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# On product of positive *L-R* fuzzy numbers and its application to multi-period portfolio selection problems

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**Abstract**
With the wide applications of fuzzy theory in optimization, fuzzy arithmetic attracts great attention due to its inevitability in solution process. However, the complexity of the Zadeh extension principle significantly reduces the practicability of fuzzy optimization technology. In this paper, we prove some important properties on positive $L$-$R$ fuzzy numbers, and propose a new calculation method for the product of multiple positive $L$-$R$ fuzzy numbers. Furthermore, a numerical integral-based simulation algorithm (NISA) is proposed to approximate the expected value, variance and skewness of the product of positive $L$-$R$ fuzzy numbers. As applications, a fuzzy multi-period utility maximization model for portfolio selection problem is considered. For handling the large number of multiplications on $L$-$R$ fuzzy numbers during the optimization process, a genetic algorithm integrating NISA is designed. Finally, some numerical experiments are presented to demonstrate the advantages of NISA. The results greatly enrich the fuzzy arithmetic methods and promote the practicability of fuzzy optimization technology.

**Keywords** Fuzzy sets · Fuzzy arithmetic · Positive $L$-$R$ fuzzy number · Multi-period portfolio selection

## 1 Introduction

Fuzzy theory has been widely applied in optimal control, liner optimization, nonlinear optimization, integer optimization and many other fields of optimization since it

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was proposed in 1965. As an important component of fuzzy theory, fuzzy arithmetic attracts great attention for its inevitability in solution process. Various fuzzy arithmetic rules were proposed in the past few years. For example, Zadeh (1978) proposed the basic fuzzy arithmetic rules based on the extension principle. Dimitar and Ronald (1997) pointed out that the binary fuzzy arithmetic could be represented as fuzzy reasoning, and proposed some simple expressions for operations on fuzzy numbers. The above arithmetic rules are all based on the Zadeh extension principle, which are computationally expensive in solving complex optimization problems.

Based on the fuzzy representation theorem, some arithmetic rules were established by treating fuzzy numbers as a collection of $\gamma$-level sets. For example, Chou (2003) studied the product arithmetic of two triangular fuzzy numbers by proposing $L$-$R$ inverse function arithmetic principle, and presented an effective method to compute the canonical representation of product on two triangular fuzzy numbers. Chang and Hung (2006) studied the $\gamma$-level fuzzy arithmetic by employing vertex method, and proposed some simplifying rules to reduce the computation cost. More $\gamma$-level fuzzy arithmetic rules could be found in Anile et al. (1995). Although those rules can reduce the computational complexity, they cannot preserve some important algebraic properties that are naturally valid for the arithmetics of real numbers. In view of this, Holčapek and Štěpnička (2014) presented a novel framework for arithmetic of extensional fuzzy numbers by introducing MI-algebras structures, which imposed an improvement from a theoretical as well as a practical perspective.

To simplify arithmetics on fuzzy numbers, some approximation methods are proposed by approximating the general fuzzy numbers with special fuzzy numbers, such as interval, triangular and trapezoidal fuzzy numbers. The current approximation methods can be classified into Euclidean distance based method and non-Euclidean distance based method. The representative studies on Euclidean distance based approximations include Abbasbandy and Asady (2004), Grzegorzewski (2002) and Ma et al. (2000). In details, Abbasbandy and Asady (2004) proposed the nearest trapezoidal approximation to a fuzzy number from the perspective of distance metric. Grzegorzewski (2002) proposed the nearest interval approximation operator to approximate fuzzy numbers. Ma et al. (2000) introduced an approach of approximating a general fuzzy number by using symmetric triangular fuzzy number. The non-Euclidean distance based methods include weighted Euclidean distance (Zeng and Li 2007), Hamming distance (Chanas 2001) and source distance (Abbasbandy and Amirfakhrian 2006). Zeng and Li (2007) applied the weighted distance between two fuzzy numbers to investigate the triangular approximations, and discussed continuity, translation invariance, scale invariance and identity in approximation process. Chanas (2001) introduced the interval approximation by minimizing the Hamming distance between the interval fuzzy number and the approximated fuzzy number. Abbasbandy and Amirfakhrian (2006) defined a source distance on the set of fuzzy numbers and applied it to approximate the generalized $L$-$R$ fuzzy numbers with trapezoidal fuzzy numbers.

Although various fuzzy arithmetic rules are proposed, there are still some limitations on product operations arithmetic. The first is the weak practicability. For example, Chou (2003) computed the canonical representation of the product of two triangular fuzzy numbers. However, the product of two triangular fuzzy numbers is not a triangular fuzzy number anymore. Therefore, this fuzzy arithmetic rule cannot be used in real applications where arithmetics on more than two fuzzy numbers are required. The second issue is the information loss. The approximation methods take special type of fuzzy numbers to approximate the general fuzzy numbers based on some distance metric. Although most of the important information are preserved, some personalized properties contained in the membership function are neglected. To overcome the above limitations, in this paper, we study the product arithmetic of multiple positive $L$-$R$ fuzzy numbers, which is meaningful on completing the fuzzy arithmetic rules, and is helpful on developing fuzzy optimization technology.

All the above studies focus on the basic fuzzy arithmetics including addition, subtraction, multiplication and division, of which the arithmetic result is still a fuzzy number. In applications, some real valued functions of fuzzy numbers need to be calculated to rank fuzzy numbers, e.g., “mathematical expectation”. One popular computing method is fuzzy simulation, which was first proposed by Liu and Iwamura (1998). Furthermore, Liu and Liu (2002) designed a stochastic discretization-based simulation algorithm to derive the expected value of functions of fuzzy variables. Based on the proposed product arithmetic methods, we develop a numerical integral-based simulation algorithm (NISA) for the product arithmetic among multiple positive $L$-$R$ fuzzy numbers to calculate the expected value of product in an acceptable computation time. As applications, we apply NISA in fuzzy portfolio selection problem, where $L$-$R$ fuzzy numbers are commonly accepted to denote the returns of risky assets due to their membership functions are convex and closer to the real return distributions. The results greatly promote the practical applications of fuzzy optimization technology.

Compared with previous fuzzy arithmetics, our proposed NISA exhibits higher practicability in solving complicated fuzzy optimization issues due to its capability of dealing with the arithmetic among large-scale fuzzy numbers without information loss. In the meanwhile, NISA outperforms some other popular fuzzy optimization techniques such as fuzzy simulation from the perspective of accuracy and running speed. The explicit contributions of this paper can be summarized as follows:

- The product of multiple positive $L$-$R$ fuzzy numbers is studied and some of its mathematical properties are proved, which assists in overcoming the limitation of information loss in the approximation arithmetics.
- The NISA is proposed to approximate the expected value, variance and skewness of fuzzy numbers, achieving a better performance than fuzzy simulation in both accuracy and running speed.
- The NIGA is firstly designed by integrating the NISA and genetic algorithm, which is able to efficiently tackle the complicated portfolio selection problems and exhibits higher practicability than fuzzy simulation-based genetic algorithm.

The rest of this paper is structured as follows. Section 2 reviews the preliminaries about fuzzy numbers. In Sect. 3, we discuss the product arithmetic of positive $L$-$R$ fuzzy numbers and present some lemmas. Section 4 proposes a NISA to approximate the expected values, variances and skewness of fuzzy numbers. Section 5 introduces a multi-period portfolio selection problem and proposes a genetic algorithm integrating NISA to tackle with multi-period portfolio selection. Section 6 provides two numerical experiments to demonstrate the effectiveness and efficiency of our proposed algorithms. Finally, Sect. 7 concludes the paper.

## 2 Preliminaries

In this section, we briefly introduce some fundamental concepts on fuzzy numbers, fuzzy expected value, fuzzy variance and fuzzy skewness.

**Definition 1** (Zadeh 1965) A fuzzy subset $\widetilde{A}$ in $X$ is defined as $\widetilde{A} = \{\langle x, \mu(x)\rangle : x \in X\}$, where $\mu : X \to [0, 1]$ and the real value $\mu(x)$ represents the degree of membership of $x$ in $\widetilde{A}$. The $\gamma$-level set of $\widetilde{A}$ is defined as $[\widetilde{A}]^\gamma = \{x \in X : \mu(x) \geq \gamma\}$ for any $\gamma \in (0, 1]$.

***Remark 1*** The $\gamma$-level set of $\widetilde{A}$ is usually expressed as $[\widetilde{A}]^\gamma = [a_1(\gamma), a_2(\gamma)]$ with $\gamma \in (0, 1]$, where $a_1(\gamma)$ and $a_2(\gamma)$ represent the infimum and supremum of $[\widetilde{A}]^\gamma$, respectively.

**Definition 2** (Dubois and Prade 1980) A fuzzy number $\xi$ is a normal and convex fuzzy subset of $\Re$. Here, normality implies that there is a point $x_0$ such that $\mu(x_0) = 1$, and convexity means that $\mu(\alpha x_1 + (1-\alpha)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$ for any $\alpha \in [0, 1]$ and $x_1, x_2 \in \Re$.

As regard to the $L$-$R$ fuzzy number, various definitions have been proposed. In this paper, we employ the most commonly used definition given by Triesch (1993).

**Definition 3** (Triesch 1993) A fuzzy number $\xi$ is a $L$-$R$ fuzzy number if its membership function satisfies

$$\mu(x) = \begin{cases} L\left(\frac{x-a}{b-a}\right), & \text{if } a \leq x \leq b \\ R\left(\frac{c-x}{c-b}\right), & \text{if } b \leq x \leq c \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where $L$ and $R$ are increasing and continuous functions: $[0, 1] \to [0, 1]$ with $L(0) = R(0) = 0$ and $L(1) = R(1) = 1$. Here $b$ is the peak of $\xi$, $b - a$ and $c - b$ are the left and right spread, respectively.

**Definition 4** (Li et al. 2015) Let $\xi$ be a fuzzy number with differentiable membership function $\mu(x)$. Then its expected value is defined as

$$E[\xi] = \int_{-\infty}^{+\infty} x\mu(x)|\mu'(x)|\mathrm{d}x. \tag{2}$$

**Definition 5** (Li et al. 2015) Let $\xi$ be a fuzzy number with differentiable membership function $\mu(x)$ and finite expected value $E[\xi]$. Then its variance is defined as

$$V[\xi] = \int_{-\infty}^{+\infty} (x - E[\xi])^2\mu(x)|\mu'(x)|\mathrm{d}x. \tag{3}$$

**Definition 6** (Li et al. 2015) Let $\xi$ be a fuzzy number with differentiable membership function $\mu(x)$ and finite expected value $E[\xi]$. Then its skewness is defined as

$$S[\xi] = \int_{-\infty}^{+\infty} (x - E[\xi])^3\mu(x)|\mu'(x)|\mathrm{d}x. \tag{4}$$

***Remark 2*** For any $L$-$R$ fuzzy number with differentiable membership function, its expected value, variance and skewness can be calculated following from Eqs. (2), (3) and (4). Here $\mu'(x)$ represents the first order derivative of $\mu(x)$ and the differentiability refers to the almost everywhere differentiability. For example, the membership function of a triangular fuzzy number $(a, b, c)$ is almost everywhere differentiable except the peak point $b$.

In addition, Li et al. (2015) proposed some equivalents of Eqs. (2), (3) and (4) by using $\gamma$-level set. Suppose that $\xi$ has $\gamma$-level set $[\xi]^\gamma$=$[a_1(\gamma), a_2(\gamma)]$, then its expected value, variance and skewness are

$$E[\xi] = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))\mathrm{d}\gamma, \tag{5}$$

$$V[\xi] = \int_0^1 \gamma[(a_1(\gamma) - E[\xi])^2 + (a_2(\gamma) - E[\xi])^2]\mathrm{d}\gamma, \tag{6}$$

$$S[\xi] = \int_0^1 \gamma[(a_1(\gamma) - E[\xi])^3 + (a_2(\gamma) - E[\xi])^3]\mathrm{d}\gamma. \tag{7}$$

## 3 Product on positive *L-R* fuzzy numbers

In this section, we study the product arithmetic of two positive $L$-$R$ fuzzy numbers and extend it to $n$ $L$-$R$ fuzzy numbers. Suppose that $\xi_1$ and $\xi_2$ are two positive $L$-$R$ fuzzy numbers with membership functions $\mu_1(x)$ and $\mu_2(x)$. Denote the peak point as $b_i$, and the left and right spreads as $b_i - a_i$ and $c_i - b_i$ with $i = 1, 2$. Assume that $\xi = \xi_1 \times \xi_2$ has membership function $\mu(x)$. It is obvious that $\xi$ takes values in $[a_1a_2, c_1c_2]$. Based on the Zadeh extension principle, $\mu(a_1a_2) = \mu(c_1c_2) = 0$ and $\mu(b_1b_2) = 1$. In addition, denote $S = \{(x_1, x_2)|a_1 \le x_1 \le c_1, a_2 \le x_2 \le c_2\}$. For any $x_0 \in (a_1a_2, c_1c_2)$, we have

$$\mu(x_0) = \sup_{(x_1,x_2)\in S} \{\min\{\mu_1(x_1), \mu_2(x_2)\}|x_1x_2 = x_0\}. \tag{8}$$

There are four scenarios on the locations of $x_1$ and $x_2$ satisfying $x_1x_2 = x_0$:

$$L_1 = \{(x_1, x_2)|a_1 \le x_1 \le b_1, a_2 \le x_2 \le b_2\},$$
$$L_2 = \{(x_1, x_2)|a_1 \le x_1 \le b_1, b_2 \le x_2 \le c_2\},$$
$$R_1 = \{(x_1, x_2)|b_1 \le x_1 \le c_1, b_2 \le x_2 \le c_2\},$$
$$R_2 = \{(x_1, x_2)|b_1 \le x_1 \le c_1, a_2 \le x_2 \le b_2\}.$$

It is obvious that $S = L_1 \cup L_2 \cup R_1 \cup R_2$ (See Fig. 1). We can prove the following properties:

(i) for any $x_0 \in (a_1a_2, b_1b_2)$, we have $\mu(x_0) = \sup_{(x_1,x_2)\in L_1} \{\min\{\mu_1(x_1), \mu_2(x_2)\}| x_1x_2 = x_0\}$ and there exist $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$ satisfying $x_1x_2 = x_0$ such that $\mu(x_0) = \mu_1(x_1) = \mu_2(x_2)$.

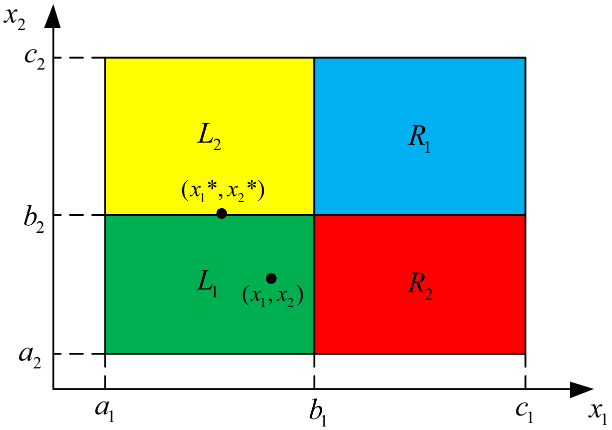


**Fig. 1** Scenarios for $x_1, x_2$ with $x_1x_2 = x_0$, $x_0 \in (a_1a_2, c_1c_2)$

(ii) for any $x_0 \in (b_1b_2, c_1c_2)$, we have $\mu(x_0) = \sup\limits_{(x_1,x_2)\in R_1} \{\min\{\mu_1(x_1), \mu_2(x_2)\}| x_1x_2 = x_0\}$ and there exist $x_1 \in [b_1, c_1]$ and $x_2 \in [b_2, c_2]$ satisfying $x_1x_2 = x_0$ such that $\mu(x_0) = \mu_1(x_1) = \mu_2(x_2)$.

The proof can be found in "Appendix A".

**Lemma 1** *Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are positive L-R fuzzy numbers with differentiable membership function $\mu_1(x), \mu_2(x), \ldots, \mu_n(x)$, then $\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n$ is also a positive L-R fuzzy number with differentiable membership function.*

***Proof*** We prove this lemma by using mathematical induction. The argument breaks down into two steps.

*Step 1* We prove this lemma holds when $n = 2$.

As is known from previous description, $\xi = \xi_1 \times \xi_2$ takes values in $[a_1a_2, c_1c_2]$, $\mu(a_1a_2) = \mu(c_1c_2) = 0$ and $\mu(b_1b_2) = 1$. First, we prove the monotonicity, continuity and differentiability of $\mu(x)$ when $x \in [a_1a_2, b_1b_2]$. Denote the $\gamma$-level set of $\xi$ as $[a_1(\gamma), a_2(\gamma)]$ and the $\gamma$-level sets of $\xi_i$ as $[a_{i1}(\gamma), a_{i2}(\gamma)]$, $i = 1, 2$, $\gamma \in (0, 1]$, where $a_1(\gamma) \in [a_1a_2, b_1b_2]$ and $a_2(\gamma) \in [b_1b_2, c_1c_2]$. It can be found that $a_{i1}(\gamma)$ is the inverse function of $\mu_i(x)$ with $x \in [a_i, b_i]$ and $a_{i2}(\gamma)$ is the inverse function of $\mu_i(x)$ with $x \in [b_i, c_i]$. According to property **(i)**, for any $x \in [a_1a_2, b_1b_2]$ with $a_1(\gamma) = x$, there exist $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$ such that $x_1x_2 = x$ and $\mu(x) = \mu_1(x_1) = \mu_2(x_2) = \gamma$, then $a_1(\gamma) = a_{11}(\gamma) \cdot a_{21}(\gamma)$. Since $a_{11}(\gamma)$ and $a_{21}(\gamma)$ are increasing, continuous and differential for $\gamma \in (0, 1]$, then $a_1(\gamma)$ is monotonic increasing, continuous and differential. Furthermore, $\mu(x)$ representing the inverse function of $a_1(\gamma)$ is differentiable when $x \in [a_1a_2, b_1b_2]$. Similarly, according to property **(ii)**, for any $x \in [b_1b_2, c_1c_2]$ with $a_2(\gamma) = x$, there exist $x_1 \in [b_1, c_1]$ and $x_2 \in [b_2, c_2]$ such that $x_1x_2 = x$ and $\mu(x) = \mu_1(x_1) = \mu_2(x_2) = \gamma$, then $a_2(\gamma) = a_{12}(\gamma) \cdot a_{22}(\gamma)$, where $a_{12}(\gamma)$ and $a_{22}(\gamma)$ are decreasing and continuous differentiable. It can be proved that $\mu(x)$ is differentiable for $x \in [b_1b_2, c_1c_2]$. According to Definition 3, $\xi_1 \times \xi_2$ is a $L$-$R$ fuzzy number with differentiable membership function $\mu(x)$.

*Step 2* We prove this lemma holds when $n > 2$.

Suppose that $\xi_1 \times \xi_2 \times \cdots \times \xi_k$ is a positive $L$-$R$ fuzzy number with differentiable membership function. Similar to Step 1, we can prove that $\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_k \times \xi_{k+1}$ is a $L$-$R$ fuzzy number with differentiable membership function. The proof is then complete. □

Following from Lemma 1, the membership function of the product of any positive $L$-$R$ fuzzy numbers with differentiable membership functions is differentiable and therefore the product's expected value, variance and skewness can be well defined.

***Example 1*** Suppose that the membership functions of positive triangular fuzzy numbers $\xi_1 = (s_1, s_2, s_3)$ and $\xi_2 = (t_1, t_2, t_3)$ are

$$\mu_1(x) = \begin{cases} \frac{x-s_1}{s_2-s_1}, & \text{if } s_1 \le x < s_2 \\ \frac{s_3-x}{s_3-s_2}, & \text{if } s_2 \le x < s_3 \\ 0, & \text{otherwise,} \end{cases} \quad \mu_2(x) = \begin{cases} \frac{x-t_1}{t_2-t_1}, & \text{if } t_1 \le x < t_2 \\ \frac{t_3-x}{t_3-t_2}, & \text{if } t_2 \le x < t_3 \\ 0, & \text{otherwise.} \end{cases}$$

Set the $\gamma$-level sets for $\xi_1$ and $\xi_2$ as $[a_{11}(\gamma), a_{12}(\gamma)]$ and $[a_{21}(\gamma), a_{22}(\gamma)]$, respectively. Then for any $\gamma \in (0, 1]$, we have

$$\begin{aligned} a_{11}(\gamma) &= s_1 + (s_2 - s_1)\gamma, \quad a_{12}(\gamma) = s_3 - (s_3 - s_2)\gamma, \\ a_{21}(\gamma) &= t_1 + (t_2 - t_1)\gamma, \quad a_{22}(\gamma) = t_3 - (t_3 - t_2)\gamma. \end{aligned}$$

Suppose that the $\gamma$-level set for $\xi = \xi_1 \times \xi_2$ is $[a_1(\gamma), a_2(\gamma)]$. From properties **(i)** and **(ii)**, we have

$$\mu(a_1(\gamma)) = \mu_1(a_{11}(\gamma)) = \mu_2(a_{21}(\gamma)), \mu(a_2(\gamma)) = \mu_1(a_{12}(\gamma)) = \mu_2(a_{22}(\gamma)).$$

Therefore, we can obtain

$$\begin{aligned} a_1(\gamma) &= a_{11}(\gamma)a_{21}(\gamma) = (s_1 + (s_2 - s_1)\gamma)(t_1 + (t_2 - t_1)\gamma), \\ a_2(\gamma) &= a_{12}(\gamma)a_{22}(\gamma) = (s_3 - (s_3 - s_2)\gamma)(t_3 - (t_3 - t_2)\gamma). \end{aligned}$$

According to Lemma 1, $\xi$ is a $L$-$R$ fuzzy number with differentiable membership function. Then its expected value, variance and skewness can be calculated as follows:

$$\begin{aligned} E[\xi] &= \frac{s_1t_1 + s_1t_2 + s_2t_1 + 6s_2t_2 + s_2t_3 + s_3t_2 + s_3t_3}{12}, \\ V[\xi] &= \frac{(s_1t_1 + 2s_2t_2)^2 + (s_3t_3 + 2s_2t_2)^2 + (s_1t_2 + s_1t_1)^2 + (s_3t_2 + s_3t_3)^2}{60} \\ &\quad + \frac{(s_2t_3 + 2s_2t_2)^2 + 4t_2^2(s_2^2 + s_1s_2 + s_2s_3) + 2s_2(s_1t_1^2 + s_3t_3^2)}{60} \end{aligned}$$

$$+\frac{(s_2t_1+2s_2t_2)^2-60E[\xi]^2}{60},$$

$$S[\xi]=\frac{3(s_1t_2+s_2t_1+s_1t_1)^3+5(s_1t_2+s_2t_1+s_2t_2)^3+3(s_1t_1+s_2t_2)^3+9s_1^3t_1^3}{840}$$
$$+\frac{3(s_3t_2+s_2t_3+s_3t_3)^3+5(s_3t_2+s_2t_3+s_2t_2)^3+3(s_3t_3+s_2t_2)^3+9s_3^3t_3^3}{840}$$
$$+\frac{s_2^2t_2^2(6s_1t_1+30s_1t_2+30s_2t_1+194s_2t_2+30s_3t_2+30s_2t_3+6s_3t_3)}{840}$$
$$+\frac{18s_2s_3t_2t_3(s_3t_2+s_2t_3)-5(s_1t_2+s_2t_1)^3-5(s_3t_2+s_2t_3)^3}{840}$$
$$+\frac{6s_1^2t_1^2(s_1t_2+s_2t_1)+6s_3^2t_3^2(s_3t_2+s_2t_3)+18s_1s_2t_1t_2(s_1t_2+s_2t_1)}{840}$$
$$-3E[\xi]V[\xi]-E[\xi]^3.$$

For example, set $\xi_1=(1,2,3)$ and $\xi_2=(2,3,6)$, then $E[\xi]=7$, $V[\xi]=311/30$ and $S[\xi]=1079/35$. In particular, if $\xi_1=\xi_2$ with $s_i=t_i$, $i=1,2,3$, we have

$$E[\xi]=\frac{s_1^2+2s_1s_2+6s_2^2+2s_2s_3+s_3^2}{12},$$
$$V[\xi]=\frac{s_1^4+10s_2^4+s_3^4+4(s_1+s_3)s_2^3+3(s_1^2+s_3^2)s_2^2+2(s_1^3+s_3^3)s_2-30E[\xi]^2}{30},$$
$$S[\xi]=\frac{s_1^6+14s_2^6+s_3^6+6(s_1+s_3)s_2^5+5(s_1^2+s_3^2)s_2^4+4(s_1^3+s_3^3)s_2^3+3s_1^4s_2^2}{56}$$
$$+\frac{3s_3^4s_2^2+2(s_1^5+s_3^5)s_2-168E[\xi]V[\xi]-56E[\xi]^3}{56}.$$

## 4 Numerical integral-based simulation algorithm

This section proposes a NISA to approximate the expected value, variance and skewness of the product and sum of multiple positive $L$-$R$ fuzzy numbers. A comparison between the approximated values and the exact values is performed to illustrate the effectiveness of NISA.

Suppose that $\xi_i$ is a positive $L$-$R$ fuzzy number taking values in interval $[a_i, c_i]$ with membership function $\mu_i(x)$, $i=1,2,\ldots,n$. According to Lemma 1, $\xi=\xi_1\times\xi_2\times\cdots\times\xi_n$ is a $L$-$R$ fuzzy number in interval $[a_1a_2\cdots a_n, c_1c_2\cdots c_n]$ with membership function $\mu(x)$. Denote the $\gamma$-level set of $\xi$ as $[a_1(\gamma), a_2(\gamma)]$ and the $\gamma$-level sets of $\xi_i$ as $[a_{i1}(\gamma), a_{i2}(\gamma)]$, $i=1,2,\ldots,n$, $\gamma\in(0,1]$. For any $\gamma\in(0,1]$, it follows from properties **(i)** and **(ii)** that

$$a_1(\gamma)=a_{11}(\gamma)a_{21}(\gamma)\cdots a_{n1}(\gamma),\quad a_2(\gamma)=a_{12}(\gamma)a_{22}(\gamma)\cdots a_{n2}(\gamma). \tag{9}$$

Based on the above analysis, we design NISA to approximate the expected value, variance and skewness of $\xi$. Firstly, uniformly select $N$ discrete points

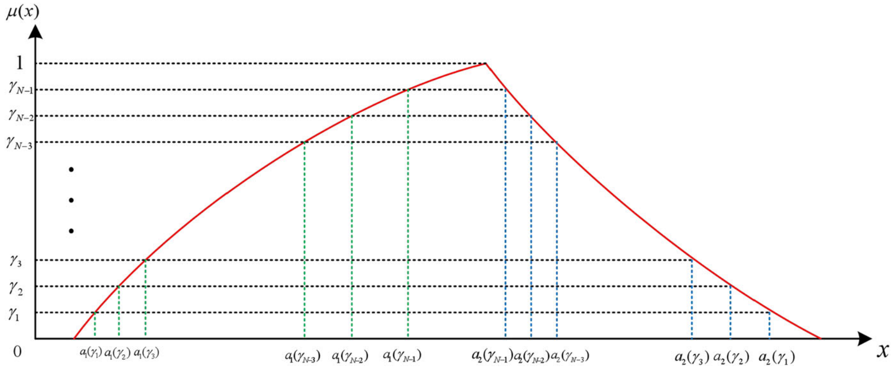


**Fig. 2** Membership function of $\xi$

$\gamma_1, \gamma_2, \ldots, \gamma_N \in (0, 1]$ (See Fig. 2) and compute the $\gamma_i$-level sets $[a_1(\gamma_i), a_2(\gamma_i)]$ following from Eq. (9) with $i = 1, 2, \ldots, N$. Secondly, approximate $E[\xi]$ by using the following numerical integration formula

$$e = \sum_{i=1}^{N} \gamma_i \left[a_1(\gamma_i) + a_2(\gamma_i)\right] (\gamma_i - \gamma_{i-1})$$

where $\gamma_0 = 0$. Finally, approximate $V[\xi]$ and $S[\xi]$ as follows

$$v = \sum_{i=1}^{N} \gamma_i[(a_1(\gamma_i) - e)^2 + (a_2(\gamma_i) - e)^2](\gamma_i - \gamma_{i-1}),$$

$$s = \sum_{i=1}^{N} \gamma_i[(a_1(\gamma_i) - e)^3 + (a_2(\gamma_i) - e)^3](\gamma_i - \gamma_{i-1}).$$

The core mechanism of NISA is generating a certain number of sample points and simulate the Riemann integrals of the expected values, variances and skewness. The steps are presented in Algorithm 4.1.

The relative errors of expected value, variance and skewness are subject to parameter $N$. Therefore, the optimal parameter $N$ needs to be determined to obtain a satisfactory approximation by varying the number of sample points in NISA.

***Example 2*** In this example, we consider the expected value, variance and skewness of $\xi = \xi_1 \times \xi_2$ with $\xi_1 = (1, 2, 3)$ and $\xi_2 = (2, 3, 6)$. According to Example 1, the exact values are $E[\xi] = 7$, $V[\xi] = 311/30$ and $S[\xi] = 1079/35$. By running NISA 20 times with $N$ varying from 10 to 200, the approximated values and relative errors are obtained (See Table 1). The relative error is defined as

$$\delta = \frac{|r - r^*|}{\max\{|r|, |r^*|\}} \times 100\%, \tag{10}$$

where $r$ is the approximated value and $r^*$ is the exact value.

**Algorithm 4.1 NISA**

Step 1.Initialize parameter $N$.
Step 2.Uniformly generate point sequence $\gamma_i = i/N$ in interval (0, 1], $i = 1, 2, \ldots, N$. Compute the $\gamma_i$-level sets: $a_1(\gamma_i) = a_{11}(\gamma_i)a_{21}(\gamma_i)\cdots a_{n1}(\gamma_i)$, $a_2(\gamma_i) = a_{12}(\gamma_i)$ $a_{22}(\gamma_i)\cdots a_{n2}(\gamma_i)$.
Step 3.Set $e = 0$, $v = 0$, $s = 0$ and $i = 1$.
Step 4.$e = e + \gamma_i a_1(\gamma_i)/N + \gamma_i a_2(\gamma_i)/N$. Set $i = i + 1$.
Step 5.If $i \leq N$, go to Step 4.
Step 6.Set $j = 1$.
Step 7.$v = v + \gamma_j(a_1(\gamma_j) - e)^2/N + \gamma_j(a_2(\gamma_j) - e)^2/N$, $s = s + \gamma_j(a_1(\gamma_j) - e)^3/N + \gamma_j(a_2(\gamma_j) - e)^3/N$. Set $j = j + 1$.
Step 8.If $j \leq N$, go to Step 7.
Step 9.Return $e$, $v$ and $s$.

**Table 1** Approximated values and relative errors as $N$ varies

| $N$ | Approximated values for $e/v/s$ | Relative errors for $e/v/s$ (%) | Times (s) |
|---|---|---|---|
| 10 | 7.5900/10.8662/11.0531 | 7.77/4.60/64.15 | 0.0071 |
| 20 | 7.2975/10.5125/21.2117 | 4.08/1.39/31.19 | 0.0088 |
| 30 | 7.1989/10.4420/24.4723 | 2.76/0.72/20.62 | 0.0067 |
| 40 | 7.1494/10.4151/26.0813 | 2.09/0.47/15.40 | 0.0068 |
| 50 | 7.1196/10.4016/27.0402 | 1.68/0.34/12.29 | 0.0062 |
| 60 | 7.0997/10.3937/27.6768 | 1.40/0.26/10.22 | 0.0154 |
| 70 | 7.0855/10.3885/28.1301 | 1.21/0.21/8.75 | 0.0149 |
| 80 | 7.0748/10.3850/28.4695 | 1.06/0.18/7.65 | 0.0163 |
| 90 | 7.0665/10.3824/28.7330 | 0.94/0.15/6.80 | 0.0083 |
| 100 | 7.0599/10.3804/28.9436 | 0.85/0.13/6.11 | 0.0130 |
| 110 | 7.0545/10.3788/29.1157 | 0.77/0.12/5.56 | 0.0128 |
| 120 | 7.0499/10.3776/29.2590 | 0.71/0.11/5.09 | 0.0143 |
| 130 | 7.0461/10.3765/29.3802 | 0.65/0.09/4.70 | 0.0173 |
| 140 | 7.0428/10.3757/29.4840 | 0.61/0.09/4.36 | 0.0145 |
| 150 | 7.0400/10.3750/29.5739 | 0.57/0.08/4.07 | 0.0077 |
| 160 | 7.0375/10.3744/29.6526 | 0.53/0.07/3.81 | 0.0129 |
| 170 | 7.0353/10.3738/29.7220 | 0.50/0.07/3.59 | 0.0158 |
| 180 | 7.0333/10.3734/29.7836 | 0.47/0.06/3.39 | 0.0073 |
| 190 | 7.0316/10.3730/29.8387 | 0.45/0.06/3.21 | 0.0083 |
| 200 | 7.0300/10.3726/29.8883 | 0.43/0.06/3.05 | 0.0114 |

The variation tendency of relative errors with regard to $N$ is depicted in Fig. 3, which shows that the relative errors for expected value, variance and skewness decrease with respect to $N$. When $N$ exceeds 160, the relative errors of expected value and variance are both less than 1% and the relative error of skewness is less than 4%, which verifies that NISA is able to converge to the exact value with a certain size of sample points. In addition, it follows from Table 1 that the average running time for different parameters is 0.0113 seconds, justifying the high running speed of NISA.

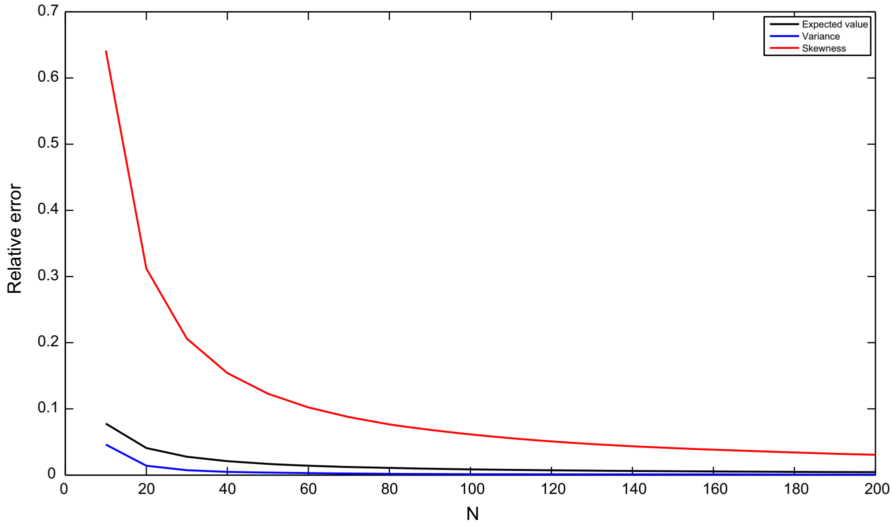


**Fig. 3** An illustration on convergence of NISA

***Example 3*** Taking $N = 2000$, we perform NISA on 12 fuzzy numbers to obtain their approximated expected values, variances and skewness and make comparisons with the exact values. The results are shown in Table 2, where the second column lists the approximated values for expected value/variance/skewness ($e/v/s$), the third column lists the exact values ($E[\xi]/V[\xi]/S[\xi]$), and the last column records the relative errors ($\delta(e)/\delta(v)/\delta(s)$).

The results show that the relative errors range from 0.00 to 2.57% and the average errors for expected values, variances and skewness are 0.48%, 0.01%, and 1.26%, which implies that NISA can obtain satisfactory approximations for the expected values, variances and skewness of fuzzy numbers.

## 5 Fuzzy multi-period portfolio selection model and numerical integral-based genetic algorithm

This section presents the application of NISA in portfolio selection. A fuzzy multi-period portfolio selection model considering different risk preferences is introduced to deal with the financial investment issues, and a hybrid algorithm integrating NISA and genetic algorithm (NIGA) is designed to solve the above model.

### 5.1 Fuzzy multi-period portfolio selection model

Portfolio selection studies the method of allocating the investor's initial capital among multiple risky assets to maximize the investment return and/or minimize the investment risk. The pioneer work of portfolio selection theory was proposed by Markowitz (1952), which inspires many subsequent portfolio selection studies (Lai et al. 2006;

**Table 2** Expected values, variances and skewness on different fuzzy numbers

| $\xi$ | $e/v/s$ | $E[\xi]/V[\xi]/S[\xi]$ | $\delta(e)/\delta(v)/\delta(s)$ (%) |
|---|---|---|---|
| $(6.5, 7.2, 9.0)^2$ | 54.8168/64.0142/464.7073 | 54.7908/64.0091/469.6980 | 0.05/0.01/1.06 |
| $(4.0, 5.0, 7.0)^2$ | 27.0958/44.8954/251.9915 | 27.0833/44.8931/253.6797 | 0.05/0.01/0.67 |
| $(7.0, 8.0, 12.0)^2$ | 73.4487/383.4053/9021.8000 | 73.4167/383.3597/9058.9816 | 0.04/0.01/0.41 |
| $(7.5, 8.0, 12.0)^2$ | 74.7195/343.2378/8622.7000 | 74.6875/343.1794/8656.2481 | 0.04/0.02/0.39 |
| $(6.5, 7.2, 9.0) \times (7.0, 8.0, 9.0)$ | 59.3038/52.8463/224.3289 | 59.2750/52.8440/228.8972 | 0.05/0.00/2.00 |
| $(6.5, 7.2, 9.0) \times (4.2, 5.6, 9.0)$ | 44.3585/120.7869/1244.3000 | 44.3383/120.7783/1251.6305 | 0.05/0.01/0.59 |
| $(6.0, 8.5, 9.6) \times (4.5, 6.5, 7.5)$ | 52.8526/92.6336/−441.9159 | 52.8250/92.6300/−434.2461 | 0.05/0.00/1.74 |
| $(4.2, 5.5, 7.8) \times (3.4, 5.5, 7.4)$ | 31.6052/77.7300/323.0716 | 31.5750/77.7273/330.1286 | 0.09/0.00/2.14 |
| $(5.0, 8.0, 10.0) \times (4.0, 6.0, 7.0)$ | 46.3573/111.8241/−313.6226 | 46.3333/111.8222/−305.5735 | 0.05/0.00/2.57 |
| $(7.5, 8.2, 11.0) \times (6.0, 7.5, 8.2)$ | 63.3132/85.6409/392.7983 | 63.2825/85.6383/400.7016 | 0.05/0.00/1.97 |
| $(7.0, 8.0, 12.0) \times (3.0, 4.0, 5.0)$ | 34.4167/63.1264/405.5515 | 34.4167/63.1264/405.5518 | 0.00/0.01/0.00 |
| $(7.5, 8.0, 11.0) \times (6.0, 7.5, 8.2)$ | 62.3258/86.0909/494.0852 | 62.2958/86.0874/501.8395 | 0.05/0.00/1.55 |

Yu et al. 2012). The above portfolio selection studies are based on probability theory, where the returns of risky assets are treated as random variables with known probability distributions derived by historical data. In case of there is no enough data, fuzzy numbers are used to characterize the asset returns. With the development of fuzzy theory, an increasing number of research works on fuzzy portfolio selection problem can be found in literature (Huang 2011; Li and Peter 2011; Li et al. 2015; Wang and Zhu 2002; Wu and Liu 2012; Zhang et al. 2017; Zhou et al. 2017), which reflect the successful applications of fuzzy theory in practical financial issues. In terms of investment horizon, the portfolio selection studies can be divided into single-period portfolio selections (Lai et al. 2006; Li and Peter 2011; Li et al. 2015; Markowitz 1952) and multi-period portfolio selections (Guo et al. 2016; Zhou et al. 2017). The representative single-period portfolio selection is Markowitz's mean-variance model, which aspires to find the best tradeoff between the return and risk in one single period. Only one decision making is required at the beginning of this short-term investment. However, for the long-term investment, it is better to adjust the investment strategies constantly since the performances of risky assets change over different periods. Therefore, multi-period portfolio selection is attracting attention from both the academic researchers and practitioners. For example, Guo et al. (2016) studied a multi-period portfolio selection model where the investment horizons of risky assets were assumed to be different. In this paper, we consider a fuzzy multi-period portfolio selection problem.

Consider a portfolio selection problem among $n$ risky assets and a risk-free asset over an investment horizon of $T$ periods. The initial wealth is $W_1$, and returns obtained at each period are reallocated in the next period. To facilitate our discussion, we introduce the following notations (See Table 3).

Various criteria are considered to achieve a high flexibility in portfolio selection, such as return, risk, skewness and transaction cost. The investment process is assumed to be self-financing, which means additional capital infusion or withdrawal is forbidden. The transaction cost is considered as a V-shaped function of differences between the $t$th period portfolio $x_t$ and the $(t-1)$th period portfolio $x_{t-1}$. Hence the unit transaction cost at period $t$ is $D_t = \sum_{i=1}^{n+1} d_{ti}|x_{ti} - x_{(t-1)i}|$ with $x_{0i} = 0$ for all

**Table 3** List of notations

| Notations | |
|---|---|
| $i$ | Index for risky asset, $i = 1, 2, \ldots, n$ |
| $t$ | Index for investment period, $t = 1, 2, \ldots, T$ |
| $r_f$ | Constant return for risk-free asset |
| $d_{ti}$ | Unit transaction cost of asset $i$ $(i = 1, 2, \ldots, n+1)$ at period $t$ $(t = 1, 2, \ldots, T)$ |
| $W_t$ | Available wealth at the beginning of period $t$ $(t = 1, 2, \ldots, T+1)$ |
| $\xi_{ti}$ | Return of risky asset $i$ $(i = 1, 2, \ldots, n)$ at period $t$ $(t = 1, 2, \ldots, T)$, which are fuzzy variables |
| $x_{ti}$ | Investment proportion on asset $i$ $(i = 1, 2, \ldots, n+1)$ at period $t$ $(t = 1, 2, \ldots, T)$, which are decision variables |

$i = 1, 2, \ldots, n+1$. Then the available wealth at the end of period $T$ can be expressed as

$$W_{T+1} = W_1 \prod_{t=1}^{T} \left( \sum_{i=1}^{n} \xi_{ti} x_{ti} + r_f x_{t(n+1)} - D_t \right). \tag{11}$$

In portfolio selection, the return, risk and skewness are three objectives that an investor concerns most, which are quantified by $E[W_{T+1}]$, $V[W_{T+1}]$ and $S[W_{T+1}]$, respectively. Although higher moments of return distributions, e.g., kurtosis, are relevant to the investment decisions, their influence on the optimal portfolio is negligible compared with the first three moments. In addition, the introduction of higher moments in portfolio selection will significantly increase the calculation burden. Therefore, in this paper, we focus on the return, risk and skewness. A good portfolio is generally able to maximize the return and skewness, and minimize the investment risk. Accordingly, the mean-variance-skewness model is formulated as

$$\begin{cases} \max \ E[W_{T+1}] \\ \min \ V[W_{T+1}] \\ \max \ S[W_{T+1}] \\ \text{s.t.} \quad x_{t1} + x_{t2} + \cdots + x_{t(n+1)} = 1, \ t = 1, 2, \ldots, T, \\ \qquad\ 0 \le x_{ti} \le 1, \ t = 1, 2, \ldots, T, \ i = 1, 2, \ldots, n+1. \end{cases} \tag{12}$$

The return, risk and skewness are of different importance for different individuals. For example, aggressive investors tend to maximize the total return rather than avert risk. However, conservative investors may exert a tremendous fascination on minimizing the total risk. In case that the return and risk are the same, both of the aggressive and conservative investors prefer a better skewness. The key to tackle with the above multi-objective portfolio selection model is to find a balance satisfying the three objectives simultaneously. To achieve this, utility function is employed to transform the multi-objective programming model into a single-objective programming problem.

Utility function of portfolio selection represents satisfaction degree of the investors with regard to the given portfolio, which differ among different individuals (See Fig. 4). The expected utility model (EUM) is employed to obtain the optimal investment strategy, which is expressed as

$$\begin{cases} \max \ E[u(W_{T+1})] \\ \text{s.t.} \quad x_{t1} + x_{t2} + \cdots + x_{t(n+1)} = 1, t = 1, 2, \ldots, T, \\ \qquad\ 0 \le x_{ti} \le 1, \ t = 1, 2, \ldots, T, i = 1, 2, \ldots, n+1, \end{cases} \tag{13}$$

where the first constraint represents that the proportions sum to one, ensuring all the available wealth is allocated to $(n+1)$ assets at the beginning of each period. The second constraint means that no short sales or borrowings is allowed.

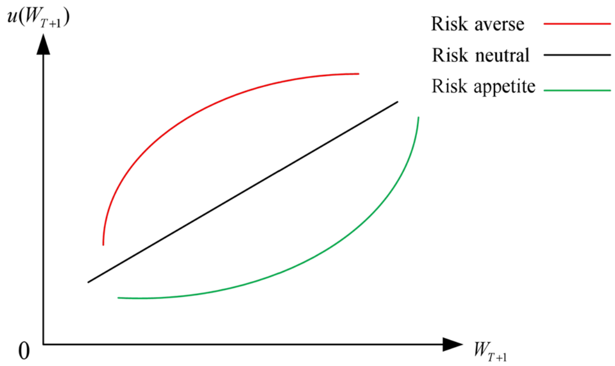


**Fig. 4** Utility functions of different investors

As is illustrated in Liu et al. (2003), it is assumed that the utility function $u(W_{T+1})$ can be approximated by the third-order Taylor's expansion around the expected value $E(W_{T+1})$. Thus, a connection between utility function, mean, variance and skewness can be established, which is beneficial to incorporate three criteria together to reduce computational complexity. The extension process is shown as follows:

$$\begin{aligned} u(W_{T+1}) = & \, u(E[W_{T+1}]) + u'(E[W_{T+1}])(W_{T+1} - E[W_{T+1}]) \\ & + \frac{1}{2}u''(E[W_{T+1}])(W_{T+1} - E[W_{T+1}])^2 \\ & + \frac{1}{6}u'''(E[W_{T+1}])(W_{T+1} - E[W_{T+1}])^3, \end{aligned}$$

where $u'(\cdot)$, $u''(\cdot)$ and $u'''(\cdot)$ represent the first, second and third order derivatives of $u(\cdot)$, respectively.

Taking the expectation in both sides of the equation, one can obtain

$$\begin{aligned} E[u(W_{T+1})] &= u(E[W_{T+1}]) + \frac{1}{2}u''(E[W_{T+1}])E[(W_{T+1} - E[W_{T+1}])^2] \\ &\quad + \frac{1}{6}u'''(E[W_{T+1}])E[(W_{T+1} - E[W_{T+1}])^3] \\ &= u(E[W_{T+1}]) + \frac{1}{2}u''(E[W_{T+1}])V[W_{T+1}] \\ &\quad + \frac{1}{6}u'''(E[W_{T+1}])S[W_{T+1}]. \end{aligned}$$

***Example 4*** Consider a 6-period portfolio selection problem with 6 risky assets, of which the returns are characterized by fuzzy numbers, and a risk-free asset with constant return $r_f$. According to Eq. (11), the available wealth at the end of period 6 is

**Table 4** EUMs for different risk preferences

| Risk preferences | Risk appetite | Risk neutral | Risk averse |
|---|---|---|---|
| Utility function | $u(x) = x^3$ | $u(x) = x$ | $u(x) = \sqrt{x}$ |
| Objective | $E[W_7]^3 + 3E[W_7]V[W_7] + S[W_7]$ | $E[W_7]$ | $E[W_7] + \frac{1}{8}E[W_7]^{-\frac{3}{2}}V[W_7] - \frac{1}{16}E[W_7]^{-\frac{5}{2}}S[W_7]$ |
| Constraint | $x_{t1} + x_{t2} + \cdots + x_{t7} = 1, 0 \leq x_{ti} \leq 1, t = 1, 2, \ldots, 6, i = 1, 2, \ldots, 7.$ | | |

$$W_7 = W_1 \prod_{t=1}^{6} \left( \sum_{i=1}^{6} \xi_{ti} x_{ti} + r_f x_{t7} - \sum_{i=1}^{7} d_{ti} |x_{ti} - x_{(t-1)i}| \right).$$

Based on different risk preferences, the EUMs can be formulated as follows (Table 4).

## 5.2 Numerical integral-based genetic algorithm

Genetic algorithm (GA) is a stochastic search method for optimization problems which is inspired by the Darwinian process of evolution. It starts by randomly generating an initial population consisting of multiple individuals, where each individual denotes a feasible solution, and then constantly update these individuals in each generation by mimicking the gene inheritance and mutation in the process of evolution. After several generations, the fittest individual survives as the optimal solution.

GA can be integrated with other optimization algorithms easily due to its strong practicability and scalability. One good illustration is the fuzzy simulation-based genetic algorithm (FSGA), which was widely applied in optimization. For example, Guo et al. (2016) employed FSGA to tackle with the multi-period portfolio selection problem with different investment horizons. Although FSGA is able to tackle with complex optimization issues, it has a significant drawback of expensive running cost. The first reason is the random selection of data points. It is theoretically reasonable to randomly select a certain number of data points to cover the whole interval such that the simulated value is closer to the exact value, but this random selection mechanism only works when the number of data points is large enough, which is caused by the complexity of the Zadeh extension principle. This explains why the number of data points $N$ in fuzzy simulation is usually larger than 2500. However, in our proposed NISA, we uniformly select the data sequence $\gamma_i$ in interval (0, 1] so that the $\gamma_i$-level sets cover every subinterval of $[a, c]$ (See Algorithm 4.1), which demonstrates a much looser constraint on $N$. As shown in Example 2, NISA can achieve a satisfactory approximation when $N$ is more than 160. The second reason is the high computational complexity of fuzzy simulation. According to the computation procedure of fuzzy simulation (See "Appendix B"), $N$ circles are performed to compute $E[\xi]$. In each circle, whether $\max\{\nu_k | y_k \geq r\}$ or $\max\{\nu_k | y_k \leq r\}$ is computed, which involves $N$ arithmetic operations. Then the computational complexity for fuzzy simulation is $O(N^2)$. For NISA, although there are also $N$ circles performed, only one arithmetic operation is needed in each circle, so its computational complexity is $O(N)$. Since the running speed of NISA is much faster than fuzzy simulation, we adopt NIGA to solve the multi-period portfolio selection problem. The whole solving process consists of five key procedures including initialization, evaluation, selection, crossover and mutation.

*Initialization* Initialization is operated to generate the initial population. First, randomly generate a chromosome $\mathbf{c} = (c_{11}, c_{12}, \ldots, c_{1(n+1)}; c_{21}, c_{22}, \ldots, c_{2(n+1)}; \ldots; c_{T1}, c_{T2}, \ldots, c_{T(n+1)})$, where $c_{ti} \in [0, 1]$, $t = 1, 2, \ldots, T$, $i = 1, 2, \ldots, n+1$. Second, denote the solution $\mathbf{x} = (x_{11}, x_{12}, \ldots, x_{1(n+1)}; x_{21}, x_{22}, \ldots, x_{2(n+1)}; \ldots; x_{T1}, x_{T2}, \ldots, x_{T(n+1)})$ by $x_{ti} = c_{ti}/(c_{t1} + c_{t2} + \cdots + c_{t(n+1)})$, which ensures the constraint $x_{t1} + x_{t2} + \cdots + x_{t(n+1)} = 1$ is satisfied. Repeat the above process $pop\_size$ times to generate an initial population consisting of $pop\_size$ feasible solutions: $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{pop\_size}$.

*Evaluation* Evaluation is used for assigning a reproduction probability to each chromosome. The chromosome $\mathbf{c}_i$ will have more chance to produce offspring when $E[u(W_{T+1}(\mathbf{x}_i))]$ is larger, $i = 1, 2, \ldots, pop\_size$. Therefore, we first calculate the value of $E[u(W_{T+1}(\mathbf{x}_i))]$ by NISA. Then, realign these chromosomes in a descending order such that

$$E[u(W_{T+1}(\mathbf{x}_1))] \geq E[u(W_{T+1}(\mathbf{x}_2))] \geq \cdots \geq E[u(W_{T+1}(\mathbf{x}_{pop\_size}))].$$

Finally, calculate the fitness of chromosome $\mathbf{c}_i$ as follows:

$$Eval(\mathbf{c}_i) = \alpha(1-\alpha)^{i-1}, \ i = 1, 2, \ldots, pop\_size,$$

where $\alpha \in (0, 1)$ is a real number.

***Remark 3*** NISA is extremely crucial in this stage for its computation accuracy determines whether GA can obtain the optimal solution, and its high computation speed can significantly reduce the running time of GA.

*Selection* Selection is intended for updating the population by spinning the roulette wheel. Set $q_i = \sum_{j=1}^{i} Eval(\mathbf{c}_j)$ for all $i = 1, 2, \ldots, pop\_size$ and $q_0 = 0$. Then, randomly generate a number $r \in (0, q_{pop\_size}]$. If $r \in (q_{i-1}, q_i]$, select the $i$th chromosome $\mathbf{c}_i$. Repeat the above procedure $pop\_size$ times.

*Crossover* Crossover is employed to reproduce the offspring by taking the arithmetic average of two selected chromosomes. Take $p_c$ as the crossover probability. For each chromosome $\mathbf{c}_i$, randomly generate a real number $r \in [0, 1]$. If $r \in [0, p_c]$, select $\mathbf{c}_i$. When the number of selected chromosomes $u$ is odd, delete the last one to make it an even number. Divide all of the selected chromosomes into pairs: $(\mathbf{v}_1, \mathbf{v}_2)$, $(\mathbf{v}_3, \mathbf{v}_4)$, $\ldots$, $(\mathbf{v}_{u-1}, \mathbf{v}_u)$. Conduct a crossover operator on each pair of chromosomes as follows

$$\mathbf{v}_i' = \lambda\mathbf{v}_i + (1-\lambda)\mathbf{v}_{i+1}, \mathbf{v}_{i+1}' = \lambda\mathbf{v}_{i+1} + (1-\lambda)\mathbf{v}_i,$$

where $\mathbf{v}_i'$ denotes the updated chromosome, $\lambda \in (0, 1]$ is a random number and odd number $i$ satisfying $1 \leq i \leq u-1$ .

*Mutation* Mutation is designed to ensure the diversity of population and prevent the local convergence arising from crossover operation. Define $\mathbf{d}$ as the mutation direction, $p_m$ as mutation probability, and $\beta$ as an approximate large positive number. For each chromosome $\mathbf{c}_i$, randomly generate a number $r \in [0, 1]$. If $r \in [0, p_m]$, generate a feasible chromosome $\mathbf{c}_i + \beta\mathbf{d}$ by selecting an appropriate $\beta$ and $\mathbf{d}$, and replace $\mathbf{c}_i$ with $\mathbf{c}_i + \beta\mathbf{d}$.

The steps for NIGA are summarized in Algorithm 5.1.

**Algorithm 5.1 NIGA**

| | |
|---|---|
| Step 1. | Input the parameters: $G$, $pop\_size$, $p_c$, $p_m$, $\mathbf{d}$, $\alpha$ and $\beta$. |
| Step 2. | Initialize $pop\_size$ chromosomes. |
| Step 3. | Compute the objective value $E[u(W_{T+1}(\mathbf{x}_i))]$ for each chromosome by using NISA, reorder these chromosomes and calculate their fitness values. |
| Step 4. | Select the chromosomes by the roulette wheel selection method. |
| Step 5. | Renew the chromosomes by crossover operation. |
| Step 6. | Renew the chromosomes by mutation operation. |
| Step 7. | Repeat the third to sixth steps for $G$ cycles. |
| Step 8. | Take the best chromosome as the optimal solution. |

## 6 Numerical examples

This section proposes two numerical examples to verify the effectiveness and practicability of our proposed NISA. In Example 5, we consider a 6-year portfolio selection problem with 8 risky assets and a risk-free asset, and obtain the optimal investment strategy by NIGA. In Example 6, we solve a real-world portfolio selection problem by NIGA and FSGA respectively. The advantages of NIGA over FSGA are analysed from the perspective of robustness and running speed.

***Example 5*** Suppose that there are 8 risky assets with triangular fuzzy returns, and the risk-free asset return is $r_f = 1.0300$. An investor is prepared to construct a 6-year portfolio with the initial wealth $W_1 = 1$ unit (1 unit represents 10000 dollars). The unit transaction cost matrix is given by

$$d = \begin{pmatrix} 0.0010 & 0.0009 & 0.0030 & 0.0021 & 0.0019 & 0.0012 & 0.0025 & 0.0015 & 0.0005 \\ 0.0018 & 0.0008 & 0.0019 & 0.0024 & 0.0021 & 0.0040 & 0.0020 & 0.0019 & 0.0005 \\ 0.0022 & 0.0008 & 0.0012 & 0.0023 & 0.0021 & 0.0030 & 0.0017 & 0.0021 & 0.0005 \\ 0.0025 & 0.0007 & 0.0011 & 0.0021 & 0.0023 & 0.0056 & 0.0020 & 0.0021 & 0.0005 \\ 0.0014 & 0.0009 & 0.0014 & 0.0024 & 0.0022 & 0.0040 & 0.0018 & 0.0020 & 0.0005 \\ 0.0012 & 0.0007 & 0.0011 & 0.0020 & 0.0021 & 0.0031 & 0.0016 & 0.0023 & 0.0005 \end{pmatrix},$$

where $d_{ti}$ denotes the transaction cost of asset $i$ in the $t$th year. For example, $d_{26} = 0.0040$ means that trading one unit of asset 6 in the second year costs 0.0040 unit (40 dollars). The fuzzy returns of risky assets, expected values, variances and skewness in different periods are shown in Table 5 (The left column includes assets 1, 3, 5 and 7, and the right column refers to assets 2, 4, 6 and 8). Suppose that the investor is risk appetite with utility function $u(x) = x^3$. According to Table 4, the EUM is

$$\begin{cases} \max\ E[W_7]^3 + 3E[W_7]V[W_7] + S[W_7] \\ \text{s.t.}\quad x_{t1} + x_{t2} + \cdots + x_{t9} = 1, t = 1, 2, \ldots, 6, \\ \qquad 0 \leq x_{ti} \leq 1,\ t = 1, 2, \ldots, 6, i = 1, 2, \ldots, 9. \end{cases} \tag{14}$$

The optimal investment strategy is obtained by running the NIGA (See Table 6), where the parameters are set as follows: $pop\_size = 200$, $G = 130$, $p_m = 0.1$ and $p_c = 0.9$. Take the first year as an example. The best investment proportions for the 9 assets are 5.02%, 4.79%, 11.07%, 9.76%, 18.52%, 18.98%, 11.85%, 18.88%, 1.13%, respectively. Asset 6 has the highest proportion for its high return and largest skewness. Asset 8 has a comparatively high proportion for its return is the highest. For a risk appetite investor, high return is more preferable compared with low risk. Therefore, due to the lowest return and relatively small skewness, the investment proportion of asset 2 is only 1.72%, which is the smallest among the 8 risky assets. In addition, it is seen from Table 6 that assets 5 and 6 are superior assets for their investment proportions stably fluctuate between 15.00% and 21.00%. While asset 2 is a typical inferior asset with its investment proportion varying greatly in different periods. Therefore, when dealing with this type of inferior asset, the investor needs to consider carefully before making the investment decision. The final expected return, variance, skewness and expected utility are 2.7728, 4.0603, 13.0982 and 68.1915, respectively.

***Example 6*** Consider a 3-year portfolio selection consisting of 8 stocks selected from the Shanghai Stock Exchange: China Vanke Company (000002), BOE Technology Group Company (000725), Shanghai Pudong Development Bank (600000), CITIC Securities Company (600030), China United Telecommunications Company (600050), Chinese Universe Publishing and Media Company (600373), HNA Innovation Company (600555) and Metallurgical Corporation of China (601618). At the beginning of each year, the investor allocates part of the available wealth to the 8 stocks and deposits the rest in the bank with interest rate $r_f = 1.0300$. According to Shanghai Stock Exchange trading rules, the transaction costs for different stocks are the same (Guo et al. 2016), then the unit transaction cost matrix is

$$d = \begin{pmatrix} 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0000 \\ 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0000 \\ 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0030 & 0.0000 \end{pmatrix}$$

where $d_{t9} = 0.0000, t = 1, 2, 3$, which means no trading fee is charged when depositing or withdrawing money in the bank. Assume the returns of stocks as triangular fuzzy numbers, which are derived from analyzing the historical data. We collect the daily opening prices from January 2014 to December 2016 in the iFinD database (https://data.mendeley.com/datasets/fckp89tff7/1). The daily returns for each stock are obtained by equation:

$$r_{ij} = p_{ij}/p_{i(j-1)}, \quad i = 1, 2, \ldots, 8,\ j = 1, 2, \ldots, n_i$$

where $p_{ij}$ represents the opening price of the $j$-th trading day for stock $i$, $p_{i0}$ denotes the initial price and $n_i$ is the total number of trading days for stock $i$. Remark that the number of trading days for different stocks may differ due to the possibility of

**Table 5** The fuzzy returns, expected values, variances and skewness for risky assets of Example 5

| Period | Fuzzy return/expected value/variance/skewness | |
|---|---|---|
| 1 | $(0.90, 1.10, 1.32)/1.03/8.00\times10^{-3}/6.54\times10^{-4}$ | $(0.80, 1.00, 1.40)/1.03/1.56\times10^{-2}/1.20\times10^{-3}$ |
| | $(0.85, 1.15, 1.50)/1.16/1.76\times10^{-2}/3.46\times10^{-4}$ | $(0.84, 1.16, 1.60)/1.18/2.43\times10^{-2}/1.10\times10^{-3}$ |
| | $(0.60, 1.20, 1.64)/1.17/4.54\times10^{-2}/2.90\times10^{-3}$ | $(0.67, 1.26, 1.50)/1.20/3.04\times10^{-2}/4.10\times10^{-3}$ |
| | $(0.91, 1.14, 1.60)/1.18/2.06\times10^{-2}/1.80\times10^{-3}$ | $(0.89, 1.26, 1.55)/1.25/1.82\times10^{-2}/-5.88\times10^{-4}$ |
| 2 | $(0.95, 1.10, 1.35)/1.12/6.80\times10^{-3}/2.65\times10^{-4}$ | $(0.90, 1.10, 1.60)/1.15/2.17\times10^{-2}/2.50\times10^{-4}$ |
| | $(0.90, 1.20, 1.55)/1.21/1.76\times10^{-2}/3.46\times10^{-4}$ | $(0.97, 1.14, 1.68)/1.20/2.29\times10^{-2}/3.10\times10^{-3}$ |
| | $(0.70, 1.20, 1.60)/1.18/3.39\times10^{-2}/-1.40\times10^{-3}$ | $(0.90, 1.17, 1.58)/1.19/1.95\times10^{-2}/1.10\times10^{-3}$ |
| | $(0.92, 1.25, 1.50)/1.24/1.41\times10^{-2}/-4.54\times10^{-4}$ | $(0.90, 1.11, 1.46)/1.13/1.33\times10^{-2}/7.30\times10^{-4}$ |
| 3 | $(0.96, 1.08, 1.40)/1.11/8.60\times10^{-3}/6.50\times10^{-4}$ | $(0.94, 1.10, 1.42)/1.13/1.00\times10^{-2}/6.15\times10^{-4}$ |
| | $(0.94, 1.10, 1.38)/1.12/8.30\times10^{-3}/3.86\times10^{-4}$ | $(0.97, 1.12, 1.48)/1.16/1.15\times10^{-2}/9.15\times10^{-4}$ |
| | $(0.90, 1.22, 1.46)/1.21/1.32\times10^{-2}/-4.23\times10^{-4}$ | $(0.97, 1.20, 1.58)/1.23/1.58\times10^{-2}/9.28\times10^{-4}$ |
| | $(0.97, 1.22, 1.66)/1.25/2.03\times10^{-2}/1.50\times10^{-3}$ | $(0.95, 1.20, 1.58)/1.22/1.68\times10^{-2}/8.56\times10^{-4}$ |
| 4 | $(0.96, 1.00, 1.42)/1.06/1.08\times10^{-2}/1.40\times10^{-3}$ | $(0.99, 1.05, 1.32)/1.09/5.20\times10^{-3}/3.88\times10^{-4}$ |
| | $(1.01, 1.05, 1.34)/1.09/5.40\times10^{-3}/4.67\times10^{-4}$ | $(0.95, 1.07, 1.46)/1.12/1.19\times10^{-2}/1.20\times10^{-4}$ |
| | $(0.97, 1.03, 1.45)/1.09/1.14\times10^{-2}/1.40\times10^{-3}$ | $(0.93, 1.07, 1.53)/1.12/1.64\times10^{-2}/1.90\times10^{-3}$ |
| | $(0.93, 1.12, 1.57)/1.16/1.80\times10^{-2}/1.80\times10^{-3}$ | $(0.91, 1.17, 1.53)/1.19/1.62\times10^{-2}/6.36\times10^{-4}$ |
| 5 | $(0.98, 1.06, 1.47)/1.12/1.15\times10^{-2}/1.40\times10^{-3}$ | $(0.97, 1.04, 1.67)/1.13/2.48\times10^{-2}/4.70\times10^{-3}$ |
| | $(0.90, 1.02, 1.24)/1.04/5.00\times10^{-3}/1.92\times10^{-4}$ | $(0.88, 1.04, 1.48)/1.09/1.61\times10^{-2}/1.70\times10^{-3}$ |
| | $(0.97, 1.12, 1.54)/1.17/1.46\times10^{-2}/1.50\times10^{-3}$ | $(0.98, 1.12, 1.49)/1.16/1.16\times10^{-2}/1.00\times10^{-3}$ |
| | $(0.97, 1.02, 1.51)/1.09/1.48\times10^{-2}/2.50\times10^{-3}$ | $(0.97, 0.99, 1.47)/1.07/1.34\times10^{-2}/2.00\times10^{-3}$ |
| 6 | $(0.97, 1.04, 1.55)/1.11/1.67\times10^{-2}/2.50\times10^{-3}$ | $(0.94, 1.09, 1.44)/1.12/1.10\times10^{-2}/8.37\times10^{-6}$ |
| | $(0.96, 1.08, 1.23)/1.09/3.10\times10^{-3}/3.55\times10^{-5}$ | $(0.78, 0.98, 1.58)/1.05/2.89\times10^{-2}/4.30\times10^{-3}$ |
| | $(0.93, 1.11, 1.54)/1.15/1.64\times10^{-2}/1.60\times10^{-3}$ | $(0.87, 1.04, 1.47)/1.08/1.59\times10^{-2}/1.60\times10^{-3}$ |
| | $(1.04, 1.05, 1.42)/1.11/7.80\times10^{-3}/9.07\times10^{-4}$ | $(0.95, 1.02, 1.55)/1.10/1.79\times10^{-2}/2.80\times10^{-3}$ |

**Table 6** The optimal portfolios of Example 5

| Period | Asset | | | | | | | | Risk-free asset |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
| 1 | 0.0502 | 0.0479 | 0.1107 | 0.0976 | 0.1852 | 0.1898 | 0.1185 | 0.1888 | 0.0113 |
| 2 | 0.0667 | 0.1040 | 0.1089 | 0.1714 | 0.1537 | 0.1832 | 0.1184 | 0.0667 | 0.0270 |
| 3 | 0.0534 | 0.0531 | 0.0726 | 0.1313 | 0.1569 | 0.1584 | 0.1750 | 0.1448 | 0.0545 |
| 4 | 0.0935 | 0.0536 | 0.0652 | 0.0545 | 0.1850 | 0.1767 | 0.1774 | 0.1862 | 0.0079 |
| 5 | 0.1369 | 0.2022 | 0.0110 | 0.0406 | 0.2024 | 0.1286 | 0.1492 | 0.1124 | 0.0167 |
| 6 | 0.1483 | 0.1048 | 0.0089 | 0.1422 | 0.1578 | 0.1797 | 0.0924 | 0.1550 | 0.0109 |

**Table 7** The fuzzy returns of stocks in Example 6

| Stock code | Period | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| 000002 | (0.9454, 1.0021, 1.0851) | (0.9020, 1.0024, 1.1124) | (0.9000, 1.0002, 1.1442) |
| 000725 | (0.9094, 1.0019, 1.0579) | (0.9841, 1.0002, 1.1302) | (0.9091, 0.9999, 1.1179) |
| 600000 | (0.9320, 1.0022, 1.1012) | (0.9072, 1.0010, 1.0710) | (0.8933, 0.9995, 1.0525) |
| 600030 | (0.8961, 1.0044, 1.1734) | (0.9576, 0.9985, 1.1355) | (0.8709, 0.9994, 1.0704) |
| 600050 | (0.9271, 1.0018, 1.0838) | (0.8829, 1.0018, 1.1838) | (0.8973, 1.0012, 1.1262) |
| 600373 | (0.8274, 0.9995, 1.1315) | (0.8489, 1.0037, 1.2587) | (0.9261, 0.9997, 1.1017) |
| 600555 | (0.9376, 1.0037, 1.2127) | (0.9584, 1.0032, 1.1532) | (0.8398, 0.9989, 1.1159) |
| 601618 | (0.8919, 1.0048, 1.1798) | (0.8288, 1.0025, 1.1605) | (0.8555, 0.9994, 1.1114) |

trade suspension. By employing the granular computing method (Zhou et al. 2017), the fuzzy returns are obtained, see for instance, Table 7.

Assume the utility function is $u(x) = \sqrt{x}$. The optimal solution can be obtained easily by running NIGA. Due to the stochastic searching mechnism of GA, diverse solutions can be obtained when the parameters are different. In terms of this, the robustness is commonly applied to measure the relative stability and reliability of algorithm. To verify the robustness of NIGA, 6 sets of parameters are adopted and their corresponding relative errors are obtained by equation

$$\delta_i = (|\max\{f_1, f_2, \ldots, f_6\} - f_i|/\max\{f_1, f_2, \ldots, f_6\}) \times 100\%, i = 1, 2, \ldots, 6$$

where $f_i$ represents the expected utility value of the $i$th parameter set. In the meanwhile, for each set of parameters, we obtain the relative error and running time of NIGA and compare them with FSGA (See Table 8).

According to Table 8, the average error and running time for NIGA are 0.12% and 102.35s, respectively. In contrast, the average error and running time for FSGA are 0.52% and 35578.74s. It is obvious that NIGA has a smaller relative error compared with FSGA, which illustrates that NIGA performs better in terms of stability and

**Table 8** Comparison of solutions in Example 6

| No | $pop\_size$ | $p_c$ | $p_m$ | $N$ | $G$ | $f_i(NIGA/FSGA)$ | $\delta_i(\%)(NIGA/FSGA)$ | $Time(s)(NIGA/FSGA)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 60 | 0.9 | 0.1 | 2500 | 110 | 1.0217/1.0255 | 0.14/0.55 | 70.30/28023.66 |
| 2 | 90 | 0.8 | 0.2 | 2700 | 120 | 1.0225/1.0252 | 0.06/0.58 | 114.11/32580.11 |
| 3 | 80 | 0.8 | 0.2 | 3000 | 100 | 1.0208/1.0259 | 0.22/0.51 | 102.47/19375.40 |
| 4 | 70 | 0.7 | 0.2 | 3000 | 120 | 1.0209/1.0263 | 0.22/0.48 | 88.85/56211.25 |
| 5 | 100 | 0.7 | 0.1 | 3000 | 130 | 1.0221/1.0208 | 0.10/1.00 | 138.72/46043.05 |
| 6 | 80 | 0.8 | 0.1 | 3200 | 110 | 1.0231/1.0312 | 0.00/0.00 | 99.67/31238.97 |

**Table 9** The optimal portfolios of Example 6

| Period | Stock | | | | | | | | Risk-free asset |
|---|---|---|---|---|---|---|---|---|---|
| | 000002 | 000725 | 600000 | 600030 | 600050 | 600373 | 600555 | 601618 | |
| 1 | 0.0176 | 0.0123 | 0.0511 | 0.1766 | 0.1005 | 0.0172 | 0.2721 | 0.1250 | 0.2275 |
| 2 | 0.0248 | 0.1547 | 0.0510 | 0.0755 | 0.1081 | 0.1513 | 0.1559 | 0.0628 | 0.2160 |
| 3 | 0.1573 | 0.1315 | 0.0169 | 0.0832 | 0.1465 | 0.1310 | 0.0627 | 0.1030 | 0.1635 |

reliability. As for the running time, NIGA has better performance. The data reveals that the running speed of NIGA is 350 times faster than FSGA. This significant difference is due to NISA costs much less time in computing the expected utility values than fuzzy simulation (See Sect. 5.2). Since calculations of multiple individuals in different generations are involved in GA, the gap in running speed is amplified significantly. In real financial applications, timely decision making is extremely important for the investment opportunity fleets soon in fast-changing financial markets. In addition, more risky assets are considered in real investment, requiring a large computation capacity to obtain the optimal investment strategy in a short time. Therefore, NIGA performs better in dealing with real financial optimization issues for its advantages in accuracy, stability and running speed. This reflects the strong practicability of NISA. The optimal portfolio strategy is shown in Table 9.

## 7 Conclusions

Fuzzy arithmetic is inevitable in both modeling and solution processes for fuzzy optimization. Currently, some fuzzy arithmetics are restricted by the weak practicability and information loss in arithmetic process. In this paper, we study the product arithmetic of multiple positive $L$-$R$ fuzzy numbers and prove some important properties on membership functions. Based on the proposed arithmetic rules and properties, we design the NISA to approximate the expected values, variances and skewness of $L$-$R$ fuzzy numbers and proved its effectiveness from the perspective of accuracy and running speed. As an application, we consider a fuzzy multi-period utility maximization model. To solve the proposed model, we integrate the NISA and GA and analyze its advantages over the FSGA. The results promote the completion of fuzzy arithmetic, improve the practicability of fuzzy optimization technology, and assist the investors to achieve the optimal investment strategy timely in the fast-changing financial markets.

Frankly speaking, this study only studies the product arithmetic on positive $L$-$R$ fuzzy numbers. To widen the practicability, in the future, we can extend the study to generalized fuzzy numbers. This is challenging for the difficulty in determining the shape of the membership function of the product and proving its continuity and differentiability. The second extension is studying the division arithmetic on fuzzy numbers. Currently, there are few studies on the rules and properties of fuzzy division arithmetic, and some proposed division arithmetics are only theoretically invalid. For example, some fuzzy analytic hierarchy process related studies treat the reciprocal of triangular fuzzy number $(a, b, c)$ as $(1/c, 1/b, 1/a)$ (Ahmed and Kilic 2019), which actually contradicts with the Zadeh extension principle. Therefore, it is attractive to extend the studies to the above cases and apply them to more optimization and decision making problems.

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## Appendices

In this section, we present a proof on properties **(i)** and **(ii)**, give the computation procedure of fuzzy simulation, and introduce a granular computing method to derive fuzzy returns based on historical data.

### Appendix A

***Proof*** **(i)**: If $x_0 \in (a_1a_2, b_1b_2)$, according to the Zadeh extension principle, we have

$$\mu(x_0) = \sup_{(x_1,x_2)\in L_1\cup L_2\cup R_2} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\}. \tag{15}$$

For any $(x_1, x_2) \in L_2$ with $x_1x_2 = x_0$, take $x_1^* = x_0/b_2$ and $x_2^* = b_2$. It is obvious that $x_1 \leq x_1^* \leq b_1$ and $(x_1^*, x_2^*) \in L_1$. Since $\xi_1$ is a $L$-$R$ fuzzy number, $\mu_1(x)$ is increasing when $x \in (a_1, b_1)$ with $\mu_1(x_1^*) \geq \mu_1(x_1)$ and $\mu_2(x_2^*) \geq \mu_2(x_2)$, which implies that

$$\min\{\mu_1(x_1^*), \mu_2(x_2^*)\} \geq \min\{\mu_1(x_1), \mu_2(x_2)\}.$$

Taking all $(x_1, x_2) \in L_2$ into consideration, we have

$$\sup_{(x_1,x_2)\in L_1} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\} \geq \sup_{(x_1,x_2)\in L_2} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\}.$$

Similarly, we can prove that

$$\sup_{(x_1,x_2)\in L_1} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\} \geq \sup_{(x_1,x_2)\in R_2} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\}.$$

According to Eq. (15), we have

$$\mu(x_0) = \sup_{(x_1,x_2)\in L_1} \{\min\{\mu_1(x_1), \mu_2(x_2)\} | x_1x_2 = x_0\}.$$

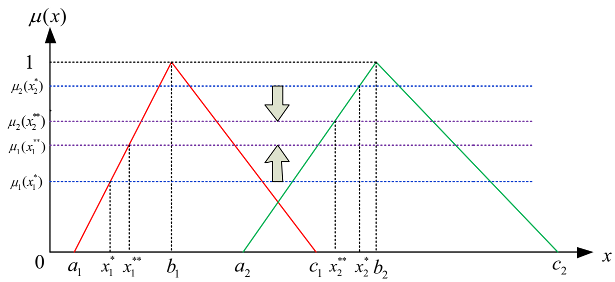


**Fig. 5** Iteration process for $\min(\mu_1(x_1), \mu_2(x_2))$ approaching $\mu(x_0)$

Now, we employ the reduction to absurdity to prove that there exist $x_1 \in [a_1, b_1]$ and $x_2 \in [a_2, b_2]$ such that $x_1x_2 = x_0$ and $\mu_1(x_1) = \mu_2(x_2) = \mu(x_0)$. Denote the $\gamma$-level sets of $\xi_i$ as $[a_{i1}(\gamma), a_{i2}(\gamma)]$, $i = 1, 2$. Suppose that $(x_1^*, x_2^*) \in L_1$ satisfying $x_1^*x_2^* = x_0$, and $\min\{\mu_1(x_1^*), \mu_2(x_2^*)\} = \mu(x_0)$. If $\mu_1(x_1^*) \neq \mu_2(x_2^*)$, without loss of generalization, assume $\mu_1(x_1^*) = \gamma_1$, $\mu_2(x_2^*) = \gamma_2$ and $\gamma_1 < \gamma_2$. Then take $x_1^{**} = (1+\varepsilon)x_1^*$ and $x_2^{**} = x_2^*/(1+\varepsilon)$, where $\varepsilon$ is a small enough positive number satisfying

$$0 < \varepsilon < \min\{-1 + a_{11}((\gamma_1 + \gamma_2)/2)/x_1^*, -1 + x_2^*/a_{21}((\gamma_1 + \gamma_2)/2)\}$$

which ensures $(x_1^{**}, x_2^{**}) \in L_1$ and $\mu_2(x_2^{**}) > (\gamma_1 + \gamma_2)/2 > \mu_1(x_1^*)$. Since $\mu_i(x)$ is increasing in interval $(a_i, b_i)$, $i = 1, 2$, then we have $\min\{\mu_1(x_1^{**}), \mu_2(x_2^{**})\} > \min\{\mu_1(x_1^*), \mu_2(x_2^*)\}$, which contradicts with $\min\{\mu_1(x_1^*), \mu_2(x_2^*)\} = \mu(x_0)$ for the reason that $\mu(x_0)$ is the supremum of $\min\{\mu_1(x_1), \mu_2(x_2)\}$ over $L_1$. This process illustrates that the value of $\min\{\mu_1(x_1), \mu_2(x_2)\}$ can be increased by getting $\mu_1(x_1)$ close to $\mu_2(x_2)$ with the above operations (See Fig. 5). If and only if $\mu_1(x_1) = \mu_2(x_2)$, $\min\{\mu_1(x_1), \mu_2(x_2)\}$ arrives at its supremum over $L_1$. Hence, for any $x_0 \in (a_1a_2, b_1b_2)$, $\mu(x_0) = \mu_1(x_1) = \mu_2(x_2)$.

**(ii)** If $x_0 \in (b_1b_2, c_1c_2)$, the conclusion can be proved in the similar way. The proof is complete. □

## Appendix B

This appendix gives a basic computation procedure of fuzzy simulation (Guo et al. 2016). Suppose that $\xi = (a, b, c)$ is a triangular fuzzy number with credibility function $\nu(x)$, where $\nu(x) = \mu(x)/2$. The steps for computing $E[\xi]$ is shown as follows. Firstly, randomly select $N$ points $y_1, y_2, \ldots, y_N$ in $[a, c]$ and calculate their credibilities $\nu_1, \nu_2, \ldots, \nu_N$. Then, set $e = 0$, $s = \min\{y_1, y_2, \ldots, y_N\}$ and $t = \max\{y_1, y_2, \ldots, y_N\}$. Secondly, randomly select a number $r$ from $[a, c]$. If $r > 0$, set $e \to e + \mathbf{Cr}\{\xi \geq r\}$. Otherwise, set $e \to e - \mathbf{Cr}\{\xi \leq r\}$. Here $\mathbf{Cr}\{\xi \geq r\}$ and $\mathbf{Cr}\{\xi \leq r\}$ are credibility measure given by

$$\mathbf{Cr}\{\xi \geq r\} = \begin{cases} \max\{\nu_k | y_k \geq r\}, & \text{if } \max\{\nu_k | y_k \geq r\} < 0.5 \\ 1 - \max\{\nu_k | y_k < r\}, & \text{if } \max\{\nu_k | y_k \geq r\} \geq 0.5, \end{cases}$$

$$\mathbf{Cr}\{\xi \le r\} = \begin{cases} \max\{\nu_k | y_k \le r\}, & \text{if} \quad \max\{\nu_k | y_k \le r\} < 0.5 \\ 1 - \max\{\nu_k | y_k > r\}, & \text{if} \quad \max\{\nu_k | y_k \le r\} \ge 0.5. \end{cases}$$

Thirdly, repeat the second operation $N$ times to update $e$ constantly and finally output $E[\xi] = \max\{s, 0\} + \min\{t, 0\} + e \cdot (t - s)/N$.

## Appendix C

This appendix introduces a granular computing method to derive fuzzy returns from historical data (Zhou et al. 2017). Denote $r_1, r_2, \ldots, r_N$ as the historical returns. We employ the following methods to generate the triangular fuzzy number $(a, b, c)$. Firstly, set $b = \sum_{i=1}^{N} r_i/N$, and calculate the membership degree of $r_i$ by $\mu(r_i) = \mathbf{1}_{(-\infty,b)}(r_i)\cdot(r_i - a)/(b - a) + (1 - \mathbf{1}_{(-\infty,b)}(r_i))\cdot(b - r_i)/(c - b)$, where $\mathbf{1}_{(-\infty,b)}(r_i) = 1$ if $r_i < b$, $\mathbf{1}_{(-\infty,b)}(r_i) = 0$ otherwise. Secondly, assume $\alpha$ is a given positive number, then determine $a$ by maximizing the value of $\sum_{a \le r_i < b} \mu(r_i) \cdot \exp(-\alpha|b - a|)$, where $\sum_{a \le r_i < b} \mu(r_i)$ is intended for covering most of the data points with $r_i < b$, while $\exp(-\alpha|b - a|)$ is applied to minimize the support length $|b - a|$. Finally, determine $c$ by maximizing $\sum_{b \le r_i \le c} \mu(r_i) \cdot \exp(-\alpha|c - b|)$, where $\sum_{b \le r_i \le c} \mu(r_i)$ is used to cover most of the data points with $r_i > b$ and $\exp(-\alpha|c - b|)$ is intended for minimizing $|c - b|$.

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