Fuzzy Optim Decis Making (2015) 14:289–310
DOI 10.1007/s10700-014-9205-1

Geometric consistency based interval weight elicitation from intuitionistic preference relations using logarithmic least square optimization

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Published online: 7 January 2015

Abstract This paper introduces the notion of intuitionistic fuzzy geometric indices to capture the central tendency of Atanassov's intuitionistic fuzzy values (A-IFVs). A ratio-based hesitation margin is defined to measure the hesitancy of an A-IFV and used to determine the geometric mean based hesitation margin of an intuitionistic preference relation (IPR). The paper defines geometric consistency of IPRs based on the intuitionistic fuzzy geometric index, and puts forward some properties for geometry consistent IPRs. A parameterized transformation formula is proposed to convert normalization interval weights into geometry consistent IPRs. A logarithmic least square model is developed for constructing the fitted geometry consistent IPR and deriving interval weights from an IPR. Based on the fitted consistent IPR, a method is provided to check the acceptable geometry consistency for IPRs, and an algorithm is further devised to solve multiple criteria decision making problems with IPRs. Two numerical examples and comparisons with existing approaches are provided to illustrate the performance and effectiveness of the proposed models.

Keywords Intuitionistic preference relation · Intuitionistic fuzzy geometric index · Geometric consistency · Logarithmic least square · Multiple criteria decision making

1 Introduction

In real-world decision problems, a decision-maker (DM) often provides his/her membership assessments on alternatives or criteria with uncertainty (Liu 2013) or hesitancy. To characterize this hesitation, Atanassov (1986) introduced the intuitionistic fuzzy

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sets (A-IFSs) by explicitly considering nonmembership where the sum of membership and nonmembership does not necessarily add up to 1. Since its inception, A-IFSs have been widely applied to decision modeling (Aggarwal et al. 2012; Gong et al. 2009; Liao and Xu 2014a, b; Park et al. 2011; Parvathi et al. 2013; Szmidt and Kacprzyk 2009; Wang 2013; Wang and Li 2012; Wu and Chiclana 2012, 2014a, b; Xu 2007, 2009, 2012; Xu et al. 2011; Xu and Liao 2014; Zhang and Xu 2012). To model DM's pairwise comparisons with hesitancy, Xu (2007) introduced the concept of intuitionistic preference relations (IPRs), in which DM's preferences are characterized by Atanassov's intuitionistic fuzzy values (A-IFVs), and the hesitation margin of a preference is determined by its intuitionistic fuzzy index.

Different transitivity conditions have been put forward to define consistent IPRs, and several approaches have been developed to derive priority weights from IPRs. By using the feasible region idea, Xu (2009) defined additive consistency of IPRs and proposed a linear-programming-based approach to estimating criterion weights from IPRs. Wang (2013) adopted intuitionistic fuzzy judgments in an IPR to define new additive consistency of IPRs and investigated how to derive priority weights by establishing goal programming models for both individual and group decision situations. Based on the converted membership intervals and the associated interval priority weights, Gong et al. (2009) defined multiplicative consistent IPRs and established goal programming models to generate interval priority weights of IPRs. Xu (2007) introduced an A-IFV based equation to define multiplicative consistent IPRs, and developed an approach to group decision making with IPRs by employing the intuitionistic fuzzy weighted averaging operator. Later, Xu et al. (2011) pointed out the drawback of the multiplicative transitivity equation given by Xu (2007), and proposed another multiplicative consistency definition for IPRs by extending the functional equation (Chiclana et al. 2009) of fuzzy preference relations. Based on this multiplicative consistency definition, they developed two algorithms to estimate missing values for incomplete IPRs. Subsequently, Xu and Liao (2014) put forward a framework of the intuitionistic fuzzy analytic hierarchy process method and applied it to handle multiple criteria decision making (MCDM) problems with a hierarchical structure.

It is well known that the consistency plays a fundamental and important role in MCDM with preference relations. The derivation of priority weights of preference relations is usually based on the consistency constraint condition. A literature review shows that the multiplicative consistency in Gong et al. (2009) is handled in an indirect manner. Although the multiplicative consistency in Xu et al. (2011) is defined by using the DM's intuitionistic fuzzy judgments, it has flaws in non-robustness for permutations of the original intuitionistic fuzzy judgments furnished by the DM (See a further study in Sect. 4). In this paper, we first introduce the notion of intuitionistic fuzzy geometric indices to capture the central tendency of an intuitionistic fuzzy judgment. A ratio-based hesitation margin is defined to measure the hesitancy index of an intuitionistic fuzzy judgment and extended to determine the geometric mean based hesitation margin of an IPR. Then, we define geometric consistency of IPRs based on the intuitionistic fuzzy geometric index, and put forward some properties for geometry consistent IPRs. Subsequently, we devise a parameterized transformation formula between normalization interval weights and geometry consistent IPRs, which is used to develop a logarithmic least square model for deriving interval weights and constructing a fitted geometry consistent IPR from any IPR. Based on the geometric-mean-based difference degree between the original IPR and the constructed geometry consistent IPR, we introduce the notion of acceptable geometry consistent IPRs and further put forward an algorithm for solving MCDM problems with IPRs.

The rest of the paper is organized as follows. Section 2 furnishes a brief review on multiplicative consistent fuzzy preference relations, A-IFSs and IPRs. Section 3 shows how to measure ratio-based amounts of A-IFVs. In Sect. 4, the geometric consistency is defined for IPRs. Sect. 5 develops a logarithmic least square model for generating a normalization interval weight vector from an IPR and put forward an algorithm for MCDM with IPRs. Two numerical examples are provided in Sect. 6 to demonstrate the performance of the proposed models, and conclusions are presented in Sect. 7.

2 Preliminaries

For an MCDM problem with a finite set of alternatives, let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of n alternatives. In eliciting his/her preference over alternatives, a DM often utilizes a pairwise comparison technique, yielding a fuzzy preference relation $R = (r_{ij})_{n\times n}$, where r_{ij} denotes a fuzzy preference degree of alternative x_i over x_j such that

$$0 \leq r_{ij} \leq 1, \quad r_{ij} + r_{ji} = 1, \quad r_{ii} = 0.5 \quad \text{for all } \ i, j = 1, 2, \ldots, n \tag{2.1}$$

Definition 2.1 (Tanino 1984) Let $R = (r_{ij})_{n\times n}$ be a fuzzy preference relation with $0 < r_{ij} < 1$ for all $i, j = 1, 2, \ldots, n$. R is said to have multiplicative consistency, if it satisfies

$$\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}} = \frac{r_{ij}}{r_{ji}} \quad \text{for all } \ i, j, k = 1, 2, \ldots, n \tag{2.2}$$

As $r_{ij} = 1 - r_{ji}$ for all i, j =1, 2, …, n, it follows from (2.2) that

$$\frac{r_{ij}}{r_{ji}}\frac{r_{jk}}{r_{kj}}\frac{r_{ki}}{r_{ik}} = \frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}}\frac{r_{ji}}{r_{ij}} \quad \text{for all } \ i, j, k = 1, 2, \ldots, n \tag{2.3}$$

For a fuzzy preference relation $R = (r_{ij})_{n\times n}$, R is multiplicative consistent if and only if there exists a weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that

$$r_{ij} = \frac{\omega_i}{\omega_i + \omega_j} \quad \text{for all } \ i, j = 1, 2, \ldots, n \tag{2.4}$$

where $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$ for $i = 1, 2, \ldots, n$.

Let Zbe a fixed nonempty universe set, an A-IFS A on Z is an object given by

$$A = \{< z, \mu_A(z), \nu_A(z) > |z \in Z\} \tag{2.5}$$

where $\mu_A : Z \to [0, 1]$, $\nu_A : Z \to [0, 1]$ such that $0 \leq \mu_A(z) + \nu_A(z) \leq 1, \forall z \in Z$.

$\mu_A(z)$ and $\nu_A(z)$ denote, respectively, the membership and nonmembership degrees of the element z to the set A. In addition, $\pi_A(z) = 1 - \mu_A(z) - \nu_A(z)$ is called the intuitionistic fuzzy index of A, representing the hesitation margin of z to A. Obviously, $0 \leq \pi_A(z) \leq 1$.

For an A-IFS A and a given z, the pair $(\mu_A(z), \nu_A(z))$ is called an Atanassov's intuitionistic fuzzy value (A-IFV) (Xu 2007). For convenience, the pair $(\mu_A(z), \nu_A(z))$is often denoted by (μ, v), where $\mu, v \in [0, 1]$ and $\mu + v \leq 1$.

Definition 2.2 (Xu 2007) An IPR $\tilde{R}$ on X is an A-IFS on the product set $X \times X$ characterized by a judgment matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$, where (μ_{ij}, v_{ij}) indicates the intuitionistic preference degree of the alternative x_i over x_j such that

$$0 \leq \mu_{ij} + v_{ij} \leq 1, \quad \mu_{ij} = v_{ji}, \quad v_{ij} = \mu_{ji}, \quad \mu_{ii} = v_{ii} = 0.5 \quad i, j = 1, 2, \ldots, n \tag{2.6}$$

3 Ratio-based measure for the amount of Atanassov's intuitionistic fuzzy values

This section discusses the problem of how to measure the amount of an A-IFV from the viewpoint of using A-IFVs to express ratio-based pairwise comparisons. The notion of intuitionistic fuzzy geometric indices is introduced to capture the central tendency of an A-IFV, and a ratio-based hesitation margin is defined to measure the hesitancy index of an A-IFV. A geometric mean based hesitation margin for a set of n A-IFVs is also defined and used to measure the hesitation index of an IPR.

For an ordinary fuzzy value μ_F with $0 < \mu_F < 1$, If $\mu_F > 0.5$, then $\frac{1}{1-\mu_F} - 1 = \frac{\mu_F}{1-\mu_F} > 1$; if $\mu_F = 0.5$, then $\frac{\mu_F}{1-\mu_F} = 1$; if $\mu_F < 0.5$, then$0 < \frac{\mu_F}{1-\mu_F} < 1$. That is, μ_F gives a multiplicative index of $\frac{\mu_F}{1-\mu_F}$ from the viewpoint of the fuzzy ratio. On the other hand, for an A-IFV (μ, v), its membership value is bounded between μ and $1-v$, and its nonmembership value is bounded between v and $1-\mu$. Therefore, the interval-valued membership $[\mu, 1 - v]$ of the A-IFV (μ, v) gives a multiplicative reciprocal interval ratio of $\left[a_\mu^-, a_\mu^+\right] = \left[\frac{\mu}{1-\mu}, \frac{1-v}{v}\right]$, and the interval-valued nonmembership $[v, 1 - \mu]$ of the A-IFV (μ, v)gives a multiplicative reciprocal interval ratio of $\left[a_v^-, a_v^+\right] = \left[\frac{v}{1-v}, \frac{1-\mu}{\mu}\right]$. Certainly, the denominators in $\left[\frac{\mu}{1-\mu}, \frac{1-v}{v}\right]$ and $\left[\frac{v}{1-v}, \frac{1-\mu}{\mu}\right]$ are assumed to be not equal to 0. In other words, the constraint $0 < \mu, v < 1$ should be considered in the multiplicative reciprocal interval ratios.

Obviously, we have $0 < a_\mu^- \leq a_\mu^+, 0 < a_v^- \leq a_v^+, a_\mu^- a_v^+ = 1, a_\mu^+ a_v^- = 1$, i.e., $\left[a_\mu^-, a_\mu^+\right]$ and $\left[a_v^-, a_v^+\right]$ are multiplicative reciprocal. If $(\mu, v) = (0.5, 0.5)$, then $\left[a_\mu^-, a_\mu^+\right] = \left[a_v^-, a_v^+\right] = [1, 1]$. Moreover, the geometric mean of the endpoints of the interval $\left[a_\mu^-, a_\mu^+\right]$ and that of the interval $\left[a_v^-, a_v^+\right]$ are also multiplicative reciprocal, i.e., $\sqrt{a_\mu^- a_\mu^+}\sqrt{a_v^- a_v^+} = 1$. Therefore, based on the geometric mean of the interval endpoints, the intuitionistic multiplicative geometric index of an A-IFV is introduced as follows.

Definition 3.1 Let $\tilde{\alpha} = (\mu, v)$ be an A-IFV with $0 < \mu, v < 1$, then the intuitionistic multiplicative geometric index of the A-IFV $\tilde{\alpha}$ is defined as

$$M_{\tilde{\alpha}}^{m} = \sqrt{\frac{\mu(1-v)}{(1-\mu)v}} \tag{3.1}$$

Clearly, for any A-IFV $\tilde{\alpha} = (\mu, v)$ with $0 < \mu, v < 1$, the intuitionistic multiplicative geometric index satisfies the following properties:

(1) $M_{\tilde{\alpha}}^{m} > 0$. Especially, $M_{\tilde{\alpha}}^{m} = 1$ if $\tilde{\alpha} = (0.5, 0.5)$;
(2) $M_{\tilde{\alpha}}^{m} M_{\tilde{\alpha}^c}^{m} = 1$, where $\tilde{\alpha}^c$ is the complement of the A-IFV $\tilde{\alpha} = (\mu, v)$, i.e., $\tilde{\alpha}^c = (v, \mu)$;
(3) If $\mu > 0.5$, then $M_{\tilde{\alpha}}^{m} > 1$;
(4) If $v > 0.5$, then $M_{\tilde{\alpha}}^{m} < 1$;
(5) Let $\tilde{\alpha}_1 = (\mu_1, v_1)$ with $0 < \mu_1, v_1 < 1$ and $\tilde{\alpha}_2 = (\mu_2, v_2)$ with $0 < \mu_2, v_2 < 1$ be two A-IFVs, if $\mu_1 \geq \mu_2$ and $v_1 \leq v_2$, then $M_{\tilde{\alpha}_1}^{m} \geq M_{\tilde{\alpha}_2}^{m}$.

The intuitionistic multiplicative geometric index expresses the multiplication-based central tendency of an A-IFV. In order to capture the fuzzy central tendency of an A-IFV, we introduce the following definition.

Definition 3.2 Let $\tilde{\alpha} = (\mu, v)$ be an A-IFV with $0 < \mu, v < 1$, then the intuitionistic fuzzy geometric index of the A-IFV $\tilde{\alpha}$ is defined as

$$M_{\tilde{\alpha}}^{f} = \frac{\sqrt{\mu(1-v)}}{\sqrt{\mu(1-v)} + \sqrt{(1-\mu)v}} \tag{3.2}$$

It is easy to prove that $M_{\tilde{\alpha}}^{f}$ satisfies the following properties for any A-IFV $\tilde{\alpha} = (\mu, v)$ with $0 < \mu < 1, 0 < v < 1$.

(i) $0 < M_{\tilde{\alpha}}^{f} < 1$. Especially, $M_{\tilde{\alpha}}^{f} = 0.5$ if $\tilde{\alpha} = (0.5, 0.5)$;
(ii) $M_{\tilde{\alpha}}^{f} + M_{\tilde{\alpha}^c}^{f} = 1$, where $\tilde{\alpha}^c$ is the complement of the A-IFV $\tilde{\alpha} = (\mu, v)$, i.e., $\tilde{\alpha}^c = (v, \mu)$;
(iii) If $\mu > 0.5$, then $M_{\tilde{\alpha}}^{f} > 0.5$;
(iv) If $v > 0.5$, then $M_{\tilde{\alpha}}^{f} < 0.5$;
(v) Let $\tilde{\alpha}_1 = (\mu_1, v_1)$ with $0 < \mu_1, v_1 < 1$ and $\tilde{\alpha}_2 = (\mu_2, v_2)$ with $0 < \mu_2, v_2 < 1$ be two A-IFVs, if $\mu_1 \geq \mu_2$ and $v_1 \leq v_2$, then $M_{\tilde{\alpha}_1}^{f} \geq M_{\tilde{\alpha}_2}^{f}$.

By (3.1) and (3.2), the relationship between the intuitionistic fuzzy geometric index and the intuitionistic multiplicative geometric index is obtained as follows.

$$M_{\tilde{\alpha}}^{f} = \frac{M_{\tilde{\alpha}}^{m}}{1 + M_{\tilde{\alpha}}^{m}}, \quad M_{\tilde{\alpha}}^{m} = \frac{M_{\tilde{\alpha}}^{f}}{1 - M_{\tilde{\alpha}}^{f}} \tag{3.3}$$

As per the property (ii) of $M_{\tilde{\alpha}}^{f}$, from (3.3), we can directly obtain the following theorem.

Theorem 3.1 *Let $\tilde{\alpha} = (\mu, v)$ be an A-IFV with $0 < \mu, v < 1$, and $\tilde{\alpha}^c$ be the complement of $\tilde{\alpha} = (\mu, v)$, then*

$$\frac{M^f_{\tilde{\alpha}}}{M^f_{\tilde{\alpha}^c}} = M^m_{\tilde{\alpha}} = \sqrt{\frac{\mu(1-v)}{(1-\mu)v}} \tag{3.4}$$

Theorem 3.1 reveals that the relation between the fuzzy central tendency and the multiplication-based central tendency, i.e., the multiplication-based central tendency of an A-IFV $\tilde{\alpha} = (\mu, v)$ can be determined by using the quotient of the intuitionistic fuzzy geometric indices of the A-IFV $\tilde{\alpha}$ and its complement $\tilde{\alpha}^c = (v, \mu)$.

The intuitionistic fuzzy geometric index of an A-IFV $\tilde{\alpha}$ captures the geometry central tendency of $\tilde{\alpha}$. However, it is unable to snatch the hesitancy level of $\tilde{\alpha}$. It is hard to directly use the Atanassov's intuitionistic fuzzy index to express the ratio-based hesitation margin of $\tilde{\alpha}$ because it is defined from the viewpoint of the interval width. For instance, two A-IFVs (0.4, 0.2) and (0.5, 0.1) have the same intuitionistic fuzzy index as per Atanassov's hesitation margin, but their hesitation ratios are not uniform. To capture the hesitation ratio of $\tilde{\alpha}$, the ratio-based hesitation margin is introduced as follows.

Definition 3.3 Let $\tilde{\alpha} = (\mu, v)$ be an A-IFV, then the ratio-based hesitation margin of the A-IFV $\tilde{\alpha}$ is defined as

$$\pi^r_{\tilde{\alpha}} = \frac{1-\mu-v}{(1-\mu)(1-v)} \tag{3.5}$$

As per (3.5), we have the following results for any A-IFV $\tilde{\alpha} = (\mu, v)$.

(a) $0 \leq \pi^r_{\tilde{\alpha}} \leq 1$;
(b) If $\mu + v = 1$, i.e., $\tilde{\alpha}$ is reduced to an ordinary fuzzy number, $\pi^r_{\tilde{\alpha}} = 0$;
(c) If $\mu = 0$ or $v = 0$, $\pi^r_{\tilde{\alpha}} = 1$;
(d) $\pi^r_{\tilde{\alpha}} = \pi^r_{\tilde{\alpha}^c}$, where $\tilde{\alpha}^c$ is the complement of $\tilde{\alpha} = (\mu, v)$, i.e., $\tilde{\alpha}^c = (v, \mu)$.

Eq. (3.5) can be equivalently expressed as:

$$\pi^r_{\tilde{\alpha}} = 1 - \frac{\mu v}{(1-\mu)(1-v)} \tag{3.6}$$

It can be seen from (3.6) that the closer the value of μv is to 0, the higher the hesitation margin of the A-IFV $\tilde{\alpha} = (\mu, v)$.

As per (3.2) and (3.5), for any two A-IFVs $\tilde{\alpha}_1 = (\mu_1, v_1)$ with $0 < \mu_1, v_1 < 1$ and $\tilde{\alpha}_2 = (\mu_2, v_2)$ with $0 < \mu_2, v_2 < 1$, if $M^f_{\tilde{\alpha}_1} = M^f_{\tilde{\alpha}_2}$ and $\pi^r_{\tilde{\alpha}_1} = \pi^r_{\tilde{\alpha}_2}$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

The aforesaid discussion indicates that the amount of an A-IFV can be measured by using its intuitionistic fuzzy geometric index and ratio-based hesitation margin, and the geometry central tendency of the A-IFV is captured by its intuitionistic fuzzy geometric index. Therefore, an alternative way of expressing the element in an A-IFS is to use the triple(μ, v, π^r), i.e., via the membership degree, the nonmembership degree and the ratio-based hesitation margin.

Eq. (3.6) gives a hesitation index for an A-IFV. In order to measure the ratio-based hesitation margin of an IPR, we need extend (3.6) to measure the hesitation index of n A-IFVs. Thus, the geometric mean based hesitation margin of n A-IFVs is defined as follows.

Definition 3.4 Let $\tilde{\alpha}_i = (\mu_i, v_i)$ (i = 1, 2, …, n) be n A-IFVs with $0 \leq \mu_i, v_i < 1$, then the geometric mean based hesitation margin of the n A-IFVs is defined as

$$\pi^r(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = 1 - \sqrt[n]{\prod_{i=1}^{n} \frac{\mu_i v_i}{(1-\mu_i)(1-v_i)}} \tag{3.7}$$

As the elements along the diagonal in an IPR are not lack of information, i.e., their ratio-based hesitation margins are always 0, the geometric mean based hesitation margin of an IPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ can be defined as follows.

Definition 3.5 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ be an IPR with $0 < \mu_{ij}, v_{ij} < 1$, then the geometric mean based hesitation margin of $\tilde{R}$ is defined as

$$\pi^r(\tilde{R}) = 1 - \left(\prod_{\substack{i,j=1,\\ i\neq j}}^{n} \frac{\mu_{ij} v_{ij}}{(1-\mu_{ij})(1-v_{ij})} \right)^{\frac{1}{n^2-n}} \tag{3.8}$$

By the intuitionistic reciprocal property of IPRs, we have $\frac{\mu_{ji} v_{ji}}{(1-\mu_{ji})(1-v_{ji})} = \frac{v_{ij}\mu_{ij}}{(1-v_{ij})(1-\mu_{ij})}$. Therefore, (3.8) can be rewritten as

$$\pi^r(\tilde{R}) = 1 - \left(\prod_{j>i=1}^{n} \frac{\mu_{ij} v_{ij}}{(1-\mu_{ij})(1-v_{ij})} \right)^{\frac{2}{(n^2-n)}} \tag{3.9}$$

Obviously, $0 \leq \pi^r(\tilde{R}) < 1$. (3.8) or (3.9) gives a way of expressing the average ratio-based hesitation margin for IPRs. If the geometric mean based hesitation margin $\pi^r(\tilde{R}) = 0$, then the IPR $\tilde{R}$ is reduced to an ordinary fuzzy preference relation; if $\pi^r(\tilde{R}) \to 1$, then $\tilde{R}$ is extremely hesitant. The larger the $\pi^r(\tilde{R})$, the more hesitant some pairwise intuitionistic judgments in $\tilde{R}$.

4 Geometric consistency of intuitionistic preference relations

This section uses the intuitionistic fuzzy geometric indices of original intuitionistic fuzzy judgments to define geometric consistency for IPRs. A numerical example is furnished to illustrate the drawback of the multiplicative consistency proposed by Xu et al. (2011) for IPRs.

As per Definition 2.2, for an IPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ with $0 < \mu_{ij}, v_{ij} < 1$ for all $i, j = 1, 2, \ldots, n$, we have $\tilde{r}^c_{ij} = \tilde{r}_{ji}, \forall i, j = 1, 2, \ldots, n$. Based on the discussion in Sect. 3, the central tendency of the intuitionistic fuzzy judgment $\tilde{r}_{ij}$ in $\tilde{R}$ can be captured by its intuitionistic fuzzy geometric index $M^f_{\tilde{r}_{ij}}$. By (3.3),

$M^f_{\tilde{r}_{ij}} / M^f_{\tilde{r}^c_{ij}} = M^m_{\tilde{r}_{ij}} = \sqrt{\frac{\mu_{ij}(1-v_{ij})}{(1-\mu_{ij})v_{ij}}}$. Therefore, the geometric consistency of an IPR can be defined as follows.

Definition 4.1 An IPR $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\mu_{ij}, v_{ij})\right)_{n\times n}$ with $0 < \mu_{ij}, v_{ij} < 1(i, j = 1, 2, \ldots, n)$ is called geometry consistent if it satisfies

$$\sqrt{\frac{\mu_{ij}(1-v_{ij})}{(1-\mu_{ij})v_{ij}}} = \sqrt{\frac{\mu_{ik}(1-v_{ik})}{(1-\mu_{ik})v_{ik}}}\sqrt{\frac{\mu_{kj}(1-v_{kj})}{(1-\mu_{kj})v_{kj}}} \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \tag{4.1}$$

If all intuitionistic fuzzy judgments $\tilde{r}_{ij} = (\mu_{ij}, v_{ij})$ are reduced to fuzzy numbers, i.e., $\mu_{ij} + v_{ij} = 1$ for all i,j =1, 2, …, n, then the IPR $\tilde{R}$ is equivalent to a fuzzy preference relation $R = (r_{ij})_{n\times n}$ with $r_{ij} = \mu_{ij}$ and $r_{ij} + r_{ji} = 1$. In this case, Eq. (4.1) is degraded to Tanino's multiplicative transitivity (2.3). Therefore, the geometric consistency generalizes the concept of the multiplicative consistency proposed by Tanino (1984).

As $\mu_{ij} = v_{ji}, v_{ij} = \mu_{ji}$ for all i,j =1, 2, …, n, (4.1) is equivalent to any of the following transitivity equations:

$$\begin{aligned}&\left(\frac{\mu_{ij}}{1-v_{ji}}\right)\left(\frac{\mu_{jk}}{1-v_{kj}}\right)\left(\frac{\mu_{ki}}{1-v_{ik}}\right)\\ &\quad=\left(\frac{\mu_{ik}}{1-v_{ki}}\right)\left(\frac{\mu_{kj}}{1-v_{jk}}\right)\left(\frac{\mu_{ji}}{1-v_{ij}}\right) \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\left(\frac{\mu_{ij}}{1-\mu_{ij}}\right)\left(\frac{\mu_{jk}}{1-\mu_{jk}}\right)\left(\frac{\mu_{ki}}{1-\mu_{ki}}\right)\\ &\quad=\left(\frac{\mu_{ik}}{1-\mu_{ik}}\right)\left(\frac{\mu_{kj}}{1-\mu_{kj}}\right)\left(\frac{\mu_{ji}}{1-\mu_{ji}}\right) \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n\end{aligned} \tag{4.3}$$

Xu et al. (2011) employed the following two equations to define multiplicative consistent IPRs (See Definition 1 on page 792).

$$\mu_{ij} = \begin{cases} 0 & (\mu_{ik}, \mu_{kj}) \in \{(0,1), (1,0)\} \\ \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj}+(1-\mu_{ik})(1-\mu_{kj})} & \text{otherwise} \end{cases} \quad \text{for all } i \le k \le j \tag{4.4}$$

$$v_{ij} = \begin{cases} 0 & (v_{ik}, v_{kj}) \in \{(0,1), (1,0)\} \\ \frac{v_{ik}v_{kj}}{v_{ik}v_{kj}+(1-v_{ik})(1-v_{kj})} & \text{otherwise} \end{cases} \quad \text{for all } i \le k \le j \tag{4.5}$$

However, the following example illustrates that this multiplicative consistency has flaws in non-robustness for permutations of the intuitionistic fuzzy judgments given by the DM.

Example 1 Let us consider an investment decision problem with three alternatives: "a car company", "a food company" and "a computer company". A DM compares each

Table 1 Intuitionistic fuzzy judgments

Car company versus	Food company	(0.2,0.5)
	Computer company	(1/7,0.2)
Food company versus	Car company	(0.5,0.2)
	Computer company	(0.4,0.2)
Computer company versus	Car company	(0.2,1/7)
	Food company	(0.2,0.4)

pair of the alternatives, and furnishes his/her intuitionistic fuzzy judgments as listed in Table 1.

Six IPRs may be used to express the DM's intuitionistic fuzzy judgments by different labeling for the three decision alternatives. If the car company, the food company and the computer company are labeled by x_1,x_2 and x_3, then the DM's intuitionistic fuzzy judgments are structured as the following IPR.

$$\tilde{R}_1 = (\tilde{r}_{ij})_{3\times 3} = \left((\mu_{ij}, v_{ij})\right)_{3\times 3} = \begin{matrix} x_1:\text{Car} \\ x_2:\text{Food} \\ x:\text{Computer} \end{matrix} \overset{\begin{matrix} x_1:\text{Car} & x_2:\text{Food} & x_3:\text{Computer} \end{matrix}}{\begin{bmatrix} (0.5,0.5) & (0.2,0.5) & (1/7,0.2) \\ (0.5,0.2) & (0.5,0.5) & (0.4,0.2) \\ (0.2,1/7) & (0.2,0.4) & (0.5,0.5) \end{bmatrix}}$$

Another possible labeling for the three alternatives is x_1: the computer company, x_2: the car company, and x_3: the food company. In this case, the intuitionistic fuzzy judgments in Table 1 are expressed as the following IPR.

$$\tilde{R}_1' = (\tilde{r}_{ij}')_{3\times 3} = \left((\mu_{ij}', v_{ij}')\right)_{3\times 3} = \begin{matrix} x_1:\text{Food} \\ x_2:\text{Computer} \\ x_3:\text{Car} \end{matrix} \overset{\begin{matrix} x_1:\text{Food} & x_2:\text{Computer} & x_3:\text{Car} \end{matrix}}{\begin{bmatrix} (0.5,0.5) & (0.4,0.2) & (0.5,0.2) \\ (0.2,0.4) & (0.5,0.5) & (0.2,1/7) \\ (0.2,0.5) & (1/7,0.2) & (0.5,0.5) \end{bmatrix}}$$

One can verify that $\mu_{13} = \frac{\mu_{12}\mu_{23}}{\mu_{12}\mu_{23}+(1-\mu_{12})(1-\mu_{23})}$, $v_{13} = \frac{v_{12}v_{23}}{v_{12}v_{23}+(1-v_{12})(1-v_{23})}$ and $\mu_{13}' \neq \frac{1}{7} = \frac{\mu_{12}'\mu_{23}'}{\mu_{12}'\mu_{23}'+(1-\mu_{12}')(1-\mu_{23}')}$, $v_{13}' \neq \frac{1}{34} = \frac{v_{12}'v_{23}'}{v_{12}'v_{23}'+(1-v_{12}')(1-v_{23}')}$. That is, $\tilde{R}_1$ satisfies (4.4) and (4.5), and $\tilde{R}_1'$ does not satisfy (4.4) and (4.5). As per Definition 1 in Xu et al. (2011), $\tilde{R}_1$ is a multiplicative consistent IPR, while $\tilde{R}_1'$ is not a multiplicative consistent IPR.

Obviously, $\tilde{R}_1'$ is a permutation of $\tilde{R}_1$, i.e., $\tilde{R}_1' = ((\mu_{ij}', v_{ij}'))_{3\times 3} = \left((\mu_{\sigma(i)\sigma(j)}, v_{\sigma(i)\sigma(j)})\right)_{3\times 3}$, where σ is a permutation of $\{1, 2, 3\}$ satisfying $\sigma(1) = 3, \sigma(2) = 1$ and $\sigma(3) = 2$. Therefore, the multiplicative consistency given in Xu et al. (2011) is non-robust to permutations of the original intuitionistic fuzzy judgments provided by the DM. In other words, it is highly dependent on the alternative labels.

By Definition 4.1, one can verify that both of $\tilde{R}_1$ and $\tilde{R}_1'$ are geometry consistent and can easily prove the following theorem.

Theorem 4.1 *Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ be an IPR with $0 < \mu_{ij}, v_{ij} < 1$ and σ be any of the $n!$ permutations of $\{1, 2, \ldots, n\}$, then $\tilde{R}$ is geometry consistent if and only if $\tilde{R}^{\sigma} = \big((\mu_{ij}^{\sigma}, v_{ij}^{\sigma})\big)_{n\times n} = \big((\mu_{\sigma(i)\sigma(j)}, v_{\sigma(i)\sigma(j)})\big)_{n\times n}$ is geometry consistent.*

Theorem 4.1 reveals that the geometric consistency has good robustness for permutations of the DM's intuitionistic fuzzy judgments. It is worth noting that we can not remove the constraint $\forall i \le k \le j$ in (4.4) and (4.5). If $\mu_{ij} = \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj}+(1-\mu_{ik})(1-\mu_{kj})}$ for all $i, j, k = 1, 2, \ldots, n$, then one can obtain $0.5 = \mu_{ii} = \frac{\mu_{ik}\mu_{ki}}{\mu_{ik}\mu_{ki}+(1-\mu_{ik})(1-\mu_{ki})} \Rightarrow \mu_{ik} + \mu_{ki} = 1$. Therefore, as $v_{ik} = \mu_{ki} \quad \forall i, k = 1, 2, \ldots, n$, we have $\mu_{ik} + v_{ik} = 1$ for all $i, k = 1, 2, \ldots, n$. This result reveals that $\tilde{R} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ is only an ordinary fuzzy preference relation.

The following theorem shows the relationship between the geometric consistency and the function equations (4.4) and (4.5).

Theorem 4.2 *Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ be an IPR with $0 < \mu_{ij}, v_{ij} < 1$, if $\tilde{R}$ satisfies (4.4) and (4.5), then $\tilde{R}$ is geometry consistent.*

Proof As $\mu_{ii} = 0.5$ for all $i = 1, 2, \ldots, n$, it is obvious that (4.3) always holds if three of the indices i, j, k are equal, or the two of them.

For all $i \ne j \ne k$, we have six possible cases. If $i < k < j$, by (4.4) and (4.5), one gets

$$
\begin{aligned}
\mu_{ij} &= \frac{\mu_{ik}\mu_{kj}}{\mu_{ik}\mu_{kj} + (1-\mu_{ik})(1-\mu_{kj})} \\
&\Rightarrow 1 - \mu_{ij} = \frac{(1-\mu_{ik})(1-\mu_{kj})}{\mu_{ik}\mu_{kj} + (1-\mu_{ik})(1-\mu_{kj})} \\
&\Rightarrow \frac{\mu_{ij}}{1-\mu_{ij}} = \frac{\mu_{ik}\mu_{kj}}{(1-\mu_{ik})(1-\mu_{kj})} \\
v_{ij} &= \frac{v_{ik}v_{kj}}{v_{ik}v_{kj} + (1-v_{ik})(1-v_{kj})} \\
&\Rightarrow \mu_{ji} = \frac{\mu_{ki}\mu_{jk}}{\mu_{ki}\mu_{jk} + (1-\mu_{ki})(1-\mu_{jk})} \\
&\Rightarrow 1 - \mu_{ji} = \frac{(1-\mu_{ki})(1-\mu_{jk})}{\mu_{ki}\mu_{jk} + (1-\mu_{ki})(1-\mu_{jk})} \\
&\Rightarrow \frac{\mu_{ji}}{1-\mu_{ji}} = \frac{\mu_{ki}\mu_{jk}}{(1-\mu_{ki})(1-\mu_{jk})}
\end{aligned}
$$

Thus, (4.3) holds for $i < k < j$. Similarly, by exchanging the subscripts, one can obtain (4.3) holds for the following five situations: $i < j < k$, $j < k < i$, $j < i < k$, $k < i < j$ and $k < j < i$. As per Definition 4.1, $\tilde{R}$ is geometry consistent. □

Theorem 4.2 indicates that if $\tilde{R}$ satisfies (4.4) and (4.5), then it is geometry consistent. However, a geometry consistent IPR does not necessarily satisfy (4.4) and (4.5).

For instance, $\tilde{R}_1'$ in Example 1 is a geometry consistent IPR, but it does not satisfy (4.4) and (4.5).

Wu and Chiclana (2014b) employed the following two Eqs. (4.6) and (4.7) to define multiplicative consistency of IPRs. Liao and Xu (2014b) proposed another multiplicative consistency definition of IPRs, where the transitivity condition is equivalent to (4.6).

$$\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji} \quad \text{for all } \; i, j, k = 1, 2, \ldots, n \tag{4.6}$$

$$(1-\upsilon_{ij})(1-\upsilon_{jk})(1-\upsilon_{ki}) = (1-\upsilon_{ik})(1-\upsilon_{kj})(1-\upsilon_{ji}) \quad \text{for all } \; i, j, k = 1, 2, \ldots, n \tag{4.7}$$

It is obvious that if (4.6) and (4.7) hold, (4.2) follows. Therefore, if an IPR has multiplicative consistency in terms of the transitivity proposed by Wu and Chiclana (2014b), then it is geometry consistent, and also multiplicative consistent under the transitivity definition by Liao and Xu (2014b). However, one can verify that the geometry consistent IPR $\tilde{R}_1$ in Example 1 does not satisfy (4.6) and (4.7). In other words, an IPR with geometric consistency does not necessarily satisfy (4.6) or/and (4.7).

5 A logarithmic least square model for generating interval priority weights

This section develops a logarithmic least square model to construct the fitted geometry consistent IPR and derive normalization interval weights from IPRs. An algorithm with acceptable geometry consistency checking is proposed for solving MCDM problems.

Denote a positive normalization interval fuzzy weight vector by $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T = ([\omega_1^-, \omega_1^+], [\omega_2^-, \omega_2^+], \ldots, [\omega_n^-, \omega_n^+])^T$ with

$$0 < \omega_i^- \leq \omega_i^+ \leq 1, \sum_{j=1, j\neq i}^{n} \omega_j^- + \omega_i^+ \leq 1, \omega_i^- + \sum_{j=1, j\neq i}^{n} \omega_j^+ \geq 1 \quad \forall i = 1, 2, \ldots, n \tag{5.1}$$

There may exist a difference between the judgment in a pairwise comparison matrix with uncertainty or hesitancy and the result obtained from an interval fuzzy weight vector $\bar{\omega}$ by interval arithmetic due to the fact that we often have $\frac{\bar{\omega}_i}{\bar{\omega}_i} \neq [1, 1]$ and $\frac{\bar{\omega}_i}{\bar{\omega}_j} \otimes \frac{\bar{\omega}_j}{\bar{\omega}_i} \neq [1, 1]$, where "–" and "⊗" denote the interval division and multiplication operations, respectively.

Moreover, preference relations with uncertainty or hesitancy are allowed by such differences. For instance, for an interval multiplicative preference relation (Saaty and Vargas 1987) $A = \left(\bar{a}_{ij}\right)_{n\times n} = \left([a_{ij}^-, a_{ij}^+]\right)_{n\times n}$, $\bar{a}_{ii} = [1, 1], \forall i = 1, 2, \ldots, n$, but we often have $\frac{\bar{\omega}_i}{\bar{\omega}_i} \neq [1, 1]$. Similarly, for IPRs, $\mu_{ii} = \upsilon_{ii} = 0.5$ for all $i = 1, 2, \ldots, n$. In order to simulate the difference, the interval multiplicative reciprocal preference intensity of x_i over x_j,$\bar{a}_{ij} = \left[a_{ij}^-, a_{ij}^+\right]$ can be defined as $\left[\frac{\omega_i^-}{\gamma_{ij}\omega_j^+}, \frac{\gamma_{ij}\omega_i^+}{\omega_j^-}\right]$ by introducing a parameterγ_{ij}, where $\sqrt{\frac{\omega_i^-\omega_j^-}{\omega_i^+\omega_j^+}} \leq \gamma_{ij} \leq 1$and $\gamma_{ji} = \gamma_{ij}$ for all $i, j = 1, 2, \ldots, n$.

Obviously, If $\gamma_{ij} = \sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}}$, we have $a_{ij}^- = a_{ij}^+$, implying that $\bar{a}_{ij}$ is degraded to a crisp judgment and the difference between $\bar{a}_{ij}$ and $\frac{\bar{\omega}_i}{\bar{\omega}_j}$is maximal. If $\gamma_{ij} = 1$, then $\bar{a}_{ij} = \frac{\bar{\omega}_i}{\bar{\omega}_j}$, indicating that $\bar{a}_{ij}$ is determined by interval arithmetic, and $\bar{a}_{ij}$ and $\frac{\bar{\omega}_i}{\bar{\omega}_j}$ are indifferent. If $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} < \gamma_{ij} < 1$, then $\bar{a}_{ij}$ is obtained by not being strictly based on interval arithmetic, and there is a difference between $\bar{a}_{ij}$ and $\frac{\bar{\omega}_i}{\bar{\omega}_j}$. The smaller the γ_{ij}, the larger the difference between $\bar{a}_{ij}$ and $\frac{\bar{\omega}_i}{\bar{\omega}_j}$.

On the other hand, one can easily prove that an interval multiplicative judgment $[a_{ij}^-, a_{ij}^+]$ can be equivalently converted to an intuitionistic judgment$\left(\frac{a_{ij}^-}{1+a_{ij}^-}, \frac{1}{1+a_{ij}^+}\right)$. Therefore, for a given interval fuzzy weight vector $\bar{\omega}$, the intuitionistic preference intensity of x_i over x_j, $\tilde{t}_{ij} = (t_{ij}^\mu, t_{ij}^v)$ is formulated as follows.

$$\tilde{t}_{ij} = (t_{ij}^\mu, t_{ij}^v) = \begin{cases} (0.5, 0.5) & i = j \\ \left(\frac{\omega_i^-}{\omega_i^- + \gamma_{ij}\omega_j^+}, \frac{\omega_j^-}{\omega_j^- + \gamma_{ij}\omega_i^+}\right) & i \neq j \end{cases} \tag{5.2}$$

where γ_{ij} is a parameter such that$\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \le \gamma_{ij} \le 1$and $\gamma_{ji} = \gamma_{ij}$ for all $i, j = 1, 2, \ldots, n, i \neq j$.

Theorem 5.1 *Let $\tilde{T} = (\tilde{t}_{ij})_{n\times n}$ be a matrix defined by (5.2), then $\tilde{T}$ is a geometry consistent IPR and $M_{\tilde{t}_{ij}}^f = \frac{\sqrt{\omega_i^- \omega_i^+}}{\sqrt{\omega_i^- \omega_i^+} + \sqrt{\omega_j^- \omega_j^+}}$ for all $i, j = 1, 2, \ldots, n$.*

Proof It is apparent that, for all $i, j = 1, 2, \ldots, n$, $(t_{ii}^\mu, t_{ii}^v) = (0.5, 0.5)$ and $(t_{ji}^\mu, t_{ji}^v) = (t_{ij}^v, t_{ij}^\mu)$. As $0 < \omega_i^- \le \omega_i^+ \le 1$, we have $0 < \frac{\omega_i^-}{\omega_i^- + \gamma_{ij}\omega_j^+} < 1$ and $0 < \frac{\omega_j^-}{\omega_j^- + \gamma_{ij}\omega_i^+} < 1$. Moreover, since $\gamma_{ij} \ge \sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}}$ for all $i, j = 1, 2, \ldots, n, i \neq j$, it follows that

$$\begin{aligned} \omega_i^- \omega_j^- &\le \left(\gamma_{ij}\right)^2 \omega_i^+ \omega_j^+ \Rightarrow 1 + \frac{\omega_j^-}{\gamma_{ij}\omega_i^+} \le 1 + \frac{\gamma_{ij}\omega_j^+}{\omega_i^-} \Rightarrow \frac{\omega_j^- + \gamma_{ij}\omega_i^+}{\gamma_{ij}\omega_i^+} \\ &\le \frac{\omega_i^- + \gamma_{ij}\omega_j^+}{\omega_i^-} \Rightarrow \frac{\omega_i^-}{\omega_i^- + \gamma_{ij}\omega_j^+} \le \frac{\gamma_{ij}\omega_i^+}{\omega_j^- + \gamma_{ij}\omega_i^+} = 1 - \frac{\omega_j^-}{\omega_j^- + \gamma_{ij}\omega_i^+} \\ &\Rightarrow t_{ij}^\mu + t_{ij}^v \le 1 \end{aligned}$$

Therefore, the elements in $\tilde{T}$ are A-IFVs and satisfy the intuitionistic reciprocal property. As per Definition 2.2, $\tilde{T}$ is an IPR.

On the other hand, for all $i \neq j \neq k$, we have

$$\left(\frac{t_{ij}^{\mu}}{1-t_{ij}^{\mu}}\right)\left(\frac{t_{jk}^{\mu}}{1-t_{jk}^{\mu}}\right)\left(\frac{t_{ki}^{\mu}}{1-t_{ki}^{\mu}}\right)=\left(\frac{\omega_i^-}{\gamma_{ij}\omega_j^+}\right)\left(\frac{\omega_j^-}{\gamma_{jk}\omega_k^+}\right)\left(\frac{\omega_k^-}{\gamma_{ki}\omega_i^+}\right)$$
$$=\frac{\omega_i^-\omega_j^-\omega_k^-}{\gamma_{ij}\gamma_{jk}\gamma_{ki}\omega_i^+\omega_j^+\omega_k^+}$$

and

$$\left(\frac{t_{ik}^{\mu}}{1-t_{ik}^{\mu}}\right)\left(\frac{t_{kj}^{\mu}}{1-t_{kj}^{\mu}}\right)\left(\frac{t_{ji}^{\mu}}{1-t_{ji}^{\mu}}\right)=\left(\frac{\omega_i^-}{\gamma_{ik}\omega_k^+}\right)\left(\frac{\omega_k^-}{\gamma_{kj}\omega_j^+}\right)\left(\frac{\omega_j^-}{\gamma_{ji}\omega_i^+}\right)$$
$$=\frac{\omega_i^-\omega_j^-\omega_k^-}{\gamma_{ji}\gamma_{kj}\gamma_{ik}\omega_i^+\omega_j^+\omega_k^+}$$

As $\gamma_{ji} = \gamma_{ij}$for all $i, j = 1, 2, \ldots, n, i \neq j$, (4.3) holds for $i \neq j \neq k$. Moreover, since $\tilde{T}$ is an IPR, (4.3) always holds for if three of the indices i, j, k are equal, or the two of them. By Definition 4.1, $\tilde{T}$ is a geometry consistent IPR.

As per (3.2), we get

$$M_{\tilde{t}_{ij}}^{f}=\frac{1}{1+\sqrt{\frac{(1-t_{ij}^{\mu})t_{ij}^{v}}{t_{ij}^{\mu}(1-t_{ij}^{v})}}}=\frac{1}{1+\sqrt{\frac{\omega_j^-\omega_j^+}{\omega_i^-\omega_i^+}}}=\frac{\sqrt{\omega_i^-\omega_i^+}}{\sqrt{\omega_i^-\omega_i^+}+\sqrt{\omega_j^-\omega_j^+}}$$

Thus, the proof of Theorem 5.1 is completed. □

Theorem 5.1 indicates that numerous geometry consistent IPRs can be obtained for a given normalization interval fuzzy weight vector by setting γ_{ij} at different values. Moreover, the intuitionistic fuzzy geometric indices of the corresponding elements in these consistent IPRs are identical, but they have different hesitation margin. The larger the γ_{ij}, the more hesitant the obtained A-IFV $\tilde{t}_{ij}$.

Corollary 5.1 *Let* $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \big((\mu_{ij}, v_{ij})\big)_{n\times n}$ *be an IPR with* $0 < \mu_{ij}, v_{ij} < 1$, *if there exists a positive normalization interval fuzzy weight vector* $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \tilde{\omega}_n)^T$ *and* γ_{ij} *(*$i, j = 1, 2, \ldots, n, i \neq j$*) such that*

$$\tilde{r}_{ij}=(\mu_{ij},v_{ij})=\begin{cases}(0.5,0.5) & i=j\\ \left(\frac{\omega_i^-}{\omega_i^-+\gamma_{ij}\omega_j^+},\frac{\omega_j^-}{\omega_j^-+\gamma_{ij}\omega_i^+}\right) & i\neq j\end{cases} \tag{5.3}$$

where γ_{ij} *(*$i, j = 1, 2, \ldots, n, i \neq j$*) satisfy* $\gamma_{ij} \geq \sqrt{\frac{\omega_i^-\omega_j^-}{\omega_i^+\omega_j^+}}$ *and* $\gamma_{ji} = \gamma_{ij}$, *then* $\tilde{R}$ *is geometry consistent.*

As per (5.3), we have

$$\frac{\mu_{ij}}{1-\mu_{ij}} = \frac{\omega_i^-}{\gamma_{ij}\omega_j^+}, \quad i \neq j \tag{5.4}$$

$$\frac{1-v_{ij}}{v_{ij}} = \frac{\gamma_{ij}\omega_i^+}{\omega_j^-}, \quad i \neq j \tag{5.5}$$

By logarithmizing (5.4) and (5.5), one can obtain

$$\ln \mu_{ij} - \ln(1-\mu_{ij}) = \ln \omega_i^- - \ln \omega_j^+ - \ln \gamma_{ij}, \quad i \neq j \tag{5.6}$$

$$\ln(1-v_{ij}) - \ln v_{ij} = \ln \gamma_{ij} + \ln \omega_i^+ - \ln \omega_j^-, \quad i \neq j \tag{5.7}$$

Eqs. (5.4)–(5.7) hold for geometry consistent IPRs. However, in real-world decision situations, it is often a challenge for a DM to furnish a geometry consistent IPR. In this case, (5.6) and (5.7) will not hold, and will have to be relaxed by allowing some deviations. Based on this modeling idea, the following logarithmic least square optimization model is established to derive interval weights from an IPR.

$$\begin{aligned} \min \quad J = \sum_{i=1}^{n} \sum_{j\neq i, j=1}^{n} & \Big((\ln \omega_i^- - \ln \omega_j^+ - \ln \gamma_{ij} + \ln(1-\mu_{ij}) - \ln \mu_{ij})^2 \\ & + (\ln \omega_i^+ - \ln \omega_j^- + \ln \gamma_{ij} + \ln v_{ij} - \ln(1-v_{ij}))^2 \Big) \end{aligned}$$
$$s.t. \begin{cases} \ln \omega_i^- + \ln \omega_j^- - \ln \omega_i^+ - \ln \omega_j^+ \leq 2\ln \gamma_{ij} \leq 0, & i \neq j = 1, 2, \ldots, n \\ \ln \gamma_{ji} = \ln \gamma_{ij}, & i \neq j = 1, 2, \ldots, n \\ \omega_i^+ + \sum_{\substack{j=1 \\ j\neq i}}^{n} \omega_j^- \leq 1, \ \omega_i^- + \sum_{\substack{j=1 \\ j\neq i}}^{n} \omega_j^+ \geq 1, & i = 1, 2, \ldots, n \\ 0 < \omega_i^- \leq \omega_i^+ \leq 1. & i = 1, 2, \ldots, n \end{cases} \tag{5.8}$$

where ω_i^-, ω_i^+ $(i = 1, 2, \ldots, n)$ and γ_{ij} $(i \neq j = 1, 2, \ldots, n)$ are decision variables. The first two line constraints come from the logarithmic expressions of $\sqrt{\frac{\omega_i^- \omega_j^-}{\omega_i^+ \omega_j^+}} \leq \gamma_{ij} \leq 1$and $\gamma_{ij} = \gamma_{ji}$, and the remaining constraints ensure that the derived weights constitute a normalization interval weight vector $\bar{\omega}$.

Since $\mu_{ji} = v_{ij}, v_{ji} = \mu_{ij}$ and $\gamma_{ji} = \gamma_{ij}$ for all $i \neq j = 1, 2, \ldots, n$, we have

$$\begin{aligned} & \ln \omega_j^- - \ln \omega_i^+ - \ln \gamma_{ji} + \ln(1-\mu_{ji}) - \ln \mu_{ji} \\ & \quad = -(\ln \omega_i^+ - \ln \omega_j^- + \ln \gamma_{ij} + \ln v_{ij} - \ln(1-v_{ij})), \quad i \neq j \end{aligned} \tag{5.9}$$

$$\begin{aligned} & \ln \omega_j^+ - \ln \omega_i^- + \ln \gamma_{ji} + \ln v_{ji} - \ln(1-v_{ji}) \\ & \quad = -(\ln \omega_i^- - \ln \omega_j^+ - \ln \gamma_{ij} + \ln(1-\mu_{ij}) - \ln \mu_{ij}), \quad i \neq j \end{aligned} \tag{5.10}$$

Thus, (5.8) can be simplified as:

$$\min J = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big((\ln\omega_i^- - \ln\omega_j^+ - \ln\gamma_{ij} + \ln(1-\mu_{ij}) - \ln\mu_{ij})^2 + (\ln\omega_i^+ - \ln\omega_j^- + \ln\gamma_{ij} + \ln v_{ij} - \ln(1-v_{ij}))^2\Big)$$

$$s.t.\begin{cases} \ln\omega_i^- + \ln\omega_j^- - \ln\omega_i^+ - \ln\omega_j^+ \le 2\ln\gamma_{ij} \le 0, & i=1,2,\ldots,n-1,\ j=i+1,i+2,\ldots,n \\ \omega_i^+ + \sum_{j=1, j\ne i}^{n}\omega_j^- \le 1,\ \omega_i^- + \sum_{j=1, j\ne i}^{n}\omega_j^+ \ge 1, & i=1,2,\ldots,n \\ 0<\omega_i^- \le \omega_i^+ \le 1 & i=1,2,\ldots,n \end{cases} \tag{5.11}$$

Solving (5.11) yields an optimal normalization interval weight vector $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \ldots, \bar{\omega}_n^*)^T = ([\omega_1^{-*}, \omega_1^{+*}], [\omega_2^{-*}, \omega_2^{+*}], \ldots, [\omega_n^{-*}, \omega_n^{+*}])^T$ and the optimal solutions γ_{ij}^* $(i=1,2,\ldots,n-1,\quad j=i+1,\ldots,n)$.

Let $\gamma_{ij}^* = \gamma_{ji}^*$ for all $i > j = 1,2,\ldots,n$, by plugging $\bar{\omega}^*$ and γ_{ij}^* $(i \ne j = 1,2,\ldots,n)$ into (5.2), the fitted geometry consistent IPR is determined as $\tilde{R}^* = (\tilde{r}_{ij}^*)_{n\times n} = \left((\mu_{ij}^*, v_{ij}^*)\right)_{n\times n}$, where

$$\tilde{r}_{ij}^* = (\mu_{ij}^*, v_{ij}^*) = \begin{cases} (0.5, 0.5) & i=j \\ \left(\frac{\omega_i^{-*}}{\omega_i^{-*}+\gamma_{ij}^*\omega_j^{+*}}, \frac{\omega_j^{-*}}{\omega_j^{-*}+\gamma_{ij}^*\omega_i^{+*}}\right) & i\ne j \end{cases} \tag{5.12}$$

Clearly, if the value of the optimal objective function in (5.11) is equal to 0, i.e., $J^*=0$, then $\tilde{R}$ can be expressed as (5.6) and (5.7) by the derived interval fuzzy weight vector $\bar{\omega}^*$ and the optimal solutions γ_{ij}^*. By Corollary 5.1, $\tilde{R}$ is geometry consistent. However, if $J^* \ne 0$, then $\tilde{R} \ne \tilde{R}^*$. In order to measure their difference, we introduce the notion of the geometric mean based difference degree.

Definition 5.1 Let $\tilde{R}_1 = (\tilde{r}_{ij}^{(1)})_{n\times n} = ((\mu_{ij}^{(1)}, v_{ij}^{(1)}))_{n\times n}$ and $\tilde{R}_2 = (\tilde{r}_{ij}^{(2)})_{n\times n} = ((\mu_{ij}^{(2)}, v_{ij}^{(2)}))_{n\times n}$ be two IPRs with $0 < \mu_{ij}^{(1)}, v_{ij}^{(1)} < 1, 0 < \mu_{ij}^{(2)}, v_{ij}^{(2)} < 1$, then the geometric mean based difference degree between $\tilde{R}_1$ and $\tilde{R}_2$ is defined as

$$GMDD(\tilde{R}_1, \tilde{R}_2) = 1 - \left(\prod_{i\ne j}\left(\frac{\min\{\mu_{ij}^{(1)}, \mu_{ij}^{(2)}\}}{\max\{\mu_{ij}^{(1)}, \mu_{ij}^{(2)}\}}\right)\left(\frac{\min\{v_{ij}^{(1)}, v_{ij}^{(2)}\}}{\max\{v_{ij}^{(1)}, v_{ij}^{(2)}\}}\right)\right)^{\frac{1}{2(n^2-n)}} \tag{5.13}$$

Obviously, $0 \le GMDD(\tilde{R}_1, \tilde{R}_2) < 1$ and $GMDD(\tilde{R}_1, \tilde{R}_2) = GMDD(\tilde{R}_2, \tilde{R}_1)$. The smaller the geometric mean based difference degree $GMDD(\tilde{R}_1, \tilde{R}_2)$, the closer $\tilde{R}_1$ is to $\tilde{R}_2$. Especially, if $GMDD(\tilde{R}_1, \tilde{R}_2) = 0$, then one can obtain $\tilde{R}_1 = \tilde{R}_2$.

Definition 5.2 Let $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\mu_{ij}, v_{ij})\right)_{n\times n}$ be an IPR with $0 < \mu_{ij}, v_{ij} < 1$, then $\tilde{R}$ is called an acceptable geometry consistent IPR, if

$$GMDD(\tilde{R}, \tilde{R}^*) \le t \tag{5.14}$$

where $\tilde{R}^*$ is the fitted geometry consistent IPR defined by (5.12), and t $(0 \leq t < 1)$ is an acceptable geometry consistency threshold given by the DM.

Based on the aforesaid analyses, an algorithm for solving MCDM problems with IPRs is now developed as follows.

Algorithm 1

Step 1. Solve the model (5.11), and derive the optimal normalization interval weight vector $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \cdots, \bar{\omega}_n^*)^T$ and the optimal solutions γ_{ij}^* $(i = 1, 2, \ldots, n-1, \quad j = i+1, \ldots, n)$.
Step 2. Calculate the fitted geometry consistent IPR $\tilde{R}^*$ as per (5.12).
Step 3. Determine the geometric mean based difference degree between $\tilde{R}$ and $\tilde{R}^*$, $GMDD(\tilde{R}, \tilde{R}^*)$, as per (5.13).
Step 4. If $GMDD(\tilde{R}, \tilde{R}^*) \leq t$, i.e., $\tilde{R}$ is an acceptable geometry consistent IPR, then go to next step; otherwise, ask the DM to adjust his/her evaluations and go to step 7.
Step 5. Construct the possibility degree matrix $P = (P(\bar{\omega}_i^* \geq \bar{\omega}_j^*))_{n\times n}$as per the following formula.

$$P(\bar{\omega}_i^* \geq \bar{\omega}_j^*) = \max\left\{1 - \max\left(\frac{\omega_j^{+*} - \omega_i^{-*}}{\omega_i^{+*} - \omega_i^{-*} + \omega_j^{+*} - \omega_j^{-*}}, 0\right), 0\right\} \quad (5.15)$$

Step 6. Add up all elements in each row of P, one gets $\theta_i = \sum_{j=1}^{n} p_{ij}$ $(i = 1, 2, \ldots, n)$. According to the decreasing order of the values θ_i $(i = 1, 2, \ldots, n)$, a ranking of decision alternatives is determined, and the alternativex_i is preferred to x_j to the possibility degree of $P(\bar{\omega}_i^* \geq \bar{\omega}_j^*)$, denoted by $x_i \overset{P(\bar{\omega}_i^* \geq \bar{\omega}_j^*))}{\succeq} x_j$.
Step 7. End.

6 Numerical examples

This section presents two numerical examples and comparisons with existing approaches to illustrate the performance and validity of the proposed models.

Example 2 Assume that a DM provides the following IPR, which has been examined by Xu and Liao (2014).

$$\tilde{R} = (\tilde{r}_{ij})_{4\times 4} = ((\mu_{ij}, v_{ij})_{4\times 4} = \begin{bmatrix} (0.5, 0.5) & (0.2, 0.6) & (0.3, 0.4) & (0.6, 0.2) \\ (0.6, 0.2) & (0.5, 0.5) & (0.5, 0.4) & (0.6, 0.4) \\ (0.4, 0.3) & (0.4, 0.5) & (0.5, 0.5) & (0.3, 0.2) \\ (0.2, 0.6) & (0.4, 0.6) & (0.2, 0.3) & (0.5, 0.5) \end{bmatrix}$$

By plugging $\tilde{R}$ into (5.11), one can obtain the optimal interval fuzzy weight vector $\bar{\omega}^*$ and the optimal solutions γ_{ij}^* as

$$\begin{aligned}\bar{\omega}^* &= (\bar{\omega}_1^*, \bar{\omega}_2^*, \bar{\omega}_3^*, \bar{\omega}_4^*)^T \\ &= ([0.2213, 0.2232], [0.2104, 0.5562], [0.1346, 0.4733], [0.0880, 0.2859])^T,\end{aligned}$$

$$\gamma_{12}^* = 1, \gamma_{13}^* = 0.9933, \gamma_{14}^* = 0.9016, \gamma_{23}^* = 0.4016, \gamma_{24}^* = 0.3411, \gamma_{34}^* = 0.9036.$$

As per (5.12), the fitted geometry consistent IPR is determined as

$$\tilde{R}^* = \begin{bmatrix} (0.5, 0.5) & (0.2846, 0.4852) & (0.3201, 0.3778) & (0.4619, 0.3042) \\ (0.4852, 0.2846) & (0.5, 0.5) & (0.5254, 0.3760) & (0.6833, 0.3169) \\ (0.3778, 0.3201) & (0.3760, 0.5254) & (0.5, 0.5) & (0.3425, 0.1707) \\ (0.3042, 0.4619) & (0.3169, 0.6833) & (0.1707, 0.3425) & (0.5, 0.5) \end{bmatrix}$$

By (5.13), the geometric mean based difference degree between $\tilde{R}$ and $\tilde{R}^*$ is determined as $GMDD(\tilde{R}, \tilde{R}^*) = 0.1629$.

If the DM expects acceptable geometry consistency threshold to be less than or equal to 0.2, then t=0.2 and $\tilde{R}$ is an acceptable geometry consistent IPR. As per the comparison method described in Sect. 5, the four interval weights are ranked as $\bar{\omega}_2^* \overset{61.59\%}{\succ} \bar{\omega}_3^* \overset{73.99\%}{\succ} \bar{\omega}_1^* \overset{67.67\%}{\succ} \bar{\omega}_4^*$.

Next, the priority methods developed by Xu and Liao (2014), Xu (2012) and Gong et al. (2009) will be applied to the same IPR $\tilde{R}$ and the obtained interval priority weights will be compared with our proposed approach.

According to Eq. (25) in Xu and Liao (2014), an interval weight vector is obtained as

$$\begin{aligned}\bar{\omega}^{xl} &= (\bar{\omega}_1^{xl}, \bar{\omega}_2^{xl}, \bar{\omega}_3^{xl}, \bar{\omega}_4^{xl})^T \\ &= ([0.1720, 0.3433], [0.2366, 0.3731], [0.1720, 0.3731], [0.1398, 0.2985])^T.\end{aligned}$$

Xu (2012) presented an error-analysis-based method to obtain interval priority weights. By employing (13) and (15) in Xu (2012), the expected priority weight vector and the corresponding error vector are determined as $(0.2458, 0.2792, 0.2542, 0.2208)^T$ and $(0.0172, 0.0093, 0.0247, 0.0224)^T$, respectively. Thus, the interval weight vector is derived as

$$\begin{aligned}\bar{\omega}^{xu} &= (\bar{\omega}_1^{xu}, \bar{\omega}_2^{xu}, \bar{\omega}_3^{xu}, \bar{\omega}_4^{xu})^T \\ &= ([0.2286, 0.2630], [0.2699, 0.2885], [0.2295, 0.2789], [0.1984, 0.2432])^T\end{aligned}$$

Gong et al. (2009) proposed a linear program to derive an interval weight vector. Using the model (21) in Gong et al. (2009), the optimal normalization interval weight vector is obtained as $\bar{\omega}^g = (\bar{\omega}_1^g, \bar{\omega}_2^g, \bar{\omega}_3^g, \bar{\omega}_4^g)^T = ([0.1, 0.1333], [0.2333, 0.2667], [0.2, 0.2333], [0.4, 0.4])^T$.

In Xu and Liao (2014), the obtained interval weights is first converted into A-IFVs, and then the following measure function proposed by Szmidt and Kacprzyk (2009) is used to rank alternatives.

$$\varphi(\tilde{\alpha}) = 0.5(1 + \pi_{\tilde{\alpha}})(1 - \mu) \tag{6.1}$$

Table 2 A comparative study for ranking interval weights

Interval weight vector	Model		
	Possibility degree (5.15)	Measure function (6.1)	Score function based method (Xu 2007)
$\bar{\omega}^*$	$\bar{\omega}_2^* \overset{61.59\%}{\succ} \bar{\omega}_3^* \overset{73.99\%}{\succ} \bar{\omega}_1^* \overset{67.67\%}{\succ} \bar{\omega}_4^*$	$\bar{\omega}_1^* \succ \bar{\omega}_2^* \succ \bar{\omega}_4^* \succ \bar{\omega}_3^*$	$\bar{\omega}_2^* \succ \bar{\omega}_3^* \succ \bar{\omega}_1^* \succ \bar{\omega}_4^*$
$\bar{\omega}^{xl}$	$\bar{\omega}_2^{xl} \overset{59.57\%}{\succ} \bar{\omega}_3^{xl} \overset{54.00\%}{\succ} \bar{\omega}_1^{xl} \overset{61.67\%}{\succ} \bar{\omega}_4^{xl}$	$\bar{\omega}_2^{xl} \succ \bar{\omega}_1^{xl} \succ \bar{\omega}_3^{xl} \succ \bar{\omega}_4^{xl}$	$\bar{\omega}_2^{xl} \succ \bar{\omega}_3^{xl} \succ \bar{\omega}_1^{xl} \succ \bar{\omega}_4^{xl}$
$\bar{\omega}^{xu}$	$\bar{\omega}_2^{xu} \overset{86.76\%}{\succ} \bar{\omega}_3^{xu} \overset{60.02\%}{\succ} \bar{\omega}_1^{xu} \overset{81.57\%}{\succ} \bar{\omega}_4^{xu}$	$\bar{\omega}_2^{xu} \succ \bar{\omega}_1^{xu} \succ \bar{\omega}_3^{xu} \succ \bar{\omega}_4^{xu}$	$\bar{\omega}_2^{xu} \succ \bar{\omega}_3^{xu} \succ \bar{\omega}_1^{xu} \succ \bar{\omega}_4^{xu}$
$\bar{\omega}^g$	$\bar{\omega}_4^g \overset{100\%}{\succ} \bar{\omega}_2^g \overset{100\%}{\succ} \bar{\omega}_3^g \overset{100\%}{\succ} \bar{\omega}_1^g$	$\bar{\omega}_4^g \succ \bar{\omega}_2^g \succ \bar{\omega}_3^g \succ \bar{\omega}_1^g$	$\bar{\omega}_4^g \succ \bar{\omega}_2^g \succ \bar{\omega}_3^g \succ \bar{\omega}_1^g$

where $\tilde{\alpha} = (\mu, v)$, and $\pi_{\tilde{\alpha}} = 1 - \mu - v$ is the Atanassov's hesitation margin of $\tilde{\alpha}$. The smaller the $\varphi(\tilde{\alpha})$, the larger the A-IFV $\tilde{\alpha}$. Therefore, the four interval weights in $\bar{\omega}^{xl}$ are ranked as $\bar{\omega}_2^{xl} \succ \bar{\omega}_1^{xl} \succ \bar{\omega}_3^{xl} \succ \bar{\omega}_4^{xl}$.

In Xu (2012), (5.15) is applied to rank the four interval weights in$\bar{\omega}^{xu}$. The ranking order is determined as $\bar{\omega}_2^{xu} \overset{86.76\%}{\succ} \bar{\omega}_3^{xu} \overset{60.02\%}{\succ} \bar{\omega}_1^{xu} \overset{81.57\%}{\succ} \bar{\omega}_4^{xu}$. In Gong et al. (2009), the derived interval weights is also first converted into A-IFVs, and then the score function based comparison method (Xu 2007) is employed to rank decision alternatives. The four interval weights in $\bar{\omega}^g$ are ranked as $\bar{\omega}_4^g \succ \bar{\omega}_2^g \succ \bar{\omega}_3^g \succ \bar{\omega}_1^g$.

The ranking results of the four interval weight vectors $\bar{\omega}^*, \bar{\omega}^{xl}, \bar{\omega}^{xu}, \bar{\omega}^g$ based on different comparison methods are summarized in Table 2.

Table 2 clearly indicates that different ranks may be obtained from the same interval weight vector based on different comparison approaches. The ranks of interval weights derived by Xu and Liao (2014) , Xu (2012) and this article are identical based on the possibility degree comparison approach, but the values of the possibility degree are not uniform. If the interval weights are converted into A-IFVs, they are ranked as different orders based on the score function and (6.1). Although the ranking orders of the interval weights obtained by Gong et al. (2009) are uniform based on the three different comparison approaches, the rank is much inconsistent with the results obtained by Xu and Liao (2014), Xu (2012) and this article. Moreover, the further comparative study (See Table 3) reveals that the interval weight vector $\bar{\omega}^g$ cannot properly reflect the original intuitionistic fuzzy judgments in $\tilde{R}$.

Szmidt and Kacprzyk (2009) gave a counter-intuitive example for ranking two Atanassov's intuitionistic fuzzy alternatives by using the score function, and proposed the function (6.1) to measure the amount of an A-IFV for ranking decision alternative. However, for the A-IFVs converted from interval fuzzy weights, the function (6.1) seems to be not a reasonable measure. For instance, for two interval fuzzy weights $\bar{\omega}_1 = [0.2, 0.2]$ and $\bar{\omega}_2 = [0.2, 0.3]$, the converted A-IFVs are determined as $\tilde{\alpha}_{\bar{\omega}_1} = (0.2, 0.8)$ and $\tilde{\alpha}_{\bar{\omega}_2} = (0.2, 0.7)$, respectively. By (6.1), one can obtain $\varphi(\tilde{\alpha}_{\bar{\omega}_1}) = 0.4$ and $\varphi(\tilde{\alpha}_{\bar{\omega}_2}) = 0.44$, and thus $\tilde{\alpha}_{\bar{\omega}_1} > \tilde{\alpha}_{\bar{\omega}_2}$, implying $\bar{\omega}_1 > \bar{\omega}_2$, which seems counterintuitive. This drawback results in an inappropriate ranking of $\bar{\omega}_1^* \succ \bar{\omega}_2^*$, .i.e, the interval weights in $\bar{\omega}^*$ are ranked as $\bar{\omega}_1^* \succ \bar{\omega}_2^* \succ \bar{\omega}_4^* \succ \bar{\omega}_3^*$ counter-intuitively.

As the interval fuzzy weight vectors obtained by the four different approaches are all assumed to be additive, an appropriate rank method for the obtained interval fuzzy

Table 3 A comparative study for the fitted IPRs

Model	Reference	IPR	Difference degree $GMDD(\tilde{R},.)$	Hesitation margin $\pi^r(.)$
Algorithm I	Xu and Liao (2014)	$\tilde{R}^{XL}$	0.1879	0.7305
(14) and (16)	Gong et al. (2009)	$\tilde{R}^{GO}$	0.3630	0.2500
(5.12)	This article	$\tilde{R}^{*}$	0.1629	0.6228

weights seems to use the possibility degree formula (5.15), and then yields a ranking order with possibility degrees. In this situation, however, it is also difficult to identify which one priority weight derivation method performs better due to the fact that the same ranking order is derived by the approaches in Xu and Liao (2014), Xu (2012) and this article. Therefore, we need to determine which the obtained interval fuzzy weight vector is the most accurate result of capturing the features of the original intuitionistic fuzzy judgments provided by the DM.

Next, the multiplicative consistent IPRs obtained from the methods in Xu and Liao (2014) and Gong et al. (2009) will be used in the further analysis by comparing their geometric mean based hesitation margins and difference degrees (Note that the priority method (Xu 2012) does not consider the consistency, hence, is omitted here for the further comparative study).

Xu and Liao (2014) employ (4.4) and (4.5) to develop an algorithm for constructing a perfect multiplicative consistent IPR. By applying Algorithm I in Xu and Liao (2014) to $\tilde{R}$, the perfect multiplicative consistent IPR is determined as

$$\tilde{R}^{XL} = \begin{bmatrix} (0.5, 0.5) & (0.2, 0.6) & (0.3333, 0.5) & (0.2079, 0.2899) \\ (0.6, 0.2) & (0.5, 0.5) & (0.5, 0.4) & (0.3956, 0.2899) \\ (0.5, 0.3333) & (0.4, 0.5) & (0.5, 0.5) & (0.3, 0.2) \\ (0.2899, 0.2079) & (0.2899, 0.3956) & (0.2, 0.3) & (0.5, 0.5) \end{bmatrix}$$

By plugging $\bar{\omega}^g$ into (14) and (16) in Gong et al. (2009), the fitted IPR with Gong et al.'s multiplicative consistency is obtained as

$$\tilde{R}^{GO} = \begin{bmatrix} (0.5, 0.5) & (0.2727, 0.6364) & (0.3, 0.6001) & (0.2, 0.75) \\ (0.6364, 0.2727) & (0.5, 0.5) & (0.5, 0.4285) & (0.3684, 0.6) \\ (0.6001, 0.3) & (0.4285, 0.5) & (0.5, 0.5) & (0.3333, 0.6316) \\ (0.75, 0.2) & (0.6, 0.3684) & (0.6316, 0.3333) & (0.5, 0.5) \end{bmatrix}$$

As per (5.13), the geometric mean based difference degrees between $\tilde{R}$ and the fitted IPRs are determined and listed in Table 3.

By (3.9), the geometric mean based hesitation margin of the original IPR is obtained as $\pi^r(\tilde{R}) = 0.6230$. Similarly, the geometric mean based hesitation margins of the others are determined and shown in the last column in Table 3.

Table 3 demonstrates that the difference degree $GMDD(\tilde{R}, \tilde{R}^*)$ is the smallest among the three difference degrees, and the hesitation margin $\pi^r(\tilde{R}^*)$ is the closest to that of the original IPR $\tilde{R}$. This result implies that the features of the original intuitionistic fuzzy judgments are accurately captured by the proposed models in this article and the obtained interval fuzzy weights in $\bar{\omega}^*$ may be employed to exactly reflect the importance degrees of decision alternatives.

Example 3 With the developing trend of the supply chain management, a core enterprise has to generate a weighting scheme for the selection of supply chain partners. There are four main criteria: product quality (c_1), cost and delivery time (c_2), supplier flexibility and responsiveness (c_3), and trust and financial status (c_4). Assume that a senior executive is asked to determine importance weights for the four criteria. The executive compares each pair of the criteria and furnishes his/her judgments by means of the following IPR.

$$\tilde{R} = (\tilde{r}_{ij})_{4\times 4} = ((\mu_{ij}, v_{ij})_{4\times 4} = \begin{bmatrix} (0.5, 0.5) & (1/3, 2/3) & (1/5, 4/5) & (1/4, 3/4) \\ (2/3, 1/3) & (0.5, 0.5) & (1/3, 2/3) & (2/5, 3/5) \\ (4/5, 1/5) & (2/3, 1/3) & (0.5, 0.5) & (4/7, 3/7) \\ (3/4, 1/4) & (3/5, 2/5) & (3/7, 4/7) & (0.5, 0.5) \end{bmatrix}$$

By plugging $\tilde{R}$ into (5.11) and solving this model, one can obtain the optimal objective value $J^* = 0$, i.e., $\tilde{R}$ is a geometry consistent IPR, and the optimal solutions as $\gamma_{ij}^* = 1$ $(i, j = 1, 2, 3, 4, i < j)$ and $\bar{\omega}^* = (\bar{\omega}_1^*, \bar{\omega}_2^*, \bar{\omega}_3^*, \bar{\omega}_4^*)^T = ([0.1, 0.1],$ $[0.2, 0.2], [0.4, 0.4], [0.3, 0.3])^T$.

Next, the priority method in Xu and Liao (2014) is applied to the same IPR $\tilde{R}$. The interval weight vector is determined as

$$\begin{aligned} \bar{\omega}^{xl} &= (\bar{\omega}_1^{xl}, \bar{\omega}_2^{xl}, \bar{\omega}_3^{xl}, \bar{\omega}_4^{xl})^T \\ &= ([0.1604, 0.1604], [0.2375, 0.2375], [0.3173, 0.3173], [0.2848, 0.2848])^T. \end{aligned}$$

By employing (13) and (15) in Xu (2012), an interval weight vector is obtained as:

$$\begin{aligned} \bar{\omega}^{xu} &= (\bar{\omega}_1^{xu}, \bar{\omega}_2^{xu}, \bar{\omega}_3^{xu}, \bar{\omega}_4^{xu})^T \\ &= ([0.1903, 0.1903], [0.2417, 0.2417], [0.2948, 0.2948], [0.2732, 0.2732])^T \end{aligned}$$

On the other hand, since $\mu_{ij} + v_{ij} = 1$ for all i, j =1, 2, 3, 4, $\tilde{R}$ is equivalent to the following ordinary fuzzy preference relation $R = (r_{ij})_{4\times 4} = (\mu_{ij})_{4\times 4}$. As per Definition 2.1, one can verify that R is Tanino's multiplicative consistent. It is obvious that the obtained three interval weight vectors $\bar{\omega}^*, \bar{\omega}^{xl}$ and $\bar{\omega}^{xu}$ are all reduced to crisp weight vectors and normalized. However, $\bar{\omega}^{xl}, \bar{\omega}^{xu}$ do not satisfy (2.4), which is the relationship between Tanino's multiplicative consistent fuzzy preference relations and crisp priority weights. This result implies that inaccurate results may be obtained when the priority weight derivation methods in Xu and Liao (2014) and Xu (2012) are applied to the reduced IPR, which is equivalent to an ordinary fuzzy preference relation with Tanino's multiplicative consistency. In other words, the priority methods in Xu and Liao (2014) and Xu (2012) seem difficult to determine an appropriate weight vector for the criteria$\{c_1, c_2, c_3, c_4\}$. One can verify that $\bar{\omega}^*$ satisfies (2.4), implying that the obtained weight vector $\bar{\omega}^*$ can be used to exactly reflect the importance degrees of the four criteria.

7 Conclusions

The research is concerned with the ratio-based measure for the amounts of A-IFVs and geometric consistency of IPRs as well as how to derive an interval weight vector from an IPR. We have defined the concepts of the intuitionistic fuzzy geometric index and the ratio-based hesitation margin for an A-IFV. Based on the intuitionistic fuzzy geometric indices, we have introduced geometric transitivity to define consistent IPRs. A numerical example has been provided to illustrate the drawback of the multiplicative consistency definition by Xu et al. (2011). We have developed a logarithmic least square model for constructing the fitted geometry consistent IPR and deriving interval priority weights from IPRs. By employing the constructed consistent IPR, we have devised an approach to check the acceptable geometry consistency for IPRs and proposed an algorithm to solve MCDM problems with IPRs. The validity of the proposed models has been shown by two numerical examples and comparisons with existing approaches.

Acknowledgments The author would like to thank the Associate Editor and the anonymous reviewers for their constructive comments and suggestions. This research is supported by the National Natural Science Foundation of China (NSFC) under Grant 71271188.

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