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A semantic study of the first-order predicate logic with uncertainty involved

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Abstract In this paper, we provide a semantic study of the first-order predicate logic for situations involving uncertainty. We introduce the concepts of uncertain predicate proposition, uncertain predicate formula, uncertain interpretation and degree of truth in the framework of uncertainty theory. Compared with classical predicate formula taking true value in {0, 1}, the degree of truth of uncertain predicate formula may take any value in the unit interval [0, 1]. We also show that the uncertain first-order predicate logic is consistent with the classical first-order predicate logic on some laws of the degree of truth.

Keywords Uncertain first-order predicate logic · Uncertain predicate formula · Degree of truth · Uncertain measure · Uncertain variable

1 Introduction

In classical logic, every proposition is either true (with truth value 1) or false (with truth value 0). However, due to incomplete information, there are some situations involving uncertainty in which propositions may be neither true nor false such that the truth values can not be determined. In order to solve this problem, the classical logic was extended to three-valued logic by Lukasiewicz. Then, probabilistic logic was proposed by Nilsson (1986), and many branches of fuzzy logics were introduced,

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such as Gödel logic (Gödel 1932), possibilistic logic (Dubois and Prade 1987), L^* logic (Wang 1997), BL logic (Hajek 1998), MTL logic (Esteva and Godo 2001) and credibilistic logic (Li and Liu 2009a). Currently, probabilistic logic and fuzzy logic have been well developed and widely applied (Hailperin 1996; Adams 1998; Coletti and Scozzafava 2002; Campos et al. 2009; Cignoli et al. 2000; Pei 2003; Wang and Wang 2006).

In practice, the experience and knowledge of professionals and experts sometimes is used for the evaluation of degrees of belief of uncertain events, due to the insufficient data under some uncertain situation. In order to quantitatively analyze this type of subjective uncertainty, an uncertainty theory was founded by Liu (2007) and renewed by Liu (2013) based on the normality, duality, subadditivity and product axioms. Within the framework of uncertainty theory, Li and Liu (2009b) proposed an uncertain propositional logic (UProL). It explains each proposition as an uncertain variable taking value in {0, 1} (1 means the proposition is true and 0 means the proposition is false), and defines its degree of truth as the uncertain measure that the uncertain variable takes value 1. Furthermore, Chen and Ralescu (2011) provided a truth value theorem for computing the degrees of belief of uncertain formulae with independent propositions. Liu (2009b) developed a reasoning methodology to calculate the degrees of truth of uncertain formulae via the maximum uncertainty principle.

This paper presents a semantic study of the first-order predicate logic for situations involving uncertainty, named as uncertain first-order predicate logic (UPreL). The rest of this paper is organized as follows. In Sect. 2, we recall some basic concepts and results about uncertainty theory. Section 3 recalls some basic concepts and results in UProL. In Sect. 4, the concepts of uncertain predicate proposition, uncertain predicate formula and uncertain interpretation are given. The degree of truth of uncertain predicate formula is also defined in this section. In Sect. 5, the consistency between UPreL and the classical first-order predicate logic is shown by proving some laws of the degree of truth.

2 Preliminaries

This section recalls some basic concepts and results about uncertainty theory.

Definition 1 (Liu 2007) Let Γ be a nonempty set, and L a σ-algebra over Γ. Each element $\Lambda \in$ L is called an event. A set function $\mathrm{M} : \mathrm{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following three axioms:

Axiom 1. (Normality) $\mathrm{M}\{\Gamma\} = 1$;

Axiom 2. (Duality) $\mathrm{M}\{\Lambda\} + \mathrm{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 3. (Subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathrm{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathrm{M}\{\Lambda_i\}.$$

If M is an uncertain measure, the triplet $(\Gamma, \mathrm{L}, \mathrm{M})$ is called an uncertainty space.

In order to obtain an uncertain measure of compound event, a product uncertain measure was defined by Liu (2009a) as the fourth axiom of uncertainty theory:

Axiom 4. Let $(\Gamma_k, \mathrm{L}_k, \mathrm{M}_k)$ be uncertainty space for $k = 1, 2, \cdots, n$. The product uncertain measure on $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \times \Gamma_n$ is an uncertain measure on the product σ-algebra $\mathrm{L} = \mathrm{L}_1 \times \mathrm{L}_2 \times \cdots \times \mathrm{L}_n$ satisfying

$$\mathrm{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2,\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathrm{M}_k\{\Lambda_k\}, & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2,\cdots\times\Lambda_n\subset\Lambda} \min\limits_{1\le k\le n} \mathrm{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2,\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathrm{M}_k\{\Lambda_k\}, & \text{if } \sup\limits_{\Lambda_1\times\Lambda_2,\cdots\times\Lambda_n\subset\Lambda^c} \min\limits_{1\le k\le n} \mathrm{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, & \text{otherwise,} \end{cases} \tag{1}$$

denoted by $\mathrm{M} = \mathrm{M}_1 \wedge \mathrm{M}_2 \wedge \cdots \wedge \mathrm{M}_n$.

Definition 2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathrm{L}, \mathrm{M})$ to the set of real numbers. That is, for any Borel set B of real numbers, we have

$$\{\gamma \in \Gamma | \xi(\gamma) \in B\} \in \mathrm{L}.$$

Definition 3 (Liu 2007) Uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if and only if

$$\mathrm{M}\left\{\bigcap_{i=1}^{m} (\xi_i \in B_i)\right\} = \min_{1\le i\le m} \mathrm{M}\{\xi_i \in B_i\} \tag{2}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

Theorem 1 (Liu 2010) *If uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are independent, then we have*

$$M\left\{\bigcup_{i=1}^{m} (\xi_i \in B_i)\right\} = \max_{1\le i\le m} M\{\xi_i \in B_i\} \tag{3}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers.

3 Uncertain propositional logic

In the section, we recall some basic concepts and results in UProL.

Definition 4 (Liu 2010) An uncertain proposition is a statement whose truth value is quantified by an uncertain measure.

An uncertain formula X was defined by Li and Liu (2009b) as a finite sequence of uncertain propositions and connective symbols which must make sense. In order to give a strict description on uncertain formula, we take the following definition in this paper. Let S be a set of finite or numerable uncertain propositions. $F(S)$ is a type $(\neg, \wedge, \vee)$ free algebra produced by S, i.e.,

(i) $\xi \in F(S)$, for each $\xi \in S$;
(ii) $\neg X, X \vee Y, X \wedge Y \in F(S)$, if $X, Y \in F(S)$;
(iii) All elements in $F(S)$ are produced by manners (i) and (ii) only.

Then each element in $F(S)$ is called an uncertain formula.

Definition 5 (Li and Liu 2009b) Let X be an uncertain formula. Then its degree of truth is defined as the uncertain measure that the uncertain formula X is true, i.e.,

$$T(X) = \mathrm{M}\{X = 1\}.$$

Example 1 Assume that uncertain propositions ξ_1 and ξ_2 denotes the statements "John will be in New York tomorrow" and "Tommy will be in New York tomorrow", respectively. Suppose that ξ_1 and ξ_2 are true with degrees 0.8 and 0.9. Take $(\Gamma_1, \mathrm{L}_1, \mathrm{M}_1)$ to be an uncertainty space $\Gamma_1 = \{\gamma_{11}, \gamma_{12}\}$ with $\mathrm{M}_1\{\gamma_{11}\} = 0.8$ and $\mathrm{M}_1\{\gamma_{12}\} = 0.2$. Then ξ_1 may be considered as an uncertain variable on uncertainty space $(\Gamma_1, \mathrm{L}_1, \mathrm{M}_1)$ defined as

$$\xi_1(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_{11} \\ 0, & \text{if } \gamma = \gamma_{12}. \end{cases}$$

It is easy to prove that $T(\xi_1) = \mathrm{M}_1\{\xi_1 = 1\} = 0.8$. Similarly, take $(\Gamma_2, \mathrm{L}_2, \mathrm{M}_2)$ to be an uncertainty space $\Gamma_2 = \{\gamma_{21}, \gamma_{22}\}$ with $\mathrm{M}_2\{\gamma_{21}\} = 0.9$ and $\mathrm{M}_2\{\gamma_{22}\} = 0.1$. Then ξ_2 may be considered as an uncertain variable on uncertainty space $(\Gamma_2, \mathrm{L}_2, \mathrm{M}_2)$ defined as

$$\xi_2(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_{21} \\ 0, & \text{if } \gamma = \gamma_{22}. \end{cases}$$

Furthermore, $\xi_1 \wedge \xi_2$ is an uncertain formula, which may be considered as an uncertain variable defined on the product uncertainty space as

$$\xi_1 \wedge \xi_2(\gamma) = \begin{cases} 1, & \text{if } \gamma = (\gamma_{11}, \gamma_{21}) \\ 0, & \text{otherwise.} \end{cases}$$

According to Axiom 4, its degree of truth is $T(\xi_1 \wedge \xi_2) = 0.8 \wedge 0.9 = 0.8$.

In the paper, we use symbols X, Y and Z to denote uncertain formulae. The implication conjunction "$\rightarrow$ " is defined as $X \rightarrow Y = \neg X \vee Y$.

Definition 6 (Li and Liu 2009b) Uncertain formulae $X_1, X_2, \cdots, X_n$ are said to be independent if they are independent uncertain variables.

Theorem 2 (Li and Liu 2009b) *If uncertain formulae X and Y are independent, then we have*

$$T(X \vee Y) = \max\{T(X), T(Y)\}, \ T(X \wedge Y) = \min\{T(X), T(Y)\}.$$

Chen and Ralescu (2011) gave the method for calculating the degree of truth of uncertain formula containing independent uncertain propositions.

Theorem 3 (Chen and Ralescu 2011) *Let X be an uncertain formula containing independent uncertain propositions $\xi_1, \xi_2, \ldots, \xi_n$ whose Boole function* (Li and Liu 2009a) *is f. Then its truth value is*

$$T(X) = \begin{cases} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), & \text{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5q \end{cases}$$

where x_i take values either 0 or 1, and $\nu_i(x_i) = M_i\{\xi_i = x_i\}$ for all $1 \le i \le n$.

For any uncertain formulae X and Y, according to the duality, subadditivity and monotonicity, Li and Liu (2009b) proved that

(i) $T(X) + T(\neg X) = 1$;
(ii) $T(X) + T(Y) - 1 \le T(X \wedge Y) \le T(Y) \wedge T(X)$;
(iii) $T(X) \vee T(Y) \le T(X \vee Y) \le T(X) + T(Y)$.

Theorem 4 *For any uncertain formulae X and Y, we have*

(i) $T(X) = 1$ if $\models X$;
(ii) $T(X) = T(Y)$ if $X \equiv Y$;
(iii) $T(X) \le T(Y)$ if $\models X \to Y$

where $\models$ denotes tautology and $\equiv$ denotes semantically equivalent in classical logic.

Proof Assume that X and Y are defined on uncertainty space $(\Gamma, \mathrm{L}, \mathrm{M})$.

(i) If $\models X$, then uncertain variable X takes a constant value 1. It follows from the normality axiom of uncertain measure that

$$T(X) = \mathrm{M}\{X = 1\} = \mathrm{M}\{\Gamma\} = 1.$$

(ii) If $X \equiv Y$, according to the definition of degree of truth, we have

$$T(X) = \mathrm{M}\{\gamma \in \Gamma | X(\gamma) = 1\} = \mathrm{M}\{\gamma \in \Gamma | Y(\gamma) = 1\} = T(Y).$$

(iii) Assume $\models X \to Y$. If $X = 1$, according to the MP rule, we have $Y = 1$, which implies that

$$\{\gamma \in \Gamma | X(\gamma) = 1\} \subseteq \{\gamma \in \Gamma | Y(\gamma) = 1\}.$$

It follows from the monotonicity of uncertain measure that

$$T(X) = \mathrm{M}\{\gamma \in \Gamma | X(\gamma) = 1\} \le \mathrm{M}\{\gamma \in \Gamma | Y(\gamma) = 1\} = T(Y).$$

The proof is completed. □

Corollary 1 *For any uncertain formulae X and Y, we have*

(i) $T(X) = 0$ *if* $\models \neg X$;
(ii) $T(X \vee \neg X) = 1,\ T(X \wedge \neg X) = 0.$

Proof (i) According to Theorem 4 (i), we have $T(\neg X) = 1$, which implies that $T(X) = 1 - T(\neg X) = 0$. (ii) Since $\models X \vee \neg X$, we have $T(X \vee \neg X) = 1$. Furthermore, it follows from $\models \neg(X \wedge \neg X)$ that $T(X \wedge \neg X) = 0$. The proof is complete. □

4 Uncertain first-order predicate logic

In this section, we introduce the uncertain first-order predicate logic. If D_i are non-empty sets for $i = 1, 2, \ldots, n$, then $\xi(x_1, x_2, \ldots, x_n)$ with $x_i \in D_i$ for $i = 1, 2, \ldots, n$ expresses multiple uncertain propositions. In what follows, ξ will be called a n-ary uncertain predicate. Now we give the concept of the language (denoted by ULa):

(i) Variable symbols: x, y, z
(ii) Individual constant symbols: a, b, c
(iii) Uncertain predicate symbols: ξ, η, τ
(iv) Conjunctions: $\neg, \wedge, \vee$
(v) Brackets: (,)
(vi) Quantifier symbol: $\forall$.

Definition 7 Let ξ be a n-ary uncertain predicate. Then $\xi(x_1, x_2, \ldots, x_n)$ is called a n-ary uncertain predicate proposition. Let G be a set of finite or numerable uncertain predicates. Then uncertain predicate formulae are defined as follows:

(i) $\xi(x_1, x_2, \ldots, x_n)$ is an uncertain predicate formula for each $\xi \in G$ and $x_i \in D_i$ for $i = 1, 2, \ldots, n$;
(ii) If X and Y are uncertain predicate formulae, then $\neg X$, $X \wedge Y$, $X \vee Y$, $(\forall x)X(x)$ are also uncertain predicate formulae;
(iii) All uncertain predicate formulae are produced by manners (i) and (ii).

The set of all uncertain predicate formulae is denoted by $F(ULa)$. If an uncertain predicate formula X contains variables $x_1, x_2, \ldots, x_n$, it will be denoted by $X(x_1, x_2, \ldots, x_n)$. Similarly, each uncertain predicate formula can also be considered as an uncertain variable taking values in $\{0, 1\}$.

Definition 8 An uncertain interpretation UI of ULa is defined as follows:

(i) Each variable x_i corresponds to a domain of discourse D_i;
(ii) Each individual constant a_i corresponds to a fixed element in D_i;
(iii) Let ξ be a n-ary uncertain predicate. Then each $(x_1, x_2, \ldots, x_n) \in D_1 \times D_2 \times \cdots \times D_n$ corresponds to an uncertain proposition $\xi(x_1, x_2, \ldots, x_n)$.

Definition 9 The triplet $\Sigma = (ULa, F(ULa), UI)$ is called a structure in UPreL.

Remark 1 If X and Y are uncertain predicate formulae, then we define $X \to Y = \neg X \vee Y$ and $(\exists x)X(x) = \neg(\forall x)\neg X(x)$.

Now we introduce the concept of degree of truth of uncertain predicate formula. Note that if uncertain predicate formula X does not contain quantifier, then for each $(x_1, x_2, \ldots, x_n) \in D_1 \times D_2 \times \cdots \times D_n$, $X(x_1, x_2, \ldots, x_n)$ is essentially an uncertain propositional formula.

Definition 10 Given a structure $\Sigma = (ULa, F(ULa), UI)$ in UPreL. Let X be an uncertain predicate formula with domains of discourse $D_1, D_2, \ldots, D_n$ for variables $x_1, x_2, \ldots, x_n$. Then for each $(x_1, x_2, \ldots, x_n) \in D_1 \times D_2 \times \cdots \times D_n$, the degree of truth of uncertain propositional formula $X(x_1, x_2, \ldots, x_n)$ is called the degree of truth of uncertain predicate formula X at $(x_1, x_2, \ldots, x_n)$.

Definition 11 Given a structure $\Sigma = (ULa, F(ULa), UI)$ in UPreL. For any uncertain predicate formula X, its degree of truth is defined as

$$T(X) = \mathrm{M}\{X = 1\}. \tag{4}$$

In what follows, we straightway say uncertain predicate formula by omitting structure Σ in case of self-evident or no confusion.

Theorem 5 *Suppose that X is an uncertain predicate with domains of discourse $D_1, D_2, \cdots, D_n$ for variables $x_1, x_2, \ldots, x_n$, respectively. If $\{X(x_1, x_2, \ldots, x_n) | x_1 \in D_1, x_2 \in D_2, \ldots, x_n \in D_n\}$ is a class of independent uncertain formulae, then*

(i) the degree of truth of formula $Y = (\forall(x_1, x_2, \ldots, x_n))X(x_1, x_2, \ldots, x_n)$ is

$$T(Y) = \inf_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n} T(X(x_1, x_2, \ldots, x_n));$$

(ii) the degree of truth of formula $Z = (\exists(x_1, x_2, \ldots, x_n))X(x_1, x_2, \ldots, x_n)$ is

$$T(Z) = \sup_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n} T(X(x_1, x_2, \ldots, x_n)).$$

Proof (i) Since $X(x_1, x_2, \ldots, x_n)$, $(x_1, x_2, \ldots, x_n) \in D_1 \times D_2 \times \cdots \times D_n$ are mutually independent uncertain propositional formulae, it follows from the Definition 3 that

$$\begin{aligned}\mathrm{M}\{Y = 1\} &= \mathrm{M}\left\{\bigcap_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n} \{X(x_1, x_2, \ldots, x_n) = 1\}\right\}\\ &= \inf_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n} \mathrm{M}\{X(x_1, x_2, \ldots, x_n) = 1\}.\end{aligned}$$

Furthermore, according to the definition of degree of truth, we have

$$T(Y) = \inf_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n} T(X(x_1, x_2, \ldots, x_n)).$$

(ii) First, it follows from Theorem 1 that

$$\mathrm{M}\{Z=1\}=\mathrm{M}\left\{\bigcup_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n}\{X(x_1,x_2,\ldots,x_n)=1\}\right\}$$
$$=\sup_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n}\mathrm{M}\{X(x_1,x_2,\ldots,x_n)=1\}.$$

Furthermore, according to the definition of degree of truth, we have

$$T(Z)=\sup_{(x_1,x_2,\ldots,x_n)\in D_1\times D_2\times\cdots\times D_n}T(X(x_1,x_2,\ldots,x_n)).$$

The proof is complete. □

5 Basic laws in UPreL

In this section, we show some basic laws on degree of truth in UPreL.

Theorem 6 *For any uncertain predicate formula $X(x)$, we have*

(i) (Law of Excluded Middle) $T((\forall x)X(x)\vee(\exists x)\neg X(x))=1$.
(ii) (Law of Contradiction) $T((\forall x)X(x)\wedge(\exists x)\neg X(x))=0$.
(iii) (Law of Truth Conservation) $T((\forall x)X(x))+T((\exists x)\neg X(x))=1$.

Proof First, it is easy to prove that $(\exists x)\neg X(x)\equiv\neg(\forall x)\neg\neg X(x)\equiv\neg(\forall x)X(x)$, which implies that $(\exists x)\neg X(x)=1$ if and only if $(\forall x)X(x)=0$. Now we prove (i),(ii) and (iii).

(i) According to the definition of degree of truth and the normality of uncertain measure, we have

$$T((\forall x)X(x)\vee(\exists x)\neg X(x))=\mathrm{M}\{\{(\forall x)X(x)=1\}\cup\{(\forall x)X(x)=0\}\}=1.$$

(ii) According to the definition of degree of truth and $\mathrm{M}\{\emptyset\}=0$, we have

$$T((\forall x)X(x)\wedge(\exists x)\neg X(x))=\mathrm{M}\{\{(\forall x)X(x)=1\}\cap\{(\forall x)X(x)=0\}\}=0.$$

(iii) According to the definition of degree of truth and the duality of uncertain measure, we have

$$T((\exists x)\neg X(x))=\mathrm{M}\{(\forall x)X(x)=0\}=1-\mathrm{M}\{(\forall x)X(x)=1\}=1-T((\forall x)X(x)).$$

The proof is complete. □

Definition 12 Assume that Σ is a given structure in UPreL and X is an uncertain predicate formula. If for all $(x_1,x_2,\ldots,x_n)\in D$, we have $T(X(x_1,x_2,\ldots,x_n))=1$ ($T(X(x_1,x_2,\ldots,x_n))=0$), then X is called an uncertain true formula (uncertain false formula) in the structure Σ.

Definition 13 Let X be an uncertain predicate formula. If it is an uncertain true formula (uncertain false formula) for any structure Σ in UPreL, then it is called an uncertain logical effective formula (uncertain contradiction).

Theorem 7 *For any uncertain predicate formula X, we have*

(i) $T((\forall x)X(x) \to X(y)) = 1$;
(ii) $T(X(y) \to (\exists x)X(x)) = 1$;
(iii) $T((\forall x)X(x) \to (\exists x)X(x)) = 1$.

Proof (i) According to Remark 1, we have $(\forall x)X(x) \to X(y) = \neg(\forall x)X(x) \vee X(y)$. If $X(y) = 1$, then $(\forall x)X(x) \vee X(y) = 1$; if $X(y) = 0$, then $(\forall x)X(x) = 0$ and $\neg(\forall x)X(x) = 1$. Hence, uncertain variable $\neg(\forall x)X(x) \vee X(y)$ takes a constant value 1, i.e., it is an uncertain logical effective formula. Similarly, we can prove (ii) and (iii). The proof is complete. □

Theorem 8 *Let X be an uncertain predicate formula, and $D_1, D_2, \ldots, D_n$ are the domains of discourses for variables $x_1, x_2, \ldots, x_n$, respectively. If the standard form of formula X is*

$$(Q_1x_1)(Q_2x_2)\cdots(Q_nx_n)Y(x_1, x_2, \ldots, x_n),$$

where $Q_1, Q_2, \ldots, Q_2$ are quantifier $\forall$ or $\exists$ and $\{Y(x_1, x_2, \ldots, x_n)|x_1 \in D_1, x_2 \in D_2, \ldots, x_n \in D_n\}$ is a class of independent uncertain formulae, then

$$T(X) = \underset{x_1\in D_1}{R}\ \underset{x_2\in D_2}{R}\cdots \underset{x\in D_n}{R}\ T(Y(x_1, x_2, \ldots, x_n))$$

where R_i is inf *if Q_i is $\forall$ and R_i is* sup *if Q_i is $\exists$ for $i = 1, 2, \ldots, n$.*

Proof Since $\{Y(x_1, x_2, \ldots, x_n)|x_1 \in D_1, x_2 \in D_2, \ldots, x_n \in D_n\}$ is a class of independent uncertain formulae, according to Theorem 5, we have

$$T(X) = \underset{x_1\in D_1}{R}\ \underset{x_2\in D_2}{R}\cdots \underset{x\in D_n}{R}\ T(Y(x_1, x_2, \ldots, x_n))$$

where R_i is inf if Q_i is $\forall$ and R_i is sup if Q_i is $\exists$ for $i = 1, 2, \ldots, n$. The proof is complete. □

6 Conclusions

The main contribution of this paper is to provide a semantic study for UPreL by defining the concepts of uncertain predicate proposition, uncertain predicate formula, uncertain interpretation and degree of truth. We showed that UPreL is consistent with the classical first-order predicate logic by proving some laws of degree of truth.

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Appendix: Classical logic

The section recalls some basic concepts and results about classical propositional logic and first-order predicate logic.

Let X be a formula containing propositions $p_1, p_2, \ldots, p_n$. Then there is a Boole function $f : \{0, 1\}^n \to \{0, 1\}$ such that $T(X) = 1$ if and only if $f(x_1, x_2, \ldots, x_n) = 1$ where $x_i = T(p_i)$ for $i = 1, 2, \ldots, n$. For simplicity, we denote the Boole function of formula X as f_X.

Definition (a) A formula X is called a tautology, denoted by $\models X$, if $f_X(x_1, x_2, \ldots, x_n) = 1$ for all $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$.

Definition (b) A formula X is said to be contradiction, denoted by $\models \neg X$, if $f_X(x_1, x_2, \ldots, x_n) = 0$ for all $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$.

Definition (c) Formulae X and Y are called semantically equivalent, denoted by $X \equiv Y$, if $f_X(x_1, x_2, \cdots, x_n) = f_Y(x_1, x_2, \ldots, x_n)$ for all $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$.

Definition (d) Let X be a formula containing propositions $p_1, p_2, \ldots, p_n$. It is said to be a disjunctive normal form if

$$X = (Q_{11} \wedge Q_{12} \wedge ... \wedge Q_{1n}) \vee (Q_{21} \wedge Q_{22} \wedge \cdots \wedge Q_{2n}) \vee \cdots \vee (Q_{m1} \wedge Q_{m2} \wedge \cdots \wedge Q_{mn}),$$

where Q_{ij} is either p_j or $\neg p_j$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Theorem (a) *Let X be a formula containing propositions $p_1, p_2, \ldots, p_n$. Then it is semantically equivalent to a disjunctive normal form as follows:*

$$\bigvee_{f_X(x_1,x_2,\ldots,x_n)=1,(x_1,x_2,\ldots,x_n)\in\{0,1\}^n} Q_{x_1} \cap Q_{x_2} \cap \cdots \cap Q_{x_n},$$

where for each $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ with $f_X(x) = 1$, $Q_{x_i} = p_i$ if $x_i = 1$ and $Q_{x_i} = \neg p_i$ if $x_i = 0$.

The disjunctive normal form of X is denoted by $G(X)$. For example, $G(p_1 \wedge p_2 \to p_3) = (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3)$.

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