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Portfolio selection problems with Markowitz's mean–variance framework: a review of literature

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Abstract Since the pioneering work of Harry Markowitz, mean–variance portfolio selection model has been widely used in both theoretical and empirical studies, which maximizes the investment return under certain risk level or minimizes the investment risk under certain return level. In this paper, we review several variations or generalizations that substantially improve the performance of Markowitz's mean–variance model, including dynamic portfolio optimization, portfolio optimization with practical factors, robust portfolio optimization and fuzzy portfolio optimization. The review provides a useful reference to handle portfolio selection problems for both researchers and practitioners. Some summaries about the current studies and future research directions are presented at the end of this paper.

Keywords Portfolio selection · Mean–variance model · Dynamic optimization · Fuzzy optimization · Robust optimization

1 Introduction

Since the pioneering work of Markowitz (1952), mean–variance (MV) methodology has been the most popular way for solving the portfolio selection problem, which is lately expanded to a seminal book (Markowitz 1959). The main idea of MV model is to handle the returns of individual security as random variables and adopt the expected value and variance to quantify the investment return and investment risk, respectively. A rational investor generally minimizes the risk for a fixed expected return level or

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maximizes the expected return for a given risk level. In the case of maximizing the return for a given level of risk, the standard formulation is,

$$\begin{cases} \max \; E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \;\text{s.t.} \; Var[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \leq \beta \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \; i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where E denotes the expected value operator, Var denotes the variance operator, x_i is the proportion of wealth invested in security i, ξ_i represents the random return for the ith security, and β is the maximum risk level that the investor can tolerate. In the case of minimizing the risk for a given return level, the formulation is,

$$\begin{cases} \min \; Var[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \;\text{s.t.} \; E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \geq \alpha \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \; i = 1, 2, \ldots, n, \end{cases} \tag{2}$$

where α represents the minimum investment return that the investor can accept. Here $x_i \geq 0$ means that short-selling is not allowed. The optimal portfolio strategy and solution scheme of classical MV model with short-selling ($x_i \in \mathcal{R}$) can be found in Merton (1972).

The Markowitz's MV model has only provided a basic idea on the optimal portfolio selection. In the complex and unpredictable financial markets, there usually exist more specific requirements, such as (1) multi-period portfolio selection should be considered to tackle the constantly dynamic financial markets; (2) the practical constraints should be considered, including transaction costs, trading rules constraints, securities types constraints, market securities constraints and some others; (3) the optimal portfolios are often sensitive to the estimation errors of sample parameters' mean value and deviation; (4) the future return and risk of a security can not be forecasted with high performance in case of there is no enough sample data. In this case, consulting experts' suggestions and financial reports can enable investors to make sound decisions on portfolio selection. This paper mainly reviews four aspects about what have been devoted to extending the Markowitz's MV model to meet the specific needs in financial markets.

The remainder of this paper is organized as follows. Section 2 introduces Markowitz's MV model in dynamic setting. Section 3 reviews some common methods to combat with the problems in real financial markets. Section 4 reviews studies on robust portfolio optimization. Section 5 reviews publications regarding fuzzy portfolio optimization. Section 6 summarizes the whole paper and shows some new directions forward.

2 Dynamic optimization

The original Markowitz's MV model is restricted to a static model which means that investors can only make a decision at the beginning of investment and have to wait for results until the investment horizon ends. This seems to be awkward in real-life financial markets. Thus, the MV model is soon extended to multi-period ones. Generalizations of these work have followed two main streams. In the first stream, the discrete-time portfolio selection problem has been studied by Samuelson (1969), Hakansson (1971), Grauer and Hakansson (1993), and Pliska (1997). For example, Samuelson (1969) proposes a discrete-time multi-period consumption investment model to fulfill the objective that maximized the expected utility of investor's terminal wealth. Grauer and Hakansson (1993) compare the MV approximation and quadratic approximation for calculating the optimal portfolios in the discrete-time dynamic investment model. In the second stream, the continuous-time portfolio selection models have been investigated by Merton (1969, 1971), Karatzas et al. (1987) and Bajeux-Besnainou and Portait (1998). Specially, Merton (1969, 1971) introduces a continuous-time model to maximize the expected utility within a fixed planning horizon. Karatzas et al. (1987) consider a general consumption/portfolio decision problem for a single agent. The objective is to maximize a linear combination of the total expected discounted utility from consumption during the continuous investment horizon and the expected utility from terminal wealth. Bajeux-Besnainou and Portait (1998) analyse the portfolio strategies when continuous rebalancing is allowed between the current data and the horizon.

For discrete-time portfolio case, the capital is reallocated at distinct, separate "points in time" in the given investment horizon T. For example, for a 5-year portfolio selection model, the investor can adjust his investment strategy at the beginning of each year. In continuous-time portfolio case, the securities are traded continuously in interval $[0, T]$, which means that the investor can reallocate the capital in securities at any time. In addition, rather than treat the mean and variance of investment portfolio as two separate objectives, the dynamic portfolio optimization considers a single objective of the expected utility of terminal wealth. The Markowitz's MV formulation enables the investor to specify a return level that he/she would like to achieve when he/she is looking for minimizing the homologous risk, or specify a risk level that he/her can accept when he/she is looking for maximizing his/her expected return. Generally speaking, it is more difficult for investors to construct a utility function of the terminal wealth than to provide this kind of subjective information. Hence, the dynamic portfolio optimization methods have the disadvantage that the relationship between the risk and the return of the derived policy is not clear.

2.1 Embedding technique

To obtain the dynamic optimal portfolio selection, it is usual for scholars to consider the dynamic programming. However, while extending the Markowitz's MV work to multi-period portfolio selection with dynamic programming, a somewhat difficult task has been stoked. Scholars find that getting efficient frontiers of MV approach encounter difficulties on account of variance minimization problem in stochastic control. More precisely, for the expected value operator, dynamic programming is applicable because of the "smoothing property": $E[E(\cdot|\mathcal{F}_j)|\mathcal{F}_k] = E(\cdot|\mathcal{F}_k), \forall j > k$, where $\mathcal{F}_t$ denotes an information set available at time t and $\mathcal{F}_{t-1} \subset \mathcal{F}_t, \forall t$, while the variance operator does not satisfy the "smoothing property": $Var[Var(\cdot|\mathcal{F}_j)|\mathcal{F}_k] \neq Var(\cdot|\mathcal{F}_k), \forall j > k$, on the grounds that the variance operator is nonlinear with respect to the expected wealth. Due to the principle of optimality no longer working, the classical dynamic programming optimal stochastic control methods will be inapplicable in such non-separable situations. Later, a few of scholars find the embedding technique can be a useful tool to address this issue. The embedding technique applied in discrete-time and continuous-time settings are introduced as follows respectively.

2.1.1 Embedding technique in discrete-time portfolio model

Li and Ng (2000) firstly solve the nonseparability difficulty by using embedding technique to construct a tractable auxiliary problem with the MV framework in discrete-time case. They consider that a capital market consisted of n risky securities with random returns and a riskless security with deterministic return. An investor with an initial wealth u_0 joins the market at time 0 and allocates his/her wealth among the $n + 1$ securities. The investor can reallocate his/her wealth among the $n + 1$ securities at the beginning of the following $T - 1$ consecutive time periods. The returns of the risky securities at time period t within the planning horizon are denoted by a vector $\mathbf{e_t} = [e_t^1, \ldots, e_t^n]'$, where e_t^i is the random return for security i at period t. Let s_t be the given deterministic return of the riskless security at period t. Let u_i be the wealth of the investor at the beginning of the tth period, and x_t^i be the amount invested in the ith risky security at the beginning of the tth period, $i = 1, 2, \ldots, n$. The multi-period MV portfolio selection is formulated as follows,

$$\begin{cases} \max\ E[u_T] - \omega Var[u_T] \\ \text{s.t. } u_{t+1} = \sum_{i=1}^{n} e_t^i x_t^i + \left(u_t - \sum_{i=1}^{n} x_t^i\right) s_t = s_t u_t + \mathbf{P_t'x_t} \\ \quad t = 0, 1, 2, \ldots, T - 1, \end{cases} \tag{3}$$

where $\mathbf{P_t} = [P_t^1, P_t^2, \ldots, P_t^n]' = [(e_t^1 - s_t), (e_t^2 - s_t), \ldots, (e_t^n - s_t)]'$, $\mathbf{x_t} = [x_t^1, x_t^2, \ldots, x_t^n]'$, $E[u_T]$ denotes the expected value of the terminal wealth u_T, $Var[u_T]$ denotes the variance of the terminal wealth, and $\omega \in [0, \infty)$ represents the risk aversion factor. The following auxiliary problem is constructed,

$$\max\ E\left[\left[-u_T^2\right] + \lambda\left[u_T\right]\right] \tag{4}$$

subjects to the constraints in model (3), where $\lambda \in (-\infty, +\infty)$. The prominent feature of formulation (4) is that it is of a separable structure in the sense of dynamic programming and also it is of a quadratic form while the system dynamic is of a linear form.

Since then, the embedding technique has been widely used in solving the discrete-time portfolio selection problem. For example, Yi et al. (2008) consider the MV formulation of discrete-time portfolio optimization for asset-liability management with an uncertain investment horizon and derive an analytical optimal strategy by using the embedding technique. Wu and Li (2011) consider a discrete-time MV portfolio selection model with regime switching under the assumption that there exit a stochastic cash flow. By using dynamic programming and embedding technique, they derive the optimal strategy and efficient frontier in closed form. Wu et al. (2014) study a discrete-time Markowitz's MV portfolio selection problem with uncertain time-horizon in a regime-switching market. And the expressions for the optimal investment strategy and the efficient frontier are derived explicitly by the embedding technique.

2.1.2 Embedding technique in continuous-time portfolio model

Following a similar embedding technique introduced by Li and Ng (2000) for discrete-time model, Zhou and Li (2000) embed the continuous-time MV portfolio selection problem into a tractable auxiliary problem as well. They show that this auxiliary problem actually is a stochastic optimal linear-quadratic (LQ) problem and can be solved explicitly by LQ theory. In their paper, $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{0 \leq t \leq T})$ is a fixed filtered complete probability space on which a standard n-dimensional Brownian motion $W(t) \equiv [W^1(t), \ldots, W^n(t)]'$ with $W(0) = 0$ is defined. It is assumed that the $\mathcal{F}_t = \sigma\{W(g) : 0 \leq g \leq t\}$ is generated by the Brownian motion and augmented by all the P-null sets. They denoted by $L^2_{\mathcal{F}}(0, T; R^n)$ the set of all R^n-valued, measurable stochastic processes $f(t)$ adapted to $\{\mathcal{F}\}_{0 \leq t \leq T}$. The riskless security's price process $P_0(t)$ is subject to the following differential equations

$$\begin{cases} \mathrm{d}P_0(t) = s(t)P_0(t)\mathrm{d}t, \, t \in [0, T] \\ P_0(0) = p_0 > 0, \end{cases} \tag{5}$$

where $s(t) > 0$ is the interest rate.

The prices of these risky securities satisfy the following stochastic differential equations

$$\begin{cases} \mathrm{d}P_i(t) = P_i(t)\left\{b_i \mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}(t)\mathrm{d}W^j(t)\right\}, t \in [0, T] \\ P_i(0) = p_i > 0, \end{cases} \tag{6}$$

where $b_i(t)$ and $(\sigma_{ij}(t))_{n \times n}$ are the return and the volatility matrix of these securities, respectively. Zhou and Li (2000) also assume that the trading of securities take place continuously. Without consideration of consumptions or transaction costs, the MV portfolio optimization problem is formulated as follows,

$$\begin{cases} \min -E[u(T)] + \omega Var[u(T)] \\ \text{ s.t. } \mathbf{V}(t) \in L^2_{\mathcal{F}}(0, T; R^n) \\ \quad \mathrm{d}u(t) = \left\{s(t)u(t) + \sum_{i=1}^{n}[b_i(t) - s(t)]u_i(t)\right\}\mathrm{d}t + \sum_{j=1}^{n}\sum_{i=1}^{n} \sigma_{ij}(t)V_i(t)\mathrm{d}W^j(t) \\ \quad u(0) = u_0 > 0, \end{cases} \tag{7}$$

where $\mathbf{V}(t) = (V_1(t), \ldots, V_n(t))'$ represents a portfolio of the investor at time t, $V_i(t) \equiv N_i(t)P_i(t),\ i = 0, 1, 2, \ldots, n$, denotes the total market value of the investor's wealth in the ith bond/stock, $u(t)$ denotes the total wealth at time t, and $N_i(t)$ denotes the shares of the ith security at time t. Then Zhou and Li (2000) propose to embed model (7) into a tractable auxiliary problem, which is a stochastic LQ problem as follows,

$$\min\ E[\lambda_1 u(T)^2 - \lambda_2 u(T)] \tag{8}$$

subjects to the constraints in model (7), where $\lambda_1, \lambda_2 \in (-\infty, +\infty)$.

Since then, a growing number of papers tender to resolve the multi-period portfolio selection problem by using the stochastic optimal LQ control technique. For example, Chiu and Li (2006) employ stochastic optimal control theory to analytically solve the asset and liability management problem in a continuous-time setting. And they derive both the optimal policy and MV efficient frontier by a stochastic LQ control framework. Xie et al. (2008) derive the optimal dynamic strategy of a continuous-time MV portfolio selection problem in an incomplete market by applying the stochastic optimal LQ control technique. Making use of the same approach, Xu and Wu (2014) investigate a continuous-time MV portfolio selection problem with inflation in an incomplete market and obtain the dynamic optimal strategy and the MV efficient frontier.

2.2 Lagrange dual method

Besides the embedding technique, another common used method, Lagrange dual method (or call Lagrange multiplier method), has been used by many authors to study dynamic MV models under some real conditions. Li et al. (2002) propose that the LQ theory typically requires the portfolio to be unconstrained on the grounds that the optimal portfolio constructed through the Riccati equation may not satisfy the portfolio constraint. To study the constrained MV portfolio problem, Li et al. (2002) for the first time introduce the lagrange dual method to investigate a continuous-time problem with no-shorting constraint. In this paper, the MV portfolio selection problem is formulated as the following optimization problem parameterized by the value of the expected terminal wealth, $d \geq u_0 e^{\int_a^b s(r)\,dr}$,

$$\begin{cases} \min\ Var[u(T)] \equiv E[u(T) - d]^2 \\ \text{s.t. } E[u(T)] = d \\ \quad (\mathbf{V}(\cdot), u(\cdot)) \text{ admissable,} \end{cases} \tag{9}$$

Since Model (9) is a convex optimization problem, the equality constraint $E[u_T] = d$ can be dealt with by introducing a Lagrange multiplier $\lambda \in \mathcal{R}$. In this way the portfolio problem (9) can be solved via the following optimal stochastic control problem,

$$\min\ E\left\{[u(T) - d]^2 + 2\lambda\{E[u(T)] - d\}\right\} \tag{10}$$

subjects to the constraints in model (7).

Comparing with the embedding technique, the lagrange dual method is quite simple in procedure settings and calculation. Lim and Zhou (2002) use Lagrange dual method to transfer the continuous-time MV portfolio selection problem into an unconstrained stochastic LQ control problem. In this paper, the authors show that embedding technique can not be used and dynamic programming is difficult to apply when the coefficients are not deterministic. Thus, the authors use Lagrange dual method and the backward stochastic differential equation (BSDE) theory to derive the MV efficient frontier and optimal policy. Wei and Ye (2007) consider a multi-period MV stochastic markets model and take the risk control over bankruptcy into consideration. Using the Lagrange dual method, the authors obtain the optimal portfolio policies. Jin and Zhou (2007) study the MV portfolio selection in a continuous-time incomplete market, with a no-shorting constraint on portfolios. Based on Lagrange dual method and BSDE theory, the MV efficient portfolio and frontiers are derived explicitly. Ma et al. (2015) consider a continuous-time MV asset-liability management problem in a market with random market parameters. The authors employ the Lagrange dual method and the BSDE theory to tackle this problem and derive optimal investment strategy as well as the MV efficient frontier.

2.3 Mean-field method

More recently, Cui et al. (2014b) propose a novel mean-field framework that offer a more efficient modeling tool in addressing the issue of nonseparable stochastic control related to the multi-period MV portfolio selection problem. With this framework, they derive a more accurate solution for optimal strategy. In particular, they develop a unified framework of mean-field formulation to investigate three discrete-time MV models (Li and Ng 2000; Zhu et al. 2004; Costa and Nabholz 2007), and improve their optimal strategies. In their paper, the conventional multi-period MV portfolio selection model can be reformulated as the following mean-field type,

$$\begin{cases} \max \ E[u_t] - \omega E[(u_t - E[u_t])^2] \\ \ \text{s.t.} \ E[u_{t+1}] = s_t E[u_t] + E[\mathbf{P}_t']E[\mathbf{x}_t] \\ \qquad u_{t+1} - E[u_{t+1}] = s_t(u_t - E[u_t]) + \mathbf{P}_t'\mathbf{x}_t - E[\mathbf{P}_t']E[\mathbf{x}_t] \\ \qquad u_0 - E[u_0] = 0. \end{cases} \tag{11}$$

The objective function becomes separable in the expanded state space $(E[u_t], u_t - E[u_t])$, which enables us to apply dynamic programming. After that, Yi et al. (2014) extend the above mean-field formulation to discrete-time MV portfolio selection problem with an uncertain exit time. Cui et al. (2015) focus on the mean-field formulation to tackle the discrete-time asset-liability MV portfolio selection with an uncertain exit time. In Cui et al. (2015), it is noted that the mean-field formulation proposed by Cui et al. (2014b) and Yi et al. (2014) is constructed for single state variable and no longer applicable, hence, Cui et al. (2015) extend their mean-field theory and propose a two dimensional mean-field formulation to derive the analytical optimal strategies and efficient frontiers.

2.4 Time-inconsistent control

Another development for dynamic MV portfolio selection problem that has received much attention is the time-inconsistent control. Some scholars argue that the optimal portfolio strategy obtained by a Bellman optimality principle is just optimal at the initial time. For instance, the volatility of terminal wealth at time $t+\tau$ may be lower than that at time t, in addition to the investors may frequently adjusts their risk preferences, thus, the investors would like to revise their time-t optimal strategy at subsequent dates. But the investors must commit the time-t “optimal strategy” even if it is not optimal in the remaining investment periods as the financial market changes in classical Markowitz’s portfolio selection problem. Therefore, dynamic portfolio selection with Markowitz’s MV framework is time-inconsistent. Basak and Chabakauri (2010) call this time-inconsistent optimal strategy as pre-commitment strategy, they firstly propose the time-consistent policy for optimal MV problem in both discrete-time and continuous-time, and tackle this problem by obtaining attractable recursive formulation for the MV objective. It is assumed that the investor is guided by MV objectives over horizon wealth u_T. The objective function is

$$\max \ E\left[u_T\right] - \frac{\omega}{2} Var\left[u_T\right], \tag{12}$$

where ω is a given constant representing the risk aversion factor. The recursive representation for maximizing the time-t objective function of the MV model is as follows,

$$\max \ U_t = E_t\left[U_{t+\tau}\right] - \frac{\omega}{2} Var_t\left[E_{t+\tau}\left[u_T\right]\right]. \tag{13}$$

This representation reveal that U_t, the decision-making at time t, involve maximizing the future expected objective function with an adjustment that quantifies the investor’s incentives to deviate from the time-t optimal policy. They also note that the same outcome can alternatively be derived as the Nash equilibrium solution of an interpersonal game model. Another feasible interpretation of the time inconsistent control is that the investors’ appetites change timely in a temporally inconsistent way. Therefore, the MV problem can be viewed as a interpersonal game, where the rules are depended on the game players’ expectation of their appetites in the future.

To our knowledge, Björk and Murgoci (2010) is the first paper that treat the game theoretic approach to time inconsistent control in more general terms both in discrete-time and continuous-time. Within the game theoretic framework, system alternatives of the subgame perfect Nash equilibrium control are derived. Wang and Forsyth (2011) study the time-consistent strategy and the pre-commitment strategy of a continuous-time MV asset allocation problem, and develop a numerical scheme to determine the strategy where any type of constraint can be applied to the investment behavior. Cui et al. (2012) claim that the dynamic MV model is time-inconsistent in efficiency in discrete-time case, but time-consistent in continuous-time case. Thus, they develop a revised MV strategy which can keep the efficiency of the portfolio strategy for all time periods. Czichowsky (2013) considers a time-consistent MV portfolio selection problem under a general semimartingale setting. Wei et al. (2013) study the time-consistent outcome of the continuous-time MV asset-liability management under Markov regime-switching setting.

Lately, within a reasonably general Markovian framework, Björk and Murgoci (2014) further present a rigorous study of the time inconsistent control in discrete-time. Bensoussan et al. (2014) study a time-consistent MV portfolio policy with short-selling prohibition in both discrete-time and continuous-time settings within the framework developed in Björk and Murgoci (2010, 2014). Li et al. (2015c) use the time-consistent investment strategy to resolve an MV portfolio selection problem under partial information within a game theoretic framework. Chiu and Wong (2015) investigate the optimal trading of cointegrated assets using the classical MV portfolio selection criterion in a continuous-time economy, and derive the time-consistent optimal trading strategy in a closed-form solution. Under the assumption that the risk aversion is allowed to depend dynamically on current wealth, which is assumed as a constant in Basak and Chabakauri (2010). Wu (2013) and Björk et al. (2014) study the time-consistent Nash equilibrium strategies for discrete-period and continuous-time MV portfolio selection problem, respectively. Wu and Chen (2015) investigate the time-consistent multi-period MV portfolio selection and assume that the risk aversion depended dynamically on the market states. In this paper, the subgame perfect Nash equilibrium strategy and closed-form equilibrium value function are derived within a game theoretic framework. For more literature about dynamic MV portfolio selection, see Tables 1, 2.

Table 1 Discrete-time MV portfolio selection with different types of constraints

	Constraints type		
	Trading rules	Market scenarios	Securities types
Zhu et al. (2004)	Bankruptcy control		
Yin and Zhou (2004)		Markov switching market	
Leippold et al. (2004)			Assets and liabilities
Costa and Araujo (2008)	Bankruptcy control	Markov switching market	
Yi et al. (2014)			Uncertain exit T assets and liabilities
Wu and Li (2011)		Markov switching market	Uncertain exit T
Wu and Li (2012)		Markov switching market stochastic cash flow	
Leippold et al. (2011)			Endogenous liabilities
Chen and Yang (2011)		Markov switching market	Assets and liabilities
Li and Li (2012)	Bankruptcy control		Assets and liabilities
Yao et al. (2013c)		Uncontrolled cash flow	Assets and liabilities uncertain exit T
Yao et al. (2013a)		Markov switching market	Uncertain exit T endogenous liabilities
Wu et al. (2014)		Markov switching market	State-dependent exit T

Table 2 Continuous-time MV portfolio selection with different types of constraints

	Constraints type		
	Trading rules	Market scenarios	Securities types
Lim and Zhou (2002)		Complete market	
Li et al. (2002)	No short-selling		
Zhou and Yin (2003)		Markov switching market	
Bielecki et al. (2005)	Bankruptcy control	Complete market	Uncertain exit T
Xia and Yan (2006)	Bankruptcy control	Incomplete market	
Chiu and Li (2006)			Assets and liabilities
Xiong and Zhou (2007)		Incomplete market	
Jin and Zhou (2007)	No short-selling	Incomplete market	Finite T horizon
Xie et al. (2008)		Incomplete market	Assets and liabilities
Chen et al. (2008)		Markov switching market	Assets and liabilities
Fu et al. (2010)		Higher borrowing rate	
Li and Xie (2010)		Incomplete market	Uncertain exit T
Chiu and Wong (2011)			Cointegrated assets
Zeng and Li (2011)		Jump diffusion market	Assets and liabilities
Chiu and Wong (2013)		Incomplete market	Cointegrated assets insurance liability
Yao et al. (2013b)			Endogenous liabilities
Chiu and Wong (2014)		Correlation risk	
Xu and Wu (2014)		Inflation	
Shen (2015)		Complete market	
Guo and Duan (2015)		Jump diffusion market	Finite T horizon
Ma et al. (2015)		Complete market	Exogenous liability
Chang (2015)		Stochastic interest rate	Assets and liabilities

2.5 Summary

This section reviews the application of dynamic optimization in portfolio selection, including discrete-time and continuous-time cases. In a dynamic setting, an investor may revise his portfolio periodically, which is more suitable for real investment situation compared with single-period case. Due to the difficulties in getting efficient frontiers, two stochastic optimal linear-quadratic methods, Lagrange dual method and time-consistent model are introduced explicitly, which has a far-reaching influence in dynamic portfolio selection problem.

It is found that for the majority of continuous-time MV portfolio selection problems, the risky securities' prices and the liability are all subject to Brownian motions, Brownian motions with drift and Lévy processes (Chiu and Li 2006; Zeng and Li 2011; Xu and Wu 2014; Guo and Duan 2015). Secondly, all security returns invested in each period can be obtained at the same exit time. However, in real-life financial markets, there are various financial derivatives with different maturity dates. For example, the return of a 2-year fund can only be obtained after 2 years. In other words, at the end of the first year, no money can be obtained. Therefore, a multi-period portfolio selection of n risky securities with different maturity dates will be attractive and should be addressed in the future. Thirdly, due to the uncertainty of financial markets, a favorable decision making method for dynamic portfolio selection should be able to update all input parameters and to react immediately to any variation from the future. In dynamic portfolio optimization, the rational investors' decisions are based upon the forecasting information for a certain periods. To our knowledge, the rolling horizon decision making proposed in Sethi and Sorger (1991) is a good approach to cope with the presence of uncertainty. Using the rolling horizon to solve the dynamic portfolio selection problems will be a challenging future research direction.

3 Practical factors

In the real world, it is noticed that the solution to the original portfolio selection problem in Models (1) and (2) is often not practical to implement. For instance, the MV optimal portfolio obtained from Models (1) and (2) may turn itself over many times, which would incur high transaction costs. Thus, despite the Markowitz's MV model has elegant theory for portfolio selection problem, ignorance of the realistic factors hinders its efficient extensions for real-life applications.

3.1 Transaction costs

In dynamic case, rational investors won't trade continuously at every instant of time with the consideration of transaction costs. In the original MV model, the transaction costs associated with buying/selling assets are ignored for the purpose of simplification. However, the lack of any transaction costs will indulge a quite unrealistic type of portfolio behavior. Pogue (1970) gives one of the first description for the MV portfolio selection problem in the presence of transaction costs. Davis and Norman (1990) further discuss the portfolio selection problem that include proportional transaction costs. Dumas and Luciano (1991), Morton and Pliska (1995) study portfolio optimization problem with proportional transaction costs and fixed transaction costs, respectively. Yoshimoto (1996) firstly assumes transaction costs to be a V-shaped function and obtain the optimal portfolio strategy.

Liu and Loewenstein (2002) study the portfolio selection problem comprised transaction costs and finite horizons where the investors wish to maximize the utility of wealth. Oksendal and Sulem (2002) consider optimal consumption and portfolio with both fixed and proportional transaction costs with the objective of maximizing the cumulative expected utility of consumption over a planning horizon. Xue et al. (2006) construct a MV portfolio selection model under concave transaction costs. Taking transaction costs into account, Lobo et al. (2007) solve the problems of single-period portfolio optimization including different types of constraints on the feasible portfolios. After that, Dai and Zhong (2008) propose a penalty method to numerically solve the continuous-time portfolio selection problem that comprise proportional transaction costs. Peng et al. (2011) present a new optimal portfolio methodology within the

Markowitz's MV framework with a quadratic form in the transaction costs. Wang and Liu (2013) investigate the multi-period MV portfolio selection problem with fixed and proportional transaction costs and define the indirect utility function for solving the problem by using dynamic programming and Lagrange multiplier.

3.2 MV portfolio selection under various constraints

Institutional policy and investors' viewpoints often lead to more complicated situations than the original formulation of the MV problem. Consequently, practical constraints are incorporated into the Markowitz's MV model to make the solution more practical. The relevant literature are listed in Tables 1, 2. In practice, the following constraints are commonly imposed.

3.2.1 Trading rules constraints

In practice, a multitude of investment policies or laws would restrict investment trading behaviors. These rules are set to regulate investors' behaviors and control risks. For example, an investor, who has bankrupted, is not allowed to keep on trading or borrowing money. Hence, putting a constraint to supervise the investor's bankrupt probability before reaching the expiration of an investment horizon is necessary. The basic idea of bankruptcy prohibition is to control the probability of investor's wealth falling below an assigned level. Zhu et al. (2004) consider an integration of discrete-time and bankruptcy control portfolio selection with a generalized MV formulation to help investors not only accomplish their expected return with an MV tradeoff, but also control the bankruptcy risk very well. Next year, Bielecki et al. (2005) study the same problem in continuous-time case. In addition, in many practical applications, the investment proportion on each asset has to take nonnegative value, which is the short-selling constraint. Jin and Zhou (2007) study the MV portfolio selection in a continuous-time incomplete market with a no short-selling constraint on portfolios. Cui et al. (2014a) consider the MV formulation in discrete-time portfolio selection under no short-selling constraint.

3.2.2 Securities constraints

There are two main restrictions in securities constraints. One is the horizon of investment time, and the other is the type of securities. Firstly, the investment time horizon contains finite time horizon (Li et al. 2006; Jin and Zhou 2007; Guo and Duan 2015) and uncertain exit time (Bielecki et al. 2005; Li and Xie 2010; Wu and Li 2011). The investment horizon may have intertemporal restrictions (Costa and Nabholz 2007). Secondly, securities in financial markets have several different types, such as risky securities, riskless securities and liabilities, etc. Riskless securities and liabilities are not considered in the Markowitz's model. However, in the real world, both of the riskless securities and liabilities are important factors which almost all investors should cope with. Hence, it will be more practical for portfolio selection model that includes the riskless securities and liabilities. In some papers, authors assume that the capital market with n risk securities and one riskless security (Zhou and Li 2000; Zhou and Yin 2003; Xu and Wu 2014). In a way, the portfolio selection with liabilities is perceived as a special type of asset-liability management (ALM). For example, Leippold et al. (2004) derive explicit expressions for the optimal portfolio policy and the efficient frontier of a discrete-time MV ALM problem by utilizing a geometric approach and stochastic LQ method. Yao et al. (2013b) investigate a continuous-time MV ALM problem in a more general market where all the securities can be risky. Furthermore, some investors would take industry limitation into consideration when making decisions. For example, a slice of investors are willing to buy stocks from the non-ferrous metal plate, and some of them are unwilling to invest in energy stocks.

3.2.3 Market scenarios constraints

It is well known that financial markets are continually changing, and the market parameters are not immutable. The market parameters include the riskless securities' interest rates, the appreciation and volatility rates of the securities. Therefore, some investors tend to characterize the market by complete, incomplete, Markovian regime-switching, and jump diffusion. Lim and Zhou (2002) consider a continuous-time MV portfolio selection problem in a complete market where interest rates, appreciation rates and volatility rates are random parameters. Yin and Zhou (2004) study a discrete-time Markowitz's MV portfolio selection model in which the market parameters are governed by Markov random regime-switching and obtain the efficient frontier of this problem. Xiong and Zhou (2007) study a continuous-time portfolio selection problem under the Markowitz's MV framework in an incomplete market. In this incomplete market, only the past prices of the stocks and the bond are available to the investors. Guo and Duan (2015) further study the continuous-time MV portfolio selection problem with finite time horizon. In their paper, the stocks' prices are assumed to satisfy stochastic differential equations with Poisson jumps, and the interest rate is also assumed to be a stochastic process.

3.2.4 Non-convex constraints

Besides all linear constraints mentioned above, there are some non-convex constraints. For example, cardinality constraints, bounding constraints, transaction lots and so on. Shaw et al. (2008) study a portfolio selection problem subject to a cardinality constraint to ensure the investment in a given number of different assets and the problem is formulated as a cardinality-constrained quadratic programming. The authors develop a dedicated Lagrangian relaxation method to solve it. Fernández and Gómez (2007) consider a generalization of the standard Markowitz MV model which includes cardinality and bounding constraints. These constraints ensure the investment in a given number of different assets and limit the amount of capital to be invested in each asset. Soleimani et al. (2009) propose a portfolio selection model based upon Markowitz's MV framework, which cover cardinality constraints, minimum transaction lots and market (sector) capitalization. Motivated by the need of developing best market timing strategy, Gao et al. (2015) consider the time cardinality constrained MV dynamic portfolio selection problem with management fees.

3.3 The heuristic approach

The underlying assumption of multivariate normality in Markowitz MV portfolio selection problem is not sustainable. The distribution of individual security returns tends to exhibit heavy tail, high peak and possible skewness according to the empirical evidence. Therefore, some studies assume that the security returns follow non-Gaussian processes. Bodnar and Schmid (2008) consider matrix elliptically contoured distributions of security returns. Goldfarb and Iyengar (2003), Lu (2011), Xiao and Valdez (2015) assume that the returns follow multivariate elliptical distributions. Liu et al. (2015) regard the distributions of security returns as interval random uncertainty sets and develop an improved particle swarm optimization algorithm to solve the proposed multi-period portfolio model. Hao and Liu (2009) assume that the membership function of security returns follow triangular fuzzy distribution. Deng and Li (2012) regard the security returns as fuzzy variables with trapezoidal membership functions. Exact solution methods may fail to solve this problem in reasonable time. In addition, for practical purposes, it may be desirable to limit the number of securities in a portfolio like cardinality constraints, which turns the problem into nonlinear mixed integer programming. Such constraints better capture the real-world trading system, but make the problem more difficult to be solved with exact methods (Chang et al. 2000). Therefore, given the computational difficulty of tackling the problem exactly, using heuristics in these cases is imperative. Heuristics do not guarantee to find the optimal solution; however, they are able to find a good solution within reasonable computation time.

Chang et al. (2000) illustrate the discontinuous nature of the MV efficient frontier in the presence of cardinality constraints and present three metaheuristic algorithms based upon a genetic algorithm, tabu search and simulated annealing for finding the cardinality constrained MV efficient frontier. Following the work of Chang et al. (2000), papers relating to heuristic approaches can be subdivided into two classes, single-objective optimization algorithm and multi-objective optimization algorithm. For Single-objective optimization algorithm, there are genetic algorithm (Yang 2006; Woodside-Oriakhi et al. 2011), simulated annealing (Crama and Schyns 2003; Ehrgott et al. 2004), tabu search (Woodside-Oriakhi et al. 2011), particle swarm (Cura 2009; Zhu et al. 2011), neural networks (Fullér and Majlender 2007), artificial bee colony (Wang et al. 2012; Tuba and Bacanin 2014), memetic algorithm (Ruiz-Torrubiano and Suárez 2015). Multi-objective optimization algorithm consist of non-dominated sorting genetic algorithm II (NSGA-II), the strength pareto evolutionary algorithm 2 (SPEA2), the e-multiobjective evolutionary algorithm (e-MOEA) and the pareto envelope-based selection algorithm (PESA) (Chiam et al. 2008; Anagnostopoulos and Mamanis 2010; Metaxiotis and Liagkouras 2012; Liagkouras and Metaxiotis 2015).

3.4 Summary

Studies considering practical factors in real-life financial markets are reviewed in this section, which are concluded into two types. One is the portfolio selection with transactions costs appearing in the objective function, and the other is MV portfolio selection under various constraints including trading rules, market scenarios, securities types and others. It is found that there are more studies considering uncertain investment exit time than finite time horizon. That is due to many factors can affect the exit time, for example, the price movements of risky assets, securities markets behavior, exogenous huge consumption such as purchasing a house or accident, which may force the investors to abandon their original investment decisions (Wu and Li 2011). Secondly, there are many real-world constraints that lead to a non-convex search space, e.g., cardinality constraints, minimum buy-in thresholds. As a consequence, the exact approaches can no longer be applied, and heuristic solutions are needed. As a result of literature survey, it can be concluded that portfolio selection problem with non-convex constraints is much researched using nature inspired metaheuristics, a physical process like simulated annealing or biologically inspired algorithms like artificial bee colony or evolutionary algorithms and so on. In addition, these constraints that imposed on portfolio selection model are more and more complex but practical to cope with the real-world challenges in financial markets (Soleimani et al. 2009; Gao et al. 2015).

4 Robust techniques

It is generally established that the distributions of security returns are not known. To implement Markowitz's portfolio model in practice, one has to estimate the mean values and variances by past sample data. However, Michaud (1989) argues that MV optimal portfolios are often "error maximization". Best and Grauer (1991) find that MV portfolio weights are highly sensitive to changes in the mean value. One study claims that MV optimization can produce extreme or non-intuitive weights for some of securities in the portfolio selection. Chopra (1993) claims that small changes to the estimates of mean values or variances can result in immensely different solutions of MV optimal portfolios. All these issues are caused by estimation errors. Typically, the estimates of those parameters are used as if they were the "true" parameter values ignoring the estimation errors. With the purpose of reducing the effect of estimation errors in the estimates of mean values and variances, more stable MV portfolio selection models are formulated by applying robust techniques. Different robust optimization techniques are introduced as follows.

4.1 Bayesian approach

Under the Bayesian approach, the estimates of mean and variance rely on the predictive distributions of security returns. The use of predictive distribution of security returns and the Bayesian optimal portfolio strategy was obtained by maximizing the expected utility. The Bayesian approach on estimation errors has been implemented in previous studies in different ways. We shall describe two common implementations in the following sections.

4.1.1 Bayesian prior portfolio

The first considered prior is diffuse prior. In portfolio choice problems, Barry (1974), Brown (1976), and Klein and Bawa (1976) are primitive Bayesian studies under parameter uncertainty that rely on diffuse priors. The Bayesian models based on the diffuse prior are commonly applied in addition to the classical methods of portfolio selection. The second is conjugate prior. Compared with the diffusion prior, the conjugate prior is an informative prior which considers a normal prior for mean and an inverse Wishart prior for variance. Frost and Savarino (1986) propose an interesting application of the conjugate prior where all securities possess identical expected returns, variances and pairwise correlation coefficients. Next, by using of stochastic approximation, Greyserman et al. (2006) consider a portfolio selection methodology using a Bayesian predictive distribution of security returns. They derive the hierarchical priors on the mean vector and covariance matrix of security returns. In order to connect the parameter estimation with economic objectives, Tu and Zhou (2010) explore a general form for priors in the presence of financial objectives. In the light of an out-of-sample loss function measure, they demonstrate that the Bayesian policy under the objective-based priors work better than the strategies under other priors in portfolio selection decisions. To allow Bayesian priors to reflect the objectives of an economic problem, Tu and Zhou (2010) propose the application of the objective-based prior to the portfolio weights of the general MV portfolio and report good results.

4.1.2 Bayes–Stein shrinkage portfolio

The Bayes–Stein portfolio is an application of the shrinkage estimation, which is pioneered by Stein (1956) and developed by James and Stein (1961). These estimators shrink the sample mean value to a common “grand mean”, which is frequently chosen to be the average expected return relying on the security volatility and the distance of its expected return from the average value. Jorion (1986) provides an interesting Bayes–Stein shrinkage estimator, and shows that the resulting portfolio rule can frequently generate high expected out-of-sample performance. In portfolio selection, one study provides a pioneering work that allow incorporate investors’ views to derive a posterior distribution of security returns. Black–Litterman model is a market-based shrinkage approach where a weighted average of the market equilibrium and the investor’s views are calculated for the estimates of expected returns. Meucci (2010) extends the Black–Litterman model to a more generic form of the market distribution. Xiao and Valdez (2015) further extend Meucci (2010)’s market-based version of the Black–Litterman model to the case where the returns distribution appertain to the kind of elliptical distribution. For the errors in estimating covariance matrixes, Ledoit and Wolf (2003, 2004) adopt Bayesian shrinkage technique to estimate covariance matrixes, and prove that this method outperformed the sample covariance matrixes as well as the global minimun variance (GMV) portfolio. Yang et al. (2014) propose a hybrid covariance matrix estimator on the base of robust M-estimation and Ledoit and Wolf (2004)’s shrinkage approach.

4.2 Global minimum variance portfolio

The global minimun variance (GMV) portfolio is a specific optimal portfolio which possesses the smallest variance among all portfolios on the efficient frontier. This portfolio corresponds to the fully-risk averse investor who aims to minimize the variance without taking the expected return into consideration. The importance of the GMV portfolio in financial applications is well motivated by Chopra and Ziemba (1993) who point out that errors in expected values are approximately ten times as significant as errors in variances and covariances. Later, a host of empirical works, such as Chan et al. (1999), Jagannathan and Ma (2002) and DeMiguel and Nogales (2009), focus on the GMV portfolio and emphasize the GMV portfolio outperformed the Markowitz MV portfolios. For example, Jagannathan and Ma (2002) claim that the GMV portfolio weights should be more stable than the standard MV portfolio weights since the estimation errors of the covariances are smaller than that of the means. Bodnar and Schmid (2008) consider the weights of the GMV portfolio with an assumption that the returns follow a matrix elliptically contoured distribution. For example, the security returns are assumed to be neither normally distributed nor independent. They also found that the stochastic properties of the GMV portfolio did not rely on the mean vector or on the distributional assumptions imposed on security returns. Candelon et al. (2012) provide that the method of adopting Bayesian shrinkage technique to estimate covariance matrixes, is not optimal for small samples (Ledoit and Wolf 2003, 2004), and introduce a new framework to solve this problem of the GMV portfolio based on shrinkage estimators of the covariance matrix. Bodnar et al. (2015) analyse the GMV portfolio within a Bayesian framework, which incorporate prior beliefs of the investors into portfolio selection decisions.

4.3 Robust optimization

Although advances have been made to enhance estimates of returns, there are still mistakes in these estimates on the grounds of the immanent stochastic nature of the security returns process. This has induced some scholars to seek alternative way to account for estimation errors directly in the decision making process for portfolio selection.

Unlike the Bayesian and GMV portfolios, robust portfolio considers the estimation errors in the decision making process directly. Generally, robust optimization refers to find good objective values to given optimization problems with the uncertain input parameters. Most robust portfolio optimization studies are stemmed from the work of Ben-Tal and Nemirovski (1998, 1999) where the unknown parameters are modeled as ellipsoidal sets instead of point estimates used in standard MV optimization. By using the robust optimization framework described in Ben-Tal and Nemirovski (1998, 1999), Goldfarb and Iyengar (2003) develop a robust factor model to resolve robust portfolio selection problem. In their paper, the uncertain market parameters are modeled as ellipsoidal uncertainty sets. Tütüncü and Koenig (2004) formulate a robust optimization problem where uncertainty is described by use of an uncertainty set which include most possible realizations of the uncertain input parameters.

Ceria and Stubbs (2006) point out the aforementioned approaches can be rather too conservative. They introduce a robust portfolio model with an alternative uncertainty region over the expected returns that create a less conservative robust problem. Garlappi et al. (2007) develop a robust model for a decision-maker, who is averse to ambiguity, with multiple priors. In Garlappi et al. (2007), the multiple priors are characterized via a "confidence interval" around the estimated expected returns and ambiguity aversion is modeled by a minimization over the priors. Differing from Goldfarb and Iyengar (2003) and Tütüncü and Koenig (2004), the model in Garlappi et al. (2007) incorporate not only parameter uncertainty but also model uncertainty. In addition, joint constraints on expected returns are considered instead of only individual constraints. Alternatively, Lu (2011) considers the same factor model for security returns as studied in Goldfarb and Iyengar (2003). For the model parameters, Lu (2011) proposes a "joint" ellipsoidal uncertainty set instead of Goldfarb and Iyengar (2003)'s "separable" uncertainty set and states that it can be structured as a confidence region related to a statistical procedure. The "joint" ellipsoidal uncertainty set indicate that each type of uncertain parameter has its own uncertainty set. Ye et al. (2012) point out that these robust models with "separable" uncertainty sets have different probability measures over the mean and the variance of returns, which lead to a worse than worst-case situation and may be over-conservative. Accordingly, they construct a robust MV model by introducing uncertainty regions over the mean vector and the second moment matrix of returns to mitigate those defects. For a decision-maker aiming to optimize a portfolio by using the global minimum-variance strategy, Maillet et al. (2015) provide a robust approach to mitigate the impact of parameter uncertainty.

For the multi-period robust portfolio optimization, Ben-Tal et al. (2000) firstly suggest using robust optimization to cope with the portfolio optimization problem in multi-period case. Based on the approach of Ben-Tal et al. (2000), Bertsimas and Pachamanova (2008) define the problem of multi-period portfolio selection with transaction costs and present different robust formulations for the multi-period robust portfolio optimization problem. Liu et al. (2015) propose a robust multi-period model for portfolio optimization that consider investors' behavioral factors by introducing dynamically updated loss aversion parameters as well as a dynamic value function based on prospect theory. For a thorough review on robust portfolio optimization, the readers can refer to Fabozzi et al. (2007, 2010); Kim et al. (2014).

4.4 Other methods reducing estimation errors

A number of other approaches have been proposed in the literature to reduce estimation errors. For instance, Chopra (1993), Jagannathan and Ma (2002) impose no short-selling constraint to reduce estimation errors in optimizing MV portfolio selection strategies. Michaud and Michaud (1998) introduces a statistical resampling technique that considers estimation errors indirectly via averaging the each optimal portfolio. Kan and Zhou (2007) provide a three-fund optimal portfolio that consist of the riskless asset, the sample tangency portfolio, and the sample global minimum-variance portfolio. They advocate that incorporating the economic objective function into parameter estimation is a good method for analyzing the MV optimal portfolio selection problem.

4.5 Summary

This section reviews the portfolio selection research from the perspective of reducing influence of the estimated errors on parameters. The Bayesian approach, global minimum variance portfolio and robust optimization methods are introduced respectively. For these studies, the security returns are mainly assumed to obey normal distribution. In fact, some empirical studies have demonstrated the disadvantages of normality in real-life applications for the reason that the distribution of security returns may have heavier tails and occasionally high peaks. One alternative distribution is multivariate elliptical distributions which can be well applied in financial markets (Fabozzi et al. 2007; Yang et al. 2014; Xiao and Valdez 2015). In addition, when estimating the covariance matrix of returns, the risky securities are assumed to be subject to the identical type of distribution (Greyserman et al. 2006; Lu 2011; Maillet et al. 2015). The future work may focus on different return distributions for individual security in formulating the MV model.

5 Fuzzy portfolio selection

Classical Markowitz's portfolio selection model has two basic assumptions: (1) all security returns are random variables; (2) investors have sufficient historical data on securities which can correctly reflect the situation of financial markets in future. However, it is invariably hard to ensure such two assumptions. Indeed, in most cases, the prediction of returns depends on experts' judgments and investors' subjective opinions. Moreover, there is no past performance information for those newly listed stocks. Numerous researchers have been devoted to finding a nonprobabilistic approach to model experts' judgements and investors' subjective opinions.

With the introduction of fuzzy set theory in Zadeh (1965) and the development in Bellman and Zadeh (1970), scholars have tried to employ fuzzy variables to manage portfolio selection problem. Various theories accounting for new facets of uncertainty have been proposed to cope with the fuzzy portfolio selection problem, such as possibility theory, credibility theory, random fuzzy theory, fuzzy random theory and others.

5.1 Possibilistic MV portfolio selection

Possibility theory is proposed by Zadeh (1978) and advanced by Dubois and Prade (1988). The family of fuzzy variables is denoted by A. Let ξ be a fuzzy variable with membership function μ, and r a real number. The possibility of fuzzy event $\xi \in A$ is defined by Zadeh (1978) as follows

$$\mathrm{Pos}\{\xi \in A\} = \sup_{x \in A} \mu(x).$$

Possibilistic portfolio selection model is initially proposed by Tanaka (1995), where fuzzy variables are associated with exponential possibility distributions. Further, Tanaka and Guo (1999) propose upper and lower possibility distributions, which can be used in portfolio selection problem to reflect experts' knowledge. Instead of conventional probability distributions in Markowitz's model, Tanaka et al. (2000) propose two kinds of portfolio selection models that based on possibility probabilities and fuzzy distributions, respectively. Afterwards, Carlsson and Fullér (2001) introduce the notions of lower and upper possibilistic mean and variance of fuzzy variables. Also they introduce the crisp possibilistic mean and variance of a continuous possibility distributions. For a fuzzy variable $\xi \in A$ with $\gamma-$level set $\left[\xi\right]^{\gamma}=[a_1(\gamma), a_2(\gamma)]$ $(0 < \gamma \leq 1)$, the possibilistic expected value and variance of ξ are defined as

$$E[\xi] = \int_0^1 \gamma\,(a_1(\gamma) + a_2(\gamma))\,\mathrm{d}\gamma, \quad Var[\xi] = \frac{1}{2}\int_0^1 \gamma\,(a_2(\gamma) - a_1(\gamma))^2\,\mathrm{d}\gamma.$$

Based on Carlsson and Fullér (2001), Carlsson et al. (2002) find an optimal MV portfolio selection model with highest utility score under the assumption that the returns of securities are trapezoidal fuzzy variables. Zhang et al. (2003) present the notions of lower and upper possibilistic variance and covariance of fuzzy variables, and Zhang and Wang (2005a, b) present a possibilistic MV portfolio selection model by integrating the managers' viewpoints and the experts' knowledge. Zhang et al. (2007) propose a lower and upper possibilistic MV model for portfolio selection when the investment proportions have lower bound constraints. The authors also present an exact algorithm to obtain the explicit expression of the possibilistic efficient frontier for this model. Zhang et al. (2009b) discuss the portfolio selection problem for bounded securities on the base of the general possibilistic MV utility function, which is a two-parameter quadratic programming problem. The authors present a sequential minimal optimization (SMO) algorithm which can derive the optimal portfolio. Subsequently, Zhang et al. (2009a) study the possibilistic MV portfolio selection problem considering the securities returns obeying LR-type possibility distributions. Based on Zhang et al. (2007), Li et al. (2015c) formulate a possibilistic MV portfolio selection model with background risk by assuming that the possibility distribution of security returns is LR-type, which can be deduced into any specific form by investors' estimation and practical situation. A genetic algorithm is developed for solving the proposed model.

In addition, a few scholars study the weighted lower and upper possibislistic MV model for portfolio selection. Fullér and Majlender (2003) propose the notations of the weighted lower and upper possibilistic mean as well as variance of fuzzy variable. The function $f : [0, 1] \to \bar{R}^{-}$ is said to be a weighting function if f is non-negative, monotonic increasing and satisfies the normalization condition as follows

$$\int_0^1 f(\gamma)\mathrm{d}\gamma = 1.$$

They define the f-weighted possibilistic expected value and variance of fuzzy variable ξ by

$$E[\xi] = \int_0^1 \frac{a_1(\gamma) + a_2(\gamma)}{2} f(\gamma)\mathrm{d}\gamma, \quad Var[\xi] = \int_0^1 \left(\frac{a_2(\gamma) - a_1(\gamma)}{2}\right)^2 f(\gamma)\mathrm{d}\gamma.$$

Based on possibility theory, Li et al. (2015b) redefine the possibilistic mean and variance of fuzzy variable and propose the concept of possibilistic skewness. For a fuzzy variable ξ with differentiable membership function $\mu(x)$, the mean and variance of ξ are defined as follows

$$E[\xi] = \int_{-\infty}^{+\infty} x\mu(x)|\mu'(x)|\mathrm{d}x, \quad Var[\xi] = \int_{-\infty}^{+\infty} (x - E[\xi])^2\mu(x)|\mu'(x)|\mathrm{d}x.$$

This definition of possibilistic mean is proved to coincide with the lower and upper possibilistic means defined by Carlsson and Fullér (2001).

Zhang and Xiao (2009) present the weighted lower and upper possibilistic MV model for portfolio selection, which can be transformed to a linear programming problem by regarding the security returns as trapezoidal fuzzy variables. Zhang et al. (2011) add transaction costs into the portfolio model and assume that the security returns are fuzzy variables by using the weighted average of possibilistic MV method introduced in Zhang and Xiao (2009). A sequential minimal optimization (SMO) algorithm is utilized to calculate the optimal strategy for this model.

In recent years, researchers have applied the possibilistic MV portfolio selection model to resolve practical problems. Deng and Li (2010) propose that the conventional MV model can be simplified as a bi-objective linear programming model on the base of possibility theory. Furthermore, the fuzzy two-stage algorithm is applied to solve the proposed bi-objective model. Sadjadi et al. (2011) present a fuzzy multi-period portfolio selection model where the the borrowing rates are higher than the lending rates. Also based on the possibility theory, Deng and Li (2012) propose a quadratic programming with inequality borrowing constraints where the security returns are assumed as trapezoid fuzzy variables. Furthermore, Lemke algorithm has been designed to solve the proposed model. Later, Deng and Li (2014) further propose a bi-objective nonlinear portfolio selection model by assuming that the returns are triangular fuzzy variables based on possibility theory, moreover, they introduce a gradually tolerant constraint method to illustrate the efficiency of the proposed model.

It is worth mentioning that quite a few scholars study the admissible errors in fuzzy circumstance using possibility theory. For instance, by assuming the expected return and variance have admissible errors, Zhang and Nie (2004) employ the definitions of fuzzy weighted average return and covariance (Tanaka et al. 2000) to study the admissible efficient portfolio problem to reflect the uncertainty in real investment world. Considering the riskless securities can be either lent or borrowed under general investment constraints, Zhang et al. (2006) present an admissible efficient portfolio model. Zhang and Wang (2008) propose an admissible efficient portfolio model concerning the borrowing case. Chen and Zhang (2010) discuss the admissible portfolio selection problem with transaction costs. And they propose a new admissible efficient portfolio selection model and produce an improved PSO algorithm for the model.

5.2 Credibilistic MV portfolio selection

Though possibility measure has been widely used in portfolio selection, it has some limitations. One limitation is that possibility measure is not self-dual (Liu and Liu 2002). However, within the framework of possibility theory, two fuzzy events with different occurring chances may have the same possibility value, which is not a favorable condition for investors. To overcome this, credibility theory is proposed (Liu and Liu 2002) and accepted by more and more scholars. Li and Liu (2006a) introduce a relation between possibility measures and credibility measures, and prove a necessary and sufficient condition for credibility measures.

In Liu and Liu (2002), the credibility is defined as

$$\mathbf{Cr}\{\xi \in A\} = \frac{1}{2}\left(\sup_{x \in A} \mu(x) + 1 - \sup_{x \in A^c} \mu(x)\right).$$

The credibility measure **Cr** is an average of possibility measure and necessity measure and is proved to be self-dual. Based on the credibility measure, Liu and Liu (2002) define the expected value as

$$E[\xi] = \int_0^{\infty} \mathbf{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathbf{Cr}\{\xi \leq r\}\mathrm{d}r.$$

This definition is not only applicable to a continuous fuzzy variable, bust also a discrete one. If a fuzzy variable ξ has finite expected value $E[\xi]$, then its variance is defined as

$$Var[\xi] = E\left[(\xi - E[\xi])^2\right].$$

This definition tells us that the variance is just the expected value of the nonnegative fuzzy variance $(\xi - E[\xi])^2$, which is expressed as

$$Var[\xi] = \int_0^{\infty} \mathbf{Cr}\{(\xi - E[\xi])^2 \geq r\}\mathrm{d}r.$$

Following Markowitz's idea of MV model, Huang (2007a) firstly proposes the credibilistic MV models as follows,

$$\begin{cases} \max \ E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{ s.t. } Var[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \leq \beta \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \ i = 1, 2, \ldots, n, \end{cases} \tag{14}$$

In the above model, ξ_i represents the return of the i-th security, which is defined as $\xi_i = (p_i' + d_i - p_i)/p_i$, $i = 1, 2, \ldots, n$, respectively, where p_i' is the estimated closing price of the security i in the next year, p_i is the closing price of the security i at present, and d_i is the estimated dividend of the security i during the coming year. A hybrid intelligent algorithm integrating fuzzy simulation and genetic algorithm is developed for giving a general solution the credibilistic portfolio selection model.

Afterwards, Chen et al. (2006) build two classes of fuzzy portfolio selection models based on credibility measure. And they give the definition of variance with triangular and trapezoidal fuzzy variables in credibilistic environment. Huang (2011) proposes two credibilistic minimax MV models for fuzzy portfolio selection problem where security returns are treated as fuzzy variables. The proposed models are converted into linear programming models in three special cases. A neuron network imbedded genetic algorithm is provided to solve the proposed optimization problems in general cases. Li et al. (2009, 2015a) design an efficient hybrid intelligent algorithm by integrating simulated annealing algorithm, neural network and fuzzy simulation techniques for solving credibilistic portfolio selection models. Huang (2009) gives a review regarding credibilistic portfolio selection approaches, where the returns of securities are assumed to be characterized by fuzzy variables with known credibility distributions.

5.3 Fuzzy random and random fuzzy MV portfolio selection

Sometimes, the investors encounter both randomness and fuzziness on the grounds of the complex financial market. In this case, a framework of hybrid theory is needed to handle the hybrid uncertainty. Therefore, some authors propose a hybrid portfolio selection containing both fuzziness and randomness. According to different requirements of measurability, different definitions of fuzzy random variable are given such as Kwarkernaak (1978, 1979), Puri and Ralescu (1986), Kruse and Meyer (1987), Liu and Liu (2003a), Li and Liu (2009) and so on.

Generally speaking, a fuzzy random variable is a measurable function from a probability space to a collection of fuzzy sets. Here, we introduce the following definition given by Liu and Liu (2003a). Let ξ be a fuzzy random variable defined on the probability space $(\Omega, \Lambda, \mathrm{Pr})$, the expected value operator of ξ is defined as

$$E[\xi] = \int_0^{\infty} \mathrm{Pr}\{\pi \in \Omega | E[\xi(\pi)] \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Pr}\{\pi \in \Omega | E[\xi(\pi)] \leq r\}\mathrm{d}r.$$

If ξ has finite expected value $E[\xi]$. The variance is defined as the expected value of $(\xi - E[\xi])^2$. That is

$$Var[\xi] = E\left[(\xi - E[\xi])^2\right].$$

Conversely, a random fuzzy variable is a function from a possibility space to a collection of random variables. Liu (2002) proposes the concept of random fuzzy variable. In Liu (2002), ξ represents a random fuzzy variable defined on the credibility space $(\Theta, P(\Theta), \mathbf{Cr})$. $E[\xi]$ is defined as

$$E[\xi] = \int_0^{\infty} \mathbf{Cr}\{\theta \in \Theta | E[\xi(\theta)] \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathbf{Cr}\{\theta \in \Theta | E[\xi(\theta)] \leq r\}\mathrm{d}r.$$

Let ξ be a random fuzzy variable with finite expected value $E[\xi]$. Then the variance of ξ is defined by

$$Var[\xi] = E\left[(\xi - E[\xi])^2\right].$$

For detailed expositions on fuzzy random theory and random fuzzy theory, the interested readers can refer to Liu and Liu (2003b), Li and Liu (2006b), Li and Liu (2009), and Liu et al. (2012).

There are a few studies about random fuzzy portfolio selection problem. Ammar (2008) investigates portfolio selection problem with a fuzzy random multi-objective quadratic programming. As a counterpart of Markowitz's MV model, Hao and Liu (2009) develop two novel classes of MV models within the framework of fuzzy random theory for portfolio selection problem, and present the variance equations for triangular fuzzy random variables. This paper designs genetic algorithm to solve the proposed problem, and verify the obtained optimal solutions via Kuhn-Tucker conditions. Li and Xu (2009) establish a new MV portfolio selection model with different perspective in fuzzy random environment. Li and Xu (2013) propose a constrained multi-objective MV portfolio selection model in which security returns are fuzzy random variables, and this model has three criteria (return, risk and liquidity) which is more practical in realistic investment environment. To avoid the difficulty of evaluating a large set of efficient solutions and to ensure the selection of the best solution, a compromise approach-based genetic algorithm is employed to solve the proposed model.

To deal with uncertainty of random fuzziness, Huang (2007b, c) employs the random fuzzy theory to discuss risk definition from different perspectives. With these new definitions of risk, both two papers present new MV models with random fuzzy returns. Huang (2007b) design a hybrid intelligent algorithm integrating random fuzzy simulation and genetic algorithm to produce a general solution method. In Huang (2007c), neural networks are utilized to calculate the expected value and the chance value in the proposed hybrid intelligent algorithm. Furthermore, Hasuike et al. (2009) propose several random fuzzy portfolio models of random fuzzy portfolio selection problems; (a) single criteria optimization model, (b) bi-criteria optimization model with a fuzzy goal of target profit. Since each problem is equivalent to a parametric nonlinear programming problem, the authors construct each efficient solution method involving the procedure of solving a parametric convex programming problem to obtain a global optimal solution.

5.4 Other models with nonprobabilistic variables

Besides the above four branches of theories mentioned above, there are also certain other methods to cope with portfolio selection problem whose returns are assumed nonprobabilistic variables. Liu (2007) proposes uncertainty theory to tackle the uncertainty which acted neither randomness nor fuzziness. Some scholars start to consider the MV portfolio selection problem in which the security returns are treated as special types of uncertain variables. Yan (2009) extends Liu (2007)'s work and provides two classes of uncertain programming models for MV portfolio selection with uncertain returns. The author discusses the crisp equivalents when the uncertain security returns are treated as uncertain variables in order to solve the proposed models by traditional methods. Huang (2012) provides a hybrid intelligent algorithm for solving the MV portfolio selection problem in which security returns are given according to experts' estimations. The author presents a method for determining the uncertainty distributions of security returns based on experts' evaluations. Lai et al. (2002), Ida (2003) and Liu et al. (2013) study the portfolio selection optimization model by using interval analysis in uncertain circumstance. Qin (2015) proposes a hybrid MV model for portfolio selection problem in the simultaneous presence of random and uncertain returns.

5.5 Summary

While it is difficult to obtain the precise probability distributions or enough data, fuzzy portfolio models behave better than probabilistic models. This section gives an introduction on portfolio selection in fuzzy environments. We review possibilistic portfolio models, credibilistic portfolio models, random fuzzy portfolio models, fuzzy random portfolio models and some other nonprobabilistic models. It is found that most of these studies focus on theoretical aspects. There are few studies offering the methods of how to transform the real-life stock data into fuzzy returns/fuzzy numbers before formulating the fuzzy portfolio models. More works (Zhang et al. 2010; Pedrycz and Song 2012) are needed to introduce the way of approximating fuzzy numbers with real data and illustrate its effectiveness. There are two main methods to obtain the optimal portfolio in uncertain environment, fuzzy portfolio selection methods, such as the fuzzy portfolio optimization method by using the quadratic programming (Hasuike et al. 2009; Zhang and Wang 2008), the multi-objective nonlinear model for fuzzy portfolio selection (Deng and Li 2014), fuzzy random multi-objective quadratic programming for portfolio selection (Ammar 2008), etc; and heuristic approach, for example, hybrid intelligent algorithm (Huang 2007b, 2012), a compromise approach-based GA (Li and Xu 2013), SMO algorithm (Zhang et al. 2009b) and so on. In addition, more real constraints should be considered in the fuzzy portfolio selection literature in the future.

6 Conclusion and future research

This paper reviews various extensions of Markowitz's MV model, which aims to help researchers find hot points and trend on portfolio selection studies and help practioners to find useful theoretical tools, including dynamic optimization, robust optimization, fuzzy optimization and so on. Despite all the efforts we have made to present a complete review on MV portfolio selection methods, it still has some limitations. Nevertheless, we are convinced that we have compiled the vast majority of the studies carried out in this field. As we have discussed, the practical financial markets are uncertain and complex, but the theoretical models are much simpler. Hence, more works should be done to reduce the gap and enhance the models' practicability in real world. For example, the difference on investment horizons of securities in dynamic portfolio selection problems should be considered. In addition, the current studies generally assume that the risky security returns obey normal distribution. However, elliptical distribution may be more suitable to tackle with the uncertainty in real-life markets.

In real-life financial market, investors want to know what is happening now, what is likely to happen next and what measures should be taken to get the optimal portfolios.

Thus, combing forecasting theory with portfolio selection is an promising research direction. The existing works (Deng and Min 2013; Xia et al. 2015) mainly focus on earning forecasting, which is an input to a portfolio optimization analysis where fundamental and statistical-based risk models are used. In addition, predicting stock's price based on financial reports is also attractive for investors. For example, the investors can make subjective judgments based on the objective technical indicators (Wu et al. 2015).

Big data ananlysis is another hot direction in portfolio optimization. Henri Waelbroeck, serves as *Global Head of Research at Portware*, published an article titled "*Big Data Techniques Can Give Institutional Portfolio Managers Upper Hand*", in which he pointed out that while Chief Information Officers (CIOs) had previously adopted quantitative methods for portfolio optimization, today the race is on to deploy big data optimization solutions to extend these capabilities to trade execution as well. There are multiple sources supporting portfolio optimization. Aside from the recorded large scale real-time data and risk rating of securities that issued by securities institutes, accumulated data from financial statements analysis of the firms also contain valuable information for evaluating profitability of the security with big data technique (Norman 2011, 2012) . In addition, information on investment environment such as economic policies, government supervision and industry situation is also an important data source. Last but not least, the specific requirements of the investors are a vital part of data source. Before investing, the investors may decide what they are planning for, how long they plan to invest, how sensitive they are to the investment risk, how much they have saved or invested currently, etc. If we can collect all these valuable information and build a database for big data analysis, we can make sound decisions on portfolio.

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