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Parameter estimation in uncertain differential equations

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Abstract
Parameter estimation is a critical problem in the wide applications of uncertain differential equations. The method of moments is employed for the first time as an approach for estimating the parameters in uncertain differential equations. Based on the difference form of an uncertain differential equation, a function of the parameters is proved to follow a standard normal uncertainty distribution. Setting the empirical moments of the functions of the parameters and the observed data equal to the moments of the standard normal uncertainty distribution, a system of equations about the parameters is obtained whose solutions are the estimates of the parameters. Analytic examples and numerical examples are given to illustrate the proposed method of moments.

Keywords Uncertain differential equation · Method of moments · Uncertainty theory · Parameter estimation

1 Introduction

The stochastic differential equations have been widely applied to modeling time evolution of dynamic systems which are influenced by random noises. The coefficients of these models sometimes contain parameters whose values are to be estimated based on some observed data of the systems. So far, many useful methods have been proposed for the task of parameter estimation in stochastic differential equations, which mainly could be divided into two categories, namely likelihood-based methods (Taraskin 1974; Kutoyant 1978) and methods of moments (Chan et al. 1992). Interested readers may consult Bishwal (2008).

With the Wiener processes describing the white noises, the stochastic differential equations may fail to model many time-varying systems. For example, two paradoxes

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were pointed out by Liu (2013) when modeling stock prices with stochastic differential equations. Hence, alternative approaches are required to describe the dynamic systems with noises. Within the framework of uncertainty theory which was founded by Liu (2007) and perfected by Liu (2009) based on the normality, duality, subadditivity and product axioms, Liu process was designed as a counterpart of Wiener process. The uncertain differential equations, which aim to model the dynamic systems with human uncertainty, are a type of differential equations driven by Liu processes. Chen and Liu (2010) showed an uncertain differential equation has a unique solution if its coefficients satisfy the linear growth condition and the Lipschitz condition, and Yao et al. (2013) showed the solution is stable in measure if the coefficients satisfy the strong Lipschitz condition. The numerical methods for solving uncertain differential equations were first designed by Yao and Chen (2013) and later extended by Yang and Ralescu (2015), Wang et al. (2015), Gao (2016), Zhang et al. (2017), etc.

The uncertain differential equations have been widely applied to the financial markets. For example, Liu (2009) assumed the short-term stock price follows the exponential Liu process, and calculated the prices of European options, and Chen (2011) calculated the prices of American options. Chen and Gao (2013) described the interest rate in the ideal market with an uncertain differential equation, and calculated the price of a zero-coupon bond. These works were later extended by other researchers such as Jiao and Yao (2015), Zhang et al. (2016), Ji and Zhou (2015), etc. In addition, the uncertain differential equations have also found many applications in optimal control (Zhu 2019), differential game (Yang and Gao 2016), population model (Sheng et al. 2017), epidemic model (Li et al. 2017), and so on.

With so many applications of uncertain differential equations, how to estimate their coefficients based on the observations is a core problem in practice. In this paper, we undertake the issue by applying the method of moments to the difference forms of the uncertain differential equations. The rest of this paper is organized as follows. In the next section, we introduce some basic concepts and theorems about uncertain variables and uncertain differential equations. Then in Sect. 3, we introduce the method of moments for the parameter estimation problems in uncertain differential equations, and give some analytic examples to illustrate the method. After that, we give some numerical examples in Sect. 4. Finally, some remarks are made in Sect. 5.

2 Preliminary

In this section, we introduce some basic concepts and formulas about uncertain variables and uncertain differential equations.

Definition 2.1 (Liu 2007, 2009) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2 (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (Subadditivity Axiom) For every countable sequence of events Λ_1, Λ_2, $\cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

Axiom 4 (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \min_{k\geq 1}\mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

An uncertain variable ξ is a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \min_{1\leq i\leq n}\mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Definition 2.2 (Liu 2007) Let ξ be an uncertain variable. Then its uncertainty distribution is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

An uncertain variable ξ is called normal if it has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(\mu - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(\mu, \sigma)$. If $\mu = 0$ and $\sigma = 1$, then ξ is called a standard normal uncertain variable. The inverse uncertainty distribution of a standard normal uncertain variable is

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad \alpha \in (0, 1).$$

Definition 2.3 (Liu 2007) Let ξ be an uncertain variable, and k be a positive integer. Then the k-th moment of ξ is defined by

$$E[\xi^k] = \int_0^{+\infty}\mathcal{M}\left\{\xi^k \geq r\right\}\mathrm{d}r - \int_{-\infty}^{0}\mathcal{M}\left\{\xi^k \leq r\right\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

Liu (2015) proved that if ξ has an inverse uncertainty distribution $\Phi^{-1}(\alpha)$, then

$$E[\xi^k] = \int_0^1 \left(\Phi^{-1}(\alpha)\right)^k \mathrm{d}\alpha.$$

Hence, for a standard normal uncertain variable $\xi \sim \mathcal{N}(0, 1)$, we have

$$E[\xi^k] = \left(\frac{\sqrt{3}}{\pi}\right)^k \int_0^1 \left(\ln \frac{\alpha}{1-\alpha}\right)^k \mathrm{d}\alpha.$$

Specially, we have $E[\xi^k] = 0$ for any positive odd number k, and

$$E[\xi^2] = 1, \quad E[\xi^4] = \frac{21}{5}, \quad E[\xi^6] = \frac{279}{7}.$$

An uncertain process is a sequence of uncertain variables indexed by the time. As an uncertain counterpart of Wiener process, Liu process is one of the most frequently used uncertain processes.

Definition 2.4 (Liu 2009) An uncertain process C_t is called a Liu process if

(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,

(ii) C_t has stationary and independent increments,

(iii) the increment $C_{s+t} - C_s$ has a normal uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Let X_t be an uncertain process. Then the uncertain integral of X_t with respect to the Liu process C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^k X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite for any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$ and

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Definition 2.5 (Liu 2008) Suppose that C_t is a Liu process, and f and g are two measurable real functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation.

An uncertain process X_t is called the solution of the uncertain differential equation (1) if it satisfies

$$X_t = X_0 + \int_0^t f(s, X_s)\mathrm{d}s + \int_0^t g(s, X_s)\mathrm{d}C_s.$$

A real-valued function X_t^α is called the α-path of the uncertain differential equation (1) if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$$

where

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}, \quad \alpha \in (0, 1)$$

is the inverse uncertainty distribution of a standard normal uncertain variable. For example, the uncertain differential equation

$$\mathrm{d}X_t = (\mu_1 - \mu_2 X_t)\mathrm{d}t + \sigma\mathrm{d}C_t, \quad \mu_2 \neq 0$$

has a solution

$$X_t = \frac{\mu_1}{\mu_2} + \left(X_0 - \frac{\mu_1}{\mu_2}\right)\cdot\exp(-\mu_2 t) + \sigma\exp(-\mu_2 t)\cdot\int_0^t \exp(\mu_2 s)\mathrm{d}C_s$$

and an α-path

$$X_t^\alpha = X_0\cdot\exp(-\mu_2 t) + \left(\frac{\mu_1}{\mu_2} + \frac{\sigma}{\mu_2}\cdot\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)(1 - \exp(-\mu_2 t)).$$

3 Parameter estimation

Consider an uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t; \boldsymbol{\mu})\mathrm{d}t + g(t, X_t; \boldsymbol{\sigma})\mathrm{d}C_t \tag{2}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are unknown parameters to be estimated. Note that the Eq. (2) has a difference form

$$X_{t_{i+1}} = X_{t_i} + f(t_i, X_{t_i}; \boldsymbol{\mu})(t_{i+1} - t_i) + g(t_i, X_{t_i}; \boldsymbol{\sigma})(C_{t_{i+1}} - C_{t_i})$$

which can be rewritten as

$$\frac{X_{t_{i+1}} - X_{t_i} - f(t_i, X_{t_i}; \boldsymbol{\mu})(t_{i+1} - t_i)}{g(t_i, X_{t_i}; \boldsymbol{\sigma})(t_{i+1} - t_i)} = \frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i}.$$

According to Definition 2.4 of Liu process, the right term

$$\frac{C_{t_{i+1}} - C_{t_i}}{t_{i+1} - t_i} \sim \mathcal{N}(0, 1)$$

is a standard normal uncertain variable with expected value 0 and variance 1, which has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1}.$$

Hence, the estimates of the parameters $\boldsymbol{\mu}$ and/or $\boldsymbol{\sigma}$ are supposed to follow the standard normal uncertainty distribution, i.e.,

$$\frac{X_{t_{i+1}} - X_{t_i} - f(t_i, X_{t_i}; \mu i)(t_{i+1} - t_i)}{g(t_i, X_{t_i}; \boldsymbol{\sigma})(t_{i+1} - t_i)} \sim \mathcal{N}(0, 1). \tag{3}$$

Assume that there are n observations $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$ of the solution X_t at the times $t_1, t_2, \cdots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively. Substituting X_{t_i} and $X_{t_{i+1}}$ with the observations x_{t_i} and $x_{t_{i+1}}$ in the Eq. (3), we write

$$h_i(\boldsymbol{\mu}, \boldsymbol{\sigma}) = \frac{x_{t_{i+1}} - x_{t_i} - f(t_i, x_{t_i}; \boldsymbol{\mu})(t_{i+1} - t_i)}{g(t_i, x_{t_i}; \boldsymbol{\sigma})(t_{i+1} - t_i)}, \quad i = 1, 2, \ldots, n-1$$

which are real functions of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$. For the estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, denoted by $\boldsymbol{\mu}^*$ and $\boldsymbol{\sigma}^*$, it follows from the Eq. (3) that the values of these functions $h_1(\boldsymbol{\mu}^*, \boldsymbol{\sigma}^*), h_2(\boldsymbol{\mu}^*, \boldsymbol{\sigma}^*), \cdots, h_{n-1}(\boldsymbol{\mu}^*, \boldsymbol{\sigma}^*)$ can be regarded as $n-1$ samples of a standard normal uncertainty distribution $\mathcal{N}(0, 1)$. The sample moments would provide good estimates of the corresponding population moments. Note that the k-th sample moments are

$$\frac{1}{n-1}\sum_{i=1}^{n-1}(h_i(\boldsymbol{\mu}^*, \boldsymbol{\sigma}^*))^k, \quad k = 1, 2, \ldots$$

and the k-th population moments are

$$\left(\frac{\sqrt{3}}{\pi}\right)^k \int_0^1 \left(\ln\frac{\alpha}{1-\alpha}\right)^k \mathrm{d}\alpha, \quad k = 1, 2, \ldots$$

Hence, we set

$$\frac{1}{n-1}\sum_{i=1}^{n-1}(h_i(\boldsymbol{\mu}^*, \boldsymbol{\sigma}^*))^k = \left(\frac{\sqrt{3}}{\pi}\right)^k \int_0^1 \left(\ln\frac{\alpha}{1-\alpha}\right)^k \mathrm{d}\alpha, \quad k = 1, 2, \ldots, K \tag{4}$$

where K is the number of unknown parameters. The solutions $\boldsymbol{\mu}^*$ and $\boldsymbol{\sigma}^*$ of the system (4) of equations are the estimates of the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$, respectively. The above method to estimate the parameters in uncertain differential equations is called the method of moments.

Example 1 Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}C_t \tag{5}$$

with two parameters μ and $\sigma > 0$ to be estimated. Assume we have n observations $x_{t_1}, x_{t_2}, \cdots, x_{t_n}$ of the solution X_t at the times $t_1, t_2, \cdots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively. Then

$$h_i(\mu, \sigma) = \frac{x_{t_{i+1}} - x_{t_i} - \mu(t_{i+1} - t_i)}{\sigma(t_{i+1} - t_i)}$$

for $i = 1, 2, \ldots, n-1$, and the system (4) of equations becomes

$$\begin{cases} \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1}\frac{x_{t_{i+1}} - x_{t_i} - \mu^*(t_{i+1} - t_i)}{\sigma^*(t_{i+1} - t_i)} = 0 \\ \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1}\left(\frac{x_{t_{i+1}} - x_{t_i} - \mu^*(t_{i+1} - t_i)}{\sigma^*(t_{i+1} - t_i)}\right)^2 = 1. \end{cases}$$

Solving the above system of equations, we have

$$\begin{cases} \mu^* = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1}\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}, \\ \sigma^* = \sqrt{\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1}\left(\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}\right)^2 - \left(\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1}\frac{x_{t_{i+1}} - x_{t_i}}{t_{i+1} - t_i}\right)^2}, \end{cases}$$

which are the estimates of μ and σ in the Eq. (5).

Example 2 Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}C_t \tag{6}$$

with two parameters μ and $\sigma > 0$ to be estimated. Assume we have n observations $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ of the solution X_t at the times $t_1, t_2, \ldots, t_n$ with $t_1 < t_2 < \cdots < t_n$, respectively. Then

$$h_i(\mu, \sigma) = \frac{x_{t_{i+1}} - x_{t_i} - \mu x_{t_i}(t_{i+1} - t_i)}{\sigma x_{t_i}(t_{i+1} - t_i)}$$

for $i = 1, 2, \cdots, n-1$, and the system (4) of equations becomes

$$\begin{cases} \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \dfrac{x_{t_{i+1}} - x_{t_i} - \mu^* x_{t_i}(t_{i+1} - t_i)}{\sigma^* x_{t_i}(t_{i+1} - t_i)} = 0 \\ \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \left(\dfrac{x_{t_{i+1}} - x_{t_i} - \mu^* x_{t_i}(t_{i+1} - t_i)}{\sigma^* x_{t_i}(t_{i+1} - t_i)}\right)^2 = 1. \end{cases}$$

Solving the above system of equations, we have

$$\begin{cases} \mu^* = \dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \dfrac{x_{t_{i+1}} - x_{t_i}}{x_{t_i}(t_{i+1} - t_i)}, \\ \sigma^* = \sqrt{\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \left(\dfrac{x_{t_{i+1}} - x_{t_i}}{x_{t_i}(t_{i+1} - t_i)}\right)^2 - \left(\dfrac{1}{n-1}\displaystyle\sum_{i=1}^{n-1} \dfrac{x_{t_{i+1}} - x_{t_i}}{x_{t_i}(t_{i+1} - t_i)}\right)^2}, \end{cases}$$

which are the estimates of μ and σ in the Eq. (6).

4 Numerical examples

In this section, we give two numerical examples to illustrate the method of moments in estimating the parameters in uncertain differential equations.

Example 3 Consider the uncertain differential equation

$$\mathrm{d}X_t = (\mu_1 - \mu_2 X_t)\mathrm{d}t + \sigma \mathrm{d}C_t$$

with three parameters μ_1, μ_2 and $\sigma > 0$ to be estimated. Assume that we have 15 groups of observed data as shown in Table 1. According to the system (4) of equations, the estimates μ_1^*, μ_2^* and σ^* solve the following system of equations

Table 1 Observed data in Example 3

i	1	2	3	4	5	6	7	8
t_i	0.4	0.7	0.9	1.2	1.5	1.9	2.1	2.4
x_{t_i}	1.33	2.20	1.52	2.30	2.37	2.49	1.90	1.65
i	9	10	11	12	13	14	15	
t_i	2.8	3.2	3.4	3.7	4.0	4.2	4.6	
x_{t_i}	1.45	2.82	1.50	2.65	2.07	2.99	1.88	

$$\begin{cases} \dfrac{1}{14}\displaystyle\sum_{i=1}^{14} \dfrac{x_{t_{i+1}} - x_{t_i} - (\mu_1^* - \mu_2^* x_{t_i})(t_{i+1} - t_i)}{\sigma^*(t_{i+1} - t_i)} = 0 \\ \dfrac{1}{14}\displaystyle\sum_{i=1}^{14} \left(\dfrac{x_{t_{i+1}} - x_{t_i} - (\mu_1^* - \mu_2^* x_{t_i})(t_{i+1} - t_i)}{\sigma^*(t_{i+1} - t_i)}\right)^2 = 1 \\ \dfrac{1}{14}\displaystyle\sum_{i=1}^{14} \left(\dfrac{x_{t_{i+1}} - x_{t_i} - (\mu_1^* - \mu_2^* x_{t_i})(t_{i+1} - t_i)}{\sigma^*(t_{i+1} - t_i)}\right)^3 = 0, \end{cases}$$

which are

$$\mu_1^* = 6.4357, \quad \mu_2^* = 3.1190, \quad \sigma^* = 2.1735.$$

Hence, the uncertain differential equation is

$$\mathrm{d}X_t = (6.4357 - 3.1190X_t)\mathrm{d}t + 2.1735\,\mathrm{d}C_t. \tag{7}$$

As shown in Fig. 1, all the observed data fall in the area between the 0.17-path and the 0.93-path of the uncertain differential equation (7), so the estimates

$$\mu_1^* = 6.4357, \quad \mu_2^* = 3.1190, \quad \sigma^* = 2.1735$$

are acceptable.

Example 4 Consider the uncertain differential equation

$$\mathrm{d}X_t = \cos(\mu_1 t + \mu_2 X_t)\mathrm{d}t + \sin(\sigma X_t)\mathrm{d}C_t$$

with three parameters μ_1, μ_2 and $\sigma > 0$ to be estimated. Assume that we have 12 groups of observed data as shown in Table 2.

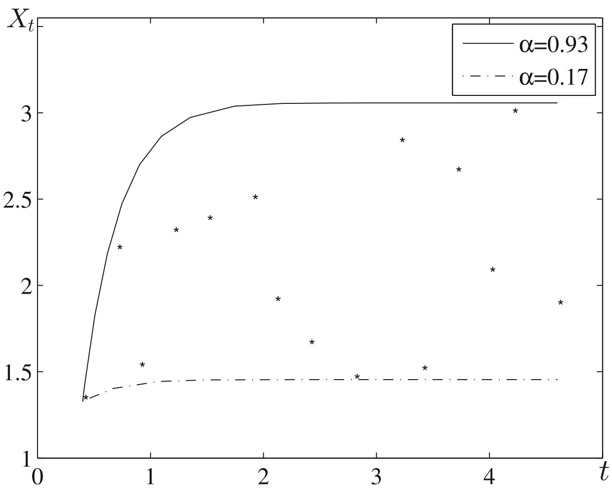

Fig. 1 Observed data and α-paths of X_t in Example 3

Table 2 Observed Data in Example 4

i	1	2	3	4	5	6	7	8	9	10	11	12
t_i	0.27	0.92	1.47	1.99	2.53	3.05	3.50	4.55	5.12	6.36	6.88	7.60
x_{t_i}	1.40	1.60	1.71	1.83	1.76	1.67	1.49	1.24	1.03	1.56	1.47	0.15

According to the system (4) of equations, the estimates μ_1^*, μ_2^* and σ^* solve the following system of equations

$$\begin{cases} \dfrac{1}{11}\displaystyle\sum_{i=1}^{11} \dfrac{x_{t_{i+1}} - x_{t_i} - \cos(\mu_1^* t + \mu_2^* x_{t_i}) \cdot (t_{i+1} - t_i)}{\sin(\sigma^* x_{t_i}) \cdot (t_{i+1} - t_i)} = 0 \\ \dfrac{1}{11}\displaystyle\sum_{i=1}^{11} \left(\dfrac{x_{t_{i+1}} - x_{t_i} - \cos(\mu_1^* t + \mu_2^* x_{t_i}) \cdot (t_{i+1} - t_i)}{\sin(\sigma^* x_{t_i}) \cdot (t_{i+1} - t_i)} \right)^2 = 1 \\ \dfrac{1}{11}\displaystyle\sum_{i=1}^{11} \left(\dfrac{x_{t_{i+1}} - x_{t_i} - \cos(\mu_1^* t + \mu_2^* x_{t_i}) \cdot (t_{i+1} - t_i)}{\sin(\sigma^* x_{t_i}) \cdot (t_{i+1} - t_i)} \right)^3 = 0, \end{cases}$$

which are

$$\mu_1^* = 0.4209, \quad \mu_2^* = 0.3699, \quad \sigma^* = 1.6618.$$

Hence, the uncertain differential equation is

$$\mathrm{d}X_t = \cos(0.4209t + 0.3699X_t)\mathrm{d}t + \sin(1.6618X_t)\mathrm{d}C_t. \tag{8}$$

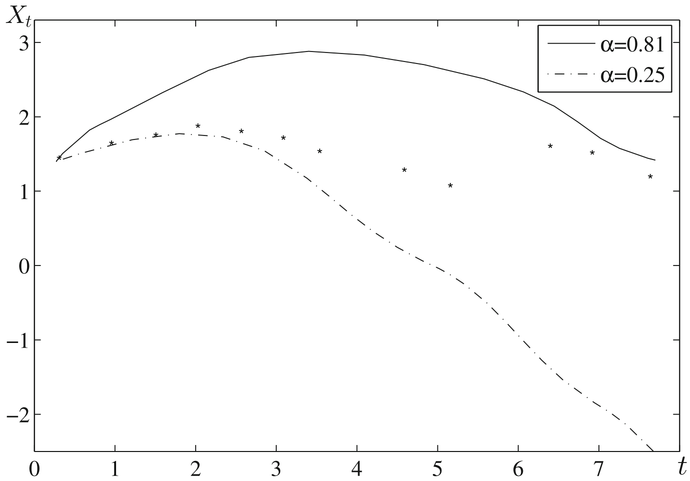

Fig. 2 Observed Data and α-paths of X_t in Example 4

As shown in Fig. 2, all the observed data fall in the area between the 0.25-path and the 0.81-path of the uncertain differential equation (8), so the estimates

$$\mu_1^* = 0.4209, \quad \mu_2^* = 0.3699, \quad \sigma^* = 1.6618$$

are acceptable.

5 Conclusion

The method of moments was employed in this paper to estimate the parameters in uncertain differential equations based on some observations. As an approximation of the uncertain differential equation, a difference equation was obtained and transformed to a special expression in order to derive a system of the equations about the parameters. By solving the system of the equations, the estimates of the parameters could be found. Some analytic examples as well as numerical examples were given to illustrate the method of moments in uncertain differential equations. Future researches may consider the hypothesis test and interval estimation of the parameters in uncertain differential equations.

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