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Semideviations of reduced fuzzy variables: a possibility approach

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Abstract This paper proposes new methods to reduce the uncertain information embedded in the secondary possibility distribution of a type-2 fuzzy variable. Based on possibility measure, we define the lower value-at-risk (VaR) and upper VaR of a regular fuzzy variable, and develop the VaR-based reduction methods for type-2 fuzzy variables. The proposed VaR-based reduction methods generalize some existing reduction methods by introducing possibility level parameter in distribution functions. For VaR reduced fuzzy variables, we employ Lebesgue–Stieltjes (L–S) integral to define three nth semideviations to gauge the risk resulted from asymmetric fuzzy uncertainty. Furthermore, we compute the mean values and semideviations of the VaR reduced fuzzy variables, and derive some useful analytical expressions. The theoretical results obtained in this paper have potential applications in practical risk management and engineering optimization problems.

Keywords Type-2 fuzzy variable · Reduction method · Parametric possibility distribution · Semideviation

1 Introduction

Type-2 fuzzy set was first proposed by Zadeh (1975) to overcome the difficulty of determining the crisp membership function of a fuzzy set. Since then, type-2 fuzzy set theory has been well-developed in the literature to handle linguistic and numerical

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uncertainties (Atacak and Bay 2012; Chen 2013; Liu et al. 2012; Liu and Liu 2010; Ngan 2013; Rajati and Mendel 2013; Wu et al. 2012). Among them, an axiomatic approach, called fuzzy possibility theory, was developed to handle type-2 fuzziness (Liu and Liu 2010). In this theory, fuzzy possibility measure, type-2 fuzzy variable, type-2 possibility distribution and secondary possibility distribution are fundamental concepts. The authors presented these concepts with the intention of adopting a variable-based approach to dealing with type-2 fuzziness, which facilitate the use of modern mathematical tools to investigate fuzzy possibility theory. Some new results about the arithmetic of type-2 fuzzy variables were given in (Chen and Zhang 2011).

In fuzzy possibility theory, it is required to reduce the uncertain information embedded in the secondary possibility distribution. By reduction methods, some reduced fuzzy variables are obtained for practical purposes. For example, based on fuzzy integrals, Qin et al. (2011a,b) proposed the mean value reduction and critical reduction methods and applied them to fuzzy data envelopment analysis; On the basis of Lebesgue-Stieltjes integral, Wu and Liu (2012) presented the equivalent value reduction methods and applied them to portfolio optimization problems. In this paper, we develop new reduction methods in fuzzy possibility theory. We first define the lower and upper VaRs of a regular fuzzy variable based on possibility measure. Then, we develop VaR-based reduction methods for type-2 fuzzy variables. The proposed methods reduce uncertain information embedded in the secondary possibility distribution, and retain the most important information in its parametric possibility distributions. The variables obtained by our reduction methods are referred to as the VaR reduced fuzzy variables. There are two types of parameters in the possibility distributions of the VaR reduced fuzzy variables. The first type parameter θ describes the degree of uncertainty that a type-2 fuzzy variable takes its values, while the second type parameter α represents the possibility level in the support of a type-2 fuzzy variable. From the geometrical viewpoint, the parameter θ determines the support's shape of a type-2 fuzzy variable, while α determines the location of possibility distribution in the support of a type-2 fuzzy variable. Given the first type parameter θ, the possibility distributions of the reduced fuzzy variables run over the entire support of a type-2 fuzzy variable as the parameter α varies in the unit interval $[0, 1]$. As a consequence, our VaR reduction methods generalize the existing reduction methods by introducing the parameter α in the possibility distribution.

In fuzzy decision systems, mean-risk is a frequently used method for modeling the choice among uncertain outcomes, which quantifies a risk management problem by two criteria, mean and risk. Since VaR reduced fuzzy variables have parametric possibility distributions, the computation about their numerical characteristics like mean value and deviations is an important issue for research. In this paper, we first compute the mean values of VaR reduced fuzzy variables. Then, using L–S integral, we define three nth semideviations for VaR reduced fuzzy variables, which are used to gauge the risk resulted from asymmetric fuzzy uncertainty. For reduced trapezoidal, normal and gamma fuzzy variables, we derive their useful analytical expressions about the mean values and semideviations. The theoretical results obtained in this paper have potential applications in engineering optimization and decision making problems (Chen et al. 2013; Huang 2012; Li et al. 2013; Liu and Bai 2013; Yang et al. 2013).

The rest of this paper is organized as follows. Section 2 defines the VaRs of a regular fuzzy variable by possibility measure, and derives some useful VaR formulas for common regular fuzzy variables. In Sect. 3, we develop the VaR-based reduction methods for type-2 fuzzy variables. For common VaR reduced fuzzy variables, Sect. 4 discusses the computation of mean values, and Sect. 5 deals with the computation of semideviations. Finally, Sect. 6 concludes the paper.

2 Regular fuzzy variables

Let Γ be an abstract space of generic elements $\gamma \in \Gamma$. An ample field $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections, and complement in Γ, and Pos is a possibility measure defined on $\mathcal{A}$. We refer to the triplet $(\Gamma, \mathcal{A}, \mathrm{Pos})$ as a possibility space, in which a regular fuzzy vector is defined as follows:

Definition 1 If $(\Gamma, \mathcal{A}, \mathrm{Pos})$ is a possibility space, then an m-ary regular fuzzy vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ is defined as a measurable map from Γ to the space $[0, 1]^m$ in the sense that for every $t = (t_1, t_2, \ldots, t_m) \in [0, 1]^m$, one has

$$\{\gamma \in \Gamma \mid \xi(\gamma) \leq t\} = \{\gamma \in \Gamma \mid \xi_1(\gamma) \leq t_1, \xi_2(\gamma) \leq t_2, \ldots, \xi_m(\gamma) \leq t_m\} \in \mathcal{A}.$$

If $m = 1$, then ξ is called a regular fuzzy variable.

In the following, we first define the VaRs of a regular fuzzy variable by possibility measure, then derive some useful VaR formulas for common regular fuzzy variables.

Definition 2 The lower VaR of a regular fuzzy variable ξ with respect to possibility, denoted by $\mathrm{VaR}^L(\xi)$, is defined as

$$\mathrm{VaR}^L_\alpha(\xi) = \inf\{x \mid \mathrm{Pos}\{\xi \leq x\} \geq \alpha\}, \tag{1}$$

while the upper VaR of ξ with respect to possibility, denoted by $\mathrm{VaR}^U(\xi)$, is defined as

$$\mathrm{VaR}^U_\alpha(\xi) = \sup\{x \mid \mathrm{Pos}\{\xi \geq x\} \geq \alpha\}. \tag{2}$$

The VaRs of a trapezoidal regular fuzzy variable are discussed in the following theorem.

Theorem 1 *Let ξ be a trapezoidal regular fuzzy variable (r_1, r_2, r_3, r_4) with $r_1 < r_2 \leq r_3 < r_4$ and $r_i \in [0, 1]$ for $i = 1, 2, 3, 4$. Then we have*

(i) The lower VaR of ξ is $\mathrm{VaR}^L_\alpha(\xi) = r_1 + \alpha(r_2 - r_1)$.

(ii) The upper VaR of ξ is $VaR^U_\alpha(\xi) = r_4 - \alpha(r_4 - r_3)$.

Proof We only prove assertion (i), and assertion (ii) can be proved similarly. According to the possibility distribution of ξ, we have

$$\mathrm{Pos}\{\xi \leq x\} = \begin{cases} 0, & \text{if } x \leq r_1 \\ \frac{x - r_1}{r_2 - r_1}, & \text{if } r_1 < x \leq r_2 \\ 1, & \text{if } x > r_2. \end{cases}$$

By the definition of $\mathrm{VaR}^L_\alpha(\xi)$, the value $\mathrm{VaR}^L_\alpha(\xi)$ is the solution of the following equation

$$\frac{x - r_1}{r_2 - r_1} - \alpha = 0.$$

Therefore, we have $\mathrm{VaR}^L_\alpha(\xi) = r_1 + \alpha(r_2 - r_1)$, which completes the proof of assertion (i). □

Corollary 1 *Let ξ be a triangular regular fuzzy variable (r_1, r_2, r_3) with $r_1 < r_2 < r_3$ and $r_i \in [0, 1]$ for $i = 1, 2, 3$. Then we have*

(i) The lower VaR of ξ is $VaR^L_\alpha(\xi) = r_1 + \alpha(r_2 - r_1)$.
(ii) The upper VaR of ξ is $VaR^U_\alpha(\xi) = r_3 - \alpha(r_3 - r_2)$.

Proof The proof of corollary is similar to that of Theorem 1. □

The VaRs of a normal regular fuzzy variable are obtained in the following theorem.

Theorem 2 *Let ξ be a normal regular fuzzy variable with the following possibility function*

$$\mu_\xi(x) = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in [0, 1],$$

where $0 \le \mu \le 1$ and $\sigma > 0$. If we denote $a = \exp(-\mu^2/2\sigma^2)$ and $b = \exp(-(1-\mu)^2/2\sigma^2)$, then we have

(i) The lower VaR of ξ is $VaR^L_\alpha(\xi) = \mu - \sqrt{-2\sigma^2 \ln \alpha}, \quad \alpha \in [a, 1)$.
(ii) The upper VaR of ξ is $VaR^U_\alpha(\xi) = \mu + \sqrt{-2\sigma^2 \ln \alpha}, \quad \alpha \in [b, 1)$.

Proof We only prove assertion (i), and assertion (ii) can be proved similarly. According to the possibility distribution of ξ, we have

$$\mathrm{Pos}\{\xi \le x\} = \begin{cases} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } 0 \le x \le \mu \\ 1, & \text{if } x > \mu. \end{cases}$$

By the definition of $\mathrm{VaR}^L_\alpha(\xi)$, we know that $\mathrm{VaR}^L_\alpha(\xi)$ is the solution of the following equation

$$\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \alpha = 0, \quad \alpha \in [a, 1), \text{and } 0 \le x \le \mu.$$

Therefore, we have $\mathrm{VaR}^L_\alpha(\xi) = \mu - \sqrt{-2\sigma^2 \ln \alpha}$. The proof of assertion (i) is complete. □

The following theorem deals with the VaRs of a gamma regular fuzzy variable.

Theorem 3 *Let ξ be a gamma regular fuzzy variable with the following possibility distribution*

$$\mu_{\xi}(x) = \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), \quad x \in [0, 1],$$

where r is a positive integer, and $0 < \lambda \leq 1/r$. If we denote $c = (1/\lambda r)^r \exp(r - 1/\lambda)$, then we have

(i) The $VaR_{\alpha}^{L}(\xi)$ of ξ is the solution of the following equation

$$\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad \alpha \in [0, 1], x \in [0, \lambda r].$$

(ii) The $VaR_{\alpha}^{U}(\xi)$ of ξ is the solution of the following equation

$$\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad \alpha \in [c, 1], x \in [\lambda r, 1].$$

Proof We only prove assertion (i), and assertion (ii) can be proved similarly. According to the possibility distribution of ξ, we have

$$\mathrm{Pos}\{\xi \leq x\} = \begin{cases} \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right), & \text{if } 0 \leq x \leq \lambda r \\ 1, & \text{if } \lambda r < x \leq 1. \end{cases}$$

By the definition of $\mathrm{VaR}_{\alpha}^{L}(\xi)$, we know that $\mathrm{VaR}_{\alpha}^{L}(\xi)$ is the solution of the following equation

$$\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - \alpha = 0, \quad \alpha \in [0, 1], x \in [0, \lambda r],$$

which completes the proof of assertion (i). □

3 VaR reduction methods for type-2 fuzzy variables

Let $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}\mathrm{os})$ be a fuzzy possibility space (Liu and Liu 2010), and $\tilde{\xi}$ a type-2 fuzzy variable with secondary possibility distribution $\mu_{\tilde{\xi}}(x)$. To reduce the uncertain information embedded in $\mu_{\tilde{\xi}}(x)$, we will give a new representation for regular fuzzy variable $\mu_{\tilde{\xi}}(x)$. Precisely, we employ the lower and upper VaRs of $\mu_{\tilde{\xi}}(x)$ as its two representing values. The method is referred to as the VaR reduction. The variables obtained by the VaR reduction methods are called the lower and upper VaR reduced fuzzy variables, and denoted by ξ^L and ξ^U, respectively.

Let $r_i \in \Re, i = 1, 2, 3, 4$ with $r_1 < r_2 \leq r_3 < r_4$, and $\mu(x)$ be the following function

$$\mu(x) = \begin{cases} \frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{r_4-x}{r_4-r_3}, & \text{if } x \in (r_3, r_4]. \end{cases}$$

Then a type-2 fuzzy variable $\tilde{\xi}$ is called trapezoidal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is the following regular triangular fuzzy variable

$$(\mu(x) - \theta_l \min\{1 - \mu(x), \mu(x)\}, \mu(x), \mu(x) + \theta_r \min\{1 - \mu(x), \mu(x)\})$$

for any $x \in [r_1, r_4]$, where $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\xi}$ takes the value x, and they may be the functions of x. For simplicity, we denote the type-2 trapezoidal fuzzy variable $\tilde{\xi}$ with the above second possibility distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$.

For a type-2 trapezoidal fuzzy variable, its VaR reduced fuzzy variables have the following parametric possibility distributions.

Theorem 4 *Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable. If we denote $\theta = (\theta_l, \theta_r)$, then the VaR reduced fuzzy variables ξ^L and ξ^U have the following parametric possibility distributions*

$$\mu_{\xi^L}(x; \theta, \alpha) = \begin{cases} (1 - \theta_l + \alpha\theta_l)\frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-\alpha\theta_l)x-(1-\alpha)\theta_l r_2 - r_1}{r_2-r_1}, & \textit{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{-(1+\theta_l-\alpha\theta_l)x+(1-\alpha)\theta_l r_3 + r_4}{r_4-r_3}, & \textit{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ (1 - \theta_l + \alpha\theta_l)\frac{r_4-x}{r_4-r_3}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4], \end{cases} \tag{3}$$

$$\mu_{\xi^U}(x; \theta, \alpha) = \begin{cases} (1 + \theta_r - \alpha\theta_r)\frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1-\theta_r+\alpha\theta_r)x+(1-\alpha)\theta_r r_2 - r_1}{r_2-r_1}, & \textit{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \textit{if } x \in (r_2, r_3] \\ \frac{-(1-\theta_r+\alpha\theta_r)x-(1-\alpha)\theta_r r_3 + r_4}{r_4-r_3}, & \textit{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ (1 + \theta_r - \alpha\theta_r)\frac{r_4-x}{r_4-r_3}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4]. \end{cases} \tag{4}$$

Proof We only prove Eqs. (3), and (4) can be proved similarly. By the supposition of theorem, the secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ of $\tilde{\xi}$ is the following triangular regular fuzzy variable

$$\left(\frac{x-r_1}{r_2-r_1}-\theta_l \min\left\{\frac{x-r_1}{r_2-r_1},\frac{r_2-x}{r_2-r_1}\right\},\frac{x-r_1}{r_2-r_1},\frac{x-r_1}{r_2-r_1}+\theta_r \min\left\{\frac{x-r_1}{r_2-r_1},\frac{r_2-x}{r_2-r_1}\right\}\right)$$

for any $x \in [r_1, r_2]$, the regular fuzzy variable $\tilde{1}$ for any $x \in [r_2, r_3]$, and

$$\left(\frac{r_4-x}{r_4-r_3}-\theta_l \min\left\{\frac{r_4-x}{r_4-r_3},\frac{x-r_3}{r_4-r_3}\right\},\frac{r_4-x}{r_4-r_3},\frac{r_4-x}{r_4-r_3}+\theta_r \min\left\{\frac{r_4-x}{r_4-r_3},\frac{x-r_3}{r_4-r_3}\right\}\right)$$

for any $x \in [r_3, r_4]$. Since ξ^L is the lower VaR reduction of $\tilde{\xi}$, we have

$$\begin{aligned}
\mu_{\xi^L}(x;\theta,\alpha) &= \mathrm{Pos}\{\xi^L = x\} \\
&= \begin{cases} \frac{x-r_1}{r_2-r_1} - (1-\alpha)\theta_l \min\left\{\frac{x-r_1}{r_2-r_1},\frac{r_2-x}{r_2-r_1}\right\}, & \text{if } x \in [r_1, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{r_4-x}{r_4-r_3} - (1-\alpha)\theta_l \min\left\{\frac{r_4-x}{r_4-r_3},\frac{x-r_3}{r_4-r_3}\right\}, & \text{if } x \in (r_3, r_4] \end{cases} \\
&= \begin{cases} (1-\theta_l+\alpha\theta_l)\frac{x-r_1}{r_2-r_1}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-\alpha\theta_l)x-(1-\alpha)\theta_l r_2 - r_1}{r_2-r_1}, & \text{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{-(1+\theta_l-\alpha\theta_l)x+(1-\alpha)\theta_l r_3 + r_4}{r_4-r_3}, & \text{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ (1-\theta_l+\alpha\theta_l)\frac{r_4-x}{r_4-r_3}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4], \end{cases}
\end{aligned}$$

which completes the proof of Eq. (3). □

The following example shows that VaR reduction methods generalize equivalent value reduction methods introduced in (Wu and Liu 2012).

Example 1 Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable. If we denote $\theta = (\theta_l, \theta_r)$, then the OEV, PEV and EV reduced fuzzy variables ξ_1, ξ_2 and ξ_3 have the following parametric possibility distributions

$$\mu_{\xi_1}(x;\theta) = \begin{cases} \frac{(2+\theta_r)(x-r_1)}{2(r_2-r_1)}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(2-\theta_r)x+\theta_r r_2-2r_1}{2(r_2-r_1)}, & \text{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{(-2+\theta_r)x-\theta_r r_3+2r_4}{2(r_4-r_3)}, & \text{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ \frac{(2+\theta_r)(r_4-x)}{2(r_4-r_3)}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4], \end{cases}$$

$$\mu_{\xi_2}(x;\theta) = \begin{cases} \frac{(2-\theta_l)(x-r_1)}{2(r_2-r_1)}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(2+\theta_l)x-\theta_l r_2-2r_1}{2(r_2-r_1)}, & \text{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{(-2-\theta_l)x+\theta_l r_3+2r_4}{2(r_4-r_3)}, & \text{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ \frac{(2-\theta_l)(r_4-x)}{2(r_4-r_3)}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4], \end{cases}$$

$$\mu_{\xi_3}(x;\theta) = \begin{cases} \frac{(4+\theta_r-\theta_l)(x-r_1)}{4(r_2-r_1)}, & \text{if } x \in [r_1, \frac{r_1+r_2}{2}] \\ \frac{(4-\theta_r+\theta_l)x+(\theta_r-\theta_l)r_2-4r_1}{4(r_2-r_1)}, & \text{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ 1, & \text{if } x \in (r_2, r_3] \\ \frac{(-4+\theta_r-\theta_l)x-(\theta_r-\theta_l)r_3+4r_4}{4(r_4-r_3)}, & \text{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ \frac{(4+\theta_r-\theta_l)(r_4-x)}{4(r_4-r_3)}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4], \end{cases}$$

where OEV, PEV and EV are the optimistic equivalent value, pessimistic equivalent value and equivalent value of a regular fuzzy variable, respectively.

According to Corollary 1 and Theorem 4, the following relations among parametric possibility distributions hold:

(i) $\mu_{\xi_1}(x;\theta) = \mu_{\xi^U}(x;\theta,\alpha)$ with $\alpha = \frac{1}{2}$.
(ii) $\mu_{\xi_2}(x;\theta) = \mu_{\xi^L}(x;\theta,\alpha)$ with $\alpha = \frac{1}{2}$.
(iii) If $\theta_l \le \theta_r$, then $\mu_{\xi_3}(x;\theta) = \mu_{\xi^U}(x;\theta,\alpha)$ with $\alpha = \frac{\theta_l+3\theta_r}{4\theta_r}$.
(iv) If $\theta_l \ge \theta_r$, then $\mu_{\xi_3}(x;\theta) = \mu_{\xi^L}(x;\theta,\alpha)$ with $\alpha = \frac{3\theta_l+\theta_r}{4\theta_l}$.

If we denote $\mu(x) = \exp(-\frac{(x-\mu)^2}{2\sigma^2})$, $x \in \Re$, then a type-2 fuzzy variable $\tilde{\eta}$ is called normal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\eta}}(x)$ is the following regular triangular fuzzy variable

$$(\mu(x) - \theta_l \min\{1-\mu(x), \mu(x)\}, \quad \mu(x), \mu(x) + \theta_r \min\{1-\mu(x), \mu(x)\})$$

for any $x \in \Re$, where $\mu \in \Re$, $\sigma > 0$, and $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\eta}$ takes the value x, and they may be the functions of x. For simplicity, we denote the type-2 normal fuzzy variable $\tilde{\eta}$ with the above second possibility distribution by $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$.

The parametric possibility distributions of the VaR reduced normal fuzzy variables are discussed in the following theorem.

Theorem 5 *Let $\tilde{\eta} = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a type-2 normal fuzzy variable. If we denote $\theta = (\theta_l, \theta_r)$, then the VaR reduced fuzzy variables η^L and η^U have the following parametric possibility distributions*

$$\mu_{\eta^L}(x;\theta,\alpha) = \begin{cases} (1-\theta_l+\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \\ \qquad \textit{if } x \le \mu - \sigma\sqrt{2\ln 2} \textit{ or } x > \mu + \sigma\sqrt{2\ln 2} \\ (1+\theta_l-\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - (1-\alpha)\theta_l, \\ \qquad \textit{if } \mu - \sigma\sqrt{2\ln 2} < x \le \mu + \sigma\sqrt{2\ln 2}, \end{cases} \tag{5}$$

$$\mu_{\eta^U}(x;\theta,\alpha) = \begin{cases} (1+\theta_r-\alpha\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \\ \qquad \textit{if } x \le \mu - \sigma\sqrt{2\ln 2} \textit{ or } x > \mu + \sigma\sqrt{2\ln 2} \\ (1-\theta_r+\alpha\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + (1-\alpha)\theta_r, \\ \qquad \textit{if } \mu - \sigma\sqrt{2\ln 2} < x \le \mu + \sigma\sqrt{2\ln 2}. \end{cases} \tag{6}$$

Proof We only prove Eqs. (5), and (6) can be proved similarly. By the supposition of theorem, the secondary possibility distribution $\tilde{\mu}_{\tilde{\eta}}(x)$ of $\tilde{\eta}$ is the following triangular regular fuzzy variable

$$\Bigg(\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)-\theta_l\min\Big\{1-\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big),\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\Big\},$$
$$\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big),$$
$$\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)+\theta_r\min\Big\{1-\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big),\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\Big\}\Bigg).$$

Since η^L is the lower VaR reduction of $\tilde{\eta}$, we have

$$\begin{aligned}
&\mu_{\eta^L}(x;\theta,\alpha)=\mathrm{Pos}\{\eta^L=x\}\\
&=\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)-(1-\alpha)\theta_l\min\Big\{1-\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big),\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\Big\}\\
&=\begin{cases}(1-\theta_l+\alpha\theta_l)\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big), & \text{if } x\le\mu-\sigma\sqrt{2\ln 2} \text{ or } x>\mu+\sigma\sqrt{2\ln 2}\\ (1+\theta_l-\alpha\theta_l)\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)-(1-\alpha)\theta_l, & \text{if } \mu-\sigma\sqrt{2\ln 2}<x\le\mu+\sigma\sqrt{2\ln 2},\end{cases}
\end{aligned}$$

which completes the proof of Eq. (5). □

The following example shows that VaR reduction methods generalize mean value reduction methods defined in (Qin et al. 2011a).

Example 2 Let $\tilde{\eta}=\tilde{n}(\mu,\sigma^2;\theta_l,\theta_r)$ be a type-2 normal fuzzy variable. If we denote $\theta=(\theta_l,\theta_r)$, then the E^*, E_* and E reduced fuzzy variables η_1, η_2 and η_3 have the following parametric possibility distributions

$$\mu_{\eta_1}(x;\theta)=\begin{cases}\frac{2+\theta_r}{2}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big), & \text{if } x\le\mu-\sigma\sqrt{2\ln 2} \text{ or } x>\mu+\sigma\sqrt{2\ln 2}\\ \frac{2-\theta_r}{2}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)+\frac{\theta_r}{2}, & \text{if } \mu-\sigma\sqrt{2\ln 2}<x\le\mu+\sigma\sqrt{2\ln 2},\end{cases}$$

$$\mu_{\eta_2}(x;\theta)=\begin{cases}\frac{2-\theta_l}{2}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big), & \text{if } x\le\mu-\sigma\sqrt{2\ln 2} \text{ or } x>\mu+\sigma\sqrt{2\ln 2}\\ \frac{2+\theta_l}{2}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)-\frac{\theta_l}{2}, & \text{if } \mu-\sigma\sqrt{2\ln 2}<x\le\mu+\sigma\sqrt{2\ln 2},\end{cases}$$

$$\mu_{\eta_3}(x;\theta)=\begin{cases}\frac{4+\theta_r-\theta_l}{4}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big), & \text{if } x\le\mu-\sigma\sqrt{2\ln 2} \text{ or } x>\mu+\sigma\sqrt{2\ln 2}\\ \frac{4-\theta_r+\theta_l}{4}\exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big)+\frac{\theta_r-\theta_l}{4}, & \text{if } \mu-\sigma\sqrt{2\ln 2}<x\le\mu+\sigma\sqrt{2\ln 2},\end{cases}$$

where E^*, E_* and E represent the upper mean value, lower mean value and mean value of a regular fuzzy variable, respectively.

According to Corollary 1 and Theorem 5, the following relations among parametric possibility distributions hold:

(i) $\mu_{\eta_1}(x;\theta) = \mu_{\eta^U}(x;\theta,\alpha)$ with $\alpha = \frac{1}{2}$.
(ii) $\mu_{\eta_2}(x;\theta) = \mu_{\eta^L}(x;\theta,\alpha)$ with $\alpha = \frac{1}{2}$.
(iii) If $\theta_l \le \theta_r$, then $\mu_{\eta_3}(x;\theta) = \mu_{\eta^U}(x;\theta,\alpha)$ with $\alpha = \frac{\theta_l+3\theta_r}{4\theta_r}$.
(iv) If $\theta_l \ge \theta_r$, then $\mu_{\eta_3}(x;\theta) = \mu_{\eta^L}(x;\theta,\alpha)$ with $\alpha = \frac{3\theta_l+\theta_r}{4\theta_l}$.

If we introduce the following three real-valued functions

$$r_1(x;\theta_l) = r_2(x) - \theta_l \min\{1 - r_2(x), r_2(x)\},$$
$$r_2(x) = \left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right),$$

and $r_3(x;\theta_r) = r_2(x)+\theta_r \min\{1-r_2(x), r_2(x)\}$, then a type-2 fuzzy variable $\tilde{\zeta}$ is called gamma if its secondary possibility distribution $\tilde{\mu}_{\tilde{\zeta}}(x)$ is the regular triangular fuzzy variable $(r_1(x;\theta_l), r_2(x), r_3(x;\theta_r))$ for any $x \in \Re^+$, where $\lambda > 0$, r is a fixed positive integer, and $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\zeta}$ takes the value x, and they may be the functions of x. For simplicity, we denote the type-2 gamma fuzzy variable $\tilde{\zeta}$ with the above second possibility distribution by $\tilde{\gamma}(\lambda, r;\theta_l,\theta_r)$.

The parametric possibility distributions of the VaR reduced gamma fuzzy variables are obtained in the following theorem.

Theorem 6 *Let $\tilde{\zeta} = \tilde{\gamma}(\lambda, r;\theta_l,\theta_r)$ be a type-2 gamma fuzzy variable. If we denote $\theta = (\theta_l,\theta_r)$, then the VaR reduced fuzzy variables ζ^L and ζ^U have the following parametric possibility distributions*

$$\mu_{\zeta^L}(x;\theta,\alpha) = \begin{cases} (1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) \le \frac{1}{2} \\ (1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l, & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) > \frac{1}{2}, \end{cases} \tag{7}$$

$$\mu_{\zeta^U}(x;\theta,\alpha) = \begin{cases} (1+\theta_r-\alpha\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) \le \frac{1}{2} \\ (1-\theta_r+\alpha\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)+(1-\alpha)\theta_r, & \text{if } \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) > \frac{1}{2}. \end{cases} \tag{8}$$

Proof We only prove Eqs. (7), and (8) can be proved similarly. By the supposition of theorem, the secondary possibility distribution $\tilde{\mu}_{\tilde{\zeta}}(x)$ of $\tilde{\zeta}$ is the following triangular regular fuzzy variable

$$\Bigg(\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-\theta_l \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}, \left(\frac{x}{\lambda r}\right)^r$$
$$\times \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)$$
$$+\theta_r \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}\Bigg).$$

Since ζ^L is the lower VaR reduction of $\tilde{\zeta}$, we have

$$\mu_{\zeta^L}(x;\theta,\alpha)=\mathrm{Pos}\{\zeta^L=x\}=\left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l$$
$$\times\min\left\{1-\left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right),\left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right)\right\}$$
$$=\begin{cases}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right), & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right)\le\frac{1}{2}\\ (1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp\left(r-\frac{x}{\lambda}\right)>\frac{1}{2},\end{cases}$$

which completes the proof of Eq. (7). □

The following example demonstrates that VaR reduction methods generalize critical value reduction methods introduced in (Qin et al. 2011b).

Example 3 Let $\tilde{\zeta}=\tilde{\gamma}(\lambda,r;\theta_l,\theta_r)$ be a type-2 gamma fuzzy variable. If we denote $\theta=(\theta_l,\theta_r)$, then the CV*, CV$_*$ and CV reduced fuzzy variables ζ_1,ζ_2 and ζ_3 have the following parametric possibility distributions

$$\mu_{\zeta_1}(x;\theta)=\begin{cases}\frac{(1+\theta_r)\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}{1+\theta_r\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})\le\frac{1}{2}\\ \frac{\theta_r+(1-\theta_r)\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}{1+\theta_r-\theta_r\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})>\frac{1}{2},\end{cases}$$
$$\mu_{\zeta_2}(x;\theta)=\begin{cases}\frac{\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}{1+\theta_l\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})\le\frac{1}{2}\\ \frac{\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}{1+\theta_l-\theta_l\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})>\frac{1}{2},\end{cases}$$
$$\mu_{\zeta_3}(x;\theta)=\begin{cases}\frac{(1+\theta_r)\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}{1+2\theta_r\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})\le\frac{1}{2}\\ \frac{\theta_l+(1-\theta_l)\left(\frac{x}{\lambda r}\right)^r\exp(1-\frac{x}{\lambda})}{1+2\theta_l-2\theta_l\left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})}, & \text{if } \left(\frac{x}{\lambda r}\right)^r\exp(r-\frac{x}{\lambda})>\frac{1}{2},\end{cases}$$

where CV*, CV$_*$ and CV represent the optimistic critical value, pessimistic critical value and critical value of a regular fuzzy variable, respectively.

Using the functions $r_1(x,\theta_l), r_2(x)$ and $r_3(x;\theta_r)$ defined above, it follows from Corollary 1 and Theorem 6 that the following relations among parametric possibility distributions hold:

(i) $\mu_{\zeta_1}(x;\theta)=\mu_{\zeta^U}(x;\theta,\alpha)$ with $\alpha=\frac{r_3}{1+r_3-r_2}$.

(ii) $\mu_{\zeta_2}(x;\theta)=\mu_{\zeta^L}(x;\theta,\alpha)$ with $\alpha=\frac{1-r_1}{1+r_2-r_1}$.

(iii) If $r_2>\frac{1}{2}$, then $\mu_{\zeta_3}(x;\theta)=\mu_{\zeta^L}(x;\theta,\alpha)$ with $\alpha=\frac{2(1-r_1)}{1+2(r_2-r_1)}$.

(iv) If $r_2\le\frac{1}{2}$, then $\mu_{\zeta_3}(x;\theta)=\mu_{\zeta^U}(x;\theta,\alpha)$ with $\alpha=\frac{2r_3}{1+2(r_3-r_2)}$.

Remark 1 The variables obtained by VaR reduction methods are referred to as the VaR reduced fuzzy variables. There are two types of parameters included in the possibility distributions of VaR reduced fuzzy variables. The first type parameters θ_l and θ_r

describe the degree of uncertainty that a type-2 fuzzy variable $\tilde{\xi}$ takes the value x, and they may be the functions of x. The second type parameter α represents the possibility level in the support of a type-2 fuzzy variable, and it may be the function of both x and θ. From the geometrical viewpoint, the parameters θ_l and θ_r determine the lower and upper boundaries of possibility distribution, while α determines the location of possibility distribution between the lower boundary and upper boundary. When parameter α varies in the interval [0, 1], the possibility distribution varies between the lower and upper boundaries. Therefore, our VaR-based reduction methods are different from other reduction methods in the literature. As shown in **Examples 1–3**, the proposed VaR reduction methods generalize mean value reduction methods, critical reduction methods and equivalent value reduction methods. As a consequence, our VaR reduction methods have some advantages over other existing reduction methods by introducing the parameter α in possibility distributions.

4 The mean values of VaR reduced fuzzy variables

In this section, we derive some useful analytical expressions about the mean values of VaR reduced fuzzy variables, which generalize the related concepts defined in (Liu and Liu 2002).

The following theorem deals with the mean values of the VaR reduced trapezoidal fuzzy variables.

Theorem 7 *Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable, and ξ^L and ξ^U its VaR reduced fuzzy variables. Then we have*

$$\mathrm{E}[\xi^L] = \frac{r_1 + r_2 + r_3 + r_4}{4} - \frac{r_1 - r_2 - r_3 + r_4}{8}(1-\alpha)\theta_l,$$
$$\mathrm{E}[\xi^U] = \frac{r_1 + r_2 + r_3 + r_4}{4} + \frac{r_1 - r_2 - r_3 + r_4}{8}(1-\alpha)\theta_r.$$

Proof We only prove the first assertion, and the second assertion can be proved similarly.

Since ξ^L is the lower reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (3). By (Liu and Liu 2003), the mean value of ξ^L can be represented by

$$\mathrm{E}[\xi^L] = \frac{1}{2}\int_0^1 \left(\xi^L_{\inf}(\beta) + \xi^L_{\sup}(\beta)\right) \mathrm{d}\beta,$$

where $\xi^L_{\inf}(\beta) = \inf\{r \mid \mathrm{Pos}\{\xi^L \le r\} \ge \beta\}$, and $\xi^L_{\sup}(\beta) = \sup\{r \mid \mathrm{Pos}\{\xi^L \ge r\} \ge \beta\}$.

Note that $\mu_{\xi^L}((r_1 + r_2)/2) = \mu_{\xi^L}((r_3 + r_4)/2) = (1 - \theta_l + \alpha\theta_l)/2$. Then, for $\beta \in (0, (1 - \theta_l + \alpha\theta_l)/2]$, $\xi^L_{\inf}(\beta)$ and $\xi^L_{\sup}(\beta)$ are the solutions of the following two equations, respectively,

$$(1-\theta_l+\alpha\theta_l)\tfrac{x-r_1}{r_2-r_1}=\beta \quad \text{and} \quad (1-\theta_l+\alpha\theta_l)\tfrac{r_4-x}{r_4-r_3}=\beta.$$

By solving the above two equations, we have

$$\xi^L_{\inf}(\beta)=\tfrac{(r_2-r_1)\beta+r_1(1-\theta_l+\alpha\theta_l)}{1-\theta_l+\alpha\theta_l} \quad \text{and} \quad \xi^L_{\sup}(\beta)=\tfrac{-(r_4-r_3)\beta+r_4(1-\theta_l+\alpha\theta_l)}{1-\theta_l+\alpha\theta_l}.$$

Therefore, for $\beta\in(0,(1-\theta_l+\alpha\theta_l)/2]$, we have

$$\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)=\frac{(r_2+r_3-r_1-r_4)\beta+(1-\theta_l+\alpha\theta_l)(r_1+r_4)}{1-\theta_l+\alpha\theta_l}.$$

On the other hand, for $\beta\in((1-\theta_l+\alpha\theta_l)/2,1]$, we can obtain the following result

$$\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)=\frac{(r_2+r_3-r_1-r_4)\beta+(1-\alpha)(r_2+r_3)\theta_l+(r_1+r_4)}{1+\theta_l-\alpha\theta_l}.$$

As a consequence, we have

$$\begin{aligned}\mathrm{E}[\xi^L]&=\frac{1}{2}\left(\int_0^{\frac{1-\theta_l+\alpha\theta_l}{2}}\left(\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)\right)\mathrm{d}\beta+\int_{\frac{1-\theta_l+\alpha\theta_l}{2}}^{1}\left(\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)\right)\mathrm{d}\beta\right)\\&=\frac{r_1+r_2+r_3+r_4}{4}-\frac{r_1-r_2-r_3+r_4}{8}(1-\alpha)\theta_l.\end{aligned}$$

The proof of theorem is complete. □

The mean values of the VaR reduced normal fuzzy variables are discussed in the following theorem.

Theorem 8 *Let $\tilde{\xi}=\tilde{n}(\mu,\sigma^2;\theta_l,\theta_r)$ be a type-2 normal fuzzy variable, and ξ^L and ξ^U its VaR reduced fuzzy variables. Then we have* $\mathrm{E}[\xi^L]=\mathrm{E}[\xi^U]=\mu$.

Proof We only prove assertion $\mathrm{E}[\xi^L]=\mu$, and assertion $\mathrm{E}[\xi^U]=\mu$ can be proved similarly.

Since ξ^L is the reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (5).

Note that $\mu_{\xi^L}\left(\mu-\sigma\sqrt{2\ln 2}\right)=\mu_{\xi^L}\left(\mu+\sigma\sqrt{2\ln 2}\right)=(1-\theta_l+\alpha\theta_l)/2$. Then, for $\beta\in(0,(1-\theta_l+\alpha\theta_l)/2]$, $\xi^L_{\inf}(\beta)$ and $\xi^L_{\sup}(\beta)$ are the solutions of the following equation

$$(1-\theta_l+\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)=\beta.$$

The solutions of the equation are as follows

$$\xi^L_{\inf}(\beta)=\mu-\sqrt{-2\sigma^2\ln\tfrac{\beta}{1-\theta_l+\alpha\theta_l}} \quad \text{and} \quad \xi^L_{\sup}(\beta)=\mu+\sqrt{-2\sigma^2\ln\tfrac{\beta}{1-\theta_l+\alpha\theta_l}}.$$

Therefore, for $\beta\in(0,(1-\theta_l+\alpha\theta_l)/2]$, we have $\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)=2\mu$.

On the other hand, for $\beta \in ((1-\theta_l+\alpha\theta_l)/2,\,1]$, we can deduce $\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta) = 2\mu$. As a consequence, we have

$$\mathrm{E}[\xi^L] = \frac{1}{2}\left(\int_0^{\frac{1-\theta_l+\alpha\theta_l}{2}} \left(\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)\right)\mathrm{d}\beta + \int_{\frac{1-\theta_l+\alpha\theta_l}{2}}^{1} \left(\xi^L_{\inf}(\beta)+\xi^L_{\sup}(\beta)\right)\mathrm{d}\beta\right) = \mu.$$

The proof of theorem is complete. □

The following theorem gives the mean values of the VaR reduced gamma fuzzy variables.

Theorem 9 *Let $\tilde{\xi} = \tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$ be a type-2 gamma fuzzy variable, and ξ^L and ξ^U its VaR reduced fuzzy variables. Then we have*

$$\begin{aligned}
\mathrm{E}[\xi^L] &= \lambda r - \frac{\lambda r!}{2r^r}\exp(r) + \lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!} - \frac{(1-\alpha)\theta_l}{2}\Bigg\{x_1 + x_2 - 2\lambda r \\
&\quad - \frac{\lambda r!}{r^r}\exp(r) + \lambda\sum_{n=0}^{r}\frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n}\right)\Bigg\},\\
\mathrm{E}[\xi^U] &= \lambda r - \frac{\lambda r!}{2r^r}\exp(r) + \lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!} + \frac{(1-\alpha)\theta_r}{2}\Bigg\{x_1 + x_2 - 2\lambda r \\
&\quad - \frac{\lambda r!}{r^r}\exp(r) + \lambda\sum_{n=0}^{r}\frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n}\right)\Bigg\}.
\end{aligned}$$

Proof We only prove the first assertion, and the second assertion can be proved similarly.

Noting that $\mathrm{Cr}\{\xi^L \le x\} = 0$ for any $x < 0$, then the mean value of fuzzy variable ξ^L is represented by

$$\mathrm{E}[\xi^L] = \int_0^{+\infty} \mathrm{Cr}\{\xi^L \ge x\}\mathrm{d}x.$$

Since ξ^L is the lower reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (7). By calculation, the credibility distribution of ξ^L is

$$\mathrm{Cr}\{\xi^L \ge x\} = \begin{cases}
1-\frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } 0 \le x \le x_1\\
1-\frac{1}{2}[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l], & \text{if } x_1 < x \le \lambda r\\
\frac{1}{2}[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l], & \text{if } \lambda r < x \le x_2\\
\frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } x > x_2,
\end{cases}$$

where $x_1, x_2 \in \Re^+$ satisfy

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r - \frac{x_1}{\lambda}\right) = \frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r - \frac{x_2}{\lambda}\right) = \frac{1}{2}.$$

As a consequence, we have

$$\begin{aligned}
\mathrm{E}[\xi^L] = &\int_0^{x_1} \left(1 - \frac{1}{2}(1 - \theta_l + \alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right)\right) \mathrm{d}x \\
&+ \int_{x_1}^{\lambda r} \left(1 - \frac{1}{2}[(1 + \theta_l - \alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1-\alpha)\theta_l]\right) \mathrm{d}x \\
&+ \int_{\lambda r}^{x_2} \left(\frac{1}{2}[(1 + \theta_l - \alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right) - (1-\alpha)\theta_l]\right) \mathrm{d}x \\
&+ \int_{x_2}^{+\infty} \left(\frac{1}{2}(1 - \theta_l + \alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r - \frac{x}{\lambda}\right)\right) \mathrm{d}x \\
= &\ \lambda r - \frac{\lambda r!}{2r^r}\exp(r) + \lambda \sum_{n=0}^{r} \frac{r!}{r^n(r-n)!} - \frac{(1-\alpha)\theta_l}{2}\left\{x_1 + x_2 - 2\lambda r\right. \\
&\left. - \frac{\lambda r!}{r^r}\exp(r) + \lambda \sum_{n=0}^{r} \frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n} + \frac{\lambda^n}{x_2^n} - \frac{2}{r^n}\right)\right\}.
\end{aligned}$$

The proof of theorem is complete. □

5 The semideviations of VaR reduced fuzzy variables

In this section, we introduce a new risk measure related to VaR reduced fuzzy variables. It is known that variance as a risk measure penalizes deviations in both directions. In many modeling situations, however, the direction of deviation matters. In such cases one of them is favorable gain and the other is disadvantageous loss. Therefore, to gauge the risk resulted from asymmetric fuzzy uncertainty, we first define semideviation variables in the following.

Let ξ be a VaR reduced fuzzy variable with finite mean value $\mathrm{E}[\xi]$. Then the lower semideviation variable of ξ with respect to its mean value is defined as

$$(\xi - \mathrm{E}[\xi])^- = \begin{cases} (\mathrm{E}[\xi] - \xi), & \text{if } \xi \le \mathrm{E}[\xi] \\ 0, & \text{if } \xi > \mathrm{E}[\xi], \end{cases}$$

while the upper semideviation variable of ξ with respect to its mean value is defined by

$$(\xi - \mathrm{E}[\xi])^+ = \begin{cases} (\xi - \mathrm{E}[\xi]), & \text{if } \xi \geq \mathrm{E}[\xi] \\ 0, & \text{if } \xi < \mathrm{E}[\xi]. \end{cases}$$

Let ξ be a VaR reduced fuzzy variable with a parametric possibility distribution $\mu_\xi(x;\theta,\alpha)$. Then for every $x \in \Re$, the possibility, necessity and credibility distributions of ξ are represented by

$$\mathrm{Pos}\{\xi \leq x\} = \sup_{t \leq x} \mu_\xi(t;\theta,\alpha),$$

$$\mathrm{Nec}\{\xi \leq x\} = \sup_{t \in \Re} \mu_\xi(t;\theta,\alpha) - \sup_{t > x} \mu_\xi(t;\theta,\alpha),$$

$$\mathrm{Cr}\{\xi \leq x\} = \frac{1}{2}\left(\sup_{t \in \Re} \mu_\xi(t;\theta,\alpha) + \sup_{t \leq x} \mu_\xi(t;\theta,\alpha) - \sup_{t > x} \mu_\xi(t;\theta,\alpha)\right).$$

The possibility, necessity and credibility are functions of the parameters θ and α, and they are monotone increasing functions with respect to $x \in \Re$. So, we can define the following semideviations of VaR reduced fuzzy variables based on L–S integrals.

Definition 3 Let ξ be a VaR reduced fuzzy variable with finite mean value $\mathrm{E}[\xi]$. Then the nth lower semideviation of ξ with respect to possibility is defined by the following L–S integral

$$\mathrm{SD}^-_{n,*}[\xi] = \int_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Pos}\{\xi \leq x\}),$$

and the nth lower semideviation of ξ with respect to necessity is defined by

$$\mathrm{SD}^{-,*}_{n}[\xi] = \int_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Nec}\{\xi \leq x\}).$$

The nth lower semideviation of ξ with respect to credibility is defined by

$$\mathrm{SD}^-_{n}[\xi] = \int_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Cr}\{\xi \leq x\}).$$

Theorem 10 *Let ξ be a VaR reduced fuzzy variable. Then we have*

$$\mathrm{SD}^-_{n}[\xi] = \frac{1}{2}(\mathrm{SD}^-_{n,*}[\xi] + \mathrm{SD}^{-,*}_{n}[\xi]).$$

Proof Note that possibility, necessary and credibility have the following relation

$$\mathrm{Cr}\{\xi \leq x\} = \frac{1}{2}(\mathrm{Pos}\{\xi \leq x\} + \mathrm{Nec}\{\xi \leq x\}),$$

where $\mathrm{Pos}\{\xi \le x\}$ and $\mathrm{Nec}\{\xi \le x\}$ are both monotone increasing functions with respect to $x \in \Re$. Therefore, according to the property of L–S integrals, we have

$$
\begin{aligned}
\mathrm{SD}_n^-[\xi] &= \int\limits_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Cr}\{\xi \le x\}) \\
&= \frac{1}{2} \int\limits_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Pos}\{\xi \le x\}) \\
&\quad + \frac{1}{2} \int\limits_{(-\infty,+\infty)} ((\xi - \mathrm{E}[\xi])^-)^n \mathrm{d}(\mathrm{Nec}\{\xi \le x\}) \\
&= \frac{1}{2}(\mathrm{SD}_{n,*}^-[\xi] + \mathrm{SD}_n^{-,*}[\xi]).
\end{aligned}
$$

The proof of theorem is complete. □

In the following, we will derive some useful formulas for the first order semideviations of VaR reduced fuzzy variables. The following results are the semideviations of the lower VaR reduced trapezoidal fuzzy variable.

Theorem 11 *Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable, and ξ^L its lower VaR reduced fuzzy variable.*

(i) If $r_2 \le \mathrm{E}[\xi^L] \le r_3$, then $\mathrm{SD}_1^-[\xi^L] = (-r_1 - r_2 + r_3 + r_4)/8 - (-r_1 + r_2 - r_3 + r_4)(1-\alpha)\theta_l/16$.

(ii) If $r_1 \le \mathrm{E}[\xi^L] \le (r_1 + r_2)/2$, then

$$
\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \frac{(-3r_1 + r_2 + r_3 + r_4)^2}{64(r_2 - r_1)} \\
&\quad - \frac{3r_1^2 + r_4^2 - r_1 r_2 - r_1 r_3 - 4r_1 r_4 + r_2 r_4 + r_3 r_4}{32(r_2 - r_1)}(1-\alpha)\theta_l \\
&\quad + \frac{(r_1 - r_2 - r_3 + r_4)(-11r_1 + 3r_2 + 3r_3 + 5r_4)}{256(r_2 - r_1)}(1-\alpha)^2\theta_l^2 \\
&\quad - \frac{(r_1 - r_2 - r_3 + r_4)^2}{256(r_2 - r_1)}(1-\alpha)^3\theta_l^3.
\end{aligned}
$$

(iii) If $(r_1 + r_2)/2 \le \mathrm{E}[\xi^L] \le r_2$, then

$$
\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \frac{(-3r_1 + r_2 + r_3 + r_4)^2}{64(r_2 - r_1)} \\
&\quad - \frac{2r_1^2 - r_2^2 - r_3^2 - 3r_1 r_2 + r_1 r_3 - 2r_1 r_4 + 2r_2 r_3 + 3r_2 r_4 - r_3 r_4}{32(r_2 - r_1)}(1-\alpha)\theta_l \\
&\quad - \frac{(r_1 - r_2 - r_3 + r_4)(3r_1 - 11r_2 + 5r_3 + 3r_4)}{256(r_2 - r_1)}(1-\alpha)^2\theta_l^2 \\
&\quad + \frac{(r_1 - r_2 - r_3 + r_4)^2}{256(r_2 - r_1)}(1-\alpha)^3\theta_l^3.
\end{aligned}
$$

(iv) If $r_3 \le \mathrm{E}[\xi^L] \le (r_3+r_4)/2$, *then*

$$\begin{aligned}\mathrm{SD}_1^-[\xi^L] = {} & \frac{(-r_1-r_2-r_3+3r_4)^2}{64(r_4-r_3)} \\ & - \frac{2r_4^2-r_2^2-r_3^2-r_1r_2+3r_1r_3-2r_1r_4+2r_2r_3+r_2r_4-3r_3r_4}{32(r_4-r_3)}(1-\alpha)\theta_l \\ & - \frac{(r_1-r_2-r_3+r_4)(3r_1+5r_2-11r_3+3r_4)}{256(r_4-r_3)}(1-\alpha)^2\theta_l^2 \\ & + \frac{(r_1-r_2-r_3+r_4)^2}{256(r_4-r_3)}(1-\alpha)^3\theta_l^3.\end{aligned}$$

(v) If $(r_3+r_4)/2 \le \mathrm{E}[\xi^L] \le r_4$, *then*

$$\begin{aligned}\mathrm{SD}_1^-[\xi^L] = {} & \frac{(-r_1-r_2-r_3+3r_4)^2}{64(r_4-r_3)} \\ & - \frac{r_1^2+3r_4^2+r_1r_2+r_1r_3-4r_1r_4-r_2r_4-r_3r_4}{32(r_4-r_3)}(1-\alpha)\theta_l \\ & + \frac{(r_1-r_2-r_3+r_4)(5r_1+3r_2+3r_3-11r_4)}{256(r_4-r_3)}(1-\alpha)^2\theta_l^2 \\ & - \frac{(r_1-r_2-r_3+r_4)^2}{256(r_4-r_3)}(1-\alpha)^3\theta_l^3.\end{aligned}$$

Proof We only prove assertion (v), and assertions (i)-(iv) can be proved similarly.

Since ξ^L is the lower VaR reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (3). By calculation, the credibility distribution of ξ^L is

$$\mathrm{Cr}\{\xi^L \le x\} = \begin{cases} 0, & \text{if } x \le r_1 \\ \frac{(1-\theta_l+\alpha\theta_l)(x-r_1)}{2(r_2-r_1)}, & \text{if } x \in (r_1, \frac{r_1+r_2}{2}] \\ \frac{(1+\theta_l-\alpha\theta_l)x-(1-\alpha)\theta_l r_2-r_1}{2(r_2-r_1)}, & \text{if } x \in (\frac{r_1+r_2}{2}, r_2] \\ \frac{1}{2}, & \text{if } x \in (r_2, r_3] \\ 1-\frac{-(1+\theta_l-\alpha\theta_l)x+(1-\alpha)\theta_l r_3+r_4}{2(r_4-r_3)}, & \text{if } x \in (r_3, \frac{r_3+r_4}{2}] \\ 1-\frac{(1-\theta_l+\alpha\theta_l)(r_4-x)}{2(r_4-r_3)}, & \text{if } x \in (\frac{r_3+r_4}{2}, r_4] \\ 1, & \text{if } x > r_4, \end{cases}$$

and the mean value of ξ^L is

$$\mathrm{E}[\xi^L] = \frac{r_1+r_2+r_3+r_4}{4} - \frac{r_1-r_2-r_3+r_4}{8}(1-\alpha)\theta_l.$$

As a consequence, the lower semideviation of ξ^L is calculated by

$$
\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \int_{(-\infty,+\infty)} (x-\mathrm{E}[\xi^L])^- \mathrm{dCr}\{\xi^L \le x\} = \int_{(-\infty,\,\mathrm{E}[\xi^L])} (\mathrm{E}[\xi^L]-x)\mathrm{dCr}\{\xi^L \le x\} \\
&= \int_{\left(r_1, \frac{r_1+r_2}{2}\right)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(\frac{(1-\theta_l+\alpha\theta_l)(x-r_1)}{2(r_2-r_1)}\right) \\
&\quad + \int_{\left(\frac{r_1+r_2}{2}, r_2\right)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(\frac{(1+\theta_l-\alpha\theta_l)x-(1-\alpha)\theta_l r_2 - r_1}{2(r_2-r_1)}\right) \\
&\quad + \int_{\left(r_3, \frac{r_3+r_4}{2}\right)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(1-\frac{-(1+\theta_l-\alpha\theta_l)x+(1-\alpha)\theta_l r_3 + r_4}{2(r_4-r_3)}\right) \\
&\quad + \int_{\left(\frac{r_3+r_4}{2}, \mathrm{E}[\xi^L]\right)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(1-\frac{(1-\theta_l+\alpha\theta_l)(r_4-x)}{2(r_4-r_3)}\right) \\
&= \frac{(-r_1-r_2-r_3+3r_4)^2}{64(r_4-r_3)} \\
&\quad - \frac{r_1^2+3r_4^2+r_1r_2+r_1r_3-4r_1r_4-r_2r_4-r_3r_4}{32(r_4-r_3)}(1-\alpha)\theta_l \\
&\quad + \frac{(r_1-r_2-r_3+r_4)(5r_1+3r_2+3r_3-11r_4)}{256(r_4-r_3)}(1-\alpha)^2\theta_l^2 \\
&\quad - \frac{(r_1-r_2-r_3+r_4)^2}{256(r_4-r_3)}(1-\alpha)^3\theta_l^3.
\end{aligned}
$$

The proof of assertion (v) is complete. □

Remark 2 For upper VaR reduced fuzzy variable ξ^U, we have the similar results about their semideviations, and the proofs are similar to those of Theorem 11.

For the semideviations of the VaR reduced normal fuzzy variable, we have the following results:

Theorem 12 *Let $\tilde{\xi} = \tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$ be a type-2 normal fuzzy variables. Then the lower semideviations of VaR reduced fuzzy variables ξ^L and ξ^U are represented by*

$$
\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \frac{\sqrt{2\pi}}{4}\sigma - \frac{1}{4}(1-\alpha)\theta_l\left[2\sqrt{2\ln 2}\sigma + 3\sqrt{2\pi}\sigma - 4\sqrt{2\pi}\sigma\Phi(\sqrt{2\ln 2})\right], \\
\mathrm{SD}_1^-[\xi^U] &= \frac{\sqrt{2\pi}}{4}\sigma + \frac{1}{4}(1-\alpha)\theta_r\left[2\sqrt{2\ln 2}\sigma + 3\sqrt{2\pi}\sigma - 4\sqrt{2\pi}\sigma\Phi(\sqrt{2\ln 2})\right].
\end{aligned}
$$

Proof We only prove the first assertion, and the second assertion can be proved similarly.

Since ξ^L is the lower reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (5). By calculation, the credibility distribution of ξ^L is

$$\mathrm{Cr}\{\xi^L \le x\} = \begin{cases} \frac{1}{2}(1-\theta_l+\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } x \le \mu-\sigma\sqrt{2\ln 2} \\ \frac{1}{2}[(1+\theta_l-\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)-(1-\alpha)\theta_l], & \text{if } \mu-\sigma\sqrt{2\ln 2} < x \le \mu \\ 1-\frac{1}{2}[(1+\theta_l-\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)-(1-\alpha)\theta_l], & \text{if } \mu \le x < \mu+\sigma\sqrt{2\ln 2} \\ 1-\frac{1}{2}(1-\theta_l+\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), & \text{if } x > \mu+\sigma\sqrt{2\ln 2}, \end{cases}$$

and the mean value of ξ^L is $\mathrm{E}[\xi^L]=\mu$. As a consequence, the lower semideviation of ξ^L is calculated by

$$\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \int_{(-\infty,+\infty)} (x-\mathrm{E}[\xi^L])^- \mathrm{dCr}\{\xi^L \le x\} = \int_{(-\infty,\,\mu)} (\mu-x)\mathrm{dCr}\{\xi^L \le x\} \\
&= \int_{\left(-\infty,\,\mu-\sigma\sqrt{2\ln 2}\right)} (\mu-x)\mathrm{d}\left(\frac{1}{2}(1-\theta_l+\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right) \\
&\quad+ \int_{\left(\mu-\sigma\sqrt{2\ln 2},\,\mu\right)} (\mu-x)\mathrm{d}\left(\frac{1}{2}\left[(1+\theta_l-\alpha\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)-(1-\alpha)\theta_l\right]\right) \\
&= \frac{1-\theta_l+\alpha\theta_l}{2}\int_{-\infty}^{\mu-\sigma\sqrt{2\ln 2}} (\mu-x)\mathrm{d}\left(\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right) \\
&\quad+\frac{1+\theta_l-\alpha\theta_l}{2}\int_{\mu-\sigma\sqrt{2\ln 2}}^{\mu} (\mu-x)\mathrm{d}\left(\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right) \\
&= \frac{\sqrt{2\pi}}{4}\sigma - \frac{1}{4}(1-\alpha)\theta_l\left[2\sqrt{2\ln 2}\sigma + 3\sqrt{2\pi}\sigma - 4\sqrt{2\pi}\sigma\,\Phi(\sqrt{2\ln 2})\right].
\end{aligned}$$

The proof of the first assertion is complete. □

Before ending this section, we establish the analytical expressions for the semideviations of the lower VaR reduced gamma fuzzy variable.

Theorem 13 *Let $\tilde{\xi} = \tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$ be a type-2 gamma fuzzy variable, and ξ^L its lower VaR reduced fuzzy variable.*

(i) If $0 \le \mathrm{E}[\xi^L] \le x_1$, then

$$\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \frac{\lambda r!}{2r^r}\exp(r) - \frac{\lambda}{2}\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right) \\
&\quad-\frac{1}{2}(1-\alpha)\theta_l\left(\frac{\lambda r!}{r^r}\exp(r)-\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\right).
\end{aligned}$$

(ii) If $x_1 \le \mathrm{E}[\xi^L] \le \lambda r$, *then*

$$\begin{aligned}\mathrm{SD}_1^-[\xi^L] &= \frac{\lambda r!}{2r^r}\exp(r) - \frac{\lambda}{2}\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\\ &-\frac{1}{2}(1-\alpha)\theta_l\left(\frac{\lambda r!}{r^r}\exp(r)+\mathrm{E}[\xi^L]-x_1-\lambda\sum_{n=0}^{r}\frac{\lambda^n r!}{x_1^n(r-n)!}\right.\\ &\left.+\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\right).\end{aligned}$$

(iii) If $\lambda r \le \mathrm{E}[\xi^L] \le x_2$, *then*

$$\begin{aligned}\mathrm{SD}_1^-[\xi^L] &= \frac{\lambda r!}{2r^r}\exp(r)+\frac{\lambda}{2}\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)+\mathrm{E}[\xi^L]\\ &-\lambda r-\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}-\frac{1}{2}(1-\alpha)\theta_l\left(2\lambda r-x_1-\lambda\sum_{n=0}^{r}\frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n}-\frac{2}{r^n}\right)\right.\\ &\left.-\mathrm{E}[\xi^L]+\frac{\lambda r!}{r^r}\exp(r)+\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\right).\end{aligned}$$

(iv) If $\mathrm{E}[\xi^L] > x_2$, *then*

$$\begin{aligned}\mathrm{SD}_1^-[\xi^L] &= \frac{\lambda r!}{2r^r}\exp(r)+\frac{\lambda}{2}\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)+\mathrm{E}[\xi^L]\\ &-\lambda r-\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}-\frac{1}{2}(1-\alpha)\theta_l\left(2\lambda r-x_1-\lambda\sum_{n=0}^{r}\frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n}+\frac{\lambda^n}{x_2^n}-\frac{2}{r^n}\right)\right.\\ &\left.-\mathrm{E}[\xi^L]-x_2+\frac{\lambda r!}{r^r}\exp(r)+\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\right).\end{aligned}$$

Here the analytical expression of $\mathrm{E}[\xi^L]$ *is given in Theorem 9 and* $x_1, x_2 \in \Re^+$ *satisfy the following conditions*

$$\left(\frac{x_1}{\lambda r}\right)^r\exp\left(r-\frac{x_1}{\lambda}\right)=\frac{1}{2},\quad \left(\frac{x_2}{\lambda r}\right)^r\exp\left(r-\frac{x_2}{\lambda}\right)=\frac{1}{2}.$$

Proof We only prove assertion (iv), and assertions (i)-(iii) can be proved similarly.

Since ξ^L is the lower reduced fuzzy variable of $\tilde{\xi}$, its parametric possibility distribution $\mu_{\xi^L}(x)$ is given by Eq. (7). By calculation, the credibility distribution of ξ^L is

$$\mathrm{Cr}\{\xi^L \le x\} = \begin{cases} \frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } 0 \le x \le x_1 \\ \frac{1}{2}[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l], & \text{if } x_1 < x \le \lambda r \\ 1-\frac{1}{2}[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l], & \text{if } \lambda r < x \le x_2 \\ 1-\frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), & \text{if } x > x_2, \end{cases}$$

where $x_1, x_2 \in \Re^+$ satisfy the following conditions

$$\left(\frac{x_1}{\lambda r}\right)^r \exp\left(r-\frac{x_1}{\lambda}\right)=\frac{1}{2}, \quad \left(\frac{x_2}{\lambda r}\right)^r \exp\left(r-\frac{x_2}{\lambda}\right)=\frac{1}{2}.$$

The mean value $\mathrm{E}[\xi^L]$ of the reduced fuzzy variable ξ is given in Theorem 9. As a consequence, the lower semideviation of ξ^L is calculated by

$$\begin{aligned}
\mathrm{SD}_1^-[\xi^L] &= \int_{(0,+\infty)} (x-\mathrm{E}[\xi^L])^- \mathrm{dCr}\{\xi^L \le x\} = \int_{(0,\,\mathrm{E}[\xi^L])} (\mathrm{E}[\xi^L]-x)\mathrm{dCr}\{\xi^L \le x\} \\
&= \int_{(0,\,x_1)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(\frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right) \\
&\quad + \int_{(x_1,\,\lambda r)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(\frac{1}{2}\left[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l\right]\right) \\
&\quad + \int_{(\lambda r,\,x_2)} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(1-\frac{1}{2}\left[(1+\theta_l-\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)-(1-\alpha)\theta_l\right]\right) \\
&\quad + \int_{(x_2,\,\mathrm{E}[\xi^L])} (\mathrm{E}[\xi^L]-x)\mathrm{d}\left(1-\frac{1}{2}(1-\theta_l+\alpha\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right) \\
&= \frac{\lambda r!}{2r^r}\exp(r) + \frac{\lambda}{2}\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)+\mathrm{E}[\xi^L]-\lambda r \\
&\quad -\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}-\frac{1}{2}(1-\alpha)\theta_l\left(2\lambda r-x_1-\lambda\sum_{n=0}^{r}\frac{r!}{(r-n)!}\left(\frac{\lambda^n}{x_1^n}+\frac{\lambda^n}{x_2^n}-\frac{2}{r^n}\right)\right. \\
&\quad \left. -\mathrm{E}[\xi^L]-x_2+\frac{\lambda r!}{r^r}\exp(r)+\lambda\sum_{n=0}^{r}\frac{r!}{r^n(r-n)!}\left(\frac{\mathrm{E}[\xi^L]}{\lambda r}\right)^{r-n}\exp\left(r-\frac{\mathrm{E}[\xi^L]}{\lambda}\right)\right).
\end{aligned}$$

The proof of assertion (iv) is complete. □

Remark 3 For upper VaR reduced gamma fuzzy variable ξ^U, we have similar results about their semideviations, and the proofs are similar to those of Theorem 13.

6 Conclusions

In fuzzy decision systems, this paper presented the semideviations of reduced fuzzy variables to gauge the risk resulted from asymmetric fuzzy uncertainty. The reduced fuzzy variables are obtained by using new VaR reduction methods for type-2 fuzzy variables. The major contributions to fuzzy possibility theory in present paper include the following four aspects.

(i) On the basis of possibility measure, we defined the lower and upper VaRs of a regular fuzzy variable, and established some VaR formulas for common regular fuzzy variables.
(ii) We developed the VaR-based reduction methods for type-2 fuzzy variables. For the VaR reduced trapezoidal, normal and gamma fuzzy variables, we established their parametric possibility distributions.
(iii) According to the parametric possibility distributions of the VaR reduced trapezoidal, normal and gamma fuzzy variables, we derived the analytical expressions of mean values.
(iv) For VaR reduced fuzzy variables, we employed L–S integrals to define three nth semideviations to measure the variations of parametric possibility distributions with respect to their mean values. Furthermore, for the VaR reduced trapezoidal, normal and gamma fuzzy variables, we derived the analytical expressions of the first order semideviations.

In many practical decision making problems, the exact possibility distributions of fuzzy coefficients are usually hard to be determined due to the lack of historical data. In this situation, parametric possibility distributions have some advantages over fixed possibility distributions. For example, there are two types of parameters included in possibility distributions. The parameters θ_l and θ_r describe the degree of uncertainty that fuzzy coefficients take their values, while the parameter α represents the possibility level in the supports of fuzzy coefficients. The proposed VaR reduction method generalizes some existing reduction methods by introducing possibility level parameter in the possibility distribution. The decision makers may adjust the values of distribution parameters according to their attitudes towards risk.

Fuzzy possibility theory is a fertile area of research. Although some issues have been resolved in this paper, some new ones have been exposed. In our future research, we will further study the properties of parametric possibility distributions like skewness and kurtosis, and consider the potential applications of the semideviations to some practical risk management and engineering optimization problems.

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