Fuzzy Optim Decis Making (2016) 15:1–20
DOI 10.1007/s10700-015-9214-8

Natural negation of interval-valued *t*-(co) norms and implications

Li Dechao[1,2] · Xie Yongjian[3] · Jiang Youfu[1,2]

Published online: 24 April 2015

Abstract In this paper, we investigate interval-valued fuzzy negations induced by interval-valued t-norms, t-conorms or implications. Some properties of interval-valued fuzzy negations induced by interval-valued sup-morphism t-norms, inf-morphism t-conorms or R-implications are firstly obtained. We also show interval-valued automorphisms acting on the interval-valued fuzzy negations induced by interval-valued t-norms, t-conorms or implications. Finally, the relations among the interval-valued fuzzy negations induced by interval-valued t-norms, t-conorms or implications are explored.

Keywords Natural negations · Interval-valued t-norms · Interval-valued t-conorms · Interval-valued implications · Automorphisms

1 Introduction

In many cases, it is difficult for decision-makers to provide a preference under inaccurate, uncertain, or incomplete information. To overcome this problem, fuzzy sets are introduced because of their flexibility in describing uncertain information. Today, fuzzy decision making is an important topic both in fuzzy sets theory and its

✉ Li Dechao
dch1831@163.com

1 School of Mathematics, Physics and Information Science, Zhejiang Ocean University, Zhoushan 316022, China

2 Key Laboratory of Oceanographic Big Data Mining and Application of Zhejiang Province, Zhoushan 316022, China

3 College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China

applications in engineering, technology, science and management and business. It is well known that fuzzy logic plays an essential role in decision making to deal with imprecision and uncertainty of information. It is one of the most important and interesting mathematical problems to characterize and represent fuzzy logical connectives (including fuzzy conjunction, fuzzy disjunction, fuzzy complement and fuzzy implication) in fuzzy logic. In order to apply better fuzzy sets theory in decision-making, it is very necessary to systematical study fuzzy negations from the mathematical point of view. There exist infinitely many approaches to define fuzzy negations such that the behavior in their extremes is as in the classical ones. Zadeh represented the most used fuzzy negation $N(x) = 1 - x$ for the first time Zadeh (1965). Soon after, other different fuzzy negations were defined (Baczyński and Jayaram 2008; Higashi and Klir 1982; Lowen 1978; Ovchinnikov 1983; Trillas 1979). Especially, it is well known today that the axiomatic definition of fuzzy negation can be found in Ovchinnikov (1983). What is worth to be mentioned that the natural negations of t-norms and t-conorms, i.e., negations generated by fuzzy connectives are a class of fuzzy negations, too (Baczyński and Jayaram 2008).

Due to the increasing complexity of environment and the vagueness of human thinking, it is not reasonable to describe something uncertain in many practical decision making situations using type-1 fuzzy sets. In order to strengthen the capability of modeling and manipulating inexact information in a logical manner, the concept of type-2 fuzzy implications were introduced by Zadeh (1971). In recent years, type-2 fuzzy sets became increasingly important since they seem to provide a better framework for the "computing with words" paradigm than classical ones (Herrera et al. 2009). As being special type-2 fuzzy sets with intervals as truth values, the interval-valued fuzzy sets (of which traditional [0,1]-valued membership degrees are replaced by intervals in [0,1]) address intuitively not only vagueness (lack of sharp class boundaries) but also a feature of uncertainty (lack of information). Moreover, interval-valued fuzzy sets are considerably easier to handle in practice than the similarly inspired type-2 fuzzy sets. To date, interval-valued fuzzy sets have been widely used to solve decision-making problems. In spite of interval-valued fuzzy sets' application in knowledge-based systems is widely understood and promoted, the research on interval-valued fuzzy sets theory is not perfect. So, it is useful to characterize and represent interval-valued fuzzy negations in interval-valued fuzzy sets theory. Several interval-valued negations have been provided in Bedregal (2010); Deschrijver et al. (2004); Gehrke et al. (1996); Gorzalczany (1987). In Bedregal (2010), Bedregal investigated the main properties of representable interval-valued fuzzy negations. In Deschrijver et al. (2004), the authors discussed the properties of involutive interval-valued negations. In addition, Wu and Luo (Wu and Luo 2011) showed the fixed points of the involutive interval-valued negations.

In this paper, we mainly investigate the natural negations of interval-valued t-norms, t-conorms and implications. Having this in mind, this paper is organized as follows. In Sect. 2, we give some definitions of basic notions and notations. Section 3 represents the natural negations of interval-valued t-norms and t-conorms. In Sect. 4, the natural negations of interval-valued implications are studied. In Sect. 5, interval-valued automorphisms acting on the natural interval-valued negations are showed. Section 6 gives the relations among some families of natural negations of interval-valued t-norms, t-conorms and implications based on the results from Sects. 3 and 4.

2 Preliminary

In order to make this work more self-contained, we introduce the main concepts and properties employed in the rest of the work.

2.1 Fuzzy negation

Definition 2.1 (Bustince et al. 2003; Fodor 1993) A function $N : [0, 1] \rightarrow [0, 1]$ is a fuzzy negation if

N1: $N(0) = 1$ and $N(1) = 0$;
N2: If $x \leq y$ then $N(x) \geq N(y), \forall x, y \in [0, 1]$.
A fuzzy negation is strict if it satisfies the following properties:
N3: N is continuous;
N4: If $x < y$ then $N(x) > N(y), \forall x, y \in [0, 1]$.
A fuzzy negation is called strong fuzzy negation if it satisfies the involutive property, i.e.
N5: $N(N(x)) = x, \forall x \in [0, 1]$.

Definition 2.2 (Bustince et al. 2003; Fodor 1993) $x \in [0, 1]$ is called a fixed point (or an equilibrium point) of a fuzzy negation N if $N(x) = x$.

Definition 2.3 (Klement and Navara 1999) A bijective mapping $\rho : [0, 1] \rightarrow [0, 1]$ is an automorphism if $x < y$ then $\rho(x) < \rho(y), \forall x, y \in [0, 1]$.

Remark 1 An equivalent definition was given in Bustince et al. (2003), where an automorphism is a continuous and strictly increasing function $\rho : [0, 1] \rightarrow [0, 1]$ such that $\rho(0) = 0$ and $\rho(1) = 1$.

Remark 2 If ρ_1 and ρ_2 are automorphisms then $\rho_1 \circ \rho_2$ is also an automorphism. The inverse of an automorphism is also an automorphism. Let $Aut([0, 1])$ denote the set of all automorphisms on $[0, 1]$, then it is not difficult to show that $(Aut([0, 1]), \circ)$ is a group.

Let ρ be an automorphism and N be a fuzzy negation. The action of ρ on N, denoted by N^ρ, is defined as follows: $N^\rho(x) = \rho^{-1}(N(\rho(x)))$.

Proposition 2.4 (Bustince et al. 2003) Let $N : [0, 1] \rightarrow [0, 1]$ be a fuzzy negation and $\rho : [0, 1] \rightarrow [0, 1]$ be an automorphism. Then N^ρ is a fuzzy negation, too. Moreover, if N is strict (strong) then N^ρ is also strict (strong).

Proposition 2.5 (Trillas 1979) $N : [0, 1] \rightarrow [0, 1]$ is a strict fuzzy negation if and only if there exist automorphisms ρ_1 and ρ_2 such that $N = \rho_1 \circ N_0 \circ \rho_2$, where N_0 is the standard fuzzy negation $N_0(x) = 1 - x$. Especially, N is strong if and only if there exists automorphism ρ such that $N = N_0^\rho$.

2.2 Interval-valued fuzzy negations, *t*-norms and *t*-conorms

Let $L^I = \{[x_1, x_2]|x_1 \leq x_2,\ x_1, x_2 \in [0, 1]\}$. For further usage, we also denote $x = [x_1, x_2]$ and the set $D = \{[x, x]|x \in [0, 1]\}$. Further, the first and second projection mapping pr_1 and pr_2 on L^I are defined as $pr_1 x = x_1$ and $pr_2 x = x_2$ for all $x \in L^I$. An ordering on L^I as $x \leq y$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ is called component-wise order or Kulisch–Miranker order (Deschrijver 2011). It is easy to verify that the defined ordering is a partially ordering on L^I, i.e., it is reflexive, antisymmetric and transitive. The largest and the smallest elements of L^I are denoted by $1_{L^I} = [1, 1]$ and $0_{L^I} = [0, 0]$, respectively. For any non-empty $A \subseteq L^I$, $\sup A = [\sup\{x_1|x \in A\}, \sup\{x_2|x \in A\}]$ and $\inf A = [\inf\{x_1|x \in A\}, \inf\{x_2|x \in A\}]$ hold (Deschrijver 2011). Therefore, it can be verified that the algebraic structure $(L^I, \vee, \wedge, 0_{L^I}, 1_{L^I})$ is a complete, bounded and distributive lattice. Moreover, it is dense, that is, if $x > y (x \geq y$ and $x \neq y)$, then there exists z such that $x > z > y$.

Definition 2.6 (Birkhoff 1948) Let L_1 and L_2 be complete lattices and A be a non-empty subset of L_1. A mapping $f : L_1 \to L_2$ is called an inf-morphism if $f(\inf A) = \inf f(A)$; sup-morphism if $f(\sup A) = \sup f(A)$.

Definition 2.7 (Moore metric) (Moore et al. 2009) For any $x, y \in L^I$, $d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ is referred as Moore metric.

Indeed, d fulfills the following conditions:

(1) Positive definiteness. $d(x, y) \geq 0$, $d(x, y) = 0$ if and only if $x = y$.
(2) Symmetry. $d(x, y) = d(y, x)$.
(3) Triangle inequality. $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.8 (Moore et al. 2009) In the metric space (L^I, d), $V(\subseteq L^I)$ is called a neighborhood of $x_0 \in L^I$ if there exists an open ball $B(x_0, \epsilon) = \{x|\ d(x, x_0) < \epsilon\}$ such that $B(x_0, \epsilon) \subseteq V$.

Definition 2.9 (Moore-continuity) (Moore et al. 2009) We say that a mapping $f : L^I \to L^I$ is continuous if for any neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subseteq V$ with respect to the metric d.

Remark 3 Acióly and Bedregal (1997) presented a quasi-metric $d_S(x, y) = \max\{y_2 - x_2, y_1 - x_1, 0\}$ on L^I . Moreover, they defined a d_S-continuous mapping which is said Scott-continuous, because this notion of continuity coincides with the continuity on Domain theory (Acióly and Bedregal 1997). Obviously, a function f is Scott-continuous iff it is order-continuous, that is, $f(\sup A) = \sup f(A)$ holds for each directed set $A \in L^I$ Santiago et al. (2006). However, Moore-continuity does not imply Scott-continuity while Scott-continuity does not imply Moore-continuity (Santiago et al. 2006). In order to discuss the connectedness, separation and compactness of L^I, the continuity on L^I means that both Moore-continuity and Scott-continuity in the remain of this paper.

Lemma 2.10 (Moore et al. 2009) The metric space (L^I, d) is connected and compact.

Further, an interval-valued fuzzy negation can be defined as follows:

Definition 2.11 (Deschrijver et al. 2004; Gehrke et al. 1996; Gorzalczany 1987) A function $\mathcal{N} : L^I \to L^I$ is called an interval-valued fuzzy negation if

$\mathcal{N}1$: $\mathcal{N}(0_{L^I}) = 1_{L^I}$, $\mathcal{N}(1_{L^I}) = 0_{L^I}$;
$\mathcal{N}2$: $\mathcal{N}(x) \geq \mathcal{N}(y)$ if $x \leq y$, $\forall\, x, y \in L^I$.
Further, an interval-valued fuzzy negation $\mathcal{N}$ is strict if it satisfies the following properties:
$\mathcal{N}3$: $\mathcal{N}$ is continuous;
$\mathcal{N}4$: $\mathcal{N}(x) > \mathcal{N}(y)$ if $x < y$.
An interval-valued fuzzy negation is strong if it is involutive, i.e.,
$\mathcal{N}5$: $\mathcal{N}(\mathcal{N}(x)) = x, \forall\, x \in L^I$.

Example 2.12 (Bedregal 2010) Let N_1 and N_2 be two fuzzy negations on $[0, 1]$ and $N_1 \leq N_2$. It is obvious to see that the operation $\mathcal{N}([x_1, x_2]) = [N_1(x_2), N_2(x_1)]$ is an interval-valued fuzzy negation. In general, it is called an interval-valued negation associated with N_1 and N_2.

Theorem 2.13 (Deschrijver et al. 2004) For an involutive interval-valued fuzzy negation $\mathcal{N}$, there exists an involutive fuzzy negation N on $[0, 1]$ such that $\mathcal{N}(x) = [N(x_2), N(x_1)]$ for any $x \in L^I$.

Remark 4 Similarly to Example 2.12, we can also obtain another interval-valued fuzzy negation associated with N as $\mathcal{N}'([x_1, x_2]) = [N(x_1), N(x_1)]$. We say an interval-valued negation $\mathcal{N}$ is trivial if the image of each $[x_1, x_2] \in L^I$ under $\mathcal{N}$ is a degenerate interval. In this paper, we always assume that all interval-valued negations are non-trivial.

Notice that $(L^I, \vee, \wedge, \mathcal{N}, 0_{L^I}, 1_{L^I})$ is a soft algebra when $\mathcal{N}$ is involutive, that is, $(L^I, \vee, \wedge)$ is a bounded and distributive lattice and keeps De Morgan identities.

Definition 2.14 (Deschrijver et al. 2004) An associative, symmetric and isotonic operation $\mathcal{T} : L^I \times L^I \to L^I$ is called an interval-valued t-norm if it satisfies $\mathcal{T}(x, 1_{L^I}) = x$ for any $x \in L^I$.

Example 2.15 (Deschrijver et al. 2004) Let T_1 and T_2 be two t-norms defined on $[0, 1]$ such that $T_1 \leq T_2$ (That is, $T_1(x, y) \leq T_2(x, y)$ for all $x, y \in [0, 1]$). It is obvious that the operation $\mathcal{T}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)]$ satisfies the properties of interval-valued t-norm. Such a t-norm associated with T_1 and T_2 is referred as t-representable. For instance, $\mathcal{T}(x, y) = [x_1 \wedge y_1, x_2 \wedge y_2]$ is t-representable.

Definition 2.16 (Deschrijver et al. 2004) An associative, symmetric and isotonic operation $\mathcal{S} : L^I \times L^I \to L^I$ is called an interval-valued t-conorm if it satisfies $\mathcal{S}(x, 0_{L^I}) = x$ for any $x \in L^I$.

Example 2.17 (Deschrijver et al. 2004) Similarly to the case of t-norms, we call $\mathcal{S}(x, y) = [S_1(x_1, x_2), S_2\ (y_1, y_2)]$ as s-representable t-conorm associated with S_1 and S_2, where S_1 and S_2 are two t-conorms on $[0, 1]$ and $S_1 \leq S_2$. For instance, $\mathcal{S}(x, y) = [x_1 \vee y_1, x_2 \vee y_2]$ is s-representable.

Remark 5 Notice that not all interval-valued t-norms (t-conorms) are t-representable (s-representable) (Deschrijver et al. 2004).

Remark 6 We define an inclusion order on L^I as $x \subseteq y$ if and only if $y_1 \leq x_1$ and $y_2 \geq x_2$. It is not difficult to find that interval-valued t-norms (t-conorms) $\mathcal{T}(\mathcal{S})$ are t-representable (s-representable) iff they are $\subseteq$-isotonic in both arguments.

Definition 2.18 (Deschrijver 2011) We say that an interval-valued t-norm $\mathcal{T}$ is a join-morphism if $\mathcal{T}(x, y \vee z) = \mathcal{T}(x, y) \vee \mathcal{T}(x, z)$ for any $x, y, z \in L^I$; a meet-morphism if $\mathcal{T}(x, y \wedge z) = \mathcal{T}(x, y) \wedge \mathcal{T}(x, z)$; a sup-morphism if $\mathcal{T}(x, \sup A) = \sup\{\mathcal{T}(x, y) | y \in A(\subseteq L^I)\}$; an inf-morphism if $\mathcal{T}(x, \inf A) = \inf\{\mathcal{T}(x, y) | y \in A(\subseteq L^I)\}$.

Replacing the words t-norm by t-conorm, we can obtain the proper definitions of join-morphism, meet-morphism, sup-morphism or inf-morphism t-conorms $\mathcal{S}$.

For continuous interval-valued t-norms, we have the following fact:

Theorem 2.19 (Deschrijver 2011) Let $\mathcal{T}$ be a continuous interval-valued t-norm. Then

i. $\mathcal{T}$ is a sup-morphism if and only if $\mathcal{T}$ is a join-morphism;
ii. $\mathcal{T}$ is an inf-morphism if and only if $\mathcal{T}$ is a meet-morphism.

Remark 7 It is easy to find that the above statements hold also for interval-valued t-conorms $\mathcal{S}$.

Definition 2.20 (Bedregal and Takahashi 2005) An interval-valued t-norm $\mathcal{T}$ is Archimedean if for any $x, y \in L^I - \{0_{L^I}, 1_{L^I}\}$ there exists an $n \in \mathbb{N}$ such that $x_{\mathcal{T}}^{(n)} < y$, where $x_{\mathcal{T}}^{(n)} = \mathcal{T}(\overbrace{x, \ldots, x}^{n})$.

Theorem 2.21 (Bedregal and Takahashi 2005) Let $\mathcal{T}$ be a continuous interval-valued t-norm. Then $\mathcal{T}$ is Archimedean if and only if $\mathcal{T}(x, x) < x$ for any $x \in L^I - \{0_{L^I}, 1_{L^I}\}$.

Definition 2.22 (Bedregal and Takahashi 2005) Let $\mathcal{T}$ be an interval-valued t-norms. An element $x \in L^I - \{0_{L^I}, 1_{L^I}\}$ is called a nilpotent element of $\mathcal{T}$ if there exist some $n \in \mathbf{N}$ such that $x_{\mathcal{T}}^{(n)} = 0_{L^I}$.

Definition 2.23 (Bedregal and Takahashi 2005) A continuous interval-valued t-norms $\mathcal{T}$ is called nilpotent if each $x \in L^I - \{0_{L^I}, 1_{L^I}\}$ is a nilpotent element of $\mathcal{T}$.

3 Natural negations of interval-valued *t*-(co) norms

One can associate interval-valued fuzzy negations with interval-valued t-norms or t-conorms. In this section, we investigate interval-valued fuzzy negations generated by interval-valued t-norms or t-conorms.

Definition 3.1 Let $\mathcal{T}$ be an interval-valued t-norm and $\mathcal{S}$ be an interval-valued t-conorm, respectively. A mapping $\mathcal{N}_{\mathcal{T}}: L^I \to L^I$ defined as

$$\mathcal{N}_{\mathcal{T}}(x) = \sup\{y \in L^I | \mathcal{T}(x, y) = 0_{L^I}\} \quad \text{for } x \in L^I \tag{1}$$

is called the natural negation of $\mathcal{T}$.

The mapping $\mathcal{N}_{\mathcal{S}}: L^I \to L^I$ defined as

$$\mathcal{N}_{\mathcal{S}}(x) = \inf\{y \in L^I | \mathcal{S}(x, y) = 1_{L^I}\} \quad \text{for } x \in L^I \tag{2}$$

is called the natural negation of $\mathcal{S}$.

It is easy to verify that $\mathcal{N}_{\mathcal{T}}(0_{L^I}) = \sup\{y \in L^I | \mathcal{T}(0_{L^I}, y) = 0_{L^I}\} = \sup\{y \in L^I\} = 1_{L^I}$ and $\mathcal{N}_{\mathcal{T}}(1_{L^I}) = \sup\{y \in L^I | \mathcal{T}(1_{L^I}, y) = 1_{L^I}\} = \sup\{y = 1_{L^I}\} = 1_{L^I}$. Moreover, for any $x, y \in L^I$ s.t. $x \leq y$, we have $\{z \in L^I | \mathcal{T}(x, z) = 1_{L^I}\} \subseteq \{z' \in L^I | \mathcal{T}(y, z') = 1_{L^I}\}$. So, $\mathcal{N}_{\mathcal{T}}(x) = \sup\{z \in L^I | \mathcal{T}(x, z) = 1_{L^I}\} \geq \sup\{z' \in L^I | \mathcal{T}(y, z') = 1_{L^I}\} = \mathcal{N}_{\mathcal{T}}(y)$. This implies that $\mathcal{N}_{\mathcal{T}}$ is an interval-valued fuzzy negation. Similarly, we can prove that $\mathcal{N}_{\mathcal{S}}$ is an interval-valued fuzzy negation, too.

Lemma 3.2 *If $\mathcal{S}$ is an inf-morphism, then*

i. $\mathcal{S}(x, y) = 1_{L^I} \iff \mathcal{N}_{\mathcal{S}}(x) \leq y$ *for any* $x, y \in L^I$;
ii. $\mathcal{N}_{\mathcal{S}}(x) = \min\{y \in L^I | \mathcal{S}(x, y) = 1_{L^I}\}$;
iii. $\mathcal{N}_{\mathcal{S}}$ *is an inf-morphism.*

Proof i. It is sufficient to prove $\mathcal{S}(x, y) = 1_{L^I}$ if $y \geq \mathcal{N}_{\mathcal{S}}(x)$. Assume that $y \geq \mathcal{N}_{\mathcal{S}}(x)$. We have $\mathcal{S}(x, y) \geq \mathcal{S}(x, \mathcal{N}_{\mathcal{S}}(x)) = \mathcal{S}(x, \inf\{z \in L^I | \mathcal{S}(x, z) = 1_{L^I}\}) = \inf\{\mathcal{S}(x, z) | \mathcal{S}(x, z) = 1_{L^I}, z \in L^I\} = 1_{L^I}$. Hence, $y \geq \mathcal{N}_{\mathcal{S}}(x)$ iff $\mathcal{S}(x, y) = 1_{L^I}$.

ii. Since $\mathcal{N}_{\mathcal{S}}(x) \geq \mathcal{N}_{\mathcal{S}}(x)$, we have $\mathcal{S}(x, \mathcal{N}_{\mathcal{S}}(x)) = 1_{L^I}$ according to i. This implies that $\mathcal{N}_{\mathcal{S}}(x) \in \{y \in L^I | \mathcal{S}(x, y) = 1_{L^I}\}$. So, the infimum in (2) is the minimum.

iii. Assume to the contrary that $\mathcal{N}_{\mathcal{S}}$ is not an inf-morphism, that is, there exists a nonempty set $A \subseteq L^I$ such that $\mathcal{N}_{\mathcal{S}}(\bigwedge A) \neq \bigwedge_{x \in A} \mathcal{N}_{\mathcal{S}}(x)$. Since $\mathcal{N}_{\mathcal{S}}$ is non-increasing, we have $\mathcal{N}_{\mathcal{S}}(\bigwedge A) > \bigwedge_{x \in A} \mathcal{N}_{\mathcal{S}}(x)$. By i, $\mathcal{S}(\bigwedge A, \bigwedge_{x \in A} \mathcal{N}_{\mathcal{S}}(x)) < 1_{L^I}$ holds. This implies that $\bigwedge_{x \in A} \mathcal{S}(x, \mathcal{N}_{\mathcal{S}}(x)) < 1_{L^I}$, which is a contradiction with $\mathcal{S}(x, \mathcal{N}_{\mathcal{S}}(x)) = 1_{L^I}$. Therefore, $\mathcal{N}_{\mathcal{S}}$ is an inf-morphism, too. □

Similarly, we can prove the following results.

Lemma 3.3 *If $\mathcal{T}$ is a sup-morphism, then*

i. $\mathcal{T}(x, y) = 0_{L^I} \iff \mathcal{N}_{\mathcal{T}}(x) \geq y$ *for any* $x, y \in L^I$;
ii. $\mathcal{N}_{\mathcal{T}}(x) = \max\{y \in L^I | \mathcal{T}(x, y) = 0_{L^I}\}$;
iii. $\mathcal{N}_{\mathcal{T}}$ *is a sup-morphism.*

Theorem 3.4 *Let interval-valued t-conorm $\mathcal{S}$ be an inf-morphism. We have the following statements:*

i. If $\mathcal{N}_{\mathcal{S}}$ is continuous, then it is strong;
ii. If $\mathcal{N}_{\mathcal{S}}$ is discontinuous, then it is not strictly antitonic.

Proof i. Firstly, we verify that $\mathcal{N}_{\mathcal{S}}$ is strict. Assume that there exist some x and y such that $\mathcal{N}_{\mathcal{S}}(x) = \mathcal{N}_{\mathcal{S}}(y)$. Let us consider the following three cases:

(i) $\mathcal{N}_{\mathcal{S}}(x) = \mathcal{N}_{\mathcal{S}}(y) = 1_{L^I}$. Since $\mathcal{N}_{\mathcal{S}}(0_{L^I}) = 1_{L^I}$, it is sufficient to verify $y = 0_{L^I}$. Suppose $y \neq 0_{L^I}$. For arbitrary $\varepsilon > 0$, we have $\mathcal{S}([1-\varepsilon, 1-\varepsilon], y) < 1_{L^I}$. Since $\mathcal{S}$ is an inf-morphism, we obtain $\mathcal{N}_{\mathcal{S}}([1-\varepsilon, 1-\varepsilon]) \nleq y$. On the other hand, since $\mathcal{N}_{\mathcal{S}}$ is continuous, $\lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}([1-\varepsilon, 1-\varepsilon]) = \mathcal{N}_{\mathcal{S}}(1_{L^I}) \nless y$ holds. However, $\mathcal{N}_{\mathcal{S}}(1_{L^I}) = 0_{L^I}$. This is a contradiction.

(ii) $\mathcal{N}_{\mathcal{S}}(x) = \mathcal{N}_{\mathcal{S}}(y) = 0_{L^I}$. Since $\mathcal{N}_{\mathcal{S}}(1_{L^I}) = 0_{L^I}$, it is sufficient to verify $y = 1_{L^I}$. Suppose $y < 1_{L^I}$. Since $\mathcal{N}_{\mathcal{S}}(y) = \inf\{z \in L^I | \mathcal{S}(y, z) = 1_{L^I}\} = 0_{L^I}$, there exists a $z_0 \in \{z \in L^I | \mathcal{S}(y, z) = 1_{L^I}\}$ such that $z_0 \leq [\varepsilon, \varepsilon]$ for arbitrary $\varepsilon > 0$. Therefore, $\mathcal{S}(y, [\varepsilon, \varepsilon]) \geq \mathcal{S}(y, z_0) = 1_{L^I}$ holds. This implies that $\mathcal{N}_{\mathcal{S}}([\varepsilon, \varepsilon]) \leq y$. Since $\mathcal{N}_{\mathcal{S}}$ is continuous, $\lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}([\varepsilon, \varepsilon]) = \mathcal{N}_{\mathcal{S}}(0_{L^I}) \leq y$ holds. However, $\mathcal{N}_{\mathcal{S}}(0_{L^I}) = 1_{L^I}$. This is a contradiction.

(iii) $\mathcal{N}_{\mathcal{S}}(x) = \mathcal{N}_{\mathcal{S}}(y) = a,\ 0_{L^I} < a < 1_{L^I}$. We further consider the following three cases:

(a) $x < y$. For any z which satisfies that $x \leq z < y$, by the definition of $\mathcal{N}_{\mathcal{S}}$ we can obtain $\mathcal{S}(z, [(a_1+\varepsilon) \wedge 1, (a_2+\varepsilon) \wedge 1]) = 1_{L^I}$ and $\mathcal{S}(z, [(a_1-\varepsilon) \vee 0, (a_2-\varepsilon) \vee 0]) < 1_{L^I}$, where arbitrary $\varepsilon > 0$ and $a = [a_1, a_2]$. This implies that $\mathcal{N}_{\mathcal{S}}([(a_1+\varepsilon) \wedge 1, (a_2+\varepsilon) \wedge 1]) \leq z$ and $\mathcal{N}_{\mathcal{S}}([(a_1-\varepsilon) \vee 0, (a_2-\varepsilon) \vee 0]) \nleq z$. Since $\mathcal{N}_{\mathcal{S}}$ is continuous, $\lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}([(a_1+\varepsilon) \wedge 1, (a_2+\varepsilon) \wedge 1]) = \lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}([(a_1-\varepsilon) \vee 0, (a_2-\varepsilon) \vee 0]) = \mathcal{N}_{\mathcal{S}}(a) = z$. This is a contradiction.

(b) $x \nless y$. This case implies that $x \subseteq y$ or $x \supseteq y$. Without loss of generality, we assume $x \subseteq y$. For any z satisfying $x \subseteq z \subsetneq y$, we have $\mathcal{N}_{\mathcal{S}}(z) = a$ as $\mathcal{N}_{\mathcal{S}}$ is Scott continuous. In a similar way as for case (a), we have $\mathcal{N}_{\mathcal{S}}(a) = z$. This is a contradiction.

(c) $y \nless x$. It can be proven in a similar way as for case (iii)-(b). □

Thus, $\mathcal{N}_{\mathcal{S}}$ is strict.

Next, we verify that $\mathcal{N}_{\mathcal{S}}$ is strong. Since $\mathcal{N}_{\mathcal{S}}$ is strict, for arbitrary $x \in L^I$ and $\varepsilon > 0$ we have $[(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0] \leq x \leq [(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1]$. Further, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0])) \leq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)) \leq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1]))$ holds. Notice that $\mathcal{S}(x, \mathcal{N}_{\mathcal{S}}([(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0]) = \mathcal{S}(x, \inf\{y | \mathcal{S}(y, [(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0]) = 1_{L^I}\} = \inf\{\mathcal{S}(x, y) | \mathcal{S}(y, [(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0]) = 1_{L^I}\} = 1_{L^I}$. By the definition of $\mathcal{N}_{\mathcal{S}}$, we obtain $x \geq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0]))$. Therefore, $\mathcal{N}_{\mathcal{S}}(x) \leq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0])))$. This means that $\mathcal{N}_{\mathcal{S}}(x) \leq \lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1-\varepsilon) \vee 0, (x_2-\varepsilon) \vee 0]))) = \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)))$. On the other hand, since $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)) \leq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1]))$, we have $\mathcal{S}(\mathcal{N}_{\mathcal{S}}(x), \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1]))) \geq \mathcal{S}(\mathcal{N}_{\mathcal{S}}(x), \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))) = 1_{L^I}$. Thus, $\mathcal{N}_{\mathcal{S}}(x) \geq \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1])))$. Once again, by the continuity of $\mathcal{N}_{\mathcal{S}}$, we obtain $\mathcal{N}_{\mathcal{S}}(x) \geq \lim_{\varepsilon \to 0} \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}([(x_1+\varepsilon) \wedge 1, (x_2+\varepsilon) \wedge 1]))) = \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)))$. From the above inequalities, $\mathcal{N}_{\mathcal{S}}(x) = \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)))$. By Lemma 4.6 from [16], it is not difficult to see that $\mathcal{N}_{\mathcal{S}}$ is involutive.

ii. Assume that $\mathcal{N}_{\mathcal{S}}$ is discontinuous at x_0. We define $b = \bigwedge_{x<x_0} \mathcal{N}_{\mathcal{S}}(x)$ and $a = \bigvee_{x>x_0} \mathcal{N}_{\mathcal{S}}(x)$. By the definition of $\mathcal{N}_{\mathcal{S}}$, we always have $a < b$. Now we consider the following two cases:

(i) $\mathcal{N}_{\mathcal{S}}(x_0) = a$. For any c which satisfies $a < c < b$, $\mathcal{N}_{\mathcal{S}}(x_0) < c$ holds obviously. This implies that $\mathcal{S}(x_0, c) < 1_{L^I}$. And then $\mathcal{N}_{\mathcal{S}}(c) \nleq x_0$, that is, $\mathcal{N}_{\mathcal{S}}(c) \geq x_0$, $\mathcal{N}_{\mathcal{S}}(c) \subseteq x_0$ or $x_0 \subseteq \mathcal{N}_{\mathcal{S}}(c)$. Let us consider the above three cases in detail.

(a) If $\mathcal{N}_{\mathcal{S}}(c) \geq x_0$. For any $x \leq x_0$, we have $\mathcal{N}_{\mathcal{S}}(x) \geq b$. This means that $\mathcal{S}(x, b) < 1_{L^I}$. Thus, $\mathcal{S}(x, c) < 1_{L^I}$. By Lemma 3.2, we have $\mathcal{N}_{\mathcal{S}}(c) \nleq x$, which is a contradiction to the fact that $x \leq x_0$.

(b) If $\mathcal{N}_{\mathcal{S}}(c) \subseteq x_0$. For any $x \supseteq x_0$, we have $\mathcal{N}_{\mathcal{S}}(x) \ngeq b$. This means that $\mathcal{S}(x, b) < 1_{L^I}$. Thus, $\mathcal{S}(x, c) < 1_{L^I}$. By Lemma 3.2, $\mathcal{N}_{\mathcal{S}}(c) \nleq x$ holds. Then, $x \nleq \mathcal{N}_{\mathcal{S}}(c) \subseteq x_0$. This contradicts to the fact that $x \supseteq x_0$.

(c) If $x_0 \subseteq \mathcal{N}_{\mathcal{S}}(c)$. For any $x \subseteq x_0$, we have $\mathcal{N}_{\mathcal{S}}(x) \nleq b$. This means that $\mathcal{S}(x, b) < 1_{L^I}$. Thus, $\mathcal{S}(x, c) < 1_{L^I}$. By Lemma 3.2, $\mathcal{N}_{\mathcal{S}}(c) \nleq x$ holds. Then, $x \ngeq \mathcal{N}_{\mathcal{S}}(c) \supseteq x_0$, which is a contradiction to the fact that $x \subseteq x_0$.

In a word, we can always obtain $\mathcal{N}_{\mathcal{S}}(c) = x_0$. This implies that $\mathcal{N}_{\mathcal{S}}$ is constant on the set $A = \{x | a \leq x \leq b\}$.

(ii) $\mathcal{N}_{\mathcal{S}}(x_0) = d$ with $a < d \leq b$. For any c which satisfies $a < c < d$, we have $\mathcal{N}_{\mathcal{S}}(x_0) > c$. This means that $\mathcal{S}(x_0, c) < 1_{L^I}$. Thus, $\mathcal{N}_{\mathcal{S}}(c) \nleq x_0$, that is, $\mathcal{N}_{\mathcal{S}}(c) \geq x_0$, $\mathcal{N}_{\mathcal{S}}(c) \subseteq x_0$ or $\mathcal{N}_{\mathcal{S}}(c) \supseteq x_0$. We can prove that $\mathcal{N}_{\mathcal{S}}(c) = x_0$ in a similar way as for case (i). Therefore, $\mathcal{N}_{\mathcal{S}}$ is constant on the set $B = \{x | a \leq x \leq d\}$.

Thus, $\mathcal{N}_{\mathcal{S}}$ is strictly antisotonic.

Corollary 3.5 *Let interval-valued t-conorm $\mathcal{S}$ be an inf-morphism. The following statements are equivalent:*

i. $\mathcal{N}_{\mathcal{S}}$ is strictly antitonic;
ii. $\mathcal{N}_{\mathcal{S}}$ is continuous;
iii. $\mathcal{N}_{\mathcal{S}}$ is strict;
iv. $\mathcal{N}_{\mathcal{S}}$ is strong.

Proof By Theorem 3.4, i $\Longrightarrow$ ii $\Longrightarrow$ iii, iv. Obviously, iii $\Longrightarrow$ i and iv $\Longrightarrow$ iii. □

Similarly, one can prove the following facts.

Theorem 3.6 *Let interval-valued t-norm $\mathcal{T}$ be a sup-morphism. We have*

i. If $\mathcal{N}_{\mathcal{T}}$ is continuous, then it is strong;
ii. If $\mathcal{N}_{\mathcal{T}}$ is discontinuous, then it is not strictly antisotonic.

Corollary 3.7 *For an interval-valued t-norm $\mathcal{T}$ as a sup-morphism, the following statements are equivalent:*

i. $\mathcal{N}_{\mathcal{T}}$ is strictly antitonic;
ii. $\mathcal{N}_{\mathcal{T}}$ is continuous;
iii. $\mathcal{N}_{\mathcal{T}}$ is strict;
iv. $\mathcal{N}_{\mathcal{T}}$ is strong.

Definition 3.8 Let $\mathcal{S}$ be an interval-valued t-conorm and $\mathcal{N}$ be an interval-valued fuzzy negation. We say that the pair $(\mathcal{S}, \mathcal{N})$ satisfies the law of excluded middle (LEM) if $\mathcal{S}(\mathcal{N}(x), x) = 1_{L^I}$ for any $x \in L^I$.

Lemma 3.9 *If the pair* $(\mathcal{S}, \mathcal{N})$ *satisfies LEM, then*

i. $\mathcal{N} \geq \mathcal{N}_{\mathcal{S}}$;
ii. $(\mathcal{N}_{\mathcal{S}} \circ \mathcal{N})(x) \leq x$ *for any* $x \in L^I$.

Proof i. Assume that there exist some x_0 such that $\mathcal{N}(x_0) \ngeq \mathcal{N}_{\mathcal{S}}(x_0)$. This implies that $\mathcal{S}(\mathcal{N}(x_0), x_0) < 1_{L^I}$. This is a contradiction to the fact that the pair $(\mathcal{S}, \mathcal{N})$ satisfies LEM.

ii. Since the pair $(\mathcal{S}, \mathcal{N})$ satisfies LEM, we have $x \in \{y | \mathcal{S}(\mathcal{N}(x), y) = 1_{L^I}\}$ for all $x \in L^I$. By the definition of $\mathcal{N}_{\mathcal{S}}$, $(\mathcal{N}_{\mathcal{S}} \circ \mathcal{N})(x) \leq x$ holds for any $x \in L^I$. □

Lemma 3.10 *Let an interval-valued t-conorm* $\mathcal{S}$ *be an inf-morphism and* $\mathcal{N}$ *be an interval-valued fuzzy negation, the following statements are equivalent:*

i. The pair $(\mathcal{S}, \mathcal{N})$ *satisfies LEM;*
ii. $\mathcal{N} \geq \mathcal{N}_{\mathcal{S}}$.

Proof It is sufficient to prove ii $\Longrightarrow$ i. Since $\mathcal{N}(x) \geq \mathcal{N}_{\mathcal{S}}(x)$ for any $x \in L^I$, we have $\mathcal{S}(\mathcal{N}(x), x) \geq \mathcal{S}(\mathcal{N}_{\mathcal{S}}(x), x) = 1_{L^I}$ by Lemma 3.2. Thus, the pair $(\mathcal{S}, \mathcal{N})$ satisfies LEM. □

Lemma 3.11 *Let* $\mathcal{S}$ *be a continuous interval-valued t-conorm such that*

(i) $\mathcal{S}$ *is Archimedean;*
(ii) $\mathcal{S}$ *is nilpotent;*
(iii) $\mathcal{S}$ *is a join-morphism;*
(iv) $\mathcal{S}(D, D) \subseteq D$.

and $\mathcal{N}$ be a strong interval-valued fuzzy negation, the following statements are equivalent:

i. the pair $(\mathcal{S}, \mathcal{N})$ satisfies LEM;
ii. There exists an automorphism ρ on [0, 1] such that

$$\mathcal{S}(x, y) = [\rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_1)) \wedge (\rho(y_2) + \rho(x_1))), \\ \rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_2)))]$$

and $\mathcal{N}(x) \geq \mathcal{N}_{\mathcal{S}}(x) = [\rho^{-1}(1 - \rho(x_2)), \rho^{-1}(1 - \rho(x_1))]$.

Proof i $\Longrightarrow$ ii. Since $\mathcal{N}$ is strong, $\mathcal{N}([0, 1]) = [0, 1]$ holds according to Ref. Deschrijver et al. (2004). So, $\mathcal{S}([0, 1], [0, 1]) = \mathcal{S}(\mathcal{N}([0, 1]), [0, 1]) = 1_{L^I}$. By Theorem 9.12 in Ref. Deschrijver et al. (2004), there exists an automorphism ρ on [0, 1] such that $\mathcal{S}(x, y) = [\rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_1)) \wedge (\rho(y_2) + \rho(x_1))), \rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_2)))]$. In this case, for any fixed $x = [x_1, x_2] \in L^I$, the natural negation of $\mathcal{S}$ has the form as follows:

$\mathcal{N}_{\mathcal{S}}(x) = \inf\{y = [y_1, y_2] \in L^I | \mathcal{S}(x, y) = 1_{L^I}\} = \inf\{y \in L^I | [\rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_1)) \wedge (\rho(y_2) + \rho(x_1))), \rho^{-1}(1 \wedge (\rho(x_2) + \rho(y_2)))] = 1_{L^I}\} = \inf\{y \in L^I | 1 \wedge (\rho(x_2) + \rho(y_1)) \wedge (\rho(y_2) + \rho(x_1)) = 1 \text{ and } 1 \wedge (\rho(x_2) + \rho(y_2)) = 1\} = \inf\{y \in L^I | \rho(x_2) + \rho(y_1) = 1 \text{ and } \rho(y_2) + \rho(x_1) = 1\} = [\rho^{-1}(1 - \rho(x_2)), \rho^{-1}(1 - \rho(x_1))]$. Moreover, $\mathcal{N}(x) \geq \mathcal{N}_{\mathcal{S}}(x)$ holds by Lemma 3.10.

ii $\Longrightarrow$ i. The proof in this direction is immediate. □

Definition 3.12 Let $\mathcal{T}$ be an interval-valued t-norm, $\mathcal{S}$ be an interval-valued t-conorm and $\mathcal{N}$ be a strict interval-valued fuzzy negation, we say the triple $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ is a De Morgan triple if for each $x, y \in L^I$,

$$\mathcal{T}(x, y) = \mathcal{N}(\mathcal{S}(\mathcal{N}(x), \mathcal{N}(y))), \quad \mathcal{S}(x, y) = \mathcal{N}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$$

Lemma 3.13 *Let an interval-valued t-norm $\mathcal{T}$ be a sup-morphism and $\mathcal{S}$ be an interval-valued t-conorm. If $(\mathcal{T}, \mathcal{S}, \mathcal{N}_\mathcal{T})$ is a De Morgan triple, then*

i. $\mathcal{N}_\mathcal{S} = \mathcal{N}_\mathcal{T}$;
ii. $\mathcal{S}$ is an inf-morphism.

Proof i. Assume that there exist some x_0 such that $\mathcal{N}_\mathcal{S}(x_0) \neq \mathcal{N}_\mathcal{T}(x_0)$. Let us consider the following three cases:

(i) $\mathcal{N}_\mathcal{T}(x_0) < \mathcal{N}_\mathcal{S}(x_0)$. There exists an $a \in L^I$ such that $\mathcal{N}_\mathcal{T}(x_0) < a < \mathcal{N}_\mathcal{S}(x_0)$. Since $(\mathcal{T}, \mathcal{S}, \mathcal{N}_\mathcal{T})$ is a De Morgan triple, $\mathcal{N}_\mathcal{T}$ is strong, and then it is a bijection. Therefore, there exists a $y \in L^I$ such that $\mathcal{N}_\mathcal{T}(y) = a$, that is, $\mathcal{N}_\mathcal{T}(x_0) < \mathcal{N}_\mathcal{T}(y) < \mathcal{N}_\mathcal{S}(x_0)$. This implies that $\mathcal{T}(x_0, \mathcal{N}_\mathcal{T}(y)) \geq \mathcal{T}(x_0, \mathcal{N}_\mathcal{T}(x_0))) = \mathcal{T}(x_0, \sup\{z \in L^I | \mathcal{T}(x_0, z) = 0_{L^I}\} = \sup\{\mathcal{T}(x_0, z) | \mathcal{T}(x_0, z) = 0_{L^I}\} = 0_{L^I}$ and $\mathcal{S}(x_0, \mathcal{N}_\mathcal{T}(y)) < 1_{L^I}$. So, $\mathcal{N}_\mathcal{T}(\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), \mathcal{N}_\mathcal{T}(\mathcal{N}_\mathcal{T}(y))))) < 1_{L^I}$ holds. And then $\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), y) > 0_{L^I}$. This means that $\mathcal{N}_\mathcal{T}(x_0) \nleq \mathcal{N}_\mathcal{T}(y)$, a contradiction as our assumption.

(ii) $\mathcal{N}_\mathcal{T}(x_0) > \mathcal{N}_\mathcal{S}(x_0)$. Similarly, we can find a $y \in L^I$ such that $\mathcal{N}_\mathcal{T}(x_0) > \mathcal{N}_\mathcal{T}(y) > \mathcal{N}_\mathcal{S}(x_0)$. This implies that $\mathcal{S}(x_0, \mathcal{N}_\mathcal{T}(y)) = 1_{L^I}$, and then $\mathcal{N}_\mathcal{T}(\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), \mathcal{N}_\mathcal{T}(\mathcal{N}_\mathcal{T}(y)))) = 1_{L^I}$. Thus, $\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), y) = 0_{L^I}$ holds. This means that $\mathcal{N}_\mathcal{T}(x_0) \leq \mathcal{N}_\mathcal{T}(y)$, a contradiction on our assumption.

(iii) $\mathcal{N}_\mathcal{T}(x_0) \subseteq \mathcal{N}_\mathcal{S}(x_0)$ or $\mathcal{N}_\mathcal{S}(x_0) \subseteq \mathcal{N}_\mathcal{T}(x_0)$. Without loss of generality, we only consider the case $\mathcal{N}_\mathcal{T}(x_0) \subseteq \mathcal{N}_\mathcal{S}(x_0)$. In this case, there exists an $a \in L^I$ such that $\mathcal{N}_\mathcal{T}(x_0) \subsetneq a \subsetneq \mathcal{N}_\mathcal{S}(x_0)$. Since $\mathcal{N}_\mathcal{T}$ is a bijection, we can find a $y \in L^I$ such that $\mathcal{N}_\mathcal{T}(y) = a$, i.e., $\mathcal{N}_\mathcal{T}(x_0) \subsetneq \mathcal{N}_\mathcal{T}(y) \subsetneq \mathcal{N}_\mathcal{S}(x_0)$. By the monotonicity of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S}$ being the $\mathcal{N}_\mathcal{T}$ -dual of $\mathcal{T}$ and the definitions of $\mathcal{N}_\mathcal{T}$, $\mathcal{N}_\mathcal{S}$ we have $\mathcal{S}(x_0, \mathcal{N}_\mathcal{T}(y)) < 1_{L^I}$. This implies that $\mathcal{N}_\mathcal{T}(\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), \mathcal{N}_\mathcal{T}(\mathcal{N}_\mathcal{T}(y)))) < 1_{L^I}$, i.e., $\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), y) > 0$. By Lemma 3.2 in Ref. Deschrijver (2011), there exists a t-norm T on $[0, 1]$ such that $pr_1(\mathcal{T}(\mathcal{N}_\mathcal{T}(x_0), y)) = T(N(pr_2 x_0)), y_1)$. Thus, we have $T(N(pr_2 x_0)), y_1) > 0$. This implies that $y > pr_2 x_0$ according to Theorem 2.13. This contradicts with $\mathcal{N}_\mathcal{T}(x_0) \subsetneq \mathcal{N}_\mathcal{T}(y) \subsetneq \mathcal{N}_\mathcal{S}(x_0)$.

ii. Since $\mathcal{S}$ is an inf-morphism t-conorm and $\mathcal{N}_\mathcal{T}$ is strong, the t-norm $\mathcal{T}$, as an $\mathcal{N}_\mathcal{S}$-dual of $\mathcal{S}$, is a sup-morphism. □

Lemma 3.14 *Let an interval-valued t-conorm $\mathcal{S}$ be an inf-morphism and $\mathcal{T}$ be an interval-valued t-norm. If $(\mathcal{T}, \mathcal{S}, \mathcal{N}_\mathcal{S})$ is a De Morgan triple, then*

i. $\mathcal{N}_\mathcal{T} = \mathcal{N}_\mathcal{S}$;
ii. $\mathcal{T}$ is a sup-morphism.

Proof This proof can be obtained similarly to that of Lemma 3.13. □

In the last part of this section, we discuss the natural negations of representable t-norms and t-conorms.

Definition 3.15 (Deschrijver et al. 2004) We say that an interval-valued fuzzy negation $\mathcal{N}$ is representable if there exist fuzzy negations N_1, N_2 on $[0, 1]$ such that $N_1 \leq N_2$ and $\mathcal{N}(x) = [N_1(x_2), N_2(x_1)]$.

Proposition 3.16 *An interval-valued fuzzy negation $\mathcal{N}$ is representable if and only if it is $\subseteq$-isotonic.*

Proof Straightforward. □

Theorem 3.17 *Let interval-valued t-norm $\mathcal{T}$ be representable, i.e., there exist t-norms T_1, T_2 on $[0, 1]$ such that $T_1 \leq T_2$ and $\mathcal{T}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)]$. Then the natural negation $\mathcal{N}_\mathcal{T}$ is representable, too. Moreover, $\mathcal{N}_\mathcal{T}(x) = [N_{T_1}(x_1) \wedge N_{T_2}(x_2), N_{T_2}(x_2)]$ for all $x \in L^I$.*

Proof For any $x \in L^I$, let A be the set: $A = \{y \in L^I | \mathcal{T}(x, y) = 0_{L^I}\}$. Since $\forall\ y \in A$, $T_1(x_1, y_1) = 0$ and $T_2(x_2, y_2) = 0$, we have $\sup\{z_1 | T_1(x_1, z_1) = 0\} \geq y_1$ and $\sup\{z_2 | T_2(x_2, z_2) = 0\} \geq y_2$. Therefore, $[\sup\{z_1 | T_1(x_1, z_1) = 0\} \wedge \sup\{z_2 | T_2(x_2, z_2) = 0\}, \sup\{z_2 | T_2(x_2, z_2) = 0\}] = [N_{T_1}(x_1) \wedge N_{T_2}(x_2), N_{T_2}(x_2)] \geq z$ for all $z \in A$. This implies that $[N_{T_1}(x_1) \wedge N_{T_2}(x_2), N_{T_2}(x_2)]$ is an upper bound of A. Let us show that it is the least upper bound of A. Suppose that a is another upper bound of A such that $a \ngeq [N_{T_1}(x_1) \wedge N_{T_2}(x_2), N_{T_2}(x_2)]$. Let us consider the following two cases:

i. $a_1 < N_{T_1}(x_1) \wedge N_{T_2}(x_2)$. This implies that $a_1 < \sup\{z_1 | T_1(x_1, z_1) = 0\}$ and $a_1 < \sup\{z_2 | T_2(x_2, z_2) = 0\}$. Therefore, there exist $z_1 > a_1$ and $z_2 > a_1$ such that $T_1(x_1, z_1) = 0$ and $T_2(x_2, z_2) = 0$, respectively. Let $b = z_1 \wedge z_2$. It is not difficult to verify that $[b, b] \in A$. However, $[b, b] \nleq a$, which contradicts to the fact that a is another upper bound of A.

ii. $a_1 \geq N_{T_1}(x_1)$ and $a_2 < N_{T_2}(x_2)$. For any z_1 such that $T_1(x_1, z_1) = 0$ we have $a_1 \geq z_1$. Moreover, $a_2 < N_{T_2}(x_2)$ implies that there exists a $z_2 > a_2$ such that $T_2(x_2, z_2) = 0$. Thus, we have $z_1 \leq a_1 \leq a_2 < z_2$. Consequently, $[z_1, z_2] \in A$, which contradicts to the fact that a be another upper bound of A. □

Theorem 3.18 *If an interval-valued t-conorm $\mathcal{S}$ is representable, i.e., there exist t-conorms S_1, S_2 on $[0, 1]$ such that $S_1 \leq S_2$ and $\mathcal{S}(x, y) = [S_1(x_1, y_1), S_2(x_2, y_2)]$. Then the natural negation $\mathcal{N}_\mathcal{S}$ is representable, too. Moreover, $\mathcal{N}_\mathcal{S}(x) = [N_{S_1}(x_1) \wedge N_{S_2}(x_2), N_{S_2}(x_2)]$ for all $x \in L^I$.*

Proof This proof can be obtained similarly to that of Theorem 3.17. □

4 Natural negations of interval-valued implications

In this section, our goal is to present the interval-valued fuzzy negations generated by interval-valued R- or $(\mathcal{S}, \mathcal{N})$-implications. In order to achieve this goal, we firstly recall the definition of interval-valued implications and some axioms for interval-valued implications to satisfy.

Definition 4.1 (Cornelis et al. 2004) An interval-valued fuzzy implication $\mathcal{I}$ is a mapping from $L^I \times L^I$ to L^I satisfying $\mathcal{I}(0_{L^I}, 0_{L^I}) = 1_{L^I}, \mathcal{I}(0_{L^I}, 1_{L^I}) = 1_{L^I}, \mathcal{I}(1_{L^I}, 1_{L^I}) = 1_{L^I}, \mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I}$.

In this paper, we take the extension of the axioms which stand as a milestone to test the suitability of fuzzy implications on [0,1] to the interval-valued case (Cornelis et al. 2004):

A1. Antitonic in the first variable, $\forall\, x, y, z \in L^I$, $\mathcal{I}(x, z) \leq \mathcal{I}(y, z)$ if $x \geq y$,
A2. Isotonic in the second variable, $\forall\, x, y, z \in L^I$, $\mathcal{I}(x, y) \leq \mathcal{I}(x, z)$ if $y \leq z$,
A3. Left and right boundary conditions, $\mathcal{I}(0_{L^I}, x) = 1_{L^I}$, $\mathcal{I}(x, 1_{L^I}) = 1_{L^I}$, $\forall\, x \in L^I$,
A4. Left neutrality property, $I(1_{L^I}, x) = x, \forall\, x \in L^I$,
A5. Identity principle, $I(x, x) = 1_{L^I}, \forall\, x \in L^I$,
A6. Exchange principle, $\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)), \forall\, x, y, z \in L^I$,
A7. Law of contraposition with a negation $\mathcal{N}$, $\mathcal{I}(x, y) = \mathcal{I}(\mathcal{N}(y), \mathcal{N}(x))$, $\forall\, x$, $y \in L^I$,
A8. Ordering property, $\mathcal{I}(x, y) = 1_{L^I} \Leftrightarrow x \leq y, \forall\, x,\ y \in L^I$,
A9. Continuity, $\mathcal{I}$ is a continuous function on L^I.

Definition 4.2 Let an interval-valued fuzzy implication $\mathcal{I}$ fulfill A1. The mapping $\mathcal{N}_{\mathcal{I}}$, defined by $\mathcal{N}_{\mathcal{I}}(x) = \mathcal{I}(x, 0_{L^I})$ for any $x \in L^I$, is called a natural negation of interval-valued implication $\mathcal{I}$.

Remark 8 It is easy to see that $\mathcal{N}_{\mathcal{I}}(1_{L^I}) = \mathcal{I}(1_{L^I}, 0_{L^I}) = 0_{L^I}$, $\mathcal{N}_{\mathcal{I}}(0_{L^I}) = \mathcal{I}(0_{L^I}, 0_{L^I}) = 1_{L^I}$, and then $\mathcal{N}_{\mathcal{I}}$ is an interval-valued fuzzy negation.

Lemma 4.3 (Li and Li 2012) Let $\mathcal{I} : L^I \times L^I \to L^I$ be an interval-valued implication and $\mathcal{N}$ be an interval-valued fuzzy negation. We have the following statements:

i. A4 $\wedge$ A7$\Longrightarrow \mathcal{N} = \mathcal{N}_{\mathcal{I}}$ is involutive.
ii. A4 $\wedge$ A7$'$ $\Longrightarrow \mathcal{N}_{\mathcal{I}} \circ \mathcal{N} = i$, where i is an identity mapping on L^I. Moreover, $\mathcal{N}$ is one-to-one, where A7$'$ is the law of left contraposition with $\mathcal{N}$ in Ref. Li and Li (2012).

Lemma 4.4 *If a mapping $\mathcal{I} : L^I \times L^I \to L^I$ satisfies A6 and A8, then the following statements are equivalent:*

i. $\mathcal{N}_{\mathcal{I}}$ is continuous;
ii. $\mathcal{N}_{\mathcal{I}}$ is strong.

Proof It is sufficient to show that i $\Longrightarrow$ ii. Since $\mathcal{I}$ satisfies A8, it also satisfies A5. Therefore, we have $\mathcal{I}(\mathcal{N}_{\mathcal{I}}(x), \mathcal{N}_{\mathcal{I}}(x)) = 1_{L^I}$, that is, $\mathcal{I}(\mathcal{I}(x, 0_{L^I}), \mathcal{I}(x, 0_{L^I})) = 1_{L^I}$. By A6, $\mathcal{I}(x, \mathcal{I}(\mathcal{I}(x, 0_{L^I}),\ 0_{L^I})) = 1_{L^I}$ holds. This implies that $x \leq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x))$ according to A8. By the definition of $\mathcal{N}_{\mathcal{I}}$, we obtain $\mathcal{N}_{\mathcal{I}}(x) \geq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x)))$.

On the other hand, $\mathcal{I}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x)), \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x))) = 1_{L^I}$. By A6, we have $\mathcal{I}(\mathcal{I}(x, 0_{L^I}), \mathcal{I}(\mathcal{N}_{\mathcal{I}}(\ \mathcal{N}_{\mathcal{I}}(x)), 0_{L^I})) = 1_{L^I}$, which means that $\mathcal{N}_{\mathcal{I}}(x) \leq \mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x)))$. Thus, $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(x))) = \mathcal{N}_{\mathcal{I}}(x)$. Since $\mathcal{N}_{\mathcal{I}}$ is continuous and onto, there exists a $y \in L^I$ such that $y = \mathcal{N}_{\mathcal{I}}(x)$ for any $x \in L^I$. Notice that $\mathcal{N}_{\mathcal{I}}(\mathcal{N}_{\mathcal{I}}(y)) = \mathcal{N}_{\mathcal{I}}(x) = y$. Therefore, $\mathcal{N}_{\mathcal{I}}$ is strong.

Next, we investigate the natural negations of interval-valued R-implications or $(\mathcal{S}, \mathcal{N})$-implications. □

Definition 4.5 (Deschrijver et al. 2004) A mapping $\mathcal{I}_R : L^I \times L^I \to L^I$ is called an interval-valued R-implication if there exists an interval-valued t-norm $\mathcal{T}$ such that

$$\mathcal{I}_{R_\mathcal{T}}(x, y) = \sup\{z \in L^I | \mathcal{T}(x, z) \leq y\}, \quad \forall\, x, y \in L^I \tag{3}$$

Lemma 4.6 *For an interval-valued t-norm $\mathcal{T}$, the following statements are equivalents:*

i. $\mathcal{T}$ is a sup-morphism;
ii. $\mathcal{T}$ satisfies the residuation principle, i.e., $\mathcal{T}(x, z) \leq y \Leftrightarrow z \leq \mathcal{I}_{R_\mathcal{T}}(x, y)$;
iii. The supremum in (3) is the maximum, i.e., $\mathcal{I}_{R_\mathcal{T}}(x, y) = \max\{z \in L^I | \mathcal{T}(x, z) \leq y\}$ for all $x, y \in L^I$.

Proof By Theorem 7.7 in Ref. Deschrijver et al. (2004), we have i $\Longleftrightarrow$ ii. Therefore, it is sufficient to prove ii $\Longleftrightarrow$ iii.

ii $\Longrightarrow$ iii. Since $\mathcal{I}_{R_\mathcal{T}}(x, y) \leq \mathcal{I}_{R_\mathcal{T}}(x, y)$, $\mathcal{T}(x, \mathcal{I}_{R_\mathcal{T}}(x, y)) \leq y$ hods by ii. This implies that $\mathcal{I}_{R_\mathcal{T}}(x, y) \in \{z \in L^I | \mathcal{T}(x, z) \leq y\}$. Thus, the supremum in (3) is the maximum.

iii $\Longrightarrow$ ii. Assume $\mathcal{T}(x, z) \leq y$. Obviously, $z \in \{u \in L^I | \mathcal{T}(x, u) \leq y\}$ holds. And then we obtain $z \leq \mathcal{I}_{R_\mathcal{T}}(x, y)$ according to iii. On the other hand, for any $z \leq \mathcal{I}_{R_\mathcal{T}}(x, y)$, we have $\mathcal{T}(x, z) \leq \mathcal{T}(x, \mathcal{I}_{R_\mathcal{T}}(x, y)) \leq y$. Thus, $\mathcal{T}(x, z) \leq y \Leftrightarrow z \leq \mathcal{I}_{R_\mathcal{T}}(x, y)$ □

Theorem 4.7 (Cornelis et al. 2004) Let $\mathcal{T}$ be an interval-valued t-norm (not necessary to be a sup-morphism). Then $\mathcal{I}_{R_\mathcal{T}}$ is an interval-valued implication. Moreover, $\mathcal{I}_{R_\mathcal{T}}$ satisfies A1, A2, A4 and A5.

However, without additional assumptions on an interval-valued t-norm $\mathcal{T}$, the interval-valued implication $\mathcal{I}_{R_\mathcal{T}}$ may not satisfy other axioms.

Lemma 4.8 (Cornelis et al. 2004) For an interval-valued t-norm $\mathcal{T}$, the following statements are equivalent:

i. $\mathcal{I}_{R_\mathcal{T}}$ satisfies A8.
ii. $\mathcal{T}_x(\cdot) = \mathcal{T}(x, \cdot)$ is continuous.

Theorem 4.9 *Let $\mathcal{I}_{R_\mathcal{T}}$ be an interval-valued $R-$implication and $\mathcal{N}_{\mathcal{I}_{R_\mathcal{T}}}$ be its natural negation. If $\mathcal{I}_{R_\mathcal{T}}$ satisfies A6 and A8, then $\mathcal{N}_{\mathcal{I}_{R_\mathcal{T}}}$ is strong.*

Proof This proof is deduced by Lemma 4.4 and Lemma 4.8. □

Definition 4.10 (Cornelis et al. 2004) A mapping $\mathcal{I}_{\mathcal{S},\mathcal{N}} : L^I \times L^I \to L^I$ is called an interval-valued $(\mathcal{S}, \mathcal{N})$-implication if there exist an interval-valued t-conorm $\mathcal{T}$ and an interval-valued fuzzy negation such that

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = \mathcal{S}(\mathcal{N}(x), y), \quad \forall\, x, y \in L^I \tag{4}$$

If $\mathcal{N}$ is strong, then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is called an interval-valued S-implication. □

Lemma 4.11 *For an interval-valued t-conorm $\mathcal{S}$ and an interval-valued fuzzy negation $\mathcal{N}$, the following statements are equivalent:*

i. $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ *satisfies A5.*
ii. The pair $(\mathcal{S},\mathcal{N})$ *satisfies LEM.*

Proof i $\Longrightarrow$ ii. If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies A5, then $\mathcal{S}(\mathcal{N}(x),x)=\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,x)=1_{L^I}$ for all $x\in L^I$. This means that the pair $(\mathcal{S},\mathcal{N})$ satisfies LEM.

ii $\Longrightarrow$i. Assume that the pair $(\mathcal{S},\mathcal{N})$ satisfies LEM. We have $\mathcal{I}_{\mathcal{S},\mathcal{N}}(x,x)=\mathcal{S}(\mathcal{N}(x),x))=1_{L^I}$ for all $x\in L^I$. This implies that $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies A5. □

By Lemma 3.11 and 4.11, we can obtain the following results:

Lemma 4.12 *Let* $\mathcal{S}$ *be a continuous interval-valued* t*-conorm such that*

(i) $\mathcal{S}$ *is Archimedean;*
(ii) $\mathcal{S}$ *is nilpotent;*
(iii) $\mathcal{S}$ *is a join-morphism;*
(iv) $\mathcal{S}(D,D)\subset D$.

and $\mathcal{N}$ *be a strong interval-valued fuzzy negation. Then the following statements are equivalent:*

i. $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ *satisfies A5.*
ii. There exists an automorphism ρ *on* $[0,1]$ *such that*

$$\mathcal{S}(x,y)=[\rho^{-1}(1\wedge(\rho(x_2)+\rho(y_1))\wedge(\rho(y_2)+\rho(x_1))),\rho^{-1}(1\wedge(\rho(x_2)+\rho(y_2)))]$$

and $\mathcal{N}(x)\geq\mathcal{N}_{\mathcal{S}}(x)=[\rho^{-1}(1-\rho(x_2)),\rho^{-1}(1-\rho(x_1))]$.

Lemma 4.13 *Let interval-valued* t*-conorm* $\mathcal{S}$ *be an inf-morphism and* $\mathcal{N}$ *be an interval-valued fuzzy negation. The following statements are equivalent:*

i. $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ *satisfies A8.*
ii. $\mathcal{N}=\mathcal{N}_{\mathcal{S}}$ *is strong and the pair* $(\mathcal{S},\mathcal{N}_{\mathcal{S}})$ *satisfies LEM.*

Proof i $\Longrightarrow$ ii. If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies A8, then it satisfies A5. According to Lemma 4.11, the pair $(\mathcal{S},\mathcal{N})$ satisfies LEM. This implies that $\mathcal{N}(x)\geq\mathcal{N}_{\mathcal{S}}(x)$ and $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x))\leq x$ hold for any $x\in L^I$ by Lemma 3.9. We can assert that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x))\geq x$. Assume that there exist some $x_0\in L^I$ such that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))\ngeq x_0$, that is, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))<x_0$, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))\subseteq x_0$ or $x_0\subseteq\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))$. Let us consider these cases in detail. □

(i) $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))<x_0$. Then there exists a $y\in L^I$ such that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))<y<x_0$. This implies that $\mathcal{S}(\mathcal{N}(x_0),y)=1_{L^I}$. By A8, we have $x_0\leq y$, which is a contradiction.
(ii) $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))\subseteq x_0$. Similarly, there exists a $y\in L^I$ such that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))\subsetneq y\subsetneq x_0$. This means that $\mathcal{S}(\mathcal{N}(x_0),y)=1_{L^I}$. However, $\mathcal{I}_{\mathcal{S},\mathcal{N}}(x_0,y)=1_{L^I}$ implies $x_0\leq y$ from A8. This is a contradiction.
(iii) $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x_0))\supseteq x_0$. It can be proven in a similar way as for case (ii).

Hence, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x))=x$ for all $x\in L^I$. It is not difficult to verify that $\mathcal{N}_{\mathcal{S}}$ is onto, and then it is continuous. Further, since $\mathcal{N}_{\mathcal{S}}$ is an interval-valued negation, we have $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))\geq\mathcal{N}_{\mathcal{S}}(\mathcal{N}(x))=x$ for all $x\in L^I$. By the monotonicity of $\mathcal{N}_{\mathcal{S}}$, we obtain

$\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))) \leq \mathcal{N}_{\mathcal{S}}(x)$. On the other hand, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)) \geq x$ holds for all $x \in L^I$. Therefore, $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))) \geq \mathcal{N}_{\mathcal{S}}(x)$. This means that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))) = \mathcal{N}_{\mathcal{S}}(x)$. Since $\mathcal{N}_{\mathcal{S}}$ is continuous and onto, there exists an $x \in L^I$ such that $y = \mathcal{N}_{\mathcal{S}}(x)$ for any $y \in L^I$. Notice that $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(y)) = \mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x))) = \mathcal{N}_{\mathcal{S}}(x) = y$. Thus, $\mathcal{N}_{\mathcal{S}}$ is strong. Moreover, it is obvious to find that $\mathcal{N} = \mathcal{N}_{\mathcal{S}}$.

ii $\Longrightarrow$ i. Assume that $\mathcal{N}_{\mathcal{S}}$ is strong and the pair $(\mathcal{S}, \mathcal{N}_{\mathcal{S}})$ satisfies LEM. Let $x \leq y$. By the monotonicity of $\mathcal{S}$, we can obtain $\mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = \mathcal{S}(\mathcal{N}(x), y) \geq \mathcal{S}(\mathcal{N}(x), x) = 1_{L^I}$. On the other hand, $\mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = 1_{L^I}$ implies $\mathcal{S}(\mathcal{N}(x), y) = 1_{L^I}$. According to the definition of $\mathcal{N}_{\mathcal{S}}$, we have $\mathcal{N}_{\mathcal{S}}(\mathcal{N}_{\mathcal{S}}(x)) = x \leq y$. Hence, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies A8.

In Ref. Li and Li (2012), the authors presented an expression of interval-valued R-implication generated by a representable t-norm. The following theorem shows that the representable R-implications only are generated by representable t-norms.

Theorem 4.14 *The interval-valued R-implication $\mathcal{I}_{R_{\mathcal{T}}}$ generated by a sup-morphism t-norm $\mathcal{T}$ has the form*

$$\mathcal{I}_{R_{\mathcal{T}}}(x, y) = [I_{R_{T_1}}(x_1, y_1) \wedge I_{R_{T_2}}(x_2, y_2), I_{R_{T_2}}(x_2, y_2)] \tag{5}$$

if and only if $\mathcal{T}$ is representable, where $I_{R_{T_1}}$ and $I_{R_{T_2}}$ are the R-implications generated by the left-continuous t-norms T_1 and T_2 such that $T_1 \leq T_2$, respectively.

Proof ($\Longleftarrow$) the proof comes from Theorem 3 in Ref. Liu and Wang (2006). □

($\Longrightarrow$) Since $I_{R_{T_1}}$ and $I_{R_{T_2}}$ are R-implications, $\mathcal{I}_{R_{\mathcal{T}}}$ formed as Eq.(5) satisfies A1, A2 and is an inf-morphism for the second variable. This implies that $\mathcal{T}(x, y) = \inf\{z \in L^I | x \leq \mathcal{I}_{R_{\mathcal{T}}}(y, z)\} = \inf\{z \in L^I | x \leq [I_{R_{T_1}}(y_1, z_1) \wedge I_{R_{T_2}}(y_2, z_2), I_{R_{T_2}}(y_2, z_2)]\}$. Notice that $x_1 \leq I_{R_{T_1}}(y_1, z_1)$ and $x_2 \leq I_{R_{T_2}}(y_2, z_2) \Longleftrightarrow T_1(x_1, y_1) \leq z_1$ and $T_2(x_2, y_2) \leq z_2$. Thus, $\mathcal{T}(x, y) = \inf\{z \in L^I | x \leq \mathcal{I}_{R_{\mathcal{T}}}(y, z)\} = [T_1(x_1, y_1), T_2(x_2, y_2)]$ with $T_1 \leq T_2$.

Theorem 4.15 *Let $\mathcal{N}$ be a continuous representable negation and $(\mathcal{S}, \mathcal{N})$-implication $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies A1, A4 and A6. Then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ has the form*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(x, y) = [I_{S_1,N_1}(x_1, y_1) \wedge I_{S_2,N_2}(x_2, y_2), I_{S_2,N_2}(x_2, y_2)]. \tag{6}$$

if and only if $\mathcal{S}$ and $\mathcal{N}$ are representable, where I_{S_1,N_1} and I_{S_2,N_2} are the (S, N)-implications associated to the t-conorms S_1, S_2 and negations N_1, N_2, respectively.

Proof ($\Longleftarrow$) Straightforward. □

($\Longrightarrow$) Notice that $\mathcal{S}(x, y) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}(x), y) = [I_{S_1,N_1}(N_1(x_1), y_1) \wedge I_{S_2,N_2}(N_2(x_2), y_2), I_{S_2,N_2}(N_2(x_2), y_2)]$. Since I_{S_1,N_1} and I_{S_2,N_2} satisfy A1, A2 and A6 (See Ref. Klement and Navara 1999 in detail), $\mathcal{S}(x, y) = [S_1(x_1, y_1), S_2(x_2, y_2)]$ holds with $S_1 \leq S_2$.

By Theorem 4.14, we can obtain the following result:

Corollary 4.16 *Assume the interval-valued R-implication $\mathcal{I}_{R_{\mathcal{T}}}$ is generated by a supmorphism t-norm $\mathcal{T}$. Then its natural negation $\mathcal{N}_{\mathcal{I}_{R_{\mathcal{T}}}}$ is representable if and only if $\mathcal{T}$ is representable. Moreover, for all $x \in L^I$,*

$$\mathcal{N}_{\mathcal{I}_{R_{\mathcal{T}}}}(x) = [N_{I_{R_{T_1}}}(x_2) \wedge N_{I_{R_{T_2}}}(x_1), N_{I_{R_{T_2}}}(x_1)]. \quad (7)$$

5 Automorphisms acting on the interval-valued natural negations

In this section, we will analyze the effects of interval-valued automorphisms acting on the natural negations of interval-valued t-norms, t-conorms or R-implications.

Definition 5.1 (Gehrke et al. (1996)) A mapping $\varrho : L^I \to L^I$ is called an interval-valued automorphism if it is bijective and satisfies that $\varrho(x) \leq \varrho(y)$ if and only if $x \leq y$ for any $x, y \in L^I$.

The set of all interval-valued automorphisms is denoted by $Aut(L^I)$. Obviously, $(Aut(L^I), \circ)$ is a group, too.

Proposition 5.2 (Bedregal and Takahashi 2005) If ϱ is an interval-valued automorphism then ϱ is $\subseteq$-isotonic, that is, if $x \subseteq y$ then $\varrho(x) \subseteq \varrho(y)$.

Lemma 5.3 *If ϱ is an interval-valued automorphism then $\varrho(D) = D$.*

Proof Straightforward. □

Lemma 5.4 *Let $\mathcal{S}$ be an interval-valued t-conorm and $\varrho \in Aut(L^I)$. Then $\mathcal{S}^{\varrho}$ is also an interval-valued t-conorm.*

Proof Straightforward. □

Lemma 5.5 *Let $\mathcal{T}$ be an interval-valued t-norm and $\varrho \in Aut(L^I)$. Then $\mathcal{T}^{\varrho}$ is an interval-valued t-norm, too.*

Proof Obviously. □

Lemma 5.6 (Bedregal 2010) Let $\mathcal{N}$ be an interval-valued (strict, strong) fuzzy negation and $\varrho \in Aut(L^I)$. Then $\mathcal{N}^{\varrho}$ is an interval-valued (strict, strong) fuzzy negation, too.

Lemma 5.7 *Let $\mathcal{I}$ be an interval-valued implication and $\varrho \in Aut(L^I)$. Then $\mathcal{T}^{\varrho}$ is an interval-valued implication, too. Moreover, if $\mathcal{I}$ satisfies A1-A9, then $\mathcal{I}^{\varrho}$ also satisfies A1-A9.*

Proof Straightforward. □

Lemma 5.8 *Let $\mathcal{I}_{\mathcal{T}}$ be an interval-valued R-implication generated by $\mathcal{T}$ and $\varrho \in Aut(L^I)$. Then $\mathcal{I}_{\mathcal{T}}^{\varrho}$ is also an interval-valued R-implication. Moreover, $\mathcal{I}_{\mathcal{T}}^{\varrho} = \mathcal{I}_{\mathcal{T}^{\varrho}}$.*

Proof Straightforward. □

Lemma 5.9 *Let $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ be an $(\mathcal{S},\mathcal{N})$-implication and $\varrho \in Aut(L^I)$. Then $\mathcal{I}^{\varrho}_{\mathcal{S},\mathcal{N}}$ is also an $(\mathcal{S},\mathcal{N})$-implication. Moreover, $\mathcal{I}^{\varrho}_{\mathcal{S},\mathcal{N}} = \mathcal{I}_{\mathcal{S}^{\varrho},\mathcal{N}^{\varrho}}$.*

Proof Straightforward. □

By the above lemmas, we can immediately obtain the following statements:

Theorem 5.10 *Let $\mathcal{N}_{\mathcal{T}}$ be a natural negation of interval-valued t-norm $\mathcal{T}$ and $\varrho \in Aut(L^I)$. Then $\mathcal{N}^{\varrho}_{\mathcal{T}}$ is also an interval-valued fuzzy negation. Moreover, $\mathcal{N}^{\varrho}_{\mathcal{T}} = \mathcal{N}_{\mathcal{T}^{\varrho}}$.*

Theorem 5.11 *Let $\mathcal{N}_{\mathcal{S}}$ be a natural negation of interval-valued t-conorm $\mathcal{S}$ and $\varrho \in Aut(L^I)$. Then $\mathcal{N}^{\varrho}_{\mathcal{S}}$ is also an interval-valued fuzzy negation. Moreover, $\mathcal{N}^{\varrho}_{\mathcal{S}} = \mathcal{N}_{\mathcal{S}^{\varrho}}$.*

Theorem 5.12 *Let $\mathcal{N}_{\mathcal{I}_{\mathcal{T}}}$ be a natural negation of interval-valued R-implication $\mathcal{I}_{\mathcal{T}}$ and $\varrho \in Aut(L^I)$. Then $\mathcal{N}^{\varrho}_{\mathcal{I}_{\mathcal{T}}}$ is also an interval-valued fuzzy negation. Moreover, $\mathcal{N}^{\varrho}_{\mathcal{I}_{\mathcal{T}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{T}^{\varrho}}}$.*

6 Relationships among families of interval-valued natural negations

In this section, we investigate the relationships among the families of natural negations of interval-valued t-norms, t-conorms or implications. Let us denote the different families of the natural negations of interval-valued t-norms, t-conorms or implications as follows:

- $\mathbb{N}$—family of all interval-valued fuzzy negations;
- $\mathbb{C}_{\mathbb{N}}$—family of all interval-valued continuous negations;
- $\mathbb{S}_{\mathbb{N}}$—family of all interval-valued strong negations;
- $\mathbb{N}_{\mathcal{T}}$—family of all natural negations of interval-valued t-norms;
- $\mathbb{N}_{\mathcal{T}_{\text{sup}}}$—family of all natural negations of interval-valued t-norms which are sup-morphisms;
- $\mathbb{N}_{\mathcal{S}}$—family of all natural negations of interval-valued t-conorms;
- $\mathbb{N}_{\mathcal{S}_{\text{inf}}}$—family of all natural negations of interval-valued t-conorms which are inf-morphisms;
- $\mathbb{N}_{\mathcal{I}}$—family of all natural negations of interval-valued implications which satisfies A1 and A2;
- $\mathbb{N}_{\mathcal{I}_{R_{\mathcal{T}}}}$—family of all natural negations of interval-valued R-implications;
- $\mathbb{N}_{\mathcal{I}_{\mathcal{S},\mathcal{T}}}$—family of all natural negations of interval-valued $(\mathcal{S},\mathcal{N})$-implications.

Obviously, we have the following statements:
$\mathbb{N} = \mathbb{N}_{\mathcal{I}_{\mathcal{S},\mathcal{T}}} = \mathbb{N}_{\mathcal{I}}$;
$\mathbb{N}_{\mathcal{T}} \cap \mathbb{N}_{\mathcal{S}} \neq \emptyset$;
$\mathbb{N}_{\mathcal{T}} = \mathbb{N}_{\mathcal{I}_R} \subseteq \mathbb{N}_{\mathcal{I}}$.
From Corollary 3.5 and 3.7, we obtain
$\mathbb{C}_{\mathbb{N}_{\mathcal{T}_{\text{sup}}}} = \mathbb{S}_{\mathbb{N}_{\mathcal{T}_{\text{sup}}}}$, $\mathbb{C}_{\mathbb{N}_{\mathcal{S}_{\text{inf}}}} = \mathbb{S}_{\mathbb{N}_{\mathcal{S}_{\text{inf}}}}$.
By Lemma 3.13, if $(\mathcal{T},\mathcal{S},\mathcal{N}_{\mathcal{S}})$ satisfies LEM, then
$\mathbb{N}_{\mathcal{T}_{\text{sup}}} = \mathbb{N}_{\mathcal{S}_{\text{inf}}}$. In this case, neither $\mathcal{N}_{\mathcal{S}}$ nor $\mathcal{N}_{\mathcal{T}}$ is necessary involutive.
By Lemma 4.4, if $\mathcal{I}$ satisfies A6 and A8, then

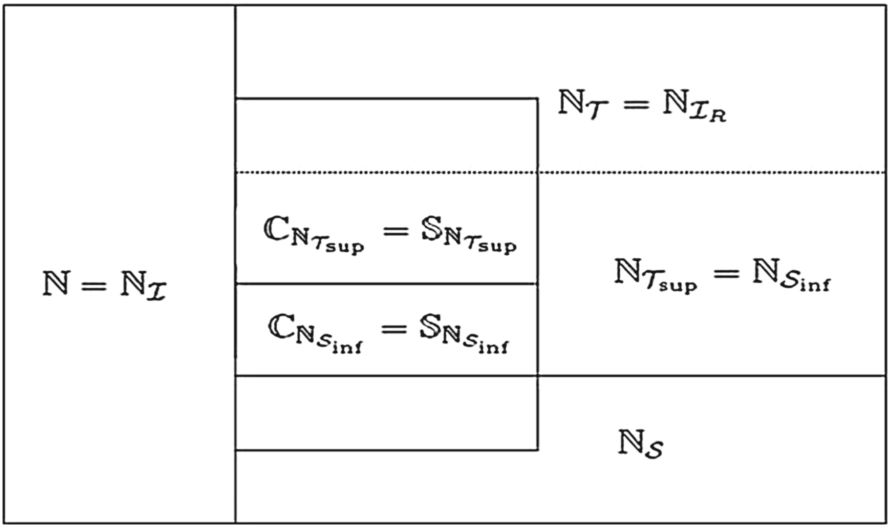

Fig. 1 Interactions among the subfamilies of interval-valued natural negations

$\mathbb{C}_{\mathbb{N}_{\mathcal{I}}} = \mathbb{S}_{\mathbb{N}_{\mathcal{I}}}$.

Finally, we can summarize the relationships among the above subfamilies which are diagrammatically represented in Fig. 1.

7 Conclusion

In this paper, we present some new results considering interval-valued fuzzy negations, mainly natural negations obtained from interval-valued t-norms, t-conorms or implications. These results may serve to develop new families of interval-valued fuzzy logic systems. Also, we show how interval-valued automorphisms act on interval-valued natural negations in order to generate new interval-valued fuzzy negations. Finally, general relationships among natural negations induced by interval-valued t-norms, t-conorms and implications are obtained.

In a future paper, we wish to investigate the natural negations generated from interval-valued non-sup-morphism t-norms and non-inf-morphism t-conorms. We will also study some other properties such as equilibrium, etc.

Acknowledgments The authors would like to thank the anonymous referees and the Editor-in-Chief for their valuable comments. This work was supported by the Natural Science Foundation of China (Grant No: 11201279), Zhejiang Provincial Natural Science Foundation (Grant No: LY12A01009), Zhejiang Province Commonweal Technique Research Project (Grant No: 2013C31001) and the Open Foundation from Marine Sciences in the Most Important Subject of Zhejiang (Grant No: 20140102).

References

Aciόly, B. M., & Bedregal, B. C. (1997). A quasi-metric topology compatible with inclusion-monotonicity property on interval space. *Reliable Computing*, *3*(3), 305–313.

Baczyński, M., & Jayaram, B. (2008). (S, N)- and R-implications: A state-of-the-art survey. *Fuzzy Sets and Systems*, *159*, 1836–1859.

Bedregal, B. C. (2010). On interval fuzzy negations. *Fuzzy Sets and Systems*, *161*(17), 2290–2313.

Bedregal, B. R. C., Takahashi, A. (2005). Interval t-norms as interval representations of t-norms, The 14th IEEE International Conference on Fuzzy Systems, May, 25–25, pp. 909–914.

Birkhof, G. (1948). L*attice theory, American Mathematical Society Colloquium Publications, vol. 25, revised edition*. New York: American Mathematical Society.

Bustince, H., Burillo, P., & Soria, F. (2003). Automorphism, negations and implication operators. *Fuzzy Sets and Systems*, *134*, 209–229.

Cornelis, C., Deschrijver, G., & Kerre, E. E. (2004). Implication in intuitionistic and interval-valued fuzzy set theory: Construction, classification and application. *International Journal of Approximate Reasoning*, *35*, 55–95.

Deschrijver, G. (2011). Triangular norms which are meet-morphisms in interval-valued fuzzy set theory. *Fuzzy Sets and Systems*, *181*, 88–101.

Deschrijver, G., Cornelis, C., & Kerre, E. E. (2004). On the representation of intuitionistic fuzzy t-norms and t-conorms. *IEEE Transactions on Fuzzy Systems*, *12*(1), 45–61.

Fodor, J. C. (1993). A new look at fuzzy connectives. *Fuzzy Sets and Systems*, *57*, 141–148.

Gehrke, M., Walker, C., & Walker, E. (1996). Some comments on interval valued fuzzy sets. *International Journal of Intelligent Systems*, *11*, 751–759.

Gorzalczany, M. B. (1987). A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems*, *21*, 1–17.

Herrera, F., Alonso, S., Chiclana, F., & Herrera-Viedma, E. (2009). Computing with words in decision making: foundations, trends and prospects. *Fuzzy Optimization and Decision Making*, *8*(4), 337–364.

Higashi, M., & Klir, G. J. (1982). On measure of fuzziness and fuzzy complements. *International Journal of General Systems*, *8*(3), 169–180.

Klement, E., & Navara, M. (1999). A survey on different traingular norm-based fuzzy logics. *Fuzzy Sets and Systems*, *101*, 241–251.

Li, D. C., & Li, Y. M. (2012). Algebraic structures of interval-valued fuzzy (S, N)-implications. *International Journal of Approximate Reasoning*, *53*, 892–900.

Liu, H. W., & Wang, G. J. (2006). A note on implicators based on binary aggregation operators in interval-valued fuzzy set theory. *Fuzzy Sets and Systems*, *157*, 3231–3236.

Lowen, R. (1978). On fuzzy complements. *Information Sciences*, *14*(2), 107–113.

Moore, R. E., Kearfott, R. B., & Cloud, M. J. (2009). *Introduction to interval analysis*. Philadelpha: SIAM.

Ovchinnikov, S. V. (1983). General negations in fuzzy set theory. *Journal of Mathematical Analysis and Applications*, *92*(1), 234–239.

Santiago, R. H. N., Bedregal, B. C., & Acióly, B. M. (2006). Formal aspects of correctness and optimality in interval computations. *Formal Aspects of Computing*, *18*(2), 231–243.

Trillas, E. (1979). Sobre funciones de negación en la teoria de conjuntos difusos. *Stochastica*, *3*, 47–59.

Wu, J. C., & Luo, M. K. (2011). Fixed points of involutive interval-valued negations. *Fuzzy Sets and Systems*, *182*(1), 110–118.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338–353.

Zadeh, L. A. (1971). Quantitative fuzzy semantics. *Information Science*, *3*, 159–176.