x}) + (1 - \lambda)F_0(\boldsymbol{y}) \times (1 - \lambda)F_1(\boldsymbol{y}) \\
&\quad = \lambda\,(F_0(\boldsymbol{x}) \times F_1(\boldsymbol{x})) + (1 - \lambda)\,(F_0(\boldsymbol{y}) \times F_1(\boldsymbol{y})) \\
&\quad = \lambda(F_0 \times F_1)(\boldsymbol{x}) + (1 - \lambda)(F_0 \times F_1)(\boldsymbol{y}).
\end{aligned}$$

Therefore, $F_0 \times F_1$ is a strictly convex mapping.

Since $F_0 \times F_1$ is a strictly convex mapping, it follows that $s_L(F_0(\cdot) \times F_1(\cdot); \boldsymbol{k})$, $s_U(F_0(\cdot) \times F_1(\cdot); \boldsymbol{k}) : \mathbb{R}^n \to \mathbb{R}$ are strictly convex functions from Theorem 2 (ii), and that the objective function of (P) is also a strictly convex function. Since the objective function of (P) is a strictly convex function and X is a convex set, it can be shown easily that $\boldsymbol{x}^*$ is a unique optimal solution of (P). Therefore, $\boldsymbol{x}^*$ is a non-dominated solution of (SOP) from Theorem 9 (i). Since $\widetilde{F}$ is order preserving on X and $\boldsymbol{x}^*$ is a non-dominated solution of (SOP), $\boldsymbol{x}^*$ is a non-dominated solution of (FOP) from Theorem 8 (i). □

The following example shows the necessity of the assumption in Theorem 10. However, whether or not we can weaken the assumption is a future work.

Example 5 In (FOP), let $n = 1$, $m = 2$, and $X = \mathbb{R}$. In addition, let $\boldsymbol{k} = (1, 1, 1, 1) \in \mathrm{int}(\mathbb{R}^4_+)$, and fix any $\lambda \in [0, 1]$. Set $A(x) = [0, |x|] \times [0, 1]$ for each $x \in \mathbb{R}$, and define $\widetilde{F} : \mathbb{R} \to \mathcal{F}(\mathbb{R}^2)$ as $\widetilde{F}(x) = c_{A(x)}$ for each $x \in \mathbb{R}$. Then, consider the following fuzzy set optimization problem:

$$\left|\begin{array}{ll} \min & \widetilde{F}(x) \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right. \tag{FOP5}$$

and the following scalarization problem:

$$\left|\begin{array}{ll} \min & \lambda s_L(F_0(x) \times F_1(x); \boldsymbol{k}) + (1 - \lambda)s_U(F_0(x) \times F_1(x); \boldsymbol{k}) \\ \text{s.t.} & x \in \mathbb{R} \end{array}\right. \tag{P5}$$

where for each $x \in \mathbb{R}$, $F_0(x) = F_1(x) = A(x)$, $s_L(F_0(x) \times F_1(x); \boldsymbol{k}) = 0$, and

$$s_U(F_0(x) \times F_1(x); \boldsymbol{k}) = \begin{cases} 1 & \text{if } x \in [-1, 1], \\ |x| & \text{otherwise.} \end{cases}$$

It can be verified that $\widetilde{F}$ is order preserving on $\mathbb{R}$, and that F_0 and F_1 are convex mappings but not strictly convex mappings. Then, it follows that each $x \in \mathbb{R}$ is an optimal solution of (P5) when $\lambda = 1$, and that each $x \in [-1, 1]$ is an optimal solution of (P5) when $\lambda \in [0, 1[$. Fix any $x_0 \in [-1, 1] \setminus \{0\}$. Since $\widetilde{F}(0) \preceq \widetilde{F}(x_0)$ and $\widetilde{F}(x_0) \not\preceq \widetilde{F}(0)$, x_0 is not a non-dominated solution of (FOP5) □

Assume that $\widetilde{F}$ is strictly order preserving on X in (FOP), and that $\boldsymbol{x}^* \in X$ is an optimal solution of (P). From Theorem 9 (ii), $\boldsymbol{x}^*$ is a weak non-dominated solution of (SOP). Then, $\boldsymbol{x}^*$ is a candidate of a weak non-dominated solution of (FOP) from Theorem 8 (ii), and $\boldsymbol{x}^*$ is also a candidate of a non-dominated solution of (FOP) from Theorem 4. Since in general, solving (SOP) is easier than solving (FOP), and solving (P) is easier than solving (SOP), the obtained results can be expected to enable us to solve (FOP) much easier by solving (SOP) or (P) under adequate assumptions.

6 Conclusion

We dealt with the fuzzy set optimization problems. First, the scalarizations of sets were introduced, and their properties were investigated. Next, the order preserving property was introduced for fuzzy sets. Based on the order preserving property, the set optimization problems associated with the fuzzy set optimization problems were derived, and the scalar optimization problems associated with the set optimization problems were derived by using the scalarizations of sets. Then, the relationships between (weak) non-dominated solutions of the fuzzy set optimization problems and those of the set optimization problems were investigated, and the relationships between non-dominated solutions of the fuzzy set optimization problems and optimal solutions of the scalar optimization problems were investigated. The obtained results enable us to solve the fuzzy set optimization problems by solving the set optimization problems or the scalar optimization problems under adequate assumptions.

References

Bellman, R. E., & Zadeh, L. A. (1970). Decision making in a fuzzy environment. *Management Science*, *17*, 141–164.

Dubois, D., Ostasiewicz, W., & Prade, H. (2000). Fuzzy sets: History and basic notions. In D. Dubois & H. Prade (Eds.), *Fundamentals of fuzzy sets* (pp. 21–124). Boston: Kluwer.

Gupta, S. K., & Dangar, D. (2014). Duality for a class of fuzzy nonlinear optimization problem under generalized convexity. *Fuzzy Optimization and Decision Making*, *13*, 131–150.

Hernández, H., & Rodríguez-Marín, L. (2007). Nonconvex scalarization in set optimization with set-valued maps. *Journal of Mathematical Analysis and Applications*, *325*, 1–18.

Inuiguchi, M. (2005). Multiple objective linear programming with fuzzy coefficients. In J. Figueira, S. Greco, & M. Ehrgott (Eds.), *Multiple criteria decision analysis: State of the art surveys* (pp. 723–760). New York: Springer.

Inuiguchi, M. (2007). On possibilistic/fuzzy optimization. In P. Melin, O. Castillo, L. T. Aguilar, & W. Pedrycz (Eds.), *Foundations of fuzzy logic and soft computing* (pp. 351–360). Berlin Heidelberg: Springer.

Inuiguchi, M., & Ramík, J. (2000). Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems*, *111*, 3–28.

Jahn, J., & Ha, T. X. D. (2011). New order relations in set optimization. *Journal of Optimization Theory and Applications*, *148*, 209–236.

Jamison, K. D., & Lodwick, W. A. (1999). Minimizing unconstrained fuzzy functions. *Fuzzy Sets and Systems*, *103*, 457–464.

Kon, M. (2014). Operation and ordering of fuzzy sets, and fuzzy set-valued convex mappings, *Journal of Fuzzy Set Valued Analysis*, Article ID jfsva-00202.

Kon, M. (2013). On degree of non-convexity of fuzzy sets. *Scientiae Mathematicae Japonicae*, *76*, 417–425.

Kurano, M., Yasuda, M., Nakagami, J., & Yoshida, Y. (2000). Ordering of convex fuzzy sets—A brief survey and new results. *Journal of the Operations Research Society of Japan*, *43*, 138–148.

Kuroiwa, D., Tanaka, T., & Ha, T. X. D. (1997). On cone convexity of set-valued maps. *Nonlinear Analysis: Theory, Methods & Applications*, *30*, 1487–1496.

Maeda, T. (2008). On characterization of fuzzy vectors and its applications to fuzzy mathematical programming problems. *Fuzzy Sets and Systems*, *159*, 3333–3346.

Maeda, T. (2012). On optimization problems with set-valued objective maps: Existence and optimality. *Journal of Optimization Theory and Applications*, *153*, 263–279.

Ramík, J., & Řimánek, J. (1985). Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets and Systems*, *16*, 123–138.

Wu, H.-C. (2006). Scalarization of the fuzzy optimization problems. *Fuzzy Optimization and Decision Making*, *5*, 331–353.

Wu, H.-C. (2007). Duality theory in fuzzy optimization problems formulated by the Wolfe's primal and dual pair. *Fuzzy Optimization and Decision Making*, *6*, 179–198.

Wu, H.-C. (2008). The optimality conditions for optimization problems with fuzzy-valued objective functions. *Optimization*, *57*, 473–489.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338–353.

Zadeh, L. A. (1975). The concept of a linguistic variable and its application to approximate reasoning-I. *Information Sciences*, *8*, 199–249.