Fuzzy Optim Decis Making (2017) 16:127–157
DOI 10.1007/s10700-016-9244-x

Handling imprecise evaluations in multiple criteria decision aiding and robust ordinal regression by *n*-point intervals

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Published online: 31 May 2016

Abstract We consider imprecise evaluation of alternatives in multiple criteria ranking problems. The imprecise evaluations are represented by n-point intervals which are defined by the largest interval of possible evaluations and by its subintervals sequentially nested one in another. This sequence of subintervals is associated with an increasing sequence of plausibility, such that the plausibility of a subinterval is greater than the plausibility of the subinterval containing it. We explain the intuition that stands behind this proposal, and we show the advantage of n-point intervals compared to other methods dealing with imprecise evaluations. Although n-point intervals can be applied in any multiple criteria decision aiding (MCDA) method, in this paper, we focus on their application in robust ordinal regression which, unlike other MCDA methods, takes into account all compatible instances of an adopted preference model, which reproduce an indirect preference information provided by the decision maker. An illustrative example shows how the method can be applied in practice.

Keywords Imprecise evaluations · n-point intervals · Multiple criteria decision aiding · Robust ordinal regression · Preference relations

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1 Introduction

Multiple criteria decision aiding (MCDA) [for an exhaustive collection of state-of-the-art surveys see Figueira et al. (2005)] concerns a set of alternatives $A = \{a, b, c, \ldots\}$ evaluated on a set of m criteria $G = \{g_1, \ldots, g_m\}$, and deal with three main types of problems: ranking, sorting and choice taking into account preferences of a Decision Maker (DM). To handle these problems, three different approaches are commonly used:

- assigning to each alternative a value through a value function, i.e., a real number reflecting its degree of desirability,
- comparing the evaluations of alternatives on the considered criteria using a binary relation,
- using a set of "if..., then..." decision rules induced from the DM's preference information.

In the first case, the multi-attribute value theory (MAVT) (Keeney and Raiffa 1993) is most frequently used; MAVT provides a methodology for building a value function. In the second case, the most popular are the methods which build an outranking relation on the set of alternatives to compare them pairwise (Roy 1996). In the third case, the decision rules are derived from the DM's preference information structured by the dominance-based rough set approach (DRSA) (Greco et al. 2001; Słowiński et al. 2009).

In the context of MAVT, one often uses additive value functions, that is functions obtained by adding up marginal value functions representing the degree of preference on the corresponding evaluation criteria. In order to use this approach, one needs to construct marginal value functions for all considered criteria. The construction requires some preference information elicited by the DM. An analyst can obtain it in one of two ways: asking the DM to provide this information directly, or indirectly. As direct definition of marginal value functions requires too big cognitive effort from the DM, indirect elicitation of preference information has been proposed and widely used in MCDA (see, e.g., Jacquet-Lagreze and Siskos 2001). Indirect preference information is expressed by the DM in terms of decision examples, e.g., holistic pairwise comparisons of some reference alternatives. When looking for an additive value function which is compatible with the decision examples provided by the DM, i.e., which reproduces the DM's decisions, one can find, in general, many compatible instances of such value function, and each of these instances can give a different recommendation in the considered decision context. For this reason, robust ordinal regression (ROR) (Greco et al. 2008) was proposed (for a survey on ROR see Corrente et al. 2013), that takes into consideration all compatible instances of the value function simultaneously. In the context of ROR, possible and necessary preference relations are built for each pair of alternatives a and b, such that the first one is true if a is at least as good as b for at least one compatible instance of the value function, and the second is true when a is at least as good as b for all compatible instances of the value function.

In this paper, we extend ROR on a new important issue: imprecise evaluations of alternatives. In many real world problems, alternatives are imprecisely evaluated on the considered criteria; this is due to several reasons: inexact definition of criteria, uncertainty or imprecision of data used for calculation of performances of alternatives on particular criteria, or subjective assessment of the performances. Different types of imprecise information on weights, value functions and probabilities have been dealt with in the literature (Park and Kim 1997; Weber 1987). Moreover, imprecise evaluation of alternatives on criteria has been considered in different studies. Among them, let us remember an adaptation of the DRSA to the case of multiple criteria sorting problems with imprecise evaluations and assignments (Dembczynski et al. 2009).

A review of literature on handling imprecise evaluations of alternatives in the MCDA context leads us to conclusion that there are three main approaches to this issue:

- considering imprecise evaluations by means of probability distributions, as in the decision under uncertainty (Moskowitz et al. 1993),
- stochastic multicriteria acceptability analysis (SMAA) (Lahdelma et al. 1998) which considers probability distributions on the space of evaluations as well as on the space of weights and computes for each alternative the probability of getting a given ranking position or the frequency with which it is preferred to another one,
- application of fuzzy numbers for modeling imprecise evaluations of alternatives (Zadeh 1975), so that a membership function assigns to each performance a value ranging from 0 (in case of certain non-membership) to 1 (in case of certain membership), like in the adaptation of outranking methods to fuzzy evaluations of alternatives proposed by Czyżak and Słowiński (1997) [for a survey of applications of fuzzy set theory to MCDA see, e.g., Dubois (2011); handling of imprecision due to verbal evaluation of alternatives has been considered by Dong and Herrera-Viedma (2015) and surveyed by Herrera et al. (2009)].

In this paper, we approach the issue of imprecise evaluations of alternatives in a different way than above. The basic idea is that the DM, or experts advising the DM, can specify an imprecise evaluation not only in terms of the interval of possible values, but also in terms of several gradually embedded subintervals, such that each subinterval contained in a larger subinterval includes more plausible values than the larger one. Suppose, for example, that three experts assessed an investment alternative with respect to a “profit” it may bring. The first one estimates the profit could vary between 2000 and 5000, the second estimates it between 2500 and 6000, and the third one estimates it between 4000 and 8000. If the DM would like to summarize these estimates, (s)he could say that the range of the possible profit (confirmed values by at least one expert) is equal to the interval [2000, 8000], the subinterval of more plausible values (confirmed by at least two experts) is [2500, 6000], and the subinterval of the most plausible values (confirmed by all three experts) is [4000, 5000]. This type of information can be modeled by means of the n-point interval recently proposed by Ozturk et al. (2011). In this example, the information related to the profit of the investment alternative can be represented by the 6-point interval [2000, 2500, 4000, 5000, 6000, 8000]. Handling imprecise evaluations on considered criteria by n-point intervals has several advantages over other models. These benefits are listed below:

- n-point intervals permit the DM to express imprecision in a quite easy way by using few meaningful reference values, that does not require a great cognitive effort;
- using n-point intervals, the DM can give a finer information than just intervals of possible performances of an alternative on the considered criteria;
- using n-point intervals, the DM is not obliged to give any exact probability distribution on imprecise performances of alternatives which, in general, is an information not available to her/him;
- the use of n-point intervals avoids the adoption of the linear interpolation typically considered in triangular and trapezoidal fuzzy numbers, that is an arbitrary assumption to some extent.

All above points have been detailed in Sect. 3, where we compared the n-point intervals with other ways of handling imprecise evaluations, such as probabilities, stochastic multiobjective acceptability analysis and fuzzy numbers.

The paper is structured in the following way: Sect. 2 describes basic concepts of imprecise evaluations; in Sect. 3 we compare the proposed method with other ways of handling imprecision in multiple criteria evaluations; Sect. 4 provides definitions of dominance relations and their properties; Sect. 5 shows how ROR can handle imprecise evaluations; Sect. 6 provides some properties of necessary and possible preference relations; Sect. 7 presents a didactic example, and conclusions are gathered in Sect. 8; all proofs and some further results are deferred to the Appendix.

2 Imprecise evaluations—intuition behind the model and basic definitions

We are considering a decision problem in which a finite set of alternatives, denoted by $A = \{a, b, c, \ldots\}$, can have imprecise evaluations with respect to a set of m evaluation criteria $G = \{g_1, \ldots, g_m\}$. In case of precise evaluations, a criterion $g_j,\ j \in J = \{1, \ldots, m\}$, is a function $g_j : A \rightarrow X_j$, where X_j is the set of all possible evaluations (cardinal or ordinal, depending on the evaluation scale of criterion g_j) that an alternative could assume on criterion g_j. In case of n-point intervals we assume that each criterion is a function $g_j : A \rightarrow \mathcal{I}_j$, where $\mathcal{I}_j = \big\{(x_1, \ldots, x_n) : x_{l+1} \succsim_j x_l, l = 1, \ldots, n-1\big\} \subseteq {X_j}^n$, and $\succsim_j$ is a complete preorder (transitive and complete binary relation) on $\mathcal{I}_j$, representing preferences with respect to criterion g_j. $\succsim_j$ coincides with the binary relation $\geq$ on $\mathbb{R}$ if criterion g_j has a quantitative-numerical scale, however, it is defined differently if criterion g_j has a qualitative-nominal scale. To explain this difference, let us consider the evaluation of a student regarding a certain subject denoted by g_j. If the evaluation is expressed on a quantitative-numerical scale (for example the evaluation of the student can vary between 2 and 10), then $\succsim_j$ coincides with $\geq$ because to state that student a is not worse than student b on criterion g_j, it is sufficient to check if $g_j(a) \geq g_j(b)$. Now, let us suppose that the evaluations with respect to this subject are expressed on the following qualitative-nominal scale: "very bad", "bad", "medium", "good" and "very good". Then, one needs to define an ordering $\succsim_j$ of these nominal terms that is obviously different from the inequality $\geq$ between two real numbers, that is "very good" $\succsim_j$ "good" $\succsim_j$ "medium" $\succsim_j$ "bad" $\succsim_j$ "very bad".

In the description of the proposed methodology, for the sake of simplicity and without loss of generality, we shall assume the following:

– each criterion has a quantitative scale; therefore $X_j \subseteq \mathbb{R}$, and thus $\succsim_j$ coincides with $\geq$ (indeed, if the scale X_j of criterion g_j is ordinal, one can always encode it in numerical terms, such that for all $x_j, y_j \in X_j$

$$x_j \succsim_j y_j \Leftrightarrow \overline{x}_j \geq \overline{y}_j,$$

with $\overline{x}_j$ and $\overline{y}_j$ being number codes of x_j and y_j);

– the greater $g_j^i(a)$, $a \in A$, the better is alternative a on indicator g_j^i, $j = 1, \ldots, m$, $i = 1, \ldots, n$ (in the opposite case, we can take as indicator $-g_j^i$ and we come back to the previous case).

We represent the imprecise evaluation of alternative a on criterion g by means of an n-point interval $[g^1(a), \ldots, g^n(a)]$, where $g^1(a) \leq \ldots \leq g^n(a)$. The n-points $g^1(a), \ldots, g^n(a)$ define a sequence of c nested intervals, where $c = \frac{n}{2}$ if n is even and $c = \frac{n+1}{2}$ if n is odd,

$$\left[g^1(a), g^n(a)\right] \supseteq \left[g^2(a), g^{n-1}(a)\right] \supseteq \ldots \supseteq \begin{cases} \left[g^c(a), g^{c+1}(a)\right], & \text{if } n \text{ is even,} \\ \left[g^c(a), g^c(a)\right], & \text{if } n \text{ is odd} \end{cases}$$

related to c increasing levels of plausibility $L_1, \ldots, L_c$, such that the evaluation of alternative a on criterion g belongs to the interval $\left[g^1(a), g^n(a)\right]$ with level of plausibility L_1, while the evaluation of a on g belongs to the interval $\left[g^c(a), g^{c+1}(a)\right]$ (or $\left[g^c(a), g^c(a)\right]$) with level of plausibility L_c. In general, the evaluation of a on g belongs to the interval $[g^r(a), g^{n-r+1}(a)]$ with level of plausibility L_r, $r = 1, \ldots, c$.

For example, consider an imprecise evaluation of an investment a on criterion "profit", denoted g_1, expressed by the following 6-point interval [2000, 2500, 4000, 5000, 6000, 8000] with the corresponding plausibility levels $L_1 =$"possible", $L_2 =$"fairly plausible", $L_3 =$"very plausible". This means that:

– values in the interval $[2000, 8000] = [g_1^1(a), g_1^6(a)]$ are possible, that is their plausibility is L_1,
– values in the interval $[2500, 6000] = [g_1^2(a), g_1^5(a)]$ are fairly plausible, that is their plausibility is L_2,
– values in the interval $[4000, 5000] = [g_1^3(a), g_1^4(a)]$ are very plausible, that is their plausibility is L_3.

In the following, considering the same number n of indicators for each considered criterion, the evaluations of a will be represented by the following vector:

$$g(a) = \left(\left[g_1^1(a), \ldots, g_1^n(a)\right], \ldots, \left[g_j^1(a), \ldots, g_j^n(a)\right], \ldots, \left[g_m^1(a), \ldots, g_m^n(a)\right]\right).$$

Even if in this paper we consider ROR, n-point intervals can be used with any MCDA method. Indeed, from this point of view, each one of the indicators $g_j^i(a)$,

$g_j \in G$ and $i = 1, \ldots, n$, can be seen as a specific criterion in a reformulation of the original MCDA problem, where the original set of criteria $G = \{g_1, \ldots, g_m\}$ is replaced by the new set of criteria $\overline{G} = \{g_1^1, \ldots, g_1^n, \ldots, g_m^1, \ldots, g_m^n\}$. For example, if a weighted sum would be the utility model, then for each alternative $a \in A$ the following overall value of a would be defined as:

$$U(a) = \sum_{j=1}^{m} \sum_{i=1}^{n} w_j^i g_j^i(a)$$

with w_j^i representing the weight given to the indicator g_j^i, such that $w_j^i \geq 0$, $j = 1, \ldots, m$, $i = 1, \ldots, n$, and $\sum_{j=1}^{m} \sum_{i=1}^{n} w_j^i = 1$.

3 Comparison with other ways of handling imprecision in multiple criteria evaluations

In this section, we compare the proposed way of handling imprecise evaluations with other methods known from the literature: decision under uncertainty, SMAA, and fuzzy numbers.

3.1 Decision under uncertainty

Let us begin this section observing that the values $g^1(a)$,…,$g^n(a)$ of the n-point interval $[g^1(a), \ldots, g^n(a)]$ can be interpreted as qualitative counterparts of the Hurwicz criterion (Hurwicz 1951). In case of decision under uncertainty the Hurwicz criterion suggests to evaluate the payoff of an act by the value

$$\alpha M + (1 - \alpha)m,$$

where m and M are the minimum and the maximum outcomes, and $\alpha \in [0, 1]$ is a coefficient measuring the optimism of the DM. Indeed, one can imagine that each indicator $g^1(a)$,…,$g^n(a)$ is related to an increasing degree of optimism, such that $g^1(a)$ is the most pessimistic evaluation, $g^n(a)$ is the most optimistic evaluation, and, in general, $g^r(a)$, $r = 1, \ldots, n$, are evaluations such that the greater is r the more optimistic they are.

Let us observe that our approach can easily be applied also in case of probabilistic evaluations on criteria $g_j \in G$ (see, e.g., Moskowitz et al. 1993) in the sense that imprecision is related to some probability distribution on the set of values that alternatives from A can assume on criteria $g_j \in G$. In this case, for each $a \in A$ and $g_j \in G$, one can associate to each interval $[g_j^r(a), g_j^{n-r}(a)]$ when $r \neq c$, and to the value $g_j^r(a)$, when $r = c$, the probability levels p_j^r, $r = 1, \ldots, c$, where

$$0 \leq p_j^1 \leq p_j^2 \leq \ldots \leq p_j^c \leq 1,$$

such that there is a probability p_j^r that $g_j(a)$ is not smaller than $g_j^r(a)$, and there is an analogous probability p_j^r that $g_j(a)$ is not greater than $g_j^{n-r}(a)$. For example, if one knows the probability distribution $\mathbf{P}$ on values of the profit for the investment considered in the previous section, and supposing that the DM is focusing (her)his attention on probability levels 10, 25 and 40 %, the 6-point interval [2000, 2500, 4000, 5000, 6000, 8000] can be interpreted as follows:

- there is a probability of 10 % that the profit is smaller than $g^1(a) = 2000$, and there is the same probability of 10 % that the profit is greater $g^6(a) = 8000$,
- there is a probability of 25 % that the profit is smaller than $g^2(a) = 2500$, and there is the same probability of 25 % that the profit is greater than $g^5(a) = 6000$,
- there is a probability of 40 % that the profit is smaller than $g^3(a) = 4000$, and there is the same probability of 40 % that the profit is greater than $g^4(a) = 5000$.

In our opinion, evaluations expressed as n-point intervals have some advantages over probabilistic evaluations for the following reasons:

- even if the probability distributions of the values taken by alternatives $a \in A$ on criteria $g_j \in G$ are perfectly known, n-point intervals permit to focus on the probability levels most important for the DM, so that the MCDA procedure becomes more controllable;
- if the probability distributions of the values taken by alternatives $a \in A$ on criteria $g_j \in G$ are not perfectly known, n-point intervals permit to use this imperfect information taking as indicators g_j^r the values corresponding to the probability levels for which there is some information;
- n-point intervals can also represent qualitative probability distributions by considering plausibility levels $L_1, \ldots, L_c$ with a probabilistic meaning such as
  - $L_1 =$ "at least weakly probable",
  - $L_2 =$ "at least fairly probable",
  - $L_3 =$ "very probable".

Due to the verbal ordinal scale of plausibility $\mathcal{L} = \{L_1, \ldots, L_c\}$, which is intuitively understandable for the DM, expressing the imprecision in terms of n-point intervals is relatively easy for the DM. Consequently, the use of n-point intervals is the best way of getting a reliable information about imprecise evaluations of alternatives.

3.2 Stochastic multiobjective acceptability analysis (SMAA) and fuzzy numbers

Handling of imprecision by n-point interval evaluations is related to SMAA (Lahdelma et al. 1998) and to the approach based on the use of fuzzy numbers.

Let us consider an interval characterized by two extreme values only, that is, the most pessimistic and the most optimistic evaluation of an alternative on the considered criterion. This means that the evaluation of alternative a on criterion g is represented by the 2-point interval $[g^1(a), g^2(a)]$, such that a could get whichever evaluation between $g^1(a)$ (the most pessimistic evaluation) and $g^2(a)$ (the most optimistic evaluation). This is typical for handling imprecise evaluation of alternatives on particular criteria in SMAA. However, differently from SMAA, we do not assume any probability distribution of the evaluations in the interval $[g^1(a), g^2(a)]$. In fact, within SMAA,

$$\mu_v(x) = \begin{cases} 0, & x < v_1 \\ \frac{x-v_1}{v_2-v_1}, & v_1 \le x \le v_2 \\ \frac{v_3-x}{v_3-v_2}, & v_2 \le x \le v_3 \\ 0, & x > v_3 \end{cases}$$

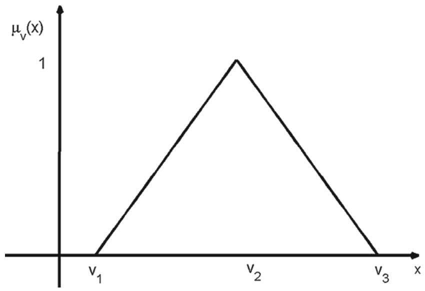

Fig. 1 Membership function of a triangular fuzzy number

a uniform probability distribution is considered, even if, in general, any probability distribution could be assumed. It is not easy, however, to select the proper probability distribution. Is it enough to justify the use of a uniform distribution by saying that it is the simplest one? It is rather more reasonable and methodologically correct to avoid consideration of any probability distribution (which is almost always the case in multiple criteria decision making), unless one has a strong evidence in favour of one specific probability distribution. In this sense, we retain the most stable and robust part of the information given by the interval $[g^1(a), g^2(a)]$, that is the two extreme values, and we do not assume any probability distribution.

Now, consider the case where each interval is characterized not only by the two extreme values but also by another point between them; in this way $g(a) = [g^1(a), g^2(a), g^3(a)]$, and this 3-point interval indicates that, with respect to criterion g, alternative a can assume whatever evaluation between $g^1(a)$ and $g^3(a)$, but, it is very plausible that the evaluation assumed by a on criterion g is around $g^2(a)$. This interpretation is coherent with an evaluation expressed by a linguistic variable represented by means of a triangular fuzzy number. If a linguistic variable v is represented by a triangular fuzzy number $\widetilde{v} = (v_1, v_2, v_3)$, then the possible values of v are between v_1 and v_3, such that other values have a null membership and the maximum membership equal to 1 is assumed in v_2. Moreover, the membership between v_1 and v_2, as well as that one between v_2 and v_3, is supposed to follow a linear interpolation. Formally, we have that the membership function $\mu_v : \mathbb{R} \rightarrow [0, 1]$ assigns to each $x \in \mathbb{R}$ a value as shown in Fig. 1.

Observe, however, that the information represented by the 3-point interval, is less arbitrary than the one represented by a triangular fuzzy number. Indeed, in the first case, the information specifies only the minimum and maximum values, v_1 and v_3, respectively, and the most plausible value, v_2. We avoid to assign a value of membership to all the values between v_1 and v_3, which would be arbitrary. The question is: why a linear interpolation and not some other interpolating function? Even if one would assume another interpolating function, how to verify that the values taken by the membership function are correct? This doubt makes our proposal more trustworthy again.

Finally, let us consider an imprecise evaluation represented by a 4-point interval $[g^1(a), g^2(a), g^3(a), g^4(a)]$, which means that the possible evaluation of a is included between $g^1(a)$ and $g^4(a)$, but the most plausible evaluation is between $g^2(a)$ and $g^3(a)$.

$$\mu_v(x) = \begin{cases} 0, & x < v_1 \\ \frac{x-v_1}{v_2-v_1}, & v_1 \le x \le v_2 \\ 1, & v_2 \le x \le v_3 \\ \frac{v_4-x}{v_4-v_3}, & v_3 \le x \le v_4 \\ 0, & x > v_4 \end{cases}$$

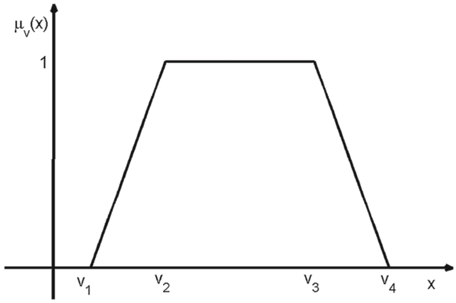

Fig. 2 Membership function of a trapezoidal fuzzy number

This interpretation is coherent with an evaluation expressed by a linguistic variable represented by a trapezoidal fuzzy number. If a linguistic variable v is represented by a trapezoidal fuzzy number $\widetilde{v} = (v_1, v_2, v_3, v_4)$, then the possible values of v are between v_1 and v_4, such that other values have a null membership and the maximum membership equal to 1 is assumed between v_2 and v_3. Moreover, the membership between v_1 and v_2, as well as that one between v_3 and v_4, is following a linear interpolation. Formally we have that the membership function $\mu_v : \mathbb{R} \rightarrow [0, 1]$ assigns to each $x \in \mathbb{R}$ a value as shown in Fig. 2.

The analogy between n-point intervals and fuzzy numbers could be continued for n greater than 4, however, the argument about the sensitivity of the end result on the choice of the interpolating function would be still in favor of our proposal.

4 Definitions of dominance relations and their properties

Various types of dominance relations stem from the formulation of multiple criteria decision problem without considering preferences. The following two concepts of dominance arise naturally in the context of imprecise evaluations of alternatives:

Definition 4.1 Given alternatives $a, b \in A$, and $i, k \in \{1, \ldots, n\}$, we say that "$a$ (i, k)-dominates b", denoted by $a\Delta^{(i,k)}b$, if $g_j^i(a) \ge g_j^k(b)$, $\forall j = 1, \ldots, m$.

Definition 4.2 Given alternatives $a, b \in A$, we say that "a normally dominates b", denoted by $a\Delta b$, if $g_j^i(a) \ge g_j^i(b)$, $\forall j = 1, \ldots, m$, and $\forall i = 1, \ldots, n$. Equivalently, we can say that "a normally dominates b" if a (i, i)-dominates b, $\forall i = 1, \ldots, n$.

Let us explain the meaning of the two above concepts of dominance taking into consideration alternatives a, b, d that get 3-point evaluations on criteria g_1, g_2 and g_3, where L_1 = "possible" and L_2 = "very plausible", as follows:

- $\left[g_1^1(a), g_1^2(a), g_1^3(a)\right] = [25, 35, 40]$, $\left[g_2^1(a), g_2^2(a), g_2^3(a)\right] = [40, 55, 65]$,

  $\left[g_3^1(a), g_3^2(a), g_3^3(a)\right] = [25, 50, 55]$,
- $\left[g_1^1(b), g_1^2(b), g_1^3(b)\right] = [10, 20, 45]$, $\left[g_2^1(b), g_2^2(b), g_2^3(b)\right] = [35, 50, 60]$,

  $\left[g_3^1(b), g_3^2(b), g_3^3(b)\right] = [20, 45, 60]$,

– $\left[g_1^1(d), g_1^2(d), g_1^3(d)\right] = [10, 15, 20]$, $\left[g_2^1(d), g_2^2(d), g_2^3(d)\right] = [30, 50, 55]$, $\left[g_3^1(d), g_3^2(d), g_3^3(d)\right] = [20, 45, 55]$.

Observe that, according to indicator g_j^1, $j = 1, 2, 3$, alternative a is at least as good as alternative b: indeed $g_1^1(a) \geq g_1^1(b)$, $g_2^1(a) \geq g_2^1(b)$ and $g_3^1(a) \geq g_3^1(b)$. Thus, considering the most pessimistic evaluation, a is not worse than b on all considered criteria. Therefore, one can say that "a (1, 1)-dominates b", denoted by $a\Delta^{(1,1)}b$. Analogously, according to indicator g_j^2, $j = 1, 2, 3$, a is at least as good as b. This means that considering the most plausible evaluation, a is not worse than b on all considered criteria. Instead, it is not true that according to indicators g_j^3, $j = 1, 2, 3$, a is at least as good as b, because $g_3^3(a) < g_3^3(b)$. This means that considering the most optimistic evaluation, a is not at least as good as b on all considered criteria. Observe, instead, that $a\Delta^{(1,1)}d$, $a\Delta^{(2,2)}d$ and $a\Delta^{(3,3)}d$, so one can conclude that $a\Delta d$, which means that a is not worse than d on all criteria considering pessimistic evaluations, the most plausible evaluations and the most optimistic evaluations on all three criteria.

Observe also that for some criterion g, one could take into account the most optimistic evaluation for a and the most plausible evaluation for b, i.e., one could consider indicator g^3 for a and indicator g^2 for b. This is consistent with preference representation by interval orders (Fishburn 1985). An interval order is a binary relation R on a set X which is reflexive and Ferrers transitive (i.e., for all $x, y, w, z \in X$, xRy and wRz imply xRz or wRy). If X is finite, then a binary relation R on X is an interval order if and only if there exists $u^+ : X \rightarrow \mathbb{R}$ and $u^- : X \rightarrow \mathbb{R}$ with $u^+(z) \geq u^-(z)$ for all $z \in X$, such that for all $x, y \in X$

$$xRy \Leftrightarrow u^+(x) \geq u^-(y).$$

Suppose that the interval order R is a weak preference relation on X, such that for each $x, y \in X$, xRy means that x is at least as good as y. In this case, $u^+(z)$ and $u^-(z)$ can be interpreted as the optimistic and the pessimistic evaluation of $z \in X$ and, consequently, one can say that x is weakly preferred to y, i.e., xRy, if the optimistic evaluation of x, i.e., $u^+(x)$, is not worse than the pessimistic evaluation of y, i.e., $u^-(y)$. Observe moreover that R^d, being the dual of R, i.e., the complement of the inverse of R, such that for all $x, y \in X$, $xR^d y$ iff not(yRx), can be interpreted as a strong preference relation on X. Thus, we have that for all $x, y \in X$,

$$xR^d y \Leftrightarrow u^-(x) > u^+(y),$$

which can be interpreted as "x is strongly preferred to y if and only if the pessimistic evaluation of x is better than the optimistic evaluation of y".

The idea of considering the pessimistic and the optimistic evaluations of alternatives to define preference relations with respect to criterion $g_j \in G$ can be easily extended to the case of n-point intervals considering indicator g_j^i for alternative x and indicator g_j^k for alternative y, which permit to say that x is (i, k)-preferred to y on criterion $g_j \in G$ (denoted by $x \succsim_j^{(i,k)} y$), if $g_j^i(x) \geq g_j^k(y)$. Considering the above example, we have

$a \succsim_3^{(3,2)} b$ because $g_3^3(a) \geq g_3^2(b)$, which means that with respect to criterion g_3 the most optimistic evaluation of a is not worse than the most plausible evaluation of b.

Using preference relations $\succsim_j^{(i,k)}$ one can say that for all $a, b \in A$, $a\Delta^{(i,k)}b$ if and only if $a \succsim_j^{(i,k)} b$ for all $g_j \in G$. For example, $a\Delta^{(3,2)}b$, because $g_1^3(a) \geq g_1^2(b)$, $g_2^3(a) \geq g_2^2(b)$ and $g_3^3(a) \geq g_3^2(b)$, which means that on all three criteria the optimistic evaluation of a is at least as good as the most plausible evaluation of b.

It is also worth noting that among all the considered pairs (i, k), an important place has to be given to the pairs $(i, n-i+1)$ with $i \leq c$[1] because $a \succsim_j^{(i,n-i+1)} b$ means that all the evaluations of $a \in A$ on $g_j \in G$ with plausibility L_i, i.e., $[g_j^i(a), g_j^{n-i+1}(a)]$, are not worse than all evaluations of $b \in A$ on g_j with the same plausibility L_i, $[g_j^i(b), g_j^{n-i+1}(b)]$. Looking at the above example, $a \succsim_1^{(1,3)} d$, are not worse than all values with plausibility L_1 for d, i.e. $[g_1^1(d), g_1^3(d)] = [10, 20]$.

The following proposition provides some basic properties of the dominance relations introduced above for n-point interval evaluations. These are the properties of reflexivity and transitivity that, if satisfied conjointly, characterize the structure of a partial preorder.

Proposition 4.1 1. *If $i \geq k$, $i, k \in \{1, \ldots, n\}$, then $\Delta^{(i,k)}$ is reflexive,*
2. *If $i \leq k$, $i, k \in \{1, \ldots, n\}$, then $\Delta^{(i,k)}$ is transitive,*
3. *For each $i \in \{1, \ldots, n\}$, $\Delta^{(i,i)}$ is a partial preorder,*
4. *If $r \geq i$ and $s \leq k$, $i, k, r, s \in \{1, \ldots, n\}$, then $\Delta^{(i,k)} \subseteq \Delta^{(r,s)}$,*
5. *Given alternatives $a, b, c \in A$, if $a\Delta^{(i,k)}b$, $b\Delta^{(i_1,k_1)}c$, and $k \geq i_1$, $i, k, i_1, k_1 \in \{1, \ldots, n\}$, then $a\Delta^{(r,s)}c$ with $r, s \in \{1, \ldots, n\}$, such that $r \geq i$ and $s \leq k_1$,*
6. *Δ is a partial preorder,*
7. *Given alternatives $a, b, c \in A$, if $a\Delta^{(i,k)}b$, $b\Delta c$, $i, k \in \{1, \ldots, n\}$, then $a\Delta^{(s,t)}c$ with $s, t \in \{1, \ldots, n\}$, such that $s \geq i$ and $t \leq k$,*
8. *Given alternatives $a, b, c \in A$, if $a\Delta b$, $b\Delta^{(i,k)}c$, $i, k \in \{1, \ldots, n\}$, then $a\Delta^{(s,t)}c$ with $s, t \in \{1, \ldots, n\}$, such that $s \geq i$ and $t \leq k$.*

Note 4.1 In the following, we shall call strong dominance, and we shall denote it by Δ^S, the dominance relation $\Delta^{(1,n)}$. Similarly, we shall call weak dominance, and we shall denote it by Δ^W, the dominance relation $\Delta^{(n,1)}$. When $n = 2$, then the dominance relation $\Delta^{(i,k)}$ boils down to strong dominance $\Delta^{(1,2)}$, weak dominance $\Delta^{(2,1)}$ and to the dominance relations $\Delta^{(1,1)}$ and $\Delta^{(2,2)}$ comparing two alternatives considering only their best values or their worst values. Using Proposition 4.1, we can state that weak dominance and normal dominance are reflexive relations, normal dominance and strong dominance are transitive relations, and so on.

Considering the strong and the weak dominance relations simultaneously, we can state the following proposition:

Proposition 4.2 1. $\Delta^{(1,n)} \subseteq \Delta \subseteq \Delta^{(n,1)}$,
2. *For $i, k = 1, \ldots, n$, $\Delta^{(1,n)} \subseteq \Delta^{(i,k)} \subseteq \Delta^{(n,1)}$.*

[1] Let us remember that by c we denote the number of nested intervals in the n-point intervals, which correspond to c levels of plausibility.

Note 4.2 Proposition 4.2 shows how important are the weak and the strong dominance relations, as they are the only two relations that can be compared directly with Δ. Indeed, in general, for any $(i,k) \in \{1,\ldots,n\} \times \{1,\ldots,n\}$, with $(i,k) \neq (1,n)$ and $(i,k) \neq (n,1)$, we can have $\Delta^{(i,k)} \nsubseteq \Delta$ and $\Delta \nsubseteq \Delta^{(i,k)}$.

5 Robust ordinal regression for imprecise evaluations—description of the methodology

The dominance relation, which is the only objective information that comes with the statement of a multiple criteria decision problem, is very poor. For this reason, in order to handle DM's preferences in one of the three typical multiple criteria decision problems, we are using the MAVT (Keeney and Raiffa 1993). MAVT considers value functions $U(g_1(a),\ldots,g_m(a))$ where

$$\mathbb{R}^m \rightarrow \mathbb{R}$$

such that:

$$\text{“}a \text{ is at least as good as } b\text{”} \;\Leftrightarrow\; U(g_1(a),\ldots,g_m(a)) \geq U(g_1(b),\ldots,g_m(b)),$$

taking into account the evaluations of alternatives on the m considered criteria. In case of imprecise evaluations, we are considering for each criterion g_j, $j \in J$, n indicators $g_j^i : A \rightarrow X_j$, $i = 1,\ldots,n$, assigning to each alternative $a \in A$ the i-th evaluation from interval $g_j(a)$. Using this notation, we can distinguish different types of value functions:

- i-th *sub-marginal value function* referring to the i-th indicator of criterion g_j, $u_j^i(g_j^i(a)) : X_j \rightarrow \mathbb{R}$, for all $j \in J$, and $i = 1,\ldots,n$,
- *marginal value function* referring to criterion g_j, such that $U_j([g_j^1(a),\ldots,g_j^n(a)]) : \mathcal{I}_j \rightarrow \mathbb{R}$, and

$$U_j\left(\left[g_j^1(a),\ldots,g_j^n(a)\right]\right) = u_j^1\left(g_j^1(a)\right) + \ldots + u_j^n\left(g_j^n(a)\right).$$

  The marginal value of alternative a with respect to criterion g_j depends on all the n indicators g_j^i because each of them takes part in the evaluation of a on criterion g_j with a different level of plausibility, which is represented by the corresponding sub-marginal value function u_j^i. In the paper, we admit that the indicators of each criterion are preferentially independent, which permits an additive aggregation of sub-marginal value functions (Keeney and Raiffa 1993). An analogous assumption is adopted with respect to aggregation of marginal value functions.
- *total additive value function*

$$U\left(\left[g_1^1(a)\ldots,g_1^n(a)\right],\ldots,\left[g_m^1(a),\ldots,g_m^n(a)\right]\right) : \mathcal{I}_1 \times \cdots \times \mathcal{I}_m \rightarrow \mathbb{R}$$

such that

$$U\left(\left[g_1^1(a)\ldots,g_1^n(a)\right],\ldots,\left[g_m^1(a),\ldots,g_m^n(a)\right]\right)=$$
$$=\sum_{j=1}^{m}U_j\left(\left[g_j^1(a),\ldots,g_j^n(a)\right]\right)=\sum_{j=1}^{m}\left[\sum_{i=1}^{n}u_j^i\left(g_j^i(a)\right)\right]. \quad (1)$$

In the following, for the sake of simplicity, for each $j \in J$ we write $U_j(a)$ instead of $U_j([g_j^1(a),\ldots,g_j^n(a)])$, and $U(a)$ instead of $U\left(\left[g_1^1(a),\ldots,g_1^n(a)\right],\ldots,\left[g_m^1(a),\ldots,g_m^n(a)\right]\right)$.

In order to take into account the imprecise nature of evaluations, we consider for each alternative $a \in A$, n fictitious alternatives $a^{(i)}$, having precise evaluations on all criteria, equal to the i-th point of interval $g_j(a)$, for each $j \in J$, i.e., $g_j^1\left(a^{(i)}\right)=\cdots=g_j^n\left(a^{(i)}\right)=g_j^i(a)$, for each $j \in J$.
For example, in case of three point intervals, $a^{(1)}$, $a^{(2)}$ and $a^{(3)}$ represent the "most pessimistic", the "most plausible" and the "most optimistic" realizations of alternative a, because these fictitious alternatives take the worst, the average and the best evaluations on all considered criteria, respectively. Note that given $a \in A$, a value function U assigns to the corresponding alternatives $a^{(i)}$ the value:

$$U\left(a^{(i)}\right)=u_1^1\left(g_1^i(a)\right)+\cdots+u_1^n\left(g_1^i(a)\right)+\ldots+u_m^1\left(g_m^i(a)\right)+\cdots+u_m^n\left(g_m^i(a)\right). \quad (2)$$

Proposition 5.1 describes the relationship between the value functions of the fictitious alternatives $a^{(i)}$, and the relationship between the total additive value function $U(a)$ and the value obtained in correspondence to the most pessimistic and the most optimistic realizations of alternative a.

Proposition 5.1 1. *For each $a \in A$, if $i \geq k$, $i,k \in \{1,\ldots,n\}$, then $U(a^{(i)}) \geq U(a^{(k)})$,*
2. *For each $a \in A$, $U(a^{(1)}) \leq U(a) \leq U(a^{(n)})$.*

Let us now discuss elicitation of preference information and application of ROR to model (1).

In order to assign to each alternative a real number representing its degree of desirability, we need to know the sub-marginal value functions $u_j^i(\cdot)$, for all $j \in J$ and for all $i \in \{1,\ldots,n\}$. They can be obtained in two different ways: asking directly the DM which is the analytical expression of functions $u_j^i(\cdot)$, or inducing them from indirect preference information elicited by the DM on a set $A^R \subseteq A$ of alternatives called *reference alternatives*. The reference alternatives will be marked with a dash, like $\overline{a}$. We propose to use the second method, and thus the DM is asked to provide some preference information regarding pairs of alternatives or intensity of preference for quadruples of alternatives, such that, for all $\overline{a},\overline{b},\overline{c},\overline{d} \in A^R$,

- $\overline{a} \succsim \overline{b}$ iff $\overline{a}$ is at least as good as $\overline{b}$,
- $\overline{a} \succsim_j \overline{b}$ iff $\overline{a}$ is at least as good as $\overline{b}$ on criterion g_j,
- $(\overline{a},\overline{b}) \succsim^* (\overline{c},\overline{d})$ iff $\overline{a}$ is preferred to $\overline{b}$ at least as much as $\overline{c}$ is preferred to $\overline{d}$,

- $(\overline{a}, \overline{b}) \succsim_j^* (\overline{c}, \overline{d})$ iff, on criterion g_j, $\overline{a}$ is preferred to $\overline{b}$ at least as much as $\overline{c}$ is preferred to $\overline{d}$.

Let us notice that $\sim$, $\sim_j$, $\sim^*$ and $\sim_j^*$ are the symmetric parts of $\succsim$, $\succsim_j$, $\succsim^*$ and $\succsim_j^*$ while $\succ$, $\succ_j$, $\succ^*$ and $\succ_j^*$ are the asymmetric parts of $\succsim$, $\succsim_j$, $\succsim^*$ and $\succsim_j^*$. That is, for example, $\overline{a} \sim \overline{b}$ iff $\overline{a} \succsim \overline{b}$ and $\overline{b} \succsim \overline{a}$, while $\overline{a} \succ \overline{b}$ iff $\overline{a} \succsim \overline{b}$ and not($\overline{b} \succsim \overline{a}$).

All this preference information can be translated into inequality constraints on the values as follows:

- $U(\overline{a}) \geq [>]\, U(\overline{b})$ iff $\overline{a} \succsim [\succ]\, \overline{b}$,
- $U_j(\overline{a}) \geq [>]\, U_j(\overline{b})$ iff $\overline{a} \succsim_j \left[\succ_j\right] \overline{b}$,
- $U(\overline{a}) - U(\overline{b}) \geq [>]\, U(\overline{c}) - U(\overline{d}) \geq 0$ iff $(\overline{a}, \overline{b}) \succsim^* \left[\succ^*\right] (\overline{c}, \overline{d})$,
- $U_j(\overline{a}) - U_j(\overline{b}) \geq [>]\, U_j(\overline{c}) - U_j(\overline{d}) \geq 0$ iff $(\overline{a}, \overline{b}) \succsim_j^* \left[\succ_j^*\right] (\overline{c}, \overline{d})$.

Strict inequality constraints are translated into weak inequality constraints by using an auxiliary positive variable ε such that, for example, $U(\overline{a}) > U(\overline{b})$ becomes $U(\overline{a}) \geq U(\overline{b}) + \varepsilon$. In the following, E^{DM} denotes the set of inequality constraints translating the preference information provided by the DM.

We shall call *compatible* a value function satisfying the set of constraints in E^{DM}, as well as some monotonicity and normalization constraints:

$$\left.\begin{array}{l} u_j^i(x_j^k) - u_j^i(x_j^{k-1}) \geq 0, \text{ for each } j \in J,\ k = 2, \ldots, m_j(A),\ i = 1, \ldots, n \\ u_j^i(x_j^1) = 0, \text{ for each } j \in J, i = 1, \ldots, n \\ \displaystyle\sum_{\substack{j \in J \\ i=1,\ldots,n}} u_j^i\left(x_j^{m_j(A)}\right) = 1 \end{array}\right\} E^{MN}$$

where, for each $j \in J$, $m_j(A) = \left|\left\{g_j^i(a),\ i = 1, \ldots, n,\ a \in A\right\}\right|$, $x_j^{m_j(A)} = \max_{a \in A} g_j^n(a)$, $x_j^1 = \min_{a \in A} g_j^1(a)$, and the values $x_j^k, k = 1, \ldots, m_j(A)$, are ordered in an increasing way, i.e., $x_j^1 < x_j^2 < \ldots < x_j^{m_j(A)-1} < x_j^{m_j(A)}$.

Denoting by $E^{A^R} = \left\{E^{DM} \cup E^{MN}\right\}$ the whole set of constraints, to check the existence of at least one compatible value function, one has to solve the following optimization problem

$$\begin{array}{l} \varepsilon^* = \max \varepsilon, \\ \text{subject to } E^{A^R} \end{array}$$

where the variables are $u_j^i(x_j^k)$, $j \in J$, $i \in \{1, \ldots, n\}$, $k = 1, \ldots, m_j(A)$, and ε. If E^{A^R} is feasible and $\varepsilon^* > 0$, then there exists at least one compatible value function $U(\cdot)$; conversely, there does not exist any compatible value function $U(\cdot)$.

Supposing that more than one compatible value function exists, we indicate by $\mathcal{U}$ the set of all compatible value functions; in general, each of these functions will produce a different ranking on the set A of alternatives. This is why ROR methods take into account all compatible value functions instead of only one.

Definition 5.1 Given two alternatives $a, b \in A$ and the set $\mathcal{U}$ of compatible value functions on $A^R \subseteq A$, we say that a is possibly preferred to b, if a is at least as good as b for at least one compatible value function:

$$a \succsim^P b \Leftrightarrow \text{ there exists } U \in \mathcal{U} : U(a) \geq U(b).$$

Definition 5.2 Given two alternatives $a, b \in A$ and the set $\mathcal{U}$ of compatible value functions on $A^R \subseteq A$, we say that a is necessarily preferred to b, if a is at least as good as b for all compatible value functions:

$$a \succsim^N b \Leftrightarrow U(a) \geq U(b), \text{ for all } U \in \mathcal{U}.$$

Following the description of the $\Delta^{(i,k)}$ dominance relation defined in Sect. 4, it is meaningful from the DM's point of view to compare $U(a^{(i)})$ and $U(b^{(k)})$ for all compatible value functions $U \in \mathcal{U}$ and for all pairs of indicators (i, k). In fact, even if $not(a \succsim^N b)$, that is a is not at least as good as b for all compatible value functions, it is interesting to check if there exists some pair of indicators (i, k) such that $U(a^{(i)}) \geq U(b^{(k)})$ for all compatible value functions in order to understand which is the degree of plausibility of the necessary preference of a over b. For example, considering two alternatives $a, b \in A$ evaluated by means of 3-point intervals, as done in Sect. 4, if one discovers that $U(a^{(2)}) \geq U(b^{(3)})$ for all $U \in \mathcal{U}$ this means that, considering the most plausible evaluations for a (i.e., $g_j^2(a)$ for all j) and the most optimistic evaluations for b (i.e., $g_j^3(b)$ for all j), a is at least as good as b for all compatible value functions. Analogously, it is interesting to understand if there exists some pair of indicators (i, k) such that $U(a^{(i)}) \geq U(b^{(k)})$ for at least one value function compatible with the preferences provided by the DM. Going back to the previous example, if one gets that there does not exist any value function such that $U(b^{(3)}) \geq U(a^{(2)})$, this means that even considering the best evaluations of b (i.e., $g_j^{(3)}(b)$ for all j) and the most plausible evaluations for a (i.e., $g_j^{(2)}(a)$ for all j), there is no compatible value function for which b is at least as good as a.

In consequence of these considerations, the following two types of necessary and possible preference relations can be considered:

Definition 5.3 Given two alternatives $a, b \in A$, the set $\mathcal{U}$ of compatible value functions on $A^R \subseteq A$, and $i, k \in \{1, \ldots, n\}$, we say that a is (i, k)-possibly preferred to b, if $a^{(i)}$ is at least as good as $b^{(k)}$ for at least one compatible value function:

$$a \succsim^P_{(i,k)} b \Leftrightarrow \text{ there exists } U \in \mathcal{U} : U\left(a^{(i)}\right) \geq U\left(b^{(k)}\right).$$

Definition 5.4 Given two alternatives $a, b \in A$, the set $\mathcal{U}$ of compatible value functions on $A^R \subseteq A$, and $i, k \in \{1, \ldots, n\}$, we say that a is (i, k)-necessarily preferred to b, if $a^{(i)}$ is at least as good as $b^{(k)}$ for all compatible value functions:

$$a \succsim^{N}_{(i,k)} b \Leftrightarrow U\left(a^{(i)}\right) \geq U\left(b^{(k)}\right), \text{ for all } U \in \mathcal{U}.$$

For all $a, b \in A$, and for all $i, k \in \{1, \ldots, n\}$, we have that:

- $a \succsim^P b$ if $E^P(a,b) = E^{A^R} \cup \{U(a) \geq U(b)\}$ is feasible and $\varepsilon^P(a,b) > 0$, where $\varepsilon^P(a,b) = \max \varepsilon, \quad \text{s.t. constraints } E^P(a,b)$,
- $a \succsim^N b$ if $E^N(a,b) = E^{A^R} \cup \{U(b) \geq U(a) + \varepsilon\}$ is infeasible or $\varepsilon^N(a,b) \leq 0$, where $\varepsilon^N(a,b) = \max \varepsilon, \quad \text{s.t. constraints } E^N(a,b)$,
- $a \succsim^P_{(i,k)} b$ if $E^P_{(i,k)}(a,b) = E^{A^R} \cup \left\{U(a^{(i)}) \geq U(b^{(k)})\right\}$ is feasible and $\varepsilon^P_{(i,k)}(a,b) > 0$, where $\varepsilon^P_{(i,k)}(a,b) = \max \varepsilon, \quad \text{s.t. constraints } E^P_{(i,k)}(a,b)$,
- $a \succsim^N_{(i,k)} b$ if $E^N_{(i,k)}(a,b) = E^{A^R} \cup \left\{U(b^{(k)}) \geq U(a^{(i)}) + \varepsilon\right\}$ is infeasible or $\varepsilon^N_{(i,k)}(a,b) \leq 0$, where $\varepsilon^N_{(i,k)}(a,b) = \max \varepsilon, \quad \text{s.t. constraints } E^N_{(i,k)}(a,b)$.

6 Properties of necessary and possible preference relations

As to the basic properties of the classical necessary and possible preference relations, they were discussed in Greco et al. (2008) and Giarlotta and Greco (2013). Let us recall that the necessary relation is included in the possible preference relation ($\succsim^N \subseteq \succsim^P$), that the necessary preference relation is a partial preorder on A, and that the possible preference relation is strongly complete and negatively transitive.

Proposition 6.1 describes the relationship between the classical dominance and necessary preference relations, and then the relationship between the dominance and necessary preference relations in case of imprecise evaluations.

Proposition 6.1 1. $\Delta \subseteq \succsim^N$,
2. *For all* $i, k \in \{1, \ldots, n\}$, $\Delta^{(i,k)} \subseteq \succsim^N_{(i,k)}$.

Proposition 6.2 gives some properties of the necessary and possible preference relations in case of imprecise evaluations. Moreover, it provides an inclusion property for $\succsim^N_{(i,k)}$ and $\succsim^P_{(i,k)}$, and a completeness property for $\succsim^N_{(i,k)}$ and $\succsim^P_{(k,i)}$.

Proposition 6.2 1. *For all* $i, k \in \{1, \ldots, n\}$, $\succsim^N_{(i,k)} \subseteq \succsim^P_{(i,k)}$,
2. *If* $i \geq k$, $i, k \in \{1, \ldots, n\}$, *then* $\succsim^N_{(i,k)}$ *is reflexive,*
3. *If* $i \leq k$, $i, k \in \{1, \ldots, n\}$, *then* $\succsim^N_{(i,k)}$ *is transitive,*
4. *For all* $a, b \in A$, *for all* $i, k \in \{1, \ldots, n\}$, *we have* $a \succsim^N_{(i,k)} b$ *or* $b \succsim^P_{(k,i)} a$,
5. *If* $i \geq k$, $i, k \in \{1, \ldots, n\}$, *then* $\succsim^P_{(i,k)}$ *is strongly complete and negatively transitive.*

Note 6.1 Let us observe that by points 1 and 3 of Proposition 6.2, $\succsim^N_{(i,i)}$ is a partial preorder for all $i = 1, \ldots, n$.

Proposition 6.3 specifies the inclusion between different necessary and possible preference relations in case of imprecise evaluations. Moreover, it links the preference relations for the case of imprecise evaluations with the preference relations for the case of precise evaluations.

Fig. 3 Relationships between all kinds of dominance relations and preference relations

$$\begin{array}{ccccc}
\Delta^S & \Rightarrow & \Delta & \Rightarrow & \Delta^W \\
\Downarrow & & \Downarrow & & \Downarrow \\
\succsim^{SN} & \Rightarrow & \succsim^{N} & \Rightarrow & \succsim^{WN} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\succsim^{SP} & \Rightarrow & \succsim^{P} & \Rightarrow & \succsim^{WP}
\end{array}$$

Proposition 6.3 1. *If* $i_1 \geq i$ *and* $k_1 \leq k$, $i, k, i_1, k_1 \in \{1, \ldots, n\}$, *then* $\succsim^N_{(i,k)} \subseteq \succsim^N_{(i_1,k_1)}$,
2. *If* $i_1 \geq i$ *and* $k_1 \leq k$, $i, k, i_1, k_1 \in \{1, \ldots, n\}$, *then* $\succsim^P_{(i,k)} \subseteq \succsim^P_{(i_1,k_1)}$,
3. $\succsim^N_{(1,n)} \subseteq \succsim^N \subseteq \succsim^N_{(n,1)}$,
4. $\succsim^P_{(1,n)} \subseteq \succsim^P \subseteq \succsim^P_{(n,1)}$.

Some propositions specifying more properties of the necessary and possible preference relations in case of imprecise evaluations are given in the Appendix.

Note 6.2 Since the necessary and possible preference relations $\succsim^N_{(1,n)}$, $\succsim^N_{(n,1)}$, $\succsim^P_{(1,n)}$ and $\succsim^P_{(n,1)}$, are the only preference relations for the case of imprecise evaluations that can be linked to the necessary and possible preference relations for the case of precise evaluations, we shall use also the following notation:

– "strongly necessary preference relation", to indicate the necessary preference relation $\succsim^N_{(1,n)}$ (denoted by $\succsim^{SN}$),
– "strongly possible preference relation", to indicate the possible preference relation $\succsim^P_{(1,n)}$ (denoted by $\succsim^{SP}$),
– "weakly necessary preference relation", to indicate the necessary preference relation $\succsim^N_{(n,1)}$ (denoted by $\succsim^{WN}$), and
– "weakly possible preference relation", to indicate the possible preference relation $\succsim^P_{(n,1)}$ (denoted by $\succsim^{WP}$).

Considering the weak, normal and strong dominance, necessary and possible preference relations, as a straightforward consequence of Propositions 4.2, 5.1, 6.1 and 6.2, and of the inclusion $\succsim^N \subseteq \succsim^P$, we obtain the set of relationships shown in Fig. 3.

7 A simple example

Let us imagine that the dean of a high school intends to give a scholarship to a good student; for this reason, she has to choose a laureate among 10 students of the school considered to be the best candidates. In order to manage this situation, the dean decides to use an MCDA approach taking into account evaluations of the student on three subjects: Mathematics (*Mat*), Physics (*Phy*) and Computer Science (*Com*). Each subject

Table 1 Evaluations of students on three criteria

Student\subject	Mat	Phy	Com
A	M	VG	VG
B	[G, G, VG]	[VB, M, M]	[B, M, G]
C	[B, G, VG]	G	[M, M, G]
D	[G, VG, VG]	[M, M, G]	[M, G, G]
E	VG	[VB, M, G]	[M, M, G]
F	[VB, M, G]	[B, B, M]	[B, B, M]
H	[M, G, G]	[M, G, G]	[M, G, G]
I	VG	[M, G, VG]	B
L	[VB, VB, B]	[B,M,M]	[VB, B, M]
M	[VB, B, B]	[G, G, VG]	VG

is thus an evaluation criterion with an ordinal scale composed of five levels ordered from the worst to the best: Very Bad (VB), Bad (B), Medium (M), Good (G), and Very Good (VG). Differently from previous cases, the dean has to face a new problem because some students got imprecise evaluations on some criteria. The students' evaluations are shown in Table 1.

One can see that in the evaluation table there are either crisp evaluations or 3-point intervals. In order to apply our method we need to consider them all as 3-point intervals. For 3-point intervals we have nested subintervals, $\left[g^1(a), g^3(a)\right]$ and $\left[g^2(a), g^2(a)\right]$, with the corresponding levels of plausibility L_1 and L_2, where $L_1 < L_2$. Remark that a crisp evaluation $g(a)$ is, formally, a 1-point interval and it can be represented by the 3-point interval $[g(a), g(a), g(a)]$ with equal subintervals $[g(a), g(a)]$, $[g(a), g(a)]$. In our example this would mean that the evaluation of student **A** on Mat is represented by the 3-point interval $[M, M, M]$. As to the levels of plausibility, we have to distinguish between 3-point intervals with three different evaluations and 3-point intervals with two different evaluations only. On one hand, $Mat(\mathbf{F}) = [VB, M, G]$ means that the evaluation of **F** with respect to Mat belongs to the interval $[VB, G]$ with plausibility L_1, and it can be equal to M with level of plausibility L_2. On the other hand, $Com(\mathbf{C}) = [M, M, G]$ and $Com(\mathbf{D}) = [M, G, G]$ mean that the evaluation of **C** and **D** on Com belongs to the interval $[M, G]$ with level of plausibility L_1, but the level of plausibility L_2 is assigned to the evaluation M for student **C** and to the evaluation G for student **D**.

The only preference relation that stems from the evaluations of the students is the normal dominance relation shown in Fig. 4a. Therefore, taking into account all different indicators, one can observe that five of the nine students dominate **L**. Referring to the weak and strong dominance relations shown in Fig. 4b, c, the first impression about the bad quality of **L** is confirmed. Indeed, considering the best evaluations of **L** and the worst evaluations for all others students, **L** weakly dominates **F** only (see Fig. 4b), while four students strongly dominate **L** (see Fig. 4c). Since two different levels of plausibility are considered in case of 3-point intervals, it is interesting to see what happens when the evaluations have the highest level of plausibility for all students.

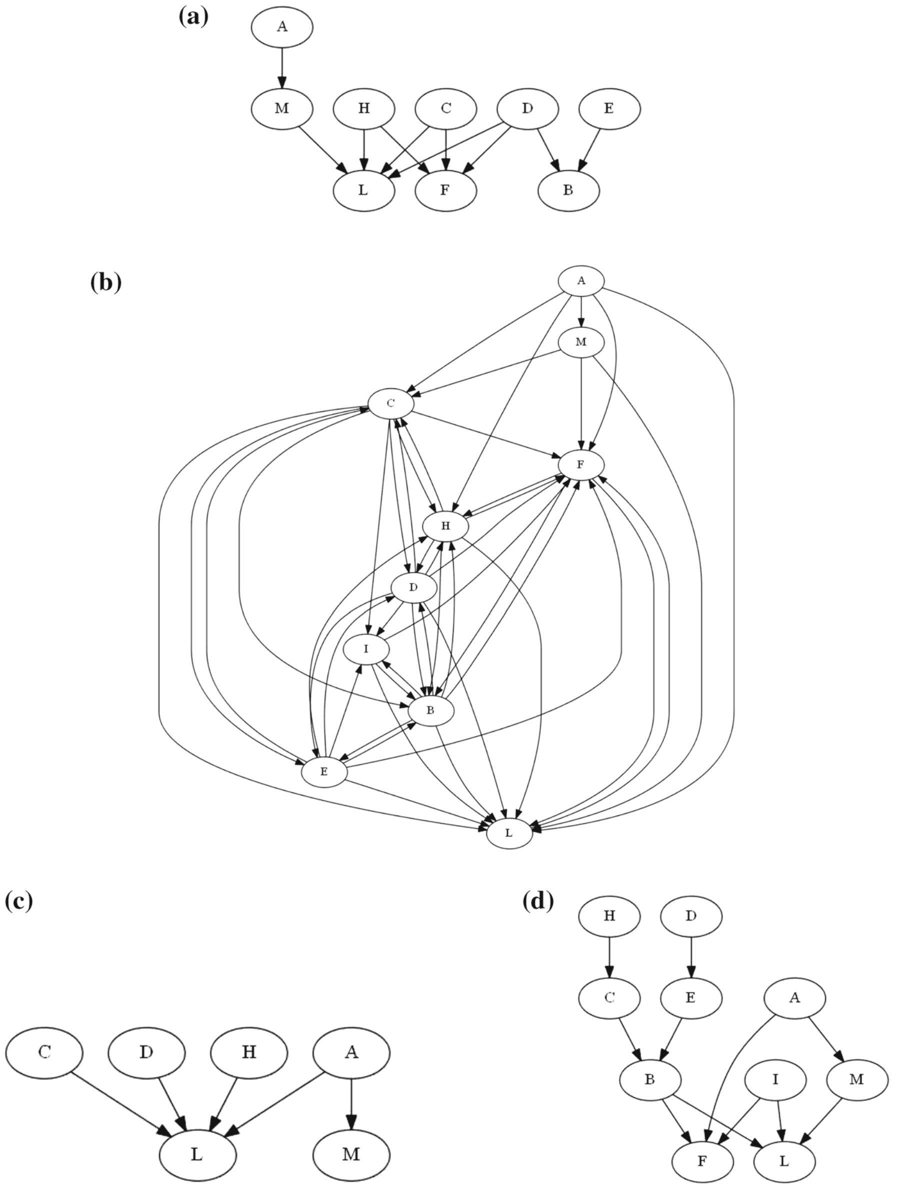

Fig. 4 Dominance relations in the set of students obtained for imprecise evaluations shown in Table 1 and according to definitions introduced in Sect. 2; (**a**) dominance relation Δ; (**b**) weak dominance relation $\Delta^{(3,1)}$; (**c**) strong dominance relation $\Delta^{(1,3)}$; (**d**) dominance relation for evaluations with the highest level of plausibility $\Delta^{(2,2)}$

This result is shown in Fig. 4d. One can observe that **L** is dominated there by all other students but **F**, and that **D** and **H** dominate four other students each. Remark that due to point 2 of Proposition 4.1, the strong dominance relation $\Delta^{(1,3)}$ and the dominance

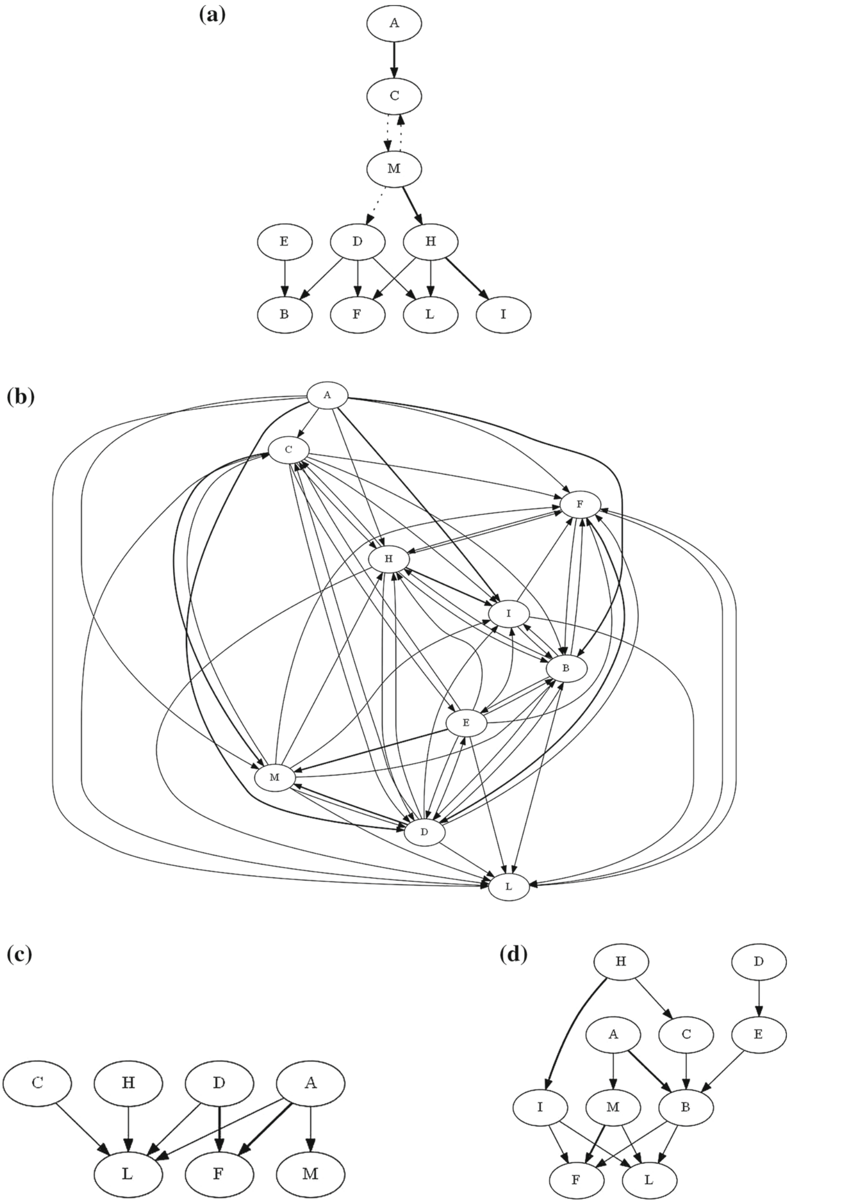

Fig. 5 Necessary preference relations in the set of students obtained after including the preference information provided by the dean and computed according to definitions given in Sect. 5. Full arrows represent the corresponding dominance relation; dotted arrows represent preference information provided by the dean; bold arrows represent new necessary preferences got in consequence of the added preference information. Note that by point 6.2 of Proposition 6.2, the strong necessary preference relation $\succsim^N_{(1,3)}$ is transitive while the weak necessary preference relation $\succsim^N_{(3,1)}$ is not; (**a**) necessary preference relation $\succsim^N$; (**b**) weak necessary preference relation $\succsim^N_{3,1}$; (**c**) strong necessary preference relation $\succsim^N_{(1,3)}$; (**d**) necessary preference relation having the highest level of plausibility $\succsim^N_{(2,2)}$

relation $\Delta^{(2,2)}$ shown in Fig. 4c, d are transitive, while the weak dominance relation shown in Fig. 4b is not transitive.

In order to get a more conclusive recommendation, the dean has expressed her preferences through three pieces of the following preference information:

- "student **M** is preferred to student **D**", which is translated into the constraint $U(\mathbf{M}) > U(\mathbf{D})$;
- "Student **M** is preferred to student **I** more than student **C** is preferred to student **H**", which is translated into the constraint $U(\mathbf{M}) - U(\mathbf{I}) > U(\mathbf{C}) - U(\mathbf{H})$;
- "Student **C** and student **M** are indifferent", which is translated into the constraint $U(\mathbf{C}) = U(\mathbf{M})$.

Taking into account this preference information, we compute the necessary preference relation $\succsim^N$, the weak necessary preference relation $\succsim^N_{(3,1)}$, the strong necessary preference relation $\succsim^N_{(1,3)}$ and the necessary preference relation $\succsim^N_{(2,2)}$ obtained for evaluations with the highest level of plausibility. These necessary preference relations are shown in Fig. 5.

One can see that the preference information provided by the dean enriched considerably the dominance preference relations. Without any preference information, **A** dominated students **L** and **M** only, while after the preference information was added, **A** is necessarily preferred to eight other students and neither necessarily nor weakly necessarily preferred only to **E** (Fig. 5a, b). Looking at Fig. 5c, d, one can observe two interesting things: **A** is strongly necessarily preferred to the largest number (3) of students, and **H** is necessarily preferred to the largest number of students (5) considering the necessary preference relation computed for evaluations with the highest level of plausibility. This means that even taking into account the worst possible evaluations, **A** is a good student, however for evaluations with the highest level of plausibility, **H** is better than **A**.

The above discussion shows that the proposed method permits to get a useful insight into the multiple criteria choice problem that became more complicated by the fact of imprecise evaluations of alternatives. Based on minimum available information about the range and tipicality of imprecise evaluations, and using few examples of exhibited preferences, the method sheds light on the preference relations in the set of alternatives and facilitates a conscious decision making.

8 Conclusions

In this paper, we dealt with one of the most important issues of multiple criteria decision aiding (MCDA), that is the imprecise evaluations of alternatives. The possible sources of this imprecision are, for example, lack of data, imprecise measurement or intangible criteria. Many authors have studied different types of imprecision regarding weights of criteria, utility functions or probabilities about the different states of the world. In our approach, we are supposing that evaluations of the alternatives with respect to the different criteria can be imprecise and expressed by n-point intervals. These intervals are characterized not only by the largest interval of possible evaluations, but also by its subintervals sequentially nested one in another. To each of these subintervals is associated an increasing level of plausibility such that the plausibility of a subinterval is not lower than the plausibility of the subinterval containing it. Due to this way of representing the imprecision, our approach permits fine modelling of imprecise multiple criteria evaluations, taking into account a whole spectrum of attitudes ranging from an extremely pessimistic one to an extremely optimistic one in the evaluations. Moreover, differently from other ways of dealing with imprecision, such as stochastic multiobjective acceptability analysis (SMAA) or fuzzy numbers, n-point intervals take into account only the most stable, robust and meaningful information carried by imprecise evaluations. n-point intervals can be applied to any MCDA method but, in this paper, we focused on additive value functions and, in order to take into account the whole set of value functions compatible with the preference information provided by the DM, we adapted robust ordinal regression (ROR). In result of applying ROR, one gets necessary and possible preference relations for all realizations of the imprecise evaluations.

The methodology proposed in this paper follows the constructivist approach (Roy 1993). This means that MCDA methods do not assume that there pre-exist some preference system in the DM's mind that need to be discovered, but the DM's preferences have to be built step by step in the course of an interaction between the DM and the analyst responsible for mathematical modeling. In other words, MCDA methods should be seen as tools for going deeper into the decision problem, for exploring various possibilities, interpreting them, debating and arguing, rather than tools able to make the decision. As a consequence, the performances of the MCDA methods cannot be tested on some benchmarks, like, for example, machine learning methods (Corrente et al. 2013). Instead, the MCDA methods are acceptable if they possess some practical and theoretical properties judged as desirable in the actual decision context (Keeney and Raiffa 1993; Roy and Słowiński 2013). Unfortunately, many researchers are tempted to compare different MCDA methods by basing their conclusions on comparison of end results obtained by these methods. As argued in Roy and Słowiński (2013) such a comparison is ill-founded.

The presented methodology can be extended in several directions that are shortlisted below:

- consideration of preference models in the form of outranking relations instead of value functions (Roy 1996);
- consideration of the decision rule preference model composed of "if..., then..." decision rules induced from the DM's preference information structured by dominance-based rough set approach (Greco et al. 2001; Słowiński et al. 2009);
- consideration of the hierarchy of criteria using the multiple criteria hierarchy process (Corrente et al. 2012).

Acknowledgements The first and the second authors wish to acknowledge funding by the "FIR of the University of Catania BCAEA3 New developments in Multiple Criteria Decision Aiding (MCDA) and their application to territorial competitiveness"

Appendix

Proof of Proposition 4.1 1. Let $a \in A$, and $i, k \in \{1, \ldots, n\}$ such that $i \geq k$; this implies that $g_j^i(a) \geq g_j^k(a)$, $\forall j = 1, \ldots, m$, and thus $a\Delta^{(i,k)}a$. Therefore $\Delta^{(i,k)}$ is reflexive. □

2. Let us suppose that $a, b, c \in A$ such that $a\Delta^{(i,k)}b$, $b\Delta^{(i,k)}c$ and $i \leq k$ with $i, k \in \{1, \ldots, n\}$;

$$\begin{cases} a\Delta^{(i,k)}b \Leftrightarrow g_j^i(a) \geq g_j^k(b), \forall j \in J, \\ b\Delta^{(i,k)}c \Leftrightarrow g_j^i(b) \geq g_j^k(c), \forall j \in J. \end{cases}$$

Being $i \leq k$ we get that $g_j^k(b) \geq g_j^i(b)$, and, consequently:

$$g_j^i(a) \geq g_j^k(b) \geq g_j^i(b) \geq g_j^k(c), \forall j = 1, \ldots, m \Rightarrow g_j^i(a) \geq g_j^k(c), \\ \forall j \in J \Leftrightarrow a\Delta^{(i,k)}c.$$

Thus $\Delta^{(i,k)}$ is transitive. □

3. It follows from points 1 and 2 of this Proposition since a partial preorder is a reflexive and transitive binary relation. □

4. Let be $a, b \in A$ and $i, k, r, s \in \{1, \ldots, n\}$ such that $a\Delta^{(i,k)}b$, $r \geq i$ and $k \geq s$. Then

$$\begin{cases} a\Delta^{(i,k)}b \Leftrightarrow g_j^i(a) \geq g_j^k(b), \quad \forall j \in J, \\ r \geq i \quad \Leftrightarrow g_j^r(a) \geq g_j^i(a), \quad \forall j \in J, \\ k \geq s \quad \Leftrightarrow g_j^k(b) \geq g_j^s(b), \quad \forall j \in J. \end{cases}$$

From this it follows that:

$$g_j^r(a) \geq g_j^i(a) \geq g_j^k(b) \geq g_j^s(b), \ \forall j \in J \Rightarrow g_j^r(a) \geq g_j^s(b), \\ \forall j \in J \Leftrightarrow a\Delta^{(r,s)}b.$$

□

5. Let $a, b, c \in A$, and $i, k, i_1, k_1 \in \{1, \ldots, n\}$ such that $a\Delta^{(i,k)}b$, $b\Delta^{(i_1,k_1)}c$ and $k \geq i_1$. Then, we have:

$$\begin{cases} a\Delta^{(i,k)}b \quad \Leftrightarrow g_j^i(a) \geq g_j^k(b), \quad \forall j \in J, \\ b\Delta^{(i_1,k_1)}c \Leftrightarrow g_j^{i_1}(b) \geq g_j^{k_1}(c), \quad \forall j \in J, \\ k \geq i_1 \quad \Leftrightarrow g_j^k(b) \geq g_j^{i_1}(b), \quad \forall j \in J. \end{cases}$$

From this it follows that:

$$g^{i}_{j}(a) \geq g^{k}_{j}(b) \geq g^{i_1}_{j}(b) \geq g^{k_1}_{j}(c), \ \forall j \in J \Rightarrow g^{i}_{j}(a) \geq g^{k_1}_{j}(c), \ \forall j \in J \Leftrightarrow a\Delta^{(i,k_1)}c.$$

For point 4 of this Proposition, if $r, s \in \{1, \ldots, n\}$ such that $r \geq i$ and $s \leq k_1$, then $a\Delta^{(r,s)}c$. □

6. We have said that $\Delta = \cap_{i=1}^{n} \Delta^{(i,i)}$; since $\Delta^{(i,i)}$ is a partial preorder for point 3 of this Proposition, and since the intersection of partial preorders is a partial preoder, Δ is a partial preorder. □
7. Let $a, b, c \in A$, and $i, k \in \{1, \ldots, n\}$, such that $a\Delta^{(i,k)}b$, $b\Delta c$, $s \geq i$ and $k \geq t$. Then, we have:

$$\begin{cases} a\Delta^{(i,k)}b \Leftrightarrow g^{i}_{j}(a) \geq g^{k}_{j}(b), \quad \forall j \in J, \\ b\Delta c \quad\ \Leftrightarrow g^{r}_{j}(b) \geq g^{r}_{j}(c), \quad \forall j \in J, \quad \forall r = 1, \ldots, n, \\ s \geq i \quad\ \Leftrightarrow g^{s}_{j}(a) \geq g^{i}_{j}(a), \quad \forall j \in J, \\ k \geq t \quad\ \Leftrightarrow g^{k}_{j}(c) \geq g^{t}_{j}(c), \quad \forall j \in J. \end{cases}$$

From this it follows that:

$$g^{s}_{j}(a) \geq g^{i}_{j}(a) \geq g^{k}_{j}(b) \geq g^{k}_{j}(c) \geq g^{t}_{j}(c), \ \forall j \in J \Rightarrow g^{s}_{j}(a) \geq g^{t}_{j}(c), \\ \forall j \in J \Leftrightarrow a\Delta^{(s,t)}c.$$

□

8. Let $a, b, c \in A$, and $i, k \in \{1, \ldots, n\}$, such that $a\Delta b$, $b\Delta^{(i,k)}c$, $s \geq i$ and $k \geq t$. Then, we have:

$$\begin{cases} a\Delta b \quad\ \Leftrightarrow g^{r}_{j}(a) \geq g^{r}_{j}(b), \quad \forall j \in J, \quad \forall r = 1, \ldots, n, \\ b\Delta^{(i,k)}c \Leftrightarrow g^{i}_{j}(b) \geq g^{k}_{j}(c), \quad \forall j \in J, \\ s \geq i \quad\ \Leftrightarrow g^{s}_{j}(a) \geq g^{i}_{j}(a), \quad \forall j \in J, \\ k \geq t \quad\ \Leftrightarrow g^{k}_{j}(c) \geq g^{t}_{j}(c), \quad \forall j \in J, \end{cases}$$

From this it follows that:

$$g^{s}_{j}(a) \geq g^{i}_{j}(a) \geq g^{i}_{j}(b) \geq g^{k}_{j}(c) \geq g^{t}_{j}(c), \quad \forall j \in J \Rightarrow g^{s}_{j}(a) \geq g^{t}_{j}(c), \\ \forall j \in J \Leftrightarrow a\Delta^{(s,t)}c.$$

□

Proof of Proposition 4.2 1. For all $i \in \{1, \ldots, n\}$, from point 4 of Proposition 4.1, we have $\Delta^{(1,n)} \subseteq \Delta^{(i,i)}$ because $i \geq 1$ and $i \leq n$; from this follows that $\Delta^{(1,n)} \subseteq \cap_{i=1}^{n} \Delta^{(i,i)} = \Delta$, and this proves the first part of the Proposition. □
For the same reason, $\forall i \in \{1, \ldots, n\}$ we have $\Delta^{(i,i)} \subseteq \Delta^{(n,1)}$ because $n \geq i$ and $1 \leq i$. From this it follows that $\Delta = \cap_{i=1}^{n} \Delta^{(i,i)} \subseteq \Delta^{(n,1)}$, and this proves the second part of the Proposition. □
2. For all $i, k \in \{1, \ldots, n\}$, from point 4 of Proposition 4.1, since $1 \leq i$ and $k \leq n$, we have $\Delta^{(1,n)} \subseteq \Delta^{(i,k)}$ and $\Delta^{(i,k)} \subseteq \Delta^{(n,1)}$. In this way we obtain the thesis. □

Proof of Proposition 5.1 1. Let $a \in A$ and $i, k \in \{1, \ldots, n\}$. From the definition, fictitious alternatives $a^{(i)}$ and $a^{(k)}$ are such that $\forall j \in J$, and $\forall r \in \{1, \ldots, n\}$, $g_j^r\left(a^{(i)}\right) = g_j^i(a)$, and $g_j^r\left(a^{(k)}\right) = g_j^k(a)$. Since $i \geq k$, $\forall r \in \{1, \ldots, n\}$ and $\forall j \in J$, $g_j^r(a^{(i)}) \geq g_j^r(a^{(k)})$, and using the monotonicity of marginal value functions $u_{j,r}(\cdot)$, we obtain $\forall r \in \{1, \ldots, n\}$, $\forall j \in J$ $u_j^r(g_j^r(a^{(i)})) \geq u_j^r(g_j^r(a^{(k)}))$; adding up with respect to j and r we obtain the thesis. □
2. We have seen that:

$$U(a^{(1)}) = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} u_j^i(g_j^1(a)) \right], \quad U(a) = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} u_j^i(g_j^i(a)) \right]$$
$$U(a^{(n)}) = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} u_j^i(g_j^n(a)) \right].$$

$\forall j \in J$, and $\forall i \in \{1, \ldots, n\}$, since $g_j^1(a) \leq g_j^i(a) \leq g_j^n(a)$ and by monotonicity of marginal value functions $u_j^i(\cdot)$, we obtain:

$$u_j^i\left(g_j^1(a)\right) \leq u_j^i\left(g_j^i(a)\right) \leq u_j^i\left(g_j^n(a)\right)$$

and therefore adding up with respect to j and i we obtain the thesis, that is

$$U\left(a^{(1)}\right) \leq U(a) \leq U\left(a^{(n)}\right).$$

□

Proof of Proposition 6.1 1. Let $a, b \in A$, such that $a \Delta b$. This implies that $g_j^i(a) \geq g_j^i(b)$, $\forall j \in J, \forall i = 1, \ldots, n$. We know that, for all $U \in \mathcal{U}$:

$$U(a) = \sum_{j=1}^{m} \left[\sum_{i=1}^{n} u_j^i(g_j^i(a)) \right].$$

From monotonicity of marginal value functions $u_j^i(\cdot)$, we have that $\forall j \in J$, and $\forall i = 1, \ldots, n$, $u_j^i(g_j^i(a)) \geq u_j^i(g_j^i(b))$ and adding up with respect to indices j

and i, we obtain $U(a) \geq U(b)$ for all compatible value functions, thus we obtain the thesis. □

2. Let $a, b \in A$, and $i, k \in \{1, \ldots, n\}$, such that $a \Delta^{(i,k)} b$. This implies that $g_j^i(a) \geq g_j^k(b), \forall j \in J$. We know that, for all $U \in \mathcal{U}$:

$$U(a^{(i)}) = \sum_{j=1}^{m} \left[\sum_{r=1}^{n} u_j^r(g_j^i(a)) \right], \quad U(b^{(k)}) = \sum_{j=1}^{m} \left[\sum_{r=1}^{n} u_j^r(g_j^k(b)) \right].$$

From the monotonicity of marginal value functions $u_j^r(\cdot)$ we have that $\forall j \in J$, and $\forall r = 1, \ldots, n$, $u_j^r(g_j^i(a)) \geq u_j^r(g_j^k(b))$ and adding up with respect to indices j and r we obtain the thesis. □

Proof of Proposition 6.2 1. $\forall a, b \in A, \forall i, k \in \{1, \ldots, n\}$, if $a^{(i)}$ is at least as good as $b^{(k)}$ for all compatible value functions ($a \succsim_{(i,k)}^N b$), then there exists at least one compatible value function for which $a^{(i)}$ is at least as good as $b^{(k)}$ ($a \succsim_{(i,k)}^P b$). □

2. It follows from point 1 of Proposition 5.1. □
3. Let $a, b, c \in A$ and $i, k \in \{1, \ldots, n\}$, $i \leq k$, such that $a \succsim_{(i,k)}^N b$ and $b \succsim_{(i,k)}^N c$. This means that for all $U \in \mathcal{U}$, $U(a^{(i)}) \geq U(b^{(k)})$ and $U(b^{(i)}) \geq U(c^{(k)})$. Since $i \leq k$, by point 1 of Proposition 5.1 we have that for all $U \in \mathcal{U}$, $U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(i)}) \geq U(c^{(k)})$ and, consequently, $U(a^{(i)}) \geq U(c^{(k)})$ for all $U \in \mathcal{U}$, that is $a \succsim_{(i,k)}^N c$. □
4. Let $a, b \in A$, and $i, k \in \{1, \ldots, n\}$, such that $a \not\succsim_{(i,k)}^N b$. This means that $\exists U \in \mathcal{U} : U(a^{(i)}) < U(b^{(k)})$. Therefore, $b \succsim_{(k,i)}^P (a)$. □
5. Let $a, b \in A$, and $i, k \in \{1, \ldots, n\}$ with $i \geq k$ such that $a \not\succsim_{(i,k)}^P b$. This means that for all $U \in \mathcal{U}, U(b^{(k)}) > U(a^{(i)})$. Since $i \geq k$, and from point 1 of Proposition 5.1, we obtain that for all $U \in \mathcal{U}$, $U(b^{(i)}) \geq U(b^{(k)}) > U(a^{(i)}) \geq U(a^{(k)})$; thus for all $U \in \mathcal{U}, U(b^{(i)}) > U(a^{(k)})$, therefore $b \succsim_{(i,k)}^N a$ implying $b \succsim_{(i,k)}^P a$ by point 1 of this Proposition. In this way, $\succsim_{(i,k)}^P$ is strongly complete.
   Let $a, b, c \in A, i, k \in \{1, \ldots, n\}$, such that $i \geq k$, $a \not\succsim_{(i,k)}^P b$ and $b \not\succsim_{(i,k)}^P c$. Then, we have:

$$\begin{cases} a \not\succsim_{(i,k)}^P b \Leftrightarrow U(a^{(i)}) < U(b^{(k)}), & \forall U \in \mathcal{U}, \\ b \not\succsim_{(i,k)}^P c \Leftrightarrow U(b^{(i)}) < U(c^{(k)}), & \forall U \in \mathcal{U}. \end{cases}$$

From this and from point 1 of Proposition 5.1 it follows that

$$U(a^{(i)}) < U(b^{(k)}) \leq U(b^{(i)}) < U(c^{(k)}), \quad \forall U \in \mathcal{U} \Rightarrow U(a^{(i)}) < U(c^{(k)}), \\ \forall U \in \mathcal{U} \Leftrightarrow a \not\succsim_{(i,k)}^P c.$$

This proves that $\succsim_{(i,k)}^P$ is negatively transitive. □

Proof of Proposition 6.3 1. Let $a, b \in A$, and $i, k, i_1, k_1 \in \{1, \ldots, n\}$, such that $i_1 \geq i, k_1 \leq k$ and $a \succsim^N_{(i,k)} b$. Then, from point 1 of Proposition 5.1, we have:

$$\begin{cases} a \succsim^N_{(i,k)} b \Leftrightarrow U(a^{(i)}) \geq U(b^{(k)}), & \forall U \in \mathcal{U}, \\ i_1 \geq i \quad\quad \Rightarrow U(a^{(i_1)}) \geq U(a^{(i)}), & \forall U \in \mathcal{U}, \\ k_1 \leq k \quad\quad \Rightarrow U(b^{(k_1)}) \leq U(b^{(k)}), & \forall U \in \mathcal{U}. \end{cases}$$

Thus:

$$U(a^{(i_1)}) \geq U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(k_1)}), \forall U \in \mathcal{U} \Rightarrow U(a^{(i_1)}) \geq U(b^{(k_1)}),$$
$$\forall U \in \mathcal{U} \Leftrightarrow a \succsim^N_{(i_1,k_1)} b.$$

□

2. Let $a, b \in A$, and $i, k, i_1, k_1 \in \{1, \ldots, n\}$ such that $i_1 \geq i, k_1 \leq k$ and $a \succsim^P_{(i,k)} b$. Then, from point 1 of Proposition 5.1, we have:

$$\begin{cases} a \succsim^P_{(i,k)} b \Leftrightarrow \exists U \in \mathcal{U} : U(a^{(i)}) \geq U(b^{(k)}), \\ i_1 \geq i \Rightarrow U(a^{(i_1)}) \geq U(a^{(i)}), \quad \forall U \in \mathcal{U}, \\ k_1 \leq k \Rightarrow U(b^{(k_1)}) \leq U(b^{(k)}), \quad \forall U \in \mathcal{U}. \end{cases}$$

Thus:

$$\exists U \in \mathcal{U} : U(a^{(i_1)}) \geq U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(k_1)}) \Rightarrow \exists U \in \mathcal{U} : U(a^{(i_1)})$$
$$\geq U(b^{(k_1)}) \Leftrightarrow a \succsim^P_{(i_1,k_1)} b.$$

□

3. Let $a, b \in A$ such that $a \succsim^N_{(1,n)} b$. This means that $U(a^{(1)}) \geq U(b^{(n)}), \forall U \in \mathcal{U}$. From point 2 of Proposition 5.1, we have that $U(a) \geq U(a^{(1)}) \geq U(b^{(n)}) \geq U(b)$, $\forall U \in \mathcal{U}$, and thus we obtain $U(a) \geq U(b), \forall U \in \mathcal{U}$, that is $a \succsim^N b$. In this way we proved that $\succsim^N_{(1,n)} \subseteq \succsim^N$.
   Analogously, $a \succsim^N b$ means that $U(a) \geq U(b)$, $\forall U \in \mathcal{U}$; from point 2 of Proposition 5.1 we obtain $U(a^{(n)}) \geq U(a) \geq U(b) \geq U(b^{(1)}), \forall U \in \mathcal{U}$, and thus we have $U(a^{(n)}) \geq U(b^{(1)}), \forall U \in \mathcal{U}$, that is $a \succsim^N_{(n,1)} b$. In this way we proved that $\succsim^N \subseteq \succsim^N_{(n,1)}$. □
4. Let $a, b \in A$ such that $a \succsim^P_{(1,n)} b$. This means that $\exists U \in \mathcal{U} : U(a^{(1)}) \geq U(b^{(n)})$. From point 2 of Proposition 5.1 we have:

$$U(a) \geq U(a^{(1)}) \geq U(b^{(n)}) \geq U(b) \Rightarrow U(a) \geq U(b) \Leftrightarrow a \succsim^P b.$$

   In this way we proved that $\succsim^P_{(1,n)} \subseteq \succsim^P$.

Analogously, $a \succsim^P b$ means that $\exists U \in \mathcal{U} : U(a) \geq U(b)$; from point 2 of Proposition 5.1 we obtain:

$$U(a^{(n)}) \geq U(a) \geq U(b) \geq U(b^{(1)}) \Rightarrow U(a^{(n)}) \geq U(b^{(1)}) \Leftrightarrow a \succsim^P_{(n,1)} b.$$

In this way we proved that $\succsim^P \subseteq \succsim^P_{(n,1)}$. □

Proposition 8.1 provides some further results regarding the imprecise necessary and possible preference relations.

Proposition 8.1 1. *If* $a \succsim^N_{(i,k)} b$, $b \succsim^N_{(i_1,k_1)} c$, *and* $k \geq i_1$, *with* $i, k, i_1, k_1 \in \{1, \ldots, n\}$, *then* $a \succsim^N_{(r,s)} c$, *for all* $r, s \in \{1, \ldots, n\}$ *such that* $r \geq i$ *and* $s \leq k_1$,
2. *If* $a \succsim^N_{(i,k)} b$, $b \succsim^P_{(i_1,k_1)} c$, *and* $k \geq i_1$, *with* $i, k, i_1, k_1 \in \{1, \ldots, n\}$, *then* $a \succsim^P_{(r,s)} c$, *for all* $r, s \in \{1, \ldots, n\}$ *such that* $r \geq i$ *and* $s \leq k_1$,
3. *If* $a \succsim^P_{(i,k)} b$, $b \succsim^N_{(i_1,k_1)} c$, *and* $k \geq i_1$, *with* $i, k, i_1, k_1 \in \{1, \ldots, n\}$, *then* $a \succsim^P_{(r,s)} c$, *for all* $r, s \in \{1, \ldots, n\}$ *such that* $r \geq i$ *and* $s \leq k_1$,

□

Proof 1. Let $a, b, c \in A$ and $i, k, i_1, k_1 \in \{1, \ldots, n\}$, such that $a \succsim^N_{(i,k)} b$, $b \succsim^N_{(i_1,k_1)} c$ and $k \geq i_1$. Then we have:

$$\begin{cases} a \succsim^N_{(i,k)} b \Leftrightarrow U(a^{(i)}) \geq U(b^{(k)}), \quad \forall U \in \mathcal{U}, \\ b \succsim^N_{(i_1,k_1)} c \Leftrightarrow U(b^{(i_1)}) \geq U(c^{(k_1)}), \quad \forall U \in \mathcal{U}, \\ k \geq i_1 \Rightarrow U(b^{(k)}) \geq U(b^{(i_1)}) \end{cases}$$

From this it follows:

$$U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(i_1)}) \geq U(c^{(k_1)}), \forall U \in \mathcal{U} \Rightarrow U(a^{(i)}) \geq U(c^{(k_1)}), \\ \forall U \in \mathcal{U} \Leftrightarrow a \succsim^N_{(i,k_1)} c.$$

Since $a \succsim^N_{(i,k_1)} c$ and $r, s \in \{1, \ldots, n\}$: $r \geq i$ and $s \leq k_1$, from point 1 of Proposition 6.3 we obtain $a \succsim^N_{(r,s)} c$. □

2. Let $a, b, c \in A$, and $i, k, i_1, k_1 \in \{1, \ldots, n\}$, such that $a \succsim^N_{(i,k)} b$, $b \succsim^P_{(i_1,k_1)} c$, and $k \geq i_1$. Then we have:

$$\begin{cases} a \succsim^N_{(i,k)} b \Leftrightarrow U(a^{(i)}) \geq U(b^{(k)}), \quad \forall U \in \mathcal{U}, \\ b \succsim^P_{(i_1,k_1)} c \Leftrightarrow \exists U \in \mathcal{U} : U(b^{(i_1)}) \geq U(c^{(k_1)}), \\ k \geq i_1 \Rightarrow U(b^{(k)}) \geq U(b^{(i_1)}), \quad \forall U \in \mathcal{U}. \end{cases}$$

It follows that:

$$\exists U \in \mathcal{U} : U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(i_1)}) \geq U(c^{(k_1)}) \Rightarrow \exists U \in \mathcal{U} : U(a^{(i)}) \geq U(c^{(k_1)}) \Leftrightarrow a \succsim^{P}_{(i,k_1)} c.$$

Since $r, s \in \{1, \ldots, n\}$: $r \geq i$ and $s \leq k_1$, from point 2 of Proposition 6.3 we obtain $a \succsim^{P}_{(r,s)} c$. □

3. Let $a, b, c \in A$, $i, k, i_1, k_1 \in \{1, \ldots, n\}$ such that $a \succsim^{P}_{(i,k)} b$, $b \succsim^{N}_{(i_1,k_1)} c$, and $k \geq i_1$. We have that:

$$\begin{cases} a \succsim^{P}_{(i,k)} b \Leftrightarrow \exists U \in \mathcal{U} : U(a^{(i)}) \geq U(b^{(k)}), \\ b \succsim^{N}_{(i_1,k_1)} c \Leftrightarrow U(b^{(i_1)}) \geq U(c^{(k_1)}), \quad \forall U \in \mathcal{U}, \\ k \geq i_1 \Rightarrow U(b^{(k)}) \geq U(b^{(i_1)}), \quad \forall U \in \mathcal{U}. \end{cases}$$

From this it follows:

$$\exists U \in \mathcal{U} : U(a^{(i)}) \geq U(b^{(k)}) \geq U(b^{(i_1)}) \geq U(c^{(k_1)}) \Rightarrow \exists U \in \mathcal{U} : U(a^{(i)}) \geq U(c^{(k_1)}) \Leftrightarrow a \succsim^{P}_{(i,k_1)} c.$$

Since $r, s \in \{1, \ldots, n\}$ such that $r \geq i$ and $s \leq k_1$, from point 2 of Proposition 6.3 we obtain $a \succsim^{P}_{(r,s)} c$. □

Proposition 8.2 describes some properties involving the imprecise necessary and possible preference relations together with the classical necessary and possible preference relations.

Proposition 8.2 1. *Given $a, b, c \in A$, $i \in \{1, \ldots, n\}$ such that $a \succsim^{N}_{(i,n)} b$ and $b \succsim^{N} c$, then $a \succsim^{N}_{(r,1)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \geq i$,*
2. *Given $a, b, c \in A$, $k \in \{1, \ldots, n\}$ such that $a \succsim^{N} b$ and $b \succsim^{N}_{(1,k)} c$, then $a \succsim^{N}_{(n,r)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \leq k$,*
3. *Given $a, b, c \in A$, $i \in \{1, \ldots, n\}$ such that $a \succsim^{P}_{(i,n)} b$ and $b \succsim^{N} c$, then $a \succsim^{P}_{(r,1)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \geq i$,*
4. *Given $a, b, c \in A$, $k \in \{1, \ldots, n\}$ such that $a \succsim^{N} b$ and $b \succsim^{P}_{(1,k)} c$, then $a \succsim^{P}_{(n,r)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \leq k$,*
5. *Given $a, b, c \in A$, $i \in \{1, \ldots, n\}$ such that $a \succsim^{N}_{(i,n)} b$ and $b \succsim^{P} c$, then $a \succsim^{P}_{(r,1)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \geq i$,*
6. *Given $a, b, c \in A$, $k \in \{1, \ldots, n\}$ such that $a \succsim^{P} b$ and $b \succsim^{N}_{(1,k)} c$, then $a \succsim^{P}_{(n,r)} c$, for all $r \in \{1, \ldots, n\}$ such that $r \leq k$.*

Proof 1. Let $a, b, c \in A$ and $i, r \in \{1, \ldots, n\}$ such that $a \succsim^N_{(i,n)} b$, $b \succsim^N c$ and $r \geq i$. Then we have:

$$\begin{cases} a \succsim^N_{(i,n)} b \Leftrightarrow U(a^{(i)}) \geq U(b^{(n)}), & \forall U \in \mathcal{U}, \\ b \succsim^N c \Leftrightarrow U(b) \geq U(c), & \forall U \in \mathcal{U}, \\ r \geq i \Rightarrow U(a^{(r)}) \geq U(a^{(i)}), & \forall U \in \mathcal{U}. \end{cases}$$

It follows that, for all $U \in \mathcal{U}$, $U(a^{(r)}) \geq U(a^{(i)}) \geq U(b^{(n)}) \geq U(b) \geq U(c) \geq U(c^{(1)})$ where $U(b^{(n)}) \geq U(b)$ and $U(c) \geq U(c^{(1)})$ hold by point 2 of Proposition 5.1. Thus, for all $U \in \mathcal{U}$ we obtain $U(a^{(r)}) \geq U(c^{(1)})$, and therefore $a \succsim^N_{(r,1)} c$. □

Points 2-6 can be proved analogously. □

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