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Multi-dimensional uncertain differential equation: existence and uniqueness of solution

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Abstract Multi-dimensional uncertain differential equation is a system of uncertain differential equations driven by a multi-dimensional Liu process. This paper first gives the analytic solutions of two special types of multi-dimensional uncertain differential equations. After that, it proves that the multi-dimensional uncertain differential equation has a unique solution provided that its coefficients satisfy the Lipschitz condition and the linear growth condition.

Keywords Uncertain differential equation · Liu process · Uncertain process · Uncertainty theory

1 Introduction

Stochastic differential equation, since it was founded by Ito in 1940s, has been widely used to model the evolution of dynamic stochastic system. Initially, it is a type of differential equation driven by a Wiener process, and it can only describe the continuous stochastic systems. However, in order to describe the evolution of discontinuous stochastic systems, stochastic differential equations driven by Poisson process, by Lévy process, and by some martingales were proposed. In addition, in order to describe a complex stochastic system with many components which all evolute with the time,

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a multi-dimensional stochastic differential equation was also proposed and studied. Essentially, it is a system of stochastic differential equations.

As we know, Wiener process is a type of stochastic process defined within the framework of probability theory, which aims to model the random phenomena. In order to model the human uncertainty which is quite different from the randomness, an uncertainty theory was founded by Liu (2007) and refined by Liu (2009) based on the normality, duality, subadditivity, and product axioms. A concept of uncertain variable is used to model the quantity associated with human uncertainty, and concepts of uncertainty distribution, expected value, variance and entropy are employed to describe the uncertain variable. Peng and Iwamura (2010) first gave a sufficient and necessary condition for a real function being an uncertainty distribution of some uncertain variable, Liu and Ha (2010) gave a formula to calculate the expected value of a function of uncertain variables, and Yao (2015a) derived a formula to calculate the variance of an uncertain variable.

Uncertain process is a sequence of uncertain variables driven by the time. As a special type of uncertain process, Liu (2009) designed a canonical Liu process within the framework of uncertainty theory as a counterpart of standard Wiener process. It has stationary and independent increments which are normal uncertain variables, and its almost all sample paths are Lipschitz continuous. Meanwhile, Liu (2009) established an uncertain calculus with respect to canonical Liu process, which was generalized by Liu and Yao (2012), and Yao (2014) later. Based on uncertain calculus, Chen and Ralescu (2013) defined a Liu process.

Uncertain differential equation, first proposed by Liu (2008) in 2008, is a type of differential equation driven by canonical Liu process, and it aims to describe the evolution of dynamic uncertain systems. It has been widely applied in finance so far. Liu (2009) assumed the stock price follows a time-homogenous uncertain differential equation, and proposed the first uncertain stock model. Then Chen (2011) derived its American option pricing formulas, and Yao (2015b) gave a sufficient and necessary condition for the stock market being no-arbitrage. After that, Peng and Yao (2011) proposed a mean-reverting stock model to describe the variation of the stock price in long time horizon. Interest rate in the uncertain market was first studied by Chen and Gao (2013), where they proposed three models to describe the variation of the interest rate in different environments. After that, Jiao and Yao (2015) calculated the price of zero-coupon bond for some uncertain interest models. Uncertain currency model was proposed by Liu et al. (2015) to describe the currency exchange rate via uncertain differential equation. For a detailed literature review about uncertain finance, interested readers may refer to Liu (2013).

The solution methods of uncertain differential equation also draw much attention from the researchers. Chen and Liu (2010) gave the analytic solution of a linear uncertain differential equation, and Liu (2012) and Yao (2013b) provided some methods to obtain the analytic solutions of some nonlinear uncertain differential equations, respectively. In addition, Yao and Chen (2013) and Yao (2013a) provided numerical methods to solve an uncertain differential equation. In 2010, Chen and Liu (2010) gave a sufficient condition for an uncertain differential equation having a unique solution. The concept of stability for uncertain differential equations was first proposed by Liu

(2009), and a sufficient condition was given by Yao et al. (2013) for an uncertain differential equation being stable.

In our daily life, a complex uncertain system may have many components, and each component may follow an uncertain differential equation. In order to model such a complex system, Yao (2014) proposed a multi-dimensional uncertain differential equation whose solution is a multi-dimensional uncertain process. In this paper, we will give the analytic solutions of some special types of multi-dimensional uncertain differential equations. Most importantly, we will provide a sufficient condition to ensure that a multi-dimensional uncertain differential equation has a unique solution, based on which we may further study the stability of a multi-dimensional uncertain differential equation. The rest of this paper is organized as follows. In Sects. 2 and 3, we introduce some basic concepts about uncertain variables and multi-dimensional canonical Liu process, respectively. Then in Sect. 4, we give the analytic solutions of some special types of multi-dimensional uncertain differential equations. In Sect. 5, we derive a sufficient condition for a multi-dimensional uncertain differential equation having a unique solution. Finally, some remarks are made in Sect. 6.

2 Uncertain variable

In this section, we introduce some basic concepts about uncertain variable, including uncertainty distribution, expected value, independence and operational law.

Definition 1 (Liu 2007) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}\colon \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1 (*Normality Axiom*) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2 (*Duality Axiom*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3 (*Subadditivity Axiom*) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

A product uncertain measure for a compound uncertain event was defined by Liu (2009), which is the fourth axiom of uncertain measure.

Axiom 4 (*Product Axiom*) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

As a measurable function on an uncertainty space, an uncertain variable is used to model an uncertain quantity.

Definition 2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set $\Re$ of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \mid \xi(\gamma) \in B\}$$

is an event.

Definition 3 (Liu 2007) The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

If the inverse function Φ^{-1} exists and is unique for each $\alpha \in (0, 1)$, then it is called the inverse uncertainty distribution of ξ. In this case, the uncertainty distribution Φ is said to be regular.

Definition 4 (Liu 2007) Let ξ be an uncertain variable. Then its expected value $E[\xi]$ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

For an uncertain variable ξ with an uncertainty distribution Φ, we have

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(r))\mathrm{d}r - \int_{-\infty}^{0} \Phi(r)\mathrm{d}r.$$

If the inverse uncertainty distribution Φ^{-1} exists, then

$$E[\xi] = \int_0^{1} \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

Definition 5 (Liu 2009) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{k=1}^{m}(\xi_i \in B_i)\right\} = \bigwedge_{k=1}^{m} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Theorem 1 (Liu 2010) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(r) = f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)\right).$$

3 Multi-dimensional uncertain calculus

In this section, we first introduce the canonical Liu process and the uncertain calculus. Then as a generalization, we introduce multi-dimensional canonical Liu process and multi-dimensional uncertain calculus.

Definition 6 (Liu 2008) Let T be an index set, and $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process X_t is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set

$$\{X_t \in B\} = \{\gamma \mid X_t(\gamma) \in B\}$$

is an event.

Definition 7 (Liu 2009) An uncertain process C_t is said to be a canonical Liu process if

(i) almost all sample paths of C_t are Lipschitz continuous,
(ii) $C_0 = 0$ and C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with an uncertainty distribution

$$\Phi_t(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Remark 1 Although both canonical Liu process and standard Wiener process are stationary and independent increment processes, they are two different types of processes. The former is essentially an uncertain process defined within the framework of uncertainty theory, while the latter is essentially a stochastic process defined within the framework of probability theory.

Definition 8 (Liu 2009) Let X_t be an uncertain process and C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \ldots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then uncertain integral of X_t with respect to C_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be uncertain integrable.

Definition 9 (Liu 2014) Uncertain processes $X_{1t}, X_{2t}, \ldots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \ldots, t_k$, the uncertain vectors

$$\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \ldots, X_{it_k}), \quad i = 1, 2, \ldots, n$$

are independent, i.e., for any Borel sets $\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_n$ of k-dimensional real vectors, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in \boldsymbol{B}_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in \boldsymbol{B}_i\}.$$

Definition 10 (Zhang and Chen 2013) Let $C_{it}, i = 1, 2, \ldots, n$ be independent canonical Liu processes on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \ldots, C_{nt})^T$ is called an n-dimensional canonical Liu process on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

The multi-dimensional canonical Liu process $\boldsymbol{C}_t$ also possesses the property of independent and stationary increment. In addition, almost all the sample paths of $\boldsymbol{C}_t$ are also Lipschitz continuous. Based on multi-dimensional canonical Liu process, multi-dimensional uncertain integral was defined as follows.

Definition 11 (Yao 2014) Let $\boldsymbol{C}_t = (C_{1t}, C_{2t}, \ldots, C_{nt})^T$ be an n-dimensional canonical Liu process, and $\boldsymbol{X}_t = [X_{ijt}]$ be an $m \times n$ uncertain matrix process whose elements X_{ijt} are integrable uncertain processes. Then the uncertain integral of $\boldsymbol{X}_t$ with respect to the n-dimensional canonical Liu process $\boldsymbol{C}_t$ is defined by

$$\int_a^b \boldsymbol{X}_t \mathrm{d}\boldsymbol{C}_t = \begin{pmatrix} \sum_{j=1}^{n} \int_a^b X_{1jt} \mathrm{d}C_{jt} \\ \sum_{j=1}^{n} \int_a^b X_{2jt} \mathrm{d}C_{jt} \\ \vdots \\ \sum_{j=1}^{n} \int_a^b X_{mjt} \mathrm{d}C_{jt} \end{pmatrix}.$$

4 Multi-dimensional uncertain differential equation

In this section, we introduce the concept of multi-dimensional uncertain differential equation, and give the analytic solutions of two special types of multi-dimensional uncertain differential equations.

Definition 12 (Yao 2014) Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process. Suppose $\boldsymbol{f}(t, \boldsymbol{x})$ is a vector-valued function from $T \times \Re^m$ to $\Re^m$, and $\boldsymbol{g}(t, \boldsymbol{x})$ is a matrix-valued function from $T \times \Re^m$ to the set of $m \times n$ matrices. Then

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{f}(t, \boldsymbol{X}_t)\mathrm{d}t + \boldsymbol{g}(t, \boldsymbol{X}_t)\mathrm{d}\boldsymbol{C}_t \tag{1}$$

is called an m-dimensional uncertain differential equation driven by an n-dimensional canonical Liu process. A solution is an m-dimensional uncertain process that satisfied (1) identically in each t.

Remark 2 The multi-dimensional uncertain differential Eq. (1) is equivalent to the multi-dimensional uncertain integral equation

$$\boldsymbol{X}_s = \boldsymbol{X}_0 + \int_0^s \boldsymbol{f}(t, \boldsymbol{X}_t)\mathrm{d}t + \int_0^s \boldsymbol{g}(t, \boldsymbol{X}_t)\mathrm{d}\boldsymbol{C}_t.$$

Theorem 2 *Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process, $\boldsymbol{U}_t$ be an m-dimensional integrable uncertain process, and $\boldsymbol{V}_t$ be an $m \times n$ integrable uncertain matrix process. Then the m-dimensional uncertain differential equation*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{U}_t\mathrm{d}t + \boldsymbol{V}_t\mathrm{d}\boldsymbol{C}_t$$

has a solution

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \boldsymbol{U}_s\mathrm{d}s + \int_0^t \boldsymbol{V}_s\mathrm{d}\boldsymbol{C}_s.$$

Proof Consider the k-th element of the solution process $\boldsymbol{X}_t$, denoted by X_{kt}. It is a solution of the uncertain differential equation

$$\mathrm{d}X_{kt} = U_{kt}\mathrm{d}t + V_{k1t}\mathrm{d}C_{1t} + \ldots + V_{knt}\mathrm{d}C_{nt} \tag{2}$$

according to the definition of multi-dimensional uncertain differential equation. Obviously, the uncertain differential Eq. (2) has a solution

$$X_{kt} = X_{k0} + \int_0^t U_{ks}\mathrm{d}s + \int_0^t V_{k1s}\mathrm{d}C_{1s} + \ldots + \int_0^t V_{kns}\mathrm{d}C_{ns}.$$

As a result, we have

$$\boldsymbol{X}_t = \begin{pmatrix} X_{1t} \\ X_{2t} \\ \vdots \\ X_{mt} \end{pmatrix} = \begin{pmatrix} X_{10} + \int_0^t U_{1s}\mathrm{d}s + \int_0^t V_{11s}\mathrm{d}C_{1s} + \ldots + \int_0^t V_{1ns}\mathrm{d}C_{ns} \\ X_{20} + \int_0^t U_{2s}\mathrm{d}s + \int_0^t V_{21s}\mathrm{d}C_{1s} + \ldots + \int_0^t V_{2ns}\mathrm{d}C_{ns} \\ \vdots \\ X_{m0} + \int_0^t U_{ms}\mathrm{d}s + \int_0^t V_{m1s}\mathrm{d}C_{1s} + \ldots + \int_0^t V_{mns}\mathrm{d}C_{ns} \end{pmatrix}.$$

In the form of matrix, we have

$$\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \boldsymbol{U}_s\mathrm{d}s + \int_0^t \boldsymbol{V}_s\mathrm{d}\boldsymbol{C}_s.$$

The proof is completed. □

Example 1 Let $\boldsymbol{C}_t = (C_{1t}, C_{2t})^T$ be a 2-dimensional canonical Liu process. Consider a 2-dimensional uncertain differential equation

$$\mathrm{d}\boldsymbol{X}_t = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} v_{11t} & v_{12t} \\ v_{21t} & v_{22t} \end{pmatrix}\mathrm{d}\boldsymbol{C}_t$$

with an initial value $\boldsymbol{X}_0 = (x_{10}, x_{20})^T$. According to Theorem 2, it has a solution

$$\boldsymbol{X}_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} x_{10} + \int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{11s}\mathrm{d}C_{1s} + \int_0^t v_{12s}\mathrm{d}C_{2s} \\ x_{20} + \int_0^t u_{2s}\mathrm{d}s + \int_0^t v_{21s}\mathrm{d}C_{1s} + \int_0^t v_{22s}\mathrm{d}C_{2s} \end{pmatrix}.$$

In other words, the system of uncertain differential equations

$$\begin{cases} \mathrm{d}X_{1t} = u_{1t}\mathrm{d}t + v_{11t}\mathrm{d}C_{1t} + v_{12t}\mathrm{d}C_{2t}, & X_{10} = x_{10} \\ \mathrm{d}X_{2t} = u_{2t}\mathrm{d}t + v_{21t}\mathrm{d}C_{1t} + v_{22t}\mathrm{d}C_{2t}, & X_{20} = x_{20} \end{cases}$$

has a solution

$$\begin{cases} X_{1t} = x_{10} + \int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{11s}\mathrm{d}C_{1s} + \int_0^t v_{12s}\mathrm{d}C_{2s}, \\ X_{2t} = x_{20} + \int_0^t u_{2s}\mathrm{d}s + \int_0^t v_{21s}\mathrm{d}C_{1s} + \int_0^t v_{22s}\mathrm{d}C_{2s}. \end{cases}$$

Theorem 3 *Let C_t be a canonical Liu process, and $\boldsymbol{U}$ and $\boldsymbol{V}$ be two $m \times m$ matrices satisfying $\boldsymbol{U}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{U}$. Then the m-dimensional uncertain differential equation*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{U}\boldsymbol{X}_t\mathrm{d}t + \boldsymbol{V}\boldsymbol{X}_t\mathrm{d}C_t \tag{3}$$

has a solution

$$\boldsymbol{X}_t = \exp(t\boldsymbol{U} + C_t\boldsymbol{V}) \cdot \boldsymbol{X}_0.$$

Proof The m-dimensional uncertain process $\boldsymbol{X}_t$ can also be written as

$$\boldsymbol{X}_t = \sum_{n=0}^{\infty} \frac{1}{n!}(t\boldsymbol{U} + C_t\boldsymbol{V})^n \cdot \boldsymbol{X}_0.$$

We will show that it satisfies the form (3). Since $\boldsymbol{U}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{U}$, taking differentiation operations on both sides, we have

$$\begin{aligned}
\mathrm{d}\boldsymbol{X}_t &= \boldsymbol{U}\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(t\boldsymbol{U} + C_t\boldsymbol{V})^{n-1} \cdot \boldsymbol{X}_0\mathrm{d}t \\
&\quad + \boldsymbol{V}\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(t\boldsymbol{U} + C_t\boldsymbol{V})^{n-1} \cdot \boldsymbol{X}_0\mathrm{d}C_t \\
&= \boldsymbol{U}\sum_{n=0}^{\infty} \frac{1}{n!}(t\boldsymbol{U} + C_t\boldsymbol{V})^n \cdot \boldsymbol{X}_0\mathrm{d}t + \boldsymbol{V}\sum_{n=0}^{\infty} \frac{1}{n!}(t\boldsymbol{U} + C_t\boldsymbol{V})^n \cdot \boldsymbol{X}_0\mathrm{d}C_t \\
&= \boldsymbol{U}\exp(t\boldsymbol{U} + C_t\boldsymbol{V}) \cdot \boldsymbol{X}_0\mathrm{d}t + \boldsymbol{V}\exp(t\boldsymbol{U} + C_t\boldsymbol{V}) \cdot \boldsymbol{X}_0\mathrm{d}C_t \\
&= \boldsymbol{U}\boldsymbol{X}_t\mathrm{d}t + \boldsymbol{V}\boldsymbol{X}_t\mathrm{d}C_t.
\end{aligned}$$

The theorem is thus proved. □

Example 2 Let C_t be a canonical Liu process. Consider a 2-dimensional uncertain differential equation

$$\mathrm{d}\boldsymbol{X}_t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{X}_t\mathrm{d}C_t \tag{4}$$

with an initial value $\boldsymbol{X}_0 = (1, 0)^T$. In this case, we have $\boldsymbol{U} = \boldsymbol{0}$ and

$$\boldsymbol{V} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $\boldsymbol{U}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{U} = \boldsymbol{0}$, the multi-dimensional uncertain differential Equation (4) has a solution

$$\begin{aligned}
\boldsymbol{X}_t &= \exp(C_t\boldsymbol{V}) \cdot \boldsymbol{X}_0 = \exp\begin{pmatrix} 0 & C_t \\ -C_t & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos(C_t) & \sin(C_t) \\ -\sin(C_t) & \cos(C_t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(C_t) \\ -\sin(C_t) \end{pmatrix}.
\end{aligned}$$

In other words, the system of uncertain differential equations

$$\begin{cases} \mathrm{d}X_{1t} = X_{2t}\mathrm{d}C_t, & X_{10} = 1 \\ \mathrm{d}X_{2t} = -X_{1t}\mathrm{d}C_t, & X_{20} = 0 \end{cases}$$

has a solution

$$\begin{cases} X_{1t} = \cos(C_t) \\ X_{2t} = -\sin(C_t). \end{cases}$$

Example 3 Let C_t be a canonical Liu process. Consider a 2-dimensional uncertain differential equation

$$\mathrm{d}\boldsymbol{X}_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{X}_t \mathrm{d}t + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \boldsymbol{X}_t \mathrm{d}C_t \tag{5}$$

with an initial value $\boldsymbol{X}_0 = (0, 1)^T$. In this case, we have

$$\boldsymbol{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{V} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Since $\boldsymbol{UV} = \boldsymbol{VU} = \boldsymbol{V}$, the multi-dimensional uncertain differential Equation (5) has a solution

$$\begin{aligned} \boldsymbol{X}_t &= \exp(t\boldsymbol{U} + C_t\boldsymbol{V}) \cdot \boldsymbol{X}_0 \\ &= \exp\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \cdot \exp\begin{pmatrix} C_t & C_t \\ -C_t & C_t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(t) \end{pmatrix} \cdot \begin{pmatrix} \exp(C_t)\cos(C_t) & \exp(C_t)\sin(C_t) \\ -\exp(C_t)\sin(C_t) & \exp(C_t)\cos(C_t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \exp(t + C_t)\sin(C_t) \\ \exp(t + C_t)\cos(C_t) \end{pmatrix}. \end{aligned}$$

In other words, the system of uncertain differential equations

$$\begin{cases} \mathrm{d}X_{1t} = X_{1t}\mathrm{d}t + (X_{1t} + X_{2t})\mathrm{d}C_t, & X_{10} = 0 \\ \mathrm{d}X_{2t} = X_{2t}\mathrm{d}t + (-X_{1t} + X_{2t})\mathrm{d}C_t, & X_{20} = 1 \end{cases}$$

has a solution

$$\begin{cases} X_{1t} = \exp(t + C_t)\sin(C_t) \\ X_{2t} = \exp(t + C_t)\cos(C_t). \end{cases}$$

5 Existence and uniqueness theorem

In this section, we consider the existence and uniqueness of the solution of a multi-dimensional uncertain differential equation. For simplicity, we employ the infinite norm, and write

$$|\boldsymbol{x}| = \max_{1 \le i \le m} |x_i|, \quad |A| = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

for an n-dimensional vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_m)$ and an $m \times n$ matrix $A = [a_{ij}]$, respectively.

Lemma 1 *Let $\boldsymbol{X}_t$ be an m-dimensional integrable uncertain process on $[a, b]$. Then*

$$\left|\int_a^b \boldsymbol{X}_t(\gamma)\mathrm{d}t\right| \leq \int_a^b |\boldsymbol{X}_t(\gamma)|\mathrm{d}t$$

for each $\gamma \in \Gamma$.

Proof It follows from the definition of the infinite norm that

$$\left|\int_a^b \boldsymbol{X}_t(\gamma)\mathrm{d}t\right| = \max_{1\leq i\leq m}\left|\int_a^b X_{it}(\gamma)\mathrm{d}t\right| \leq \max_{1\leq i\leq m}\int_a^b |X_{it}(\gamma)|\mathrm{d}t$$
$$\leq \int_a^b \max_{1\leq i\leq m}|X_{it}(\gamma)|\mathrm{d}t = \int_a^b |\boldsymbol{X}_t(\gamma)|\mathrm{d}t$$

for each $\gamma \in \Gamma$. The lemma is proved.

Lemma 2 *Let $\boldsymbol{C}_t$ be an n-dimensional canonical Liu process, and let $\boldsymbol{Y}_t$ be an $m \times n$ integrable uncertain matrix process. Then*

$$\left|\int_a^b \boldsymbol{Y}_t(\gamma)\mathrm{d}\boldsymbol{C}_t(\gamma)\right| \leq K(\gamma)\int_a^b |\boldsymbol{Y}_t(\gamma)|\mathrm{d}t$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $\boldsymbol{C}_t(\gamma)$.

Proof It follows from the definition of the infinite norm that

$$\left|\int_a^b \boldsymbol{Y}_t(\gamma)\mathrm{d}\boldsymbol{C}_t(\gamma)\right| = \max_{1\leq i\leq m}\left|\sum_{j=1}^n\int_a^b Y_{ijt}(\gamma)\mathrm{d}C_{jt}(\gamma)\right| \leq \max_{1\leq i\leq m}\sum_{j=1}^n\left|\int_a^b Y_{ijt}(\gamma)\mathrm{d}C_{jt}(\gamma)\right|$$
$$\leq \max_{1\leq i\leq m}\sum_{j=1}^n K(\gamma)\int_a^b \left|Y_{ijt}(\gamma)\right|\mathrm{d}t = K(\gamma)\max_{1\leq i\leq m}\int_a^b\sum_{j=1}^n\left|Y_{ijt}(\gamma)\right|\mathrm{d}t$$
$$\leq K(\gamma)\int_a^b \max_{1\leq i\leq m}\sum_{j=1}^n\left|Y_{ijt}(\gamma)\right|\mathrm{d}t = K(\gamma)\int_a^b |\boldsymbol{Y}_t(\gamma)|\mathrm{d}t.$$

The lemma is thus proved. □

Theorem 4 *The multi-dimensional uncertain differential equation*

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{f}(t, \boldsymbol{X}_t)\mathrm{d}t + \boldsymbol{g}(t, \boldsymbol{X}_t)\mathrm{d}\boldsymbol{C}_t \tag{6}$$

with an initial value $\boldsymbol{X}_0$ has a unique solution if the coefficients $\boldsymbol{f}(t, \boldsymbol{x})$ and $\boldsymbol{g}(t, \boldsymbol{x})$ satisfy the Lipschitz condition

$$|\boldsymbol{f}(t, \boldsymbol{x}) - \boldsymbol{f}(t, \boldsymbol{y})| + |\boldsymbol{g}(t, \boldsymbol{x}) - \boldsymbol{g}(t, \boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|, \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \Re^m, t \geq 0$$

and the linear growth condition

$$|\boldsymbol{f}(t,\boldsymbol{x})|+|\boldsymbol{g}(t,\boldsymbol{x})|\le L(1+|\boldsymbol{x}|),\quad \forall \boldsymbol{x}\in\Re^m, t\ge 0$$

for some constant L.

Proof We first prove that there exists a solution of (6) by means of successive approximation. For simplicity, we just consider the solution on $[0, T]$ for any given real number T. For each $\gamma\in\Gamma$, define

$$\boldsymbol{X}_t^{(0)}=\boldsymbol{X}_0,\quad \boldsymbol{X}_t^{(n+1)}(\gamma)=\boldsymbol{X}_0+\int_0^t f\left(s,\boldsymbol{X}_s^{(n)}(\gamma)\right)\mathrm{d}s+\int_0^t g\left(s,\boldsymbol{X}_s^{(n)}(\gamma)\right)\mathrm{d}\boldsymbol{C}_t(\gamma)$$

and

$$Q_t^{(n)}(\gamma)=\max_{0\le s\le t}\left|\boldsymbol{X}_s^{(n+1)}(\gamma)-\boldsymbol{X}_s^{(n)}(\gamma)\right|$$

for $n=1,2,\ldots$ We will prove by induction method that

$$Q_t^{(n)}(\gamma)\le(1+|\boldsymbol{X}_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}t^{n+1}\tag{7}$$

for almost every $\gamma\in\Gamma$ and for every nonnegative integer n. Since the right term of (7) satisfies

$$\sum_{n=0}^{\infty}(1+|\boldsymbol{X}_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}t^{n+1}<+\infty,$$

it follows from Weierstrass criterion that $\boldsymbol{X}_t^{(n)}(\gamma)$ converges uniformly on $[0, T]$, whose limit is denoted by $\boldsymbol{X}_t(\gamma)$. Then we have

$$\boldsymbol{X}_t(\gamma)=\boldsymbol{X}_0+\int_0^t f(s,\boldsymbol{X}_s(\gamma))\mathrm{d}s+\int_0^t g(s,\boldsymbol{X}_s(\gamma))\mathrm{d}\boldsymbol{C}_s(\gamma).$$

Therefore, the uncertain process $\boldsymbol{X}_t$ is just the solution of multi-dimensional uncertain differential Eq. (6). The inequality (7) is proved as follows. For $n=0$, we have

$$\begin{aligned}
Q_t^{(0)}(\gamma)&=\max_{0\le s\le t}\left|\int_0^s f(u,\boldsymbol{X}_0)\mathrm{d}u+\int_0^s g(u,\boldsymbol{X}_0)\mathrm{d}\boldsymbol{C}_u(\gamma)\right|\\
&\le\max_{0\le s\le t}\left|\int_0^s f(u,\boldsymbol{X}_0)\mathrm{d}u\right|+\max_{0\le s\le t}\left|\int_0^s g(u,\boldsymbol{X}_0)\mathrm{d}\boldsymbol{C}_u(\gamma)\right|\\
&\le\max_{0\le s\le t}\int_0^s|f(u,\boldsymbol{X}_0)|\,\mathrm{d}u+K(\gamma)\max_{0\le s\le t}\int_0^s|g(u,\boldsymbol{X}_0)|\,\mathrm{d}u
\end{aligned}$$

$$\leq \int_0^t |f(u, \boldsymbol{X}_0)| \,\mathrm{d}u + K(\gamma) \int_0^t |g(u, \boldsymbol{X}_0)| \,\mathrm{d}u$$
$$\leq (1 + |\boldsymbol{X}_0|) L (1 + K(\gamma)) t$$

by Lemmas 1 and 2. Assume the inequality (7) holds for the integer n, i.e.,

$$Q_t^{(n)}(\gamma) = \max_{0\leq s\leq t} \left| \boldsymbol{X}_s^{(n+1)}(\gamma) - \boldsymbol{X}_s^{(n)}(\gamma) \right| \leq (1+|\boldsymbol{X}_0|) \frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!} t^{n+1}.$$

Then we have

$$\begin{aligned}
Q_t^{(n+1)}(\gamma) &= \max_{0\leq s\leq t} \left| \boldsymbol{X}_s^{(n+2)}(\gamma) - \boldsymbol{X}_s^{(n+1)}(\gamma) \right| \\
&= \max_{0\leq s\leq t} \left| \int_0^s f\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - f\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \mathrm{d}u \right. \\
&\quad \left. + \int_0^s g\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - g\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \mathrm{d}\boldsymbol{C}_u(\gamma) \right| \\
&\leq \max_{0\leq s\leq t} \left| \int_0^s f\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - f\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \mathrm{d}u \right| \\
&\quad + \max_{0\leq s\leq t} \left| \int_0^s g\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - g\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \mathrm{d}\boldsymbol{C}_u(\gamma) \right| \\
&\leq \int_0^t \left| f\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - f\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \right| \mathrm{d}u \\
&\quad + K(\gamma) \int_0^t \left| g\left(u, \boldsymbol{X}_u^{(n+1)}(\gamma)\right) - g\left(u, \boldsymbol{X}_u^{(n)}(\gamma)\right) \right| \mathrm{d}u \\
&\leq L \int_0^t \left| \boldsymbol{X}_u^{(n+1)}(\gamma) - \boldsymbol{X}_u^{(n)}(\gamma) \right| \mathrm{d}u + K(\gamma) L \int_0^t \left| \boldsymbol{X}_u^{(n+1)}(\gamma) - \boldsymbol{X}_u^{(n)}(\gamma) \right| \mathrm{d}u \\
&\leq L(1 + K(\gamma)) \int_0^t (1 + |\boldsymbol{X}_0|) \frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!} u^{n+1} \mathrm{d}u \\
&= (1 + |\boldsymbol{X}_0|) \frac{L^{n+2}(1 + K(\gamma))^{n+2}}{(n+2)!} t^{n+2}.
\end{aligned}$$

It means the inequality (7) also holds for the integer $n + 1$. Thus the inequality (7) holds for all nonnegative integers.

Next, we prove the uniqueness of the solution under the given conditions. Assume that $\boldsymbol{X}_t$ and $\boldsymbol{X}_t^*$ are two solutions of the multi-dimensional uncertain differential Eq. (6) with the same initial value $\boldsymbol{X}_0$. Then for almost every $\gamma \in \Gamma$, we have

$$\begin{aligned}
|\boldsymbol{X}_t(\gamma) - \boldsymbol{X}_t^*(\gamma)| = &\left| \int_0^t f(s, \boldsymbol{X}_s(\gamma)) - f(s, \boldsymbol{X}_s^*(\gamma)) \mathrm{d}s \right. \\
&\left. + \int_0^t g(s, \boldsymbol{X}_s(\gamma)) - g(s, \boldsymbol{X}_s^*(\gamma)) \mathrm{d}\boldsymbol{C}_s(\gamma) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\int_0^t f(s, \boldsymbol{X}_s(\gamma)) - f(s, \boldsymbol{X}_s^*(\gamma))\mathrm{d}s\right| \\
&\quad + \left|\int_0^t g(s, \boldsymbol{X}_s(\gamma)) - g(s, \boldsymbol{X}_s^*(\gamma))\mathrm{d}\boldsymbol{C}_s(\gamma)\right| \\
&\leq \int_0^t \left|f(s, \boldsymbol{X}_s(\gamma)) - f(s, \boldsymbol{X}_s^*(\gamma))\right| \mathrm{d}s \\
&\quad + K(\gamma) \int_0^t \left|g(s, \boldsymbol{X}_s(\gamma)) - g(s, \boldsymbol{X}_s^*(\gamma))\right| \mathrm{d}s \\
&\leq L \int_0^t \left|\boldsymbol{X}_s(\gamma) - \boldsymbol{X}_s^*(\gamma)\right| \mathrm{d}s + K(\gamma) L \int_0^t \left|\boldsymbol{X}_s(\gamma) - \boldsymbol{X}_s^*(\gamma)\right| \mathrm{d}s \\
&= L(1 + K(\gamma)) \int_0^t \left|\boldsymbol{X}_s(\gamma) - \boldsymbol{X}_s^*(\gamma)\right| \mathrm{d}s.
\end{aligned}
$$

By Gronwall inequality, we obtain

$$|\boldsymbol{X}_t(\gamma) - \boldsymbol{X}_t^*(\gamma)| \leq 0 \cdot \exp\left(L(1 + K(\gamma))t\right) = 0.$$

That means $X_t = X_t^*$ almost surely. The uniqueness of the solution is verified. □

6 Conclusions

Multi-dimensional uncertain differential equation, as a generalization of uncertain differential equation, aims to model a multi-dimensional dynamic uncertain system. This paper gave the analytic solutions of two special types of multi-dimensional uncertain differential equations. Besides, it gave a sufficient condition for a multi-dimensional uncertain differential equation having a unique solution. We will work toward weakening the conditions in the future.

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