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Comment on "Duality theory in fuzzy linear programming problems with fuzzy coefficients"

Hong Mai[1] · **Bing-Yuan Cao**[1,3] · **Xue-Gang Zhou**[1,2] · **Mei-Chun Liu**[2]

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Abstract This note provides a counterexample to illustrate the incorrectness of the proof of Proposition 3.3 that was presented by Wu (Fuzzy Optim Decis Mak 2:61–73, 2003). The original proof of Proposition 3.3 by Wu can only be correct when the extra assumption $\mu_{\tilde{y}_i}(0) = 1$ is added. The correct proof of Proposition 3.3 is also presented in this note.

Keywords Fuzzy numbers · Nonnegative or nonpositive fuzzy number · Membership function

1 Introduction

The original proof of Proposition 3.3 by Wu (2003) can only be correct when the extra assumption $\mu_{\tilde{y}_i}(0) = 1$ is added. A counterexample for the case of $\mu_{\tilde{y}_i}(0) \neq 1$ is given below.

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✉ Bing-Yuan Cao
caobingy@163.com

1 School of Mathematics and Information Science, Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong, Higher Education Institutes, Guangzhou University, Guangzhou 510006, Guangdong, China

2 Department of Applied Mathematics, Guangdong University of Finance, Guangzhou 510521, Guangdong, China

3 Guangzhou Vocational College of Science and Technology, Guangzhou 510550, Guangdong, China

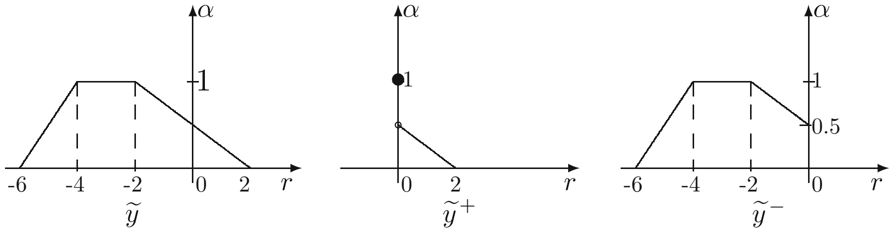

Fig. 1 $\widetilde{y}$, $\widetilde{y}^+$ and $\widetilde{y}^-$ of Example

Example Let $\widetilde{y} = (-4, -2, 2, 4)$ be trapezoidal fuzzy number and its membership function be

$$\mu_{\widetilde{y}}(r) = \begin{cases} (r+6)/2, & \text{if } r \leqslant -4, \\ 1, & \text{if } r \in [-4, -2], \\ (2-r)/4, & \text{if } r \geqslant -2, \\ 0, & \text{otherwise.} \end{cases}$$

We see that trapezoidal fuzzy number $\widetilde{y}$ is not a nonnegative or nonpositive fuzzy number. As shown in Fig. 1, $(\widetilde{y}^-)^U_\alpha = 2 - 4\alpha < 0 = (\widetilde{y}^+)^L_\alpha$ and $(\widetilde{y}^+)^U_\alpha = 0 \neq 2 - 4\alpha = \widetilde{y}^U_\alpha$ if $\alpha \in (0.5, 1]$. It implies that, for all $\alpha \in [0, 1]$, $(\widetilde{y}^+)^L_\alpha = 0 = (\widetilde{y}^-)^U_\alpha$ and $(\widetilde{y}^+)^U_\alpha = \widetilde{y}^U_\alpha$, $(\widetilde{y}^-)^L_\alpha = \widetilde{y}^L_\alpha$ is incorrect, if $\widetilde{y}$ be not a nonnegative or nonpositive fuzzy number.

2 The correct proof of Proposition 3.3

Now, we propose the correct proof of Proposition 3.3.

Proposition 3.1 *Let $\widetilde{\mathbf{x}}$ be in $\mathcal{F}^n(\mathbf{R})$. If $\widetilde{\mathbf{x}}$ is nonnegative or nonpositive, then*

$$\langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \rangle\rangle = \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^+ \rangle\rangle \oplus \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^- \rangle\rangle. \tag{1}$$

Proof Let $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \ldots, \widetilde{x}_n)$, $\widetilde{\mathbf{y}} = (\widetilde{y}_1, \ldots, \widetilde{y}_n) \in \mathcal{F}^n(\mathbf{R})$ and $\widetilde{\mathbf{y}}$ be not a nonnegative or nonpositive fuzzy number vector. By Propositions 3.1 and 3.2 in Wu (2003), we just need to show that, for any $\alpha \in [0, 1]$,

$$\langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \rangle\rangle_\alpha = (\langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^+ \rangle\rangle \oplus \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^- \rangle\rangle)_\alpha = \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^+ \rangle\rangle_\alpha + \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^- \rangle\rangle_\alpha,$$

i.e., for any $\alpha \in [0, 1]$,

$$\langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \rangle\rangle^L_\alpha = \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^+ \rangle\rangle^L_\alpha + \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^- \rangle\rangle^L_\alpha, \ \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}} \rangle\rangle^U_\alpha = \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^+ \rangle\rangle^U_\alpha + \langle\langle \widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}^- \rangle\rangle^U_\alpha,$$

which also says that

$$\begin{aligned}
&(\widetilde{x}_1 \otimes \widetilde{y}_1)_\alpha^L + \cdots + (\widetilde{x}_n \otimes \widetilde{y}_n)_\alpha^L \\
&= \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^+\right)_\alpha^L + \cdots + \left(\widetilde{x}_n \otimes \widetilde{y}_n^+\right)_\alpha^L\right] + \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^-\right)_\alpha^L + \cdots + \left(\widetilde{x}_n \otimes \widetilde{y}_n^-\right)_\alpha^L\right] \\
&= \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^+\right)_\alpha^L + \left(\widetilde{x}_1 \otimes \widetilde{y}_1^-\right)_\alpha^L\right] + \cdots + \left[\left(\widetilde{x}_n \otimes \widetilde{y}_n^+\right)_\alpha^L + \left(\widetilde{x}_n \otimes \widetilde{y}_n^-\right)_\alpha^L\right],
\end{aligned}$$

and

$$\begin{aligned}
&(\widetilde{x}_1 \otimes \widetilde{y}_1)_\alpha^U + \cdots + (\widetilde{x}_n \otimes \widetilde{y}_n)_\alpha^U \\
&= \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^+\right)_\alpha^U + \cdots + \left(\widetilde{x}_n \otimes \widetilde{y}_n^+\right)_\alpha^U\right] + \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^-\right)_\alpha^U + \cdots + \left(\widetilde{x}_n \otimes \widetilde{y}_n^-\right)_\alpha^U\right] \\
&= \left[\left(\widetilde{x}_1 \otimes \widetilde{y}_1^+\right)_\alpha^U + \left(\widetilde{x}_1 \otimes \widetilde{y}_1^-\right)_\alpha^U\right] + \cdots + \left[\left(\widetilde{x}_n \otimes \widetilde{y}_n^+\right)_\alpha^U + \left(\widetilde{x}_n \otimes \widetilde{y}_n^-\right)_\alpha^U\right].
\end{aligned}$$

In order to prove (1), it suffices to show that, for all $\alpha \in [0, 1]$ and $i = 1, 2, \ldots, n$,

$$\begin{aligned}
(\widetilde{x}_i \otimes \widetilde{y}_i)_\alpha^L &= \left(\widetilde{x}_i \otimes \widetilde{y}_i^+\right)_\alpha^L + \left(\widetilde{x}_i \otimes \widetilde{y}_i^-\right)_\alpha^L, \\
(\widetilde{x}_i \otimes \widetilde{y}_i)_\alpha^U &= \left(\widetilde{x}_i \otimes \widetilde{y}_i^+\right)_\alpha^U + \left(\widetilde{x}_i \otimes \widetilde{y}_i^-\right)_\alpha^U.
\end{aligned} \tag{2}$$

We shall prove (2) by considering the three cases below. Without loss of generality, let $\widetilde{\mathbf{x}}$ be nonnegative. The proof is similar if $\widetilde{\mathbf{x}}$ is nonpositive.

Case 1 For the case of $\mu_{\widetilde{y}_i}(0) = 1$, it implies that $(\widetilde{y}_i^+)_\alpha^L = 0 = (\widetilde{y}^-)_{i\alpha}^L$ and $(\widetilde{y}_i^+)_\alpha^U = \widetilde{y}_{i\alpha}^U$, $(\widetilde{y}_i^-)_\alpha^L = \widetilde{y}_{i\alpha}^L$ for all $\alpha \in [0, 1]$. The result follows immediately from Propositions 2.1 and 3.2 in Wu (2003).

Case 2 For the case of $\mu_{\widetilde{y}_i}(0) \neq 1$ and $a^U = \max\{r|\mu_{\widetilde{y}_i}(r) = 1\} < 0$, let $a^L = \min\{r|\mu_{\widetilde{y}_i}(r) = 1\}$. Then, there exists some $\alpha_0 \in [0, 1]$ such that $\widetilde{y}_\alpha^U < 0$ if $\alpha > \alpha_0$, otherwise, $\widetilde{y}_\alpha^U \geqslant 0$. It is obvious that $\widetilde{y}_i^L = (\widetilde{y}_i^-)^L \leqslant 0$, $(\widetilde{y}_i^+)^L = 0$. If $\alpha \geqslant \alpha_0$, then $\widetilde{y}_i^U = (\widetilde{y}_i^-)^U$ and $(\widetilde{y}_i^+)^U = 0$, otherwise, $\widetilde{y}_i^U = (\widetilde{y}_i^+)^U$ and $(\widetilde{y}_i^-)^U = 0$. For all $\alpha \in [0, 1]$, we have $\widetilde{x}_{i\alpha}^U \geqslant \widetilde{x}_{i\alpha}^L \geqslant 0$ since $\widetilde{x}_i$ is a nonnegative fuzzy number. So, we can obtain

$$\begin{gathered}
(\widetilde{x}_i \otimes \widetilde{y}_i)_\alpha^L = \widetilde{x}_{i\alpha}^U \widetilde{y}_{i\alpha}^L, \\
\left(\widetilde{x}_i \otimes \widetilde{y}_i^+\right)_\alpha^L + (\widetilde{x}_i \otimes \widetilde{y}_i^-)_\alpha^L = \widetilde{x}_{i\alpha}^L \left(\widetilde{y}_i^+\right)_\alpha^L + \widetilde{x}_{i\alpha}^U \widetilde{y}_{i\alpha}^L = \widetilde{x}_{i\alpha}^U \widetilde{y}_{i\alpha}^L, \\
(\widetilde{x}_i \otimes \widetilde{y}_i)_\alpha^U = \begin{cases} \widetilde{x}_{i\alpha}^L \widetilde{y}_{i\alpha}^U, & \text{if } \alpha \geqslant \alpha_0, \\ \widetilde{x}_{i\alpha}^U \widetilde{y}_{i\alpha}^U, & \text{if } \alpha < \alpha_0, \end{cases} \\
\left(\widetilde{x}_i \otimes \widetilde{y}_i^+\right)_\alpha^U = \begin{cases} 0, & \text{if } \alpha \geqslant \alpha_0, \\ \widetilde{x}_{i\alpha}^U \widetilde{y}_{i\alpha}^U, & \text{if } \alpha < \alpha_0, \end{cases} \\
\left(\widetilde{x}_i \otimes \widetilde{y}_i^-\right)_\alpha^U = \begin{cases} \widetilde{x}_{i\alpha}^L \widetilde{y}_{i\alpha}^U, & \text{if } \alpha \geqslant \alpha_0, \\ 0, & \text{if } \alpha < \alpha_0. \end{cases}
\end{gathered}$$

This implies that (2) holds for all $\alpha \in [0, 1]$ and $i = 1, 2, \ldots, n$.

Case 3 For the case of $\mu_{\widetilde{y}_i}(0) \neq 1$ and $a^L = \min\{r | \mu_{\widetilde{y}_i}(r) = 1\} > 0$, the similar proof of case 2 is still valid.

Reference

Wu, H.-C. (2003). Duality theory in fuzzy linear programming problems with fuzzy coefficients. *Fuzzy Optimization and Decision Making*, *2*, 61–73.