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The dual simplex method and sensitivity analysis for fuzzy linear programming with symmetric trapezoidal numbers

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Abstract In this paper, we first extend the dual simplex method to a type of fuzzy linear programming problem involving symmetric trapezoidal fuzzy numbers. The results obtained lead to a solution for fuzzy linear programming problems that does not require their conversion into crisp linear programming problems. We then study the ranges of values we can achieve so that when changes to the data of the problem are introduced, the fuzzy optimal solution remains invariant. Finally, we obtain the optimal value function with fuzzy coefficients in each case, and the results are described by means of numerical examples.

Keywords Fuzzy linear programming · Dual simplex method · Fuzzy trapezoidal numbers · Sensitivity analysis

1 Introduction

Bellman and Zadeh (1970) first proposed the basic concepts of fuzzy decision making. Based on these concepts, Zimmermann (1978) formulated Fuzzy Linear Programming (FLP) problems by the use of both the minimum operator, which is noncompensatory, and the product operator, which is compensatory. Subsequently, Tanaka et al. (1973) made use of this concept in mathematical programming. A formulation of FLP with fuzzy constraints and a solution method was put forward by Tanaka and Asai (1984).

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Delgado et al. (1989) studied a general model for FLP problems which included fuzziness both in the coefficients and also in the accomplishment of the constraints. Campos and Verdegay (1989) used the concept of comparison of fuzzy numbers for solving FLP problems. In effect, most convenient methods are based on the concept of comparison of fuzzy numbers by use of ranking functions (Fortemps and Roubens 1996; Maleki 2002). In such methods, authors define a crisp model which is equivalent to the FLP problem and then use the optimal solution of the model as that of the FLP problem. A review of some common methods for ranking fuzzy numbers can be seen in Wang and Kerre (2001). Vasant (2003) proposed a kind of FLP problem based methodology using a specific membership function, as modified logistic membership function. Some authors consider various types of single and multi-objective linear programming problems in which the variables and the right-hand-sides of the constraints are fuzzy parameters (Ganesan and Veeramani 2006; Maleki 2002). Mahdavi-Amiri and Nasseri (2007) proposed a dual simplex algorithm directly using the primal simplex table for solving linear programming problems with trapezoidal fuzzy variables. Ganesan and Veeramani (2006) introduced a type of fuzzy arithmetic for symmetric trapezoidal fuzzy numbers and then proposed a primal simplex method for solving fuzzy linear programming problems without converting them to crisp linear programming problems.

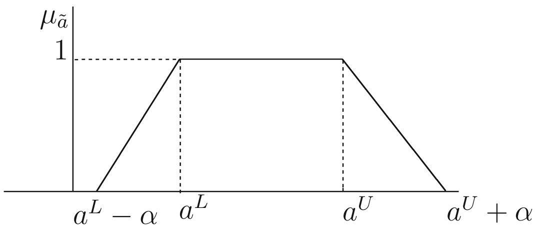
The study of duality theory for fuzzy parameter linear programming problems has attracted a number of researchers in fuzzy decision theory. The duality of fuzzy parameter linear programming was first studied by Rodder and Zimmermann (1980). Verdegay (1954) defined the fuzzy dual problem with the help of parametric linear programming and showed that fuzzy primal and dual problems both have the same fuzzy solution under certain suitable conditions. Bector and Chandra (2002) discussed duality in fuzzy linear programming based on a modification of the dual formulation put forward by Rodder and Zimmermann. Nasseri et al. (2010) discussed a concept of duality for fuzzy linear programming problems introduced by Ganesan and Veeramani, and derived the weak and strong duality theorems.

Sensitivity analysis is a basic tool for studying perturbations in optimization problems, and is considered to be one of the most interesting research areas in the field of FLP problems. Sensitivity analysis in FLP was first considered by Hamacher et al. (1978), who derived a functional relationship between changes of parameter on the right-hand-side and those of the optimal value of the primal objective function, for almost all conceivable cases. Fuller (1989) showed that the solution to FLP problems with symmetrical triangular fuzzy numbers is stable with respect to small changes to the centers of fuzzy numbers. Perturbations occur due to calculation errors or simply when answering ”What if …?” management questions.

In this paper, we first extend the dual simplex method to a type of fuzzy linear programming problem involving symmetric trapezoidal fuzzy numbers, without converting them to crisp linear programming problems. We then study sensitivity analysis for these problems and derive bounds for the values of the parameters when the data are perturbed, while the fuzzy optimal solution remains invariant.

The paper is organized as follows: In Sect. 2, we briefly recall some necessary concepts of fuzzy set theory. We then review a kind fuzzy linear programming problems with symmetric trapezoidal fuzzy numbers. We devote Sect. 3 to extension of the

Fig. 1 A symmetric trapezoidal fuzzy number.



dual simplex method for these problems and explain it by some illustrative examples. In Sect. 4 we study sensitivity analysis when given fuzzy optimal solution remains invariant. Finally, we conclude the paper in Sect. 5.

2 Preliminaries

In this section we present some notations, notions and results (Ganesan and Veeramani 2006) that will be useful in dealing with the issues addressed in this paper.

Definition 1 A fuzzy number on $\mathbb{R}$(real line) is said to be a symmetric trapezoidal fuzzy number if there exist real numbers a^L and a^U, $a^L \leq a^U$ and $\alpha > 0$, such that

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x}{\alpha} + \dfrac{\alpha - a^L}{\alpha}, & x \in [a^L - \alpha, a^L]; \\ 1, & x \in [a^L, a^U]; \\ \dfrac{-x}{\alpha} + \dfrac{a^U + \alpha}{\alpha}, & x \in [a^U, a^U + \alpha]; \\ 0, & \text{otherwise.} \end{cases}$$

We denote a symmetric trapezoidal fuzzy number $\tilde{a}$ by $\tilde{a} = (a^L, a^U, \alpha, \alpha)$, where $(a^L - \alpha, a^U + \alpha)$ is the support of $\tilde{a}$ and $[a^L, a^U]$ its core, and the set of all symmetric trapezoidal fuzzy numbers by $\mathcal{F}(\mathbb{R})$. $\mu_{\tilde{a}}(x)$ is called a membership function of $\tilde{a}$. The fuzzy number with the above membership function is shown in Fig. 1.

Let $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and $\tilde{b} = (b^L, b^U, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. The arithmetical operations on $\tilde{a}$ and $\tilde{b}$ are as follows:

1. $x > 0, \quad x \in R; \quad x\tilde{a} = (xa^L, xa^U, x\alpha, x\alpha)$,
2. $x < 0, \quad x \in R; \quad x\tilde{a} = (xa^U, xa^L, -x\alpha, -x\alpha)$,
3. $\tilde{a} + \tilde{b} = (a^L + b^L, a^U + b^U, \alpha + \beta, \alpha + \beta)$.
4. $\tilde{a}\tilde{b} = \Big(\left(\frac{a^L+a^U}{2}\right)\left(\frac{b^L+b^U}{2}\right) - w, \left(\frac{a^L+a^U}{2}\right)\left(\frac{b^L+b^U}{2}\right) + w, |a^U\beta + b^U\alpha|, |a^U\beta + b^U\alpha|\Big)$,

   where $w = \dfrac{h-k}{2}, \quad k = \min(a^Lb^L, a^Lb^U, a^Ub^L, a^Ub^U),$
   $h = \max(a^Lb^L, a^Lb^U, a^Ub^L, a^Ub^U).$

Definition 2 Let $\tilde{a} = (a^L, a^U, \alpha, \alpha)$ and $\tilde{b} = (b^L, b^U, \beta, \beta)$ be two symmetric trapezoidal fuzzy numbers. Define the relation as

$\tilde{a} \preceq \tilde{b}$ if and only if either $\frac{(a^L - \alpha) + (a^U + \alpha)}{2} < \frac{(b^L - \beta) + (b^U + \beta)}{2}$ that is

$\frac{a^L + a^U}{2} < \frac{b^L + b^U}{2}$ (in this case, we can also write $\tilde{a} \prec \tilde{b}$)
or

$\frac{a^L + a^U}{2} = \frac{b^L + b^U}{2}, b^L < a^L$ and $a^U < b^U$
or

$\frac{a^L + a^U}{2} = \frac{b^L + b^U}{2}, b^L = a^L, a^U = b^U$ and $\alpha \leq \beta$,
(in the last two cases, we can also write $\tilde{a} \approx \tilde{b}$ and say that $\tilde{a}$ and $\tilde{b}$ are equivalent).

Remark 3 Two symmetric trapezoidal fuzzy numbers $(a^L, a^U, \alpha, \alpha)$, (b^L, b^U, β, β) are equivalent if and only if

$$\frac{a^L + a^U}{2} = \frac{b^L + b^U}{2}.$$

In this case, we simply write $(a^L, a^U, \alpha, \alpha) \approx (b^L, b^U, \beta, \beta)$ and it is to be noted that a^L need not be equal to b^L or a^U need not be equal to b^U, but $(a^L, a^U, \alpha, \alpha) - (b^L, b^U, \beta, \beta) \approx (-h, h, \alpha + \beta, \alpha + \beta)$, where $h = (b^U - a^L) \geq 0$.

2.1 Fuzzy linear programming

In this section, we consider the Basic Feasible Solutions and the Complementary Slackness Conditions for fuzzy linear programming with trapezoidal fuzzy numbers, which were introduced in Ganesan and Veeramani (2006). Consider a primal fuzzy linear programming problem:

$$\begin{aligned} \max\ & \tilde{z} \simeq \tilde{c}\tilde{x} \\ \text{s.t.}\ & A\tilde{x} \simeq \tilde{b} \qquad\qquad (FLP) \\ & \tilde{x} \succeq \tilde{0}, \end{aligned}$$

with its associated dual problem (Nasseri et al. 2010)

$$\begin{aligned} \min\ & \tilde{w}_0 \simeq \tilde{w}\tilde{b} \\ \text{s.t.}\ & \tilde{w}A \succeq \tilde{c}, \qquad\qquad (FLD) \end{aligned}$$

where $\tilde{b} \in (\mathcal{F}(\mathbb{R}))^m$, $A \in \mathbb{R}^{m \times n}$, $\tilde{c}^T \in (\mathcal{F}(\mathbb{R}))^n$ are given data, and $\tilde{x} \in (\mathcal{F}(\mathbb{R}))^n$, $\tilde{w}^T \in (\mathcal{F}(\mathbb{R}))^m$ are the primal and dual variables, respectively, and $\simeq$, $\preceq$ are the fuzzy order relations.

Definition 4 A fuzzy vector $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T \in (\mathcal{F}(\mathbb{R}))^n$, where each $\tilde{x}_i \in \mathcal{F}(\mathbb{R})$, is called a fuzzy feasible solution to (FLP) if $\tilde{x} \succeq \tilde{0}$ satisfies the constraints $A\tilde{x} \simeq \tilde{b}$.

Definition 5 Let Q be the set of all fuzzy feasible solutions of (FLP). A fuzzy feasible solution $\tilde{x}^* \in Q$ is said to be a fuzzy optimal solution to (FLP) if $\tilde{c}\tilde{x} \preceq \tilde{c}\tilde{x}^*$ for all $\tilde{x} \in Q$.

Definition 6 Let A be the coefficient matrix of the FLP problem with full row rank and B be a nonsingular sub-matrix $m \times m$ of A. Let $\{B_1, \ldots, B_m\} \subset \{1, \ldots, n\}$ denote the index set of the columns of matrix B. Let $N = \{1, 2, \ldots, n\} \setminus B$. In this case, vector $\tilde{x} \simeq (\tilde{x}_B^T, \tilde{x}_N^T)^T \simeq (B^{-1}\tilde{b}, \tilde{0})$ is called a basic solution. If $\tilde{x}_B \succeq \tilde{0}$, then the fuzzy basic solution $\tilde{x}$ is called a fuzzy basic feasible solution and the corresponding fuzzy objective value will be $\tilde{z} \simeq \tilde{c}_B\tilde{x}_B$, in which $\tilde{c}_B \simeq (\tilde{c}_{B_1}, \ldots, \tilde{c}_{B_m})$.

Let $A_{.j}$ denote the jth column of A, and $\tilde{x}$ and $\tilde{w}$ the primal and dual fuzzy feasible solutions of problems (FLP) and (FLD), respectively. It should be remembered that a necessary and sufficient condition for $\tilde{x}$ and $\tilde{w}$ to be optimal is (Theorem 5.4, Nasseri et al. 2010)

$$\tilde{w}_i(\tilde{b}_i - A_{i.}\tilde{x}) \simeq \tilde{0}, \quad i = 1, 2, \ldots, m \tag{1}$$

$$(\tilde{c}_j - \tilde{w}A_{.j})\tilde{x}_j \simeq \tilde{0}, \quad j = 1, 2, \ldots, n \tag{2}$$

where $A_{i.}$ is the ith row of A, and (1) and (2) are referred to as complementary slackness conditions.

3 Dual simplex method

Consider the (FLP) problem. Suppose that a basic solution for (FLP) is given by $\tilde{x}_B \simeq B^{-1}\tilde{b}$ and $\tilde{x}_N \simeq \tilde{0}$, with the basis matrix B. Now let $\tilde{z}_j \simeq \tilde{c}_B B^{-1}A_{.j}$, $\tilde{y}_0 \simeq B^{-1}\tilde{b}$, where $\tilde{c}_B \simeq (\tilde{c}_{B_1}, \ldots, \tilde{c}_{B_m})$ and $A_{.j}$ is the jth column of the coefficient matrix A. Consider Tabel 1, where $\tilde{x}_{B_r}$ is the rth fuzzy basic variable and $y_j = B^{-1}A_{.j}$

Suppose that for $j = 1, \ldots, n$, we have

$$\tilde{z}_j - \tilde{c}_j \simeq \tilde{c}_B B^{-1}A_{.j} - \tilde{c}_j \succeq \tilde{0}, \tag{3}$$

that is, the optimality condition of the primal fuzzy problem (FLP) at $\tilde{x}$ holds true. we define $\tilde{w} \simeq \tilde{c}_B B^{-1}$, where $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_m)$. In this way, from (3), we have

$$\tilde{w}A \succeq \tilde{c},$$

that is, $\tilde{w}$ is a dual fuzzy feasible solution. If $\tilde{y}_{0r} \succeq \tilde{0}$, for all $r = 1, \ldots, m$, then we can obtain a fuzzy feasible solution for the (FLP) problem. Moreover, we will have

$$\tilde{c}\tilde{x} \simeq \tilde{c}_B\tilde{x}_B \simeq \tilde{c}_B B^{-1}\tilde{b} \simeq \tilde{w}\tilde{b}, \tag{4}$$

and thus, by Corollary 5.1 in Nasseri et al. (2010), establish the optimality of $\tilde{x}$ and $\tilde{w}$ for the (FLP) and (FLD), respectively. Therefore, we have the following result.

Table 1

Basis	…	$\tilde{x}_k$	…	$\tilde{x}_j$	…	$\tilde{x}_{B_1}$	…	$\tilde{x}_{B_r}$	…	$\tilde{x}_{B_m}$	R.H.S
$\tilde{z}$	…	$\tilde{z}_k - \tilde{c}_k$	…	$\tilde{z}_j - \tilde{c}_j$	…	0	…	0	…	0	$\tilde{c}_B B^{-1}\tilde{b}$
$\tilde{x}_{B_1}$	…	y_{1k}	…	y_{1j}	…	1	…	0	…	0	$\tilde{y}_{01}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\tilde{x}_{B_r}$	…	y_{rk}	…	y_{rj}	…	0	…	1	…	0	$\tilde{y}_{0r}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\tilde{x}_{B_m}$	…	y_{mk}	…	y_{mj}	…	0	…	0	…	1	$\tilde{y}_{0m}$

Corollary 7 *The optimality criteria $\tilde{z}_j - \tilde{c}_j \succeq \tilde{0}$ for all j, for the (FLP) problem is equivalent to the feasibility condition for the (FLD) problem. If, in addition, $\tilde{x}$ corresponding to a basis B is primal fuzzy feasible then $\tilde{x}$ is optimal for the (FLP) problem and $\tilde{w} \simeq \tilde{c}_B B^{-1}$ is optimal to the (FLD) problem.*

Now, assume that the (FLD) problem is feasible and $\tilde{x}$, corresponding to a basis B, is dual feasible but primal infeasible. That is, we have

$$\tilde{z}_j - \tilde{c}_j \succeq \tilde{0}, \ \ j = 1, 2, \ldots, n,$$

and there exists at least one r such that $\tilde{y}_{0r} \prec \tilde{0}$. Thus, according to duality theory, the (FLP) problem can be either infeasible (in which case, the (FLD) problem is unbounded), or it has an optimal solution. Next we will show how to work on row r of the above table corresponding to basis B, as the pivoting row, and either (1) detect the infeasibility of the (FLP) problem (or unboundedness of the (FLD) problem), or (2) find a column ℓ, as a pivoting column, to pivot on $y_{r\ell}$ and obtain a new dual feasible table with a non-increasing primal objective value. We explain these cases below.

Theorem 8 *If in a dual feasible simplex table an r exists such that $\tilde{y}_{0r} \prec \tilde{0}$ and $y_{rj} \geq 0$, for all j, then the (FLP) problem is infeasible.*

Proof Suppose that Table 1 is a dual feasible table, and an r exists such that $\tilde{y}_{0r} \prec \tilde{0}$ and $y_{rj} \geq 0$ for all j. Corresponding to the rth row of the table, we have

$$\tilde{x}_{B_r} + \sum_{j \in N} y_{rj}\tilde{x}_j \simeq \tilde{y}_{0r}.$$

Since, by assumption, $y_{rj} \geq 0$, $j \in N$ and $\tilde{x}_j \succeq \tilde{0}$, then $\tilde{x}_{B_r} + \sum_{j \in N} y_{rj}\tilde{x}_j \succeq \tilde{0}$ for any fuzzy basic feasible solution. However, $\tilde{y}_{0r} \prec \tilde{0}$ and this shows that the (FLP) problem is infeasible. □

Theorem 9 *If in a dual feasible simplex table, an r exists such that $\tilde{y}_{0r} \prec \tilde{0}$ and there exists a nonbasic index $k \in N$ such that $y_{rk} < 0$, then pivoting on y_{rk} will yield a dual feasible table with a corresponding non-increasing objective value.*

Proof Pivoting on the pivot y_{rk} will result in the new objective row as follows:

$$\tilde{z}_j - \tilde{c}_j - \frac{y_{rj}}{y_{rk}}(\tilde{z}_k - \tilde{c}_k), \quad j \in N. \tag{5}$$

For the new table to be dual feasible we need to have

$$\tilde{z}_j - \tilde{c}_j - \frac{y_{rj}}{y_{rk}}(\tilde{z}_k - \tilde{c}_k) \succeq \tilde{0}, \quad j \in N, \tag{6}$$

which results in

$$\frac{\tilde{z}_j - \tilde{c}_j}{y_{rj}} \preceq \frac{\tilde{z}_k - \tilde{c}_k}{y_{rk}}, \quad y_{rj} < 0. \tag{7}$$

To satisfy (7), it is sufficient to let

$$\frac{\tilde{z}_k - \tilde{c}_k}{y_{rk}} \simeq \max\left\{\frac{\tilde{z}_j - \tilde{c}_j}{y_{rj}} \mid \ y_{rj} < 0\right\}. \tag{8}$$

We note that the new objective value is non-increasing, since

$$\tilde{c}_B B^{-1}\tilde{b} - \frac{\tilde{y}_{0r}}{y_{rk}}(\tilde{z}_k - \tilde{c}_k) \preceq \tilde{c}_B B^{-1}\tilde{b},$$

based on the fact that

$$\tilde{y}_{0r} \prec \tilde{0}, \ y_{rk} < 0 \ \text{ and } \tilde{z}_k - \tilde{c}_k \succeq \tilde{0}.$$

□

Now, using the above results, we introduce a new dual algorithm to solve the (FLP) problem directly, making use of the dual feasible simplex table. Thus, we refer to the new algorithm as a dual simplex method.

Algorithm: a dual simplex method
(Dual feasibility) Let B be a basis for the (FLP) problem such that $\tilde{z}_j - \tilde{c}_j \succeq \tilde{0}$ for all j.
Compute the simplex table.
If $\tilde{y}_0 \succeq \tilde{0}$ then **Stop** (the current solution is optimal)
or else select the pivot row r with $\tilde{y}_{0r} \prec \tilde{0}$.
If $y_{rj} \geq 0$ for all j then **Stop** (the primal (FLP) is infeasible)
or else select the pivot column k by means of the following maximum ratio test:

$$\frac{\tilde{z}_k - \tilde{c}_k}{y_{rk}} \simeq \max\left\{\frac{\tilde{z}_j - \tilde{c}_j}{y_{rj}} \mid \ y_{rj} < 0\right\}.$$

Pivot on y_{rk} and **go to** step 1.

For an illustration of the dual simplex method we consider the following example.

Example 1

$$\begin{aligned} \max \tilde{z} \simeq -(13, 15, 2, 2)\tilde{x}_1 - (12, 14, 3, 3)\tilde{x}_2 - (15, 17, 2, 2)\tilde{x}_3 \\ \text{s.t.} \quad 2\tilde{x}_1 + 3\tilde{x}_2 + 2\tilde{x}_3 \succeq (45, 55, 6, 6) \\ 4\tilde{x}_1 \qquad + 3\tilde{x}_3 \succeq (60, 80, 8, 8) \\ 2\tilde{x}_1 + 5\tilde{x}_2 \qquad \succeq (65, 95, 5, 5) \\ \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3 \succeq \tilde{0}. \end{aligned}$$

We may write the first dual feasible simplex table as follows:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S*
$\tilde{z}$	(13, 15, 2, 2)	(12, 14, 3, 3)	(15, 17, 2, 2)	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$
$\tilde{x}_4$	−2	−3	−2	1	0	0	(−55, −45, 6, 6)
$\tilde{x}_5$	−4	0	−3	0	1	0	(−80, −60, 8, 8)
$\tilde{x}_6$	−2	−5	0	0	0	1	(−95, −65, 5, 5)

Since $\tilde{y}_{03} \prec \tilde{0}$, thus $\tilde{x}_6$ is a leaving variable and

$$\max\left\{\frac{\tilde{z}_j - \tilde{c}_j}{y_{3j}} : y_{3j} < 0\right\} \simeq \max\left\{\left(\frac{-15}{2}, \frac{-13}{2}, 1, 1\right), \left(\frac{-14}{5}, \frac{-12}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\} \\ \simeq \left(\frac{-14}{5}, \frac{-12}{5}, \frac{3}{5}, \frac{3}{5}\right),$$

thus $\tilde{x}_2$ is an entering variable. The new table is:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S*
$\tilde{z}$	$\left(\frac{37}{5}, \frac{51}{5}, \frac{16}{5}, \frac{16}{5}\right)$	$\tilde{0}$	(15, 17, 2, 2)	$\tilde{0}$	$\tilde{0}$	$\left(\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5}\right)$	(−263, −153, 71, 71)
$\tilde{x}_4$	$-\frac{4}{5}$	0	−2	1	0	$-\frac{3}{5}$	(−16, 12, 9, 9)
$\tilde{x}_5$	−4	0	−3	0	1	0	(−80, −60, 8, 8)
$\tilde{x}_2$	$\frac{2}{5}$	1	0	0	0	$-\frac{1}{5}$	(13, 19, 1, 1)

$\tilde{x}_5$ is a leaving variable and $\tilde{x}_1$ is an entering variable. The next table is as follows:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S*
$\tilde{z}$	$\tilde{0}$	$\tilde{0}$	$\left(\frac{147}{20}, \frac{229}{20}, \frac{22}{5}, \frac{22}{5}\right)$	$\tilde{0}$	$\left(\frac{37}{20}, \frac{51}{20}, \frac{4}{5}, \frac{4}{5}\right)$	$\left(\frac{12}{5}, \frac{14}{5}, \frac{3}{5}, \frac{3}{5}\right)$	$\left(-\frac{951}{2}, -\frac{499}{2}, \frac{157}{5}, \frac{157}{5}\right)$
$\tilde{x}_4$	0	0	$-\frac{7}{5}$	1	$-\frac{1}{5}$	$-\frac{3}{5}$	$\left(-4, 28, \frac{48}{5}, \frac{48}{5}\right)$
$\tilde{x}_1$	1	0	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	(15, 20, 2, 2)
$\tilde{x}_2$	0	1	$-\frac{3}{10}$	0	$\frac{1}{10}$	$-\frac{1}{5}$	$\left(5, 13, \frac{9}{5}, \frac{9}{5}\right)$

Therefore, the optimal solution of the (FLP) problem obtained by the dual method is $\tilde{x}_1 \simeq (15, 20, 2, 2)$, $\tilde{x}_2 \simeq \left(5, 13, \frac{9}{5}, \frac{9}{5}\right)$, $\tilde{x}_3 \simeq \tilde{0}$ with the optimal value

$$\tilde{z} \simeq \left(-\frac{951}{2}, -\frac{499}{2}, \frac{157}{5}, \frac{157}{5}\right).$$

Example 2

$$\begin{aligned} \max \tilde{z} \simeq (-1,-1,1,1)\tilde{x}_1 + (-4,-4,4,4)\tilde{x}_2 + (-2,-1,3,3)\tilde{x}_3 + (1,1,1,1)\tilde{x}_4 \\ \text{s.t.} \quad \tilde{x}_1 - 2\tilde{x}_2 + \tilde{x}_3 - \tilde{x}_4 \succeq (-2,-2,3,3) \\ 2\tilde{x}_1 + \tilde{x}_2 + 2\tilde{x}_3 - 2\tilde{x}_4 \simeq (2,3,1,1) \\ \tilde{x}_1 \qquad - 3\tilde{x}_3 + \tilde{x}_4 \succeq (1,4,3,3) \\ \tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3 \succeq \tilde{0}. \end{aligned}$$

We may write the first dual feasible simplex table as follows:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S*
$\tilde{z}$	(0, 0, 2, 2)	$\left(\frac{7}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}\right)$	(0, 1, 4, 4)	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\left(-\frac{3}{2}, -1, \frac{1}{2}, \frac{1}{2}\right)$
$\tilde{x}_5$	0	$\frac{5}{2}$	0	0	1	0	$\left(3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}\right)$
$\tilde{x}_4$	−1	$-\frac{1}{2}$	−1	1	0	0	$\left(-\frac{3}{2}, -1, \frac{1}{2}, \frac{1}{2}\right)$
$\tilde{x}_6$	−2	$-\frac{1}{2}$	2	0	0	1	$\left(-\frac{11}{2}, -2, \frac{7}{2}, \frac{7}{2}\right)$

Since $\tilde{y}_{03} \prec \tilde{0}$, thus $\tilde{x}_6$ is a leaving variable and

$$\begin{aligned} \max \left\{ \frac{\tilde{z}_j - \tilde{c}_j}{y_{3j}} : y_{3j} < 0 \right\} &\simeq \max\{(0,0,1,1), (-7,-7,9,9)\} \\ &\simeq (0,0,1,1), \end{aligned}$$

thus $\tilde{x}_1$ is an entering variable. The new table is:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	*R.H.S*
$\tilde{z}$	$\tilde{0}$	$\left(\frac{7}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}\right)$	(0, 1, 6, 6)	$\tilde{0}$	$\tilde{0}$	(0, 0, 1, 1)	$\left(-\frac{39}{8}, -\frac{17}{8}, \frac{17}{4}, \frac{17}{4}\right)$
$\tilde{x}_5$	0	$\frac{5}{2}$	0	0	1	0	$\left(3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}\right)$
$\tilde{x}_4$	0	$-\frac{1}{4}$	−2	1	0	$-\frac{1}{2}$	$\left(-\frac{1}{4}, \frac{3}{2}, \frac{7}{4}, \frac{7}{4}\right)$
$\tilde{x}_1$	1	$\frac{1}{4}$	−1	0	0	$-\frac{1}{2}$	$\left(1, \frac{11}{4}, \frac{7}{4}, \frac{7}{4}\right)$

Therefore, the optimal solution of the (FLP) problem obtained by the dual method is $\tilde{x}_1 \simeq \left(1, \frac{11}{4}, \frac{7}{4}, \frac{7}{4}\right)$, $\tilde{x}_2 \simeq \tilde{0}$, $\tilde{x}_3 \simeq \tilde{0}$ and $\tilde{x}_4 \simeq \left(-\frac{1}{4}, \frac{3}{2}, \frac{7}{4}, \frac{7}{4}\right)$ with the optimal value

$$\tilde{z} \simeq \left(-\frac{39}{8}, -\frac{17}{8}, \frac{17}{4}, \frac{17}{4}\right).$$

4 Sensitivity analysis

Consider the primal problem (FLP). Suppose that the simplex method (Ganesan and Veeramani 2006) produced an optimal basis B. We shall describe how to make use of the optimality conditions (Theorem 2.4 in Ganesan and Veeramani 2006) in order to find a new optimal solution if some of the problem data change without resolving the problem from scratch. In particular the following variations in the primal problem will be considered:

- change in the cost vector $\tilde{c}$,
- change in the right hand side vector $\tilde{b}$,
- change in the constraint matrix A,
- addition of a new activity (symmetric trapezoidal fuzzy variable),
- addition of a new constraint.

4.1 Change in the cost vector $\tilde{c}$

Given an optimal basic feasible solution, suppose that the cost coefficient of the fuzzy variable $\tilde{x}_k$ is changed from $\tilde{c}_k$ to $\tilde{c}'_k$, so that $\tilde{c}'_k := \tilde{c}_k + \lambda\tilde{\delta c}_k$. The effect of this change on the final table will occur in the cost row. We will determine the λ that make the old solution still optimal. Consider the following two separation cases:

Case 1 $\tilde{x}_k$ is a non-basic variable.
In this case $\tilde{c}_B$ is not affected, and hence $\tilde{z}_j := \tilde{c}_B B^{-1} A_{.j}$ is not changed for any j. Thus $\tilde{z}_k - \tilde{c}_k$ is replaced by $\tilde{z}_k - \tilde{c}'_k$. Now, to preserve optimality, we must have

$$\tilde{z}_k - \tilde{c}'_k = \tilde{c}_B B^{-1} A_{.k} - \tilde{c}_k - \lambda\tilde{\delta c}_k \simeq \tilde{z}_k - \tilde{c}_k - \lambda\tilde{\delta c}_k \succeq \tilde{0},$$

this implies, by Definition of the relation $\preceq$,

$$\lambda \begin{cases} \geq \dfrac{(\tilde{z}_k - \tilde{c}_k)^L + (\tilde{z}_k - \tilde{c}_k)^U}{(\tilde{\delta c}_k)^L + (\tilde{\delta c}_k)^U}, & \text{if } (\tilde{\delta c}_k)^L + (\tilde{\delta c}_k)^U < 0 \\ \leq \dfrac{(\tilde{z}_k - \tilde{c}_k)^L + (\tilde{z}_k - \tilde{c}_k)^U}{(\tilde{\delta c}_k)^L + (\tilde{\delta c}_k)^U}, & \text{if } (\tilde{\delta c}_k)^L + (\tilde{\delta c}_k)^U > 0 \end{cases} \tag{9}$$

Hence for any change in $\tilde{c}_k$, satisfying (9), the current optimal solution remains optimal and the value of the objective function also does not change since $\tilde{x}_k \simeq \tilde{0}$.

Case 2 $\tilde{x}_t$ is a basic variable, say $\tilde{x}_t := \tilde{x}_{B_k}$.
Let $\tilde{c}_{B_k}$ be replaced by $\tilde{c}'_{B_k} := \tilde{c}_{B_k} + \lambda\tilde{\delta c}_{B_k}$. In this case the evaluations of $\tilde{z}_j \simeq \tilde{c}_B B^{-1} A_{.j}$ for all non-basic variables are affected by any change in $\tilde{c}_k$ and we should have:

$$\begin{aligned}\tilde{z}'_j - \tilde{c}_j &\simeq \tilde{c}'_B B^{-1} A_{.j} - \tilde{c}_j \simeq (\tilde{c}_{B_1}, \ldots, \tilde{c}'_{B_k}, \ldots, \tilde{c}_{B_m}) B^{-1} A_{.j} - \tilde{c}_j \\ &\simeq \tilde{c}_B B^{-1} A_{.j} - \tilde{c}_j + (0, \ldots, \lambda \tilde{\delta c}_{B_k}, \ldots, 0) B^{-1} A_{.j} \\ &\simeq \tilde{z}_j - \tilde{c}_j + \lambda \tilde{\delta c}_{B_k} \sum_{i=1}^{m} \beta_{ki} A_{ij} \succeq \tilde{0}, \qquad j \in N\end{aligned}$$

where $B^{-1} = (\beta_{ij})$. This implies that

$$\lambda \begin{cases} \geq \dfrac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{\left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij}}, & \text{if } \left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \displaystyle\sum_{i=1}^m \beta_{ki} A_{ij} > 0 \\ \leq \dfrac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{\left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij}}, & \text{if } \left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \displaystyle\sum_{i=1}^m \beta_{ki} A_{ij} < 0 \end{cases} \tag{10}$$

Hence,

$$\begin{aligned}&\max_{j \in N} \left\{ \frac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{\left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij}} : \left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij} > 0 \right\} \leq \lambda \leq \\ &\quad \min_{j \in N} \left\{ \frac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{\left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij}} : \left((\tilde{\delta c}_{B_k})^L + (\tilde{\delta c}_{B_k})^U\right) \sum_{i=1}^m \beta_{ki} A_{ij} < 0 \right\}.\end{aligned}$$

Thus if the above relationship is satisfied, changes in $\tilde{c}_k$ will not affect the original optimal basis nor the value of the optimal solution. The only change will occur in the optimal value of the objective function $\tilde{z}$, and the new optimal value will be equal to:

$$\begin{aligned}\tilde{z}'_* &\simeq \tilde{c}'_B B^{-1} \tilde{b} \simeq \tilde{c}_B B^{-1} \tilde{b} + (0, \ldots, \lambda \widetilde{\delta c}_{B_k}, \ldots, 0) B^{-1} \tilde{b} \\ &\simeq \tilde{z}_* + \lambda \widetilde{\delta c}_{B_k} \sum_{i=1}^m \beta_{ki} \tilde{b}_i,\end{aligned}$$

which is a fuzzy linear function with respect to λ.

Example 3 Consider Example 2. Let $\tilde{c}_2 \simeq (-4, -4, 4, 4)$ be replaced to $\tilde{c}'_2 \simeq (-4, -4, 4, 4) + \lambda(3, 5, 6, 6)$. In this case, by using (9), we get

$$\lambda \leq \frac{7}{8}.$$

Now, suppose that $\tilde{c}_1 \simeq (-1, -1, 1, 1)$ be replaced to $\tilde{c}_1 \simeq (-1, -1, 1, 1) + \lambda(-2, 3, 4, 4)$. In this case,

$$\max\left\{-28, -\frac{1}{2}\right\} \leq \lambda \leq \min\{0\},$$

or equivalently

$$-\frac{1}{2} \leq \lambda \leq 0,$$

and the new value of the objective function is equal to:

$$\tilde{z}'_{*} \simeq \left(-\frac{39}{8}, -\frac{17}{8}, \frac{17}{4}, \frac{17}{4}\right) + \lambda\left(-\frac{95}{16}, \frac{125}{16}, \frac{65}{4}, \frac{65}{4}\right).$$

4.2 Change in the requirement vector $\tilde{b}$

Since the optimality condition, $\tilde{z}_j - \tilde{c}_j \succeq \tilde{0}, \forall j \in N$, does not depend on the requirement vector, any change in the requirement vector does not affect the optimality condition. It does, however, affect the values of the basic variables and hence the value of the objective function. Thus if the magnitude of the change in the requirement vector is such that it preserves the feasibility of the optimal basis, then the original optimal basis remains optimal.

Let the requirement vector $\tilde{b}$ be replaced by $\tilde{b}' \simeq \tilde{b} + \lambda\tilde{\delta b}$, where $\tilde{\delta b}$ is a constant fuzzy vector. Then $B^{-1}\tilde{b}$ will be replaced by $B^{-1}\tilde{b}'$. The new right-hand-side can be calculated without explicitly evaluating $B^{-1}\tilde{b}'$. This is evident by noting that:

$$B^{-1}\tilde{b}' \simeq B^{-1}\tilde{b} + \lambda B^{-1}\tilde{\delta b}. \tag{11}$$

To maintain the feasibility, we must have

$$B^{-1}\tilde{b} + \lambda B^{-1}\tilde{\delta b} \succeq \tilde{0},$$

which is equivalent to

$$\sum_{i=1}^{m} \beta_{hi}\tilde{b}_i + \lambda \sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i \succeq \tilde{0}, \quad h = 1, 2, \ldots, m.$$

The last relation implies that

$$\lambda \begin{cases} \geq -\dfrac{\left(\sum_{i=1}^{m} \beta_{hi}\tilde{b}_i\right)^L + \left(\sum_{i=1}^{m} \beta_{hi}\tilde{b}_i\right)^U}{\left(\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^L + \left(\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^U}, & \text{if } \left(\displaystyle\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^L + \left(\displaystyle\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^U > 0 \\ \leq -\dfrac{\left(\sum_{i=1}^{m} \beta_{hi}\tilde{b}_i\right)^L + \left(\sum_{i=1}^{m} \beta_{hi}\tilde{b}_i\right)^U}{\left(\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^L + \left(\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^U}, & \text{if } \left(\displaystyle\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^L + \left(\displaystyle\sum_{i=1}^{m} \beta_{hi}\tilde{\delta b}_i\right)^U < 0 \end{cases} \tag{12}$$

Thus the range for λ for which the optimal basis remains optimal is:

$$\max_{1\leq h\leq m}\left\{-\frac{\left(\sum_{i=1}^{m}\beta_{hi}\tilde{b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{b}_i\right)^U}{\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^U}:\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^U>0\right\}\leq\lambda\leq$$
$$\min_{1\leq h\leq m}\left\{-\frac{\left(\sum_{i=1}^{m}\beta_{hi}\tilde{b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{b}_i\right)^U}{\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^U}:\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^L+\left(\sum_{i=1}^{m}\beta_{hi}\tilde{\delta b}_i\right)^U<0\right\}. \tag{13}$$

The new solution of the problem is given by (11) and the value of the objective function is a fuzzy linear function with respect to λ:

$$\tilde{z}'_* \simeq \tilde{c}_B B^{-1}(\tilde{b}+\lambda\tilde{\delta b}) \simeq \tilde{c}_B B^{-1}\tilde{b}+\lambda\tilde{c}_B B^{-1}\tilde{\delta b} \simeq \tilde{z}_* + \lambda\tilde{c}_B B^{-1}\tilde{\delta b}.$$

Example 4 Consider Example 2. Let $\tilde{\delta b} \simeq ((4,2,1,1),(3,2,5,5),(1,2,3,3))^T$ be a perturbation direction, therefore by using (13) we get

$$\max\left\{-5,-\frac{15}{11}\right\}\leq\lambda\leq\min\left\{\frac{13}{7}\right\}.$$

Therefore, the stability range of the optimal solution is:

$$-\frac{15}{11}\leq\lambda\leq\frac{13}{7},$$

and the optimal value function in this region is as follows:

$$\tilde{z}(\lambda)\simeq\left(-\frac{39}{8},-\frac{17}{8},\frac{17}{4},\frac{17}{4}\right)+\lambda\left(-\frac{3}{2},-1,5,5\right).$$

4.3 Change in the coefficients matrix A

We now discuss the effect of changing some of the entries in the constraint matrix A. Two scenarios are possible, namely changes involving non-basic columns and changes involving basic columns.

Case 1 Change in the non-basic columns
Suppose that some of the non-basic columns $A_{.j},\ j\in N_1\subseteq N$ are replaced by $A'_{.j} := A_{.j}+\lambda\delta A_{.j},\ j\in N_1$, and that $\delta A_{.j}s$ are the perturbation vectors. Then the new updated columns are:

$$\begin{pmatrix}\tilde{c}_B B^{-1}A'_{.j}-\tilde{c}_j\\ B^{-1}A'_{.j}\end{pmatrix},\quad j\in N_1.$$

It is clear that the feasibility condition is not distributed. To preserve the optimality we must have:

$$\begin{aligned}\tilde{z}'_j - \tilde{c}_j &\simeq \tilde{c}_B B^{-1} A'_{.j} - \tilde{c}_j\\ &\simeq \tilde{c}_B B^{-1}(A_{.j} + \lambda\delta A_{.j}) - \tilde{c}_j\\ &\simeq (\tilde{z}_j - \tilde{c}_j) + \lambda\tilde{c}_B B^{-1}\delta A_{.j}\\ &\succeq \tilde{0}, \qquad j \in N_1.\end{aligned}$$

This implies that:

$$\lambda \begin{cases} \geq -\dfrac{(\tilde{z}_j - \tilde{c}_j)^L + (\tilde{z}_j - \tilde{c}_j)^U}{(\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U}, & \text{if } (\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U > 0,\\ \leq -\dfrac{(\tilde{z}_j - \tilde{c}_j)^L + (\tilde{z}_j - \tilde{c}_j)^U}{(\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U}, & \text{if } (\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U < 0.\end{cases} \tag{14}$$

Thus the range for λ for which the optimal basis remains optimal is:

$$\max_{j\in N_1}\left\{-\frac{(\tilde{z}_j - \tilde{c}_j)^L + (\tilde{z}_j - \tilde{c}_j)^U}{(\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U} : (\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U > 0\right\} \leq \lambda \leq$$

$$\min_{j\in N_1}\left\{-\frac{(\tilde{z}_j - \tilde{c}_j)^L + (\tilde{z}_j - \tilde{c}_j)^U}{(\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U} : (\tilde{c}_B B^{-1}\delta A_{.j})^L + (\tilde{c}_B B^{-1}\delta A_{.j})^U < 0\right\}.$$

Example 5 Consider Example 2. If $A'_{.2} = A_{.2} + \lambda\delta A_{.2}$ and $A'_{.3} = A_{.3} + \lambda\delta A_{.3}$ where $\delta A_{.2} = (1, 2, -1)^T$ and $\delta A_{.3} = (-3, 2, 2)^T$ then by using the above relationship, we have

$$\lambda \leq \frac{1}{2}.$$

Case 2 Change in the basic column
Here our goal is to determine the lower and upper bounds for λ which guarantee that the replacement $A_{.k}$ by $A'_{.k} := A_{.k} + \lambda\delta A_{.k}$, $k \in B$, does not affect the optimal basis, and the original optimal solution $\tilde{x}_*$ remains feasible and optimal. By undertaking this replacement, the optimal basis B will be replaced with $\overline{B} := B + \lambda\delta A_{.k} e_k^t$ where e_k is a unit vector. The inverse matrix $\overline{B}$ is:

$$\begin{aligned}\bar{B}^{-1} &= B^{-1} - \lambda\frac{B^{-1}\delta A_{.k} e_k^t B^{-1}}{1 + \lambda e_k^t B^{-1}\delta A_{.k}}\\ &= B^{-1} - \lambda\frac{B^{-1}\delta A_{.k}\beta_{k.}}{1 + \lambda\sum_{i=1}^m \beta_{ki}\delta A_{ik}}, \quad 1 + \lambda\sum_{i=1}^m \beta_{ki}\delta A_{ik} > 0,\end{aligned} \tag{15}$$

according to the Sherman-Morrison formulas, where $B^{-1} = (\beta_{ij})$ and $\beta_{k.}$ is the kth row B^{-1}. This change to the basis matrix will affect the feasibility of vector $\tilde{x}_*$. However, it may also affect the optimality condition and the optimal value of the objective function $\tilde{z}$. Therefore:

$$\begin{aligned}\tilde{x}_{\bar{B}} &\simeq \bar{B}^{-1}\tilde{b}\\ &\simeq \left(B^{-1} - \lambda \frac{B^{-1}\delta A_{.k}\beta_{k.}}{1+\lambda\sum_{i=1}^{m}\beta_{ki}\delta A_{ik}}\right)\tilde{b}\\ &\simeq \tilde{x}_B - \lambda \frac{B^{-1}\delta A_{.k}\beta_{k.}\tilde{b}}{1+\lambda\sum_{i=1}^{m}\beta_{ki}\delta A_{ik}}. \end{aligned} \tag{16}$$

Now the ith component of $\tilde{x}_{\overline{B}}$ is given by:

$$(\tilde{x}_{\overline{B}})_i \simeq \sum_{j=1}^{m}\beta_{ij}\tilde{b}_j - \lambda \frac{\sum_{j=1}^{m}\beta_{ij}\delta A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k}}, \quad i = 1, 2, \ldots, m. \tag{17}$$

This new basic solution $\tilde{x}_{\overline{B}}$ will be feasible if:

$$\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j - \lambda \frac{\sum_{j=1}^{m}\beta_{ij}\delta A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k}} \succeq \tilde{0}, \quad i = 1, 2, \ldots, m. \tag{18}$$

This implies:

$$\lambda\left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{j=1}^{m}\beta_{ij}\delta A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}\right) \succeq -\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j.$$

Hence to maintain feasibility, we must have:

$$\begin{aligned}\max_{1\le i\le m}&\left\{\frac{-\left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\right)^L - \left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\right)^U}{H_i} \, : \, H_i > 0\right\} \le \lambda \le \\ &\min_{1\le i\le m}\left\{\frac{-\left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\right)^L - \left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\right)^U}{H_i} \, : \, H_i < 0\right\},\end{aligned} \tag{19}$$

where

$$\begin{aligned}H_i = &\left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{j=1}^{m}\beta_{ij}\delta A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}\right)^L\\ &+\left(\sum_{j=1}^{m}\beta_{ij}\tilde{b}_j\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{j=1}^{m}\beta_{ij}\delta A_{jk}\sum_{j'=1}^{m}\beta_{kj'}\tilde{b}_{j'}\right)^U, \quad i = 1, 2, \ldots, m.\end{aligned}$$

Now, to preserve optimality, we must have:

$$\begin{aligned}\tilde{z}'_j - \tilde{c}_j &\simeq \tilde{c}_{\overline{B}}\overline{B}^{-1}A_{.j} - \tilde{c}_j \simeq \tilde{c}_B\left(B^{-1} - \lambda\frac{B^{-1}\delta A_{.k}\beta_{k.}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k}}\right)A_{.j} - \tilde{c}_j \\ &\simeq \tilde{z}_j - \tilde{c}_j - \lambda\frac{\sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m}\tilde{c}_{B_i}\beta_{ij'}\delta A_{j'k}\beta_{ki'}A_{i'j}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k}} \succeq \tilde{0}, \quad j\in N.\end{aligned} \tag{20}$$

Since $1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} > 0$, (20) reduces to:

$$\lambda\left((\tilde{z}_j - \tilde{c}_j)\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m}\tilde{c}_{B_i}\beta_{ij'}\delta A_{j'k}\beta_{ki'}A_{i'j}\right) \succeq -(\tilde{z}_j - \tilde{c}_j). \tag{21}$$

Hence in order to maintain the optimality of the new solution, λ must satisfy:

$$\max_{j\in N}\left\{\frac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{M_j} : M_j > 0\right\} \le \lambda \le \min_{j\in N}\left\{\frac{-(\tilde{z}_j - \tilde{c}_j)^L - (\tilde{z}_j - \tilde{c}_j)^U}{M_j} : M_j < 0\right\}, \tag{22}$$

where

$$\begin{aligned}M_j = &\left((\tilde{z}_j - \tilde{c}_j)\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m}\tilde{c}_{B_i}\beta_{ij'}\delta A_{j'k}\beta_{ki'}A_{i'j}\right)^L \\ &+\left((\tilde{z}_j - \tilde{c}_j)\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} - \sum_{i=1}^{m}\sum_{j'=1}^{m}\sum_{i'=1}^{m}\tilde{c}_{B_i}\beta_{ij'}\delta A_{j'k}\beta_{ki'}A_{i'j}\right)^U, \quad j\in N.\end{aligned}$$

Therefore, we have proved the following theorem:

Theorem 10 *If λ satisfies (19), (22) and $1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k} > 0$ then $\tilde{x}_*$ is an optimal solution to the perturbed problem.*

In the stability region of Theorem 10, the optimal value function is a fuzzy linear fractional function with symmetric trapezoidal fuzzy numbers as follows:

$$\tilde{z}(\lambda) \simeq \tilde{z}_* - \lambda\frac{\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{j'=1}^{m}\tilde{c}_{B_i}\beta_{ij}\delta A_{jk}\beta_{kj'}\tilde{b}_{j'}}{1+\lambda\sum_{i'=1}^{m}\beta_{ki'}\delta A_{i'k}}.$$

Example 6 Consider Example 2. If $A'_{.1} = A_{.1} + \lambda\delta A_{.1}$, where $\delta A_{.1} = (2, 0, -1)^T$, then by using Theorem 10 we obtain the following interval for λ:

$$1 \leq \lambda < 2,$$

and the optimal value function is a fuzzy linear fractional function as follows:

$$\tilde{z}(\lambda) \simeq \left(-\frac{39}{8}, -\frac{17}{8}, \frac{17}{4}, \frac{17}{4}\right) - \frac{2\lambda}{2-\lambda}\left(0, 0, \frac{11}{4}, \frac{11}{4}\right).$$

4.4 Adding a new activity

Suppose that a new activity $\tilde{x}_{n+1}$ with unit cost $\tilde{c}_{n+1}$ and consumption column $A_{.n+1}$ is considered for possible production. Without resolving the problem, we can easily determine wether producing $\tilde{x}_{n+1}$ is worthwhile or not.

It is obvious that the original optimal solution is feasible for the modified problem. It also remains optimal if $\tilde{z}_{n+1} - \tilde{c}_{n+1} \succeq \tilde{0}$. In this way, $\tilde{x}^*_{n+1} \simeq \tilde{0}$. If, however, $\tilde{z}_{n+1} - \tilde{c}_{n+1} \prec \tilde{0}$, then $\tilde{x}_{n+1}$ is introduced into the basis and the primal simplex method (Ganesan and Veeramani 2006) may be applied to find an optimal solution to the modified problem.

4.5 Adding a new constraint

Suppose that a new constraint is added to the problem after an optimal solution has already been obtained. If the optimal solution to the original problem satisfies the new constraint, it is also an optimal solution to the modified problem. If it does not satisfy the new constraint, a new optimal solution has to be found.

Suppose that B is the optimal basis before adding constraint $A_{m+1.}\tilde{x} \preceq \tilde{b}_{m+1}$. The corresponding problem to basis B is shown below:

$$\begin{aligned} \tilde{z} + (\tilde{z}_N - \tilde{c}_N)\tilde{x}_N &\simeq \tilde{c}_B B^{-1}\tilde{b} \\ \tilde{x}_B + B^{-1}N\tilde{x}_N &\simeq B^{-1}\tilde{b}. \end{aligned} \tag{23}$$

The constraint $A_{m+1.}\tilde{x} \preceq \tilde{b}_{m+1}$ is rewritten as $(A_{m+1.})_B\tilde{x}_B + (A_{m+1.})_N\tilde{x}_N + \tilde{x}_{n+1} \simeq \tilde{b}_{m+1}$, where $A_{m+1.}$ is decomposed into $((A_{m+1.})_B \ \ (A_{m+1.})_N)$ and $\tilde{x}_{n+1}$ is a slack variable. Multiplying the Eq. (23) by $(A_{m+1.})_B$ and subtracting from the new constraint gives the following system:

$$\begin{aligned} \tilde{z} + (\tilde{z}_N - \tilde{c}_N)\tilde{x}_N &\simeq \tilde{c}_B B^{-1}\tilde{b} \\ \tilde{x}_B + B^{-1}N\tilde{x}_N &\simeq B^{-1}\tilde{b} \\ \Big((A_{m+1.})_N - (A_{m+1.})_B B^{-1}N\Big)\tilde{x}_N + \tilde{x}_{n+1} &\simeq \tilde{b}_{m+1} - (A_{m+1.})_B B^{-1}\tilde{b}. \end{aligned}$$

These equations give us a basic solution of the new problem. The only possible violation of optimality of the new problem is the sign of $\tilde{b}_{m+1} - (A_{m+1.})_B B^{-1}\tilde{b}$, if

$\tilde{b}_{m+1} - (A_{m+1.})_B B^{-1}\tilde{b} \succeq \tilde{0}$, then the current solution is optimal. Otherwise, if $\tilde{b}_{m+1} - (A_{m+1.})_B B^{-1}\tilde{b} \prec \tilde{0}$ then the dual simplex method (Sect. 3) is used to restore feasibility.

Example 7 Consider Example 2. Suppose that the constraint $\tilde{x}_1 - 3\tilde{x}_2 + 2\tilde{x}_4 \preceq [-1, 3, 4, 4]$ is added to the problem, then

$$(A_{4.})_N - (A_{4.})_B B^{-1}N = \left(-\frac{11}{4}, 5, \frac{3}{2}\right),$$
$$\tilde{b}_4 - (A_{4.})_B B^{-1}\tilde{b} \simeq \left(-\frac{27}{4}, \frac{5}{2}, \frac{37}{4}, \frac{37}{4}\right).$$

So we have the following table:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	*R.H.S*
$\tilde{z}$	$\tilde{0}$	$\left(\frac{7}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}\right)$	$(0, 1, 6, 6)$	$\tilde{0}$	$\tilde{0}$	$(0, 0, 1, 1)$	$\tilde{0}$	$\left(-\frac{39}{8}, -\frac{17}{8}, \frac{17}{4}, \frac{17}{4}\right)$
$\tilde{x}_5$	0	$\frac{5}{2}$	0	0	1	0	0	$\left(3, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}\right)$
$\tilde{x}_4$	0	$-\frac{1}{4}$	-2	1	0	$-\frac{1}{2}$	0	$\left(-\frac{1}{4}, \frac{3}{2}, \frac{7}{4}, \frac{7}{4}\right)$
$\tilde{x}_1$	1	$\frac{1}{4}$	-1	0	0	$-\frac{1}{2}$	0	$\left(1, \frac{11}{4}, \frac{7}{4}, \frac{7}{4}\right)$
$\tilde{x}_7$	0	$-\frac{11}{4}$	5	0	0	$\frac{3}{2}$	1	$\left(-\frac{27}{4}, \frac{5}{2}, \frac{37}{4}, \frac{37}{4}\right)$

Since $\tilde{y}_{04} \prec \tilde{0}$, thus $\tilde{x}_7$ is a leaving variable and $\tilde{x}_2$ is an entering variable. The new table is:

Basis	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{x}_7$	*R.H.S*
$\tilde{z}$	$\tilde{0}$	$\tilde{0}$	$\left(\frac{70}{11}, \frac{81}{11}, \frac{146}{11}, \frac{146}{11}\right)$	$\tilde{0}$	$\tilde{0}$	$\left(\frac{21}{11}, \frac{21}{11}, \frac{35}{11}, \frac{35}{11}\right)$	$\left(\frac{14}{11}, \frac{14}{11}, \frac{18}{11}, \frac{18}{11}\right)$	[]
$\tilde{x}_5$	0	0	$\frac{50}{11}$	0	1	$\frac{15}{11}$	$\frac{10}{11}$	$\left(-\frac{34}{11}, \frac{61}{11}, \frac{126}{11}, \frac{126}{11}\right)$
$\tilde{x}_4$	0	0	$-\frac{27}{11}$	1	0	$-\frac{7}{11}$	$-\frac{1}{11}$	$\left(-\frac{5}{11}, \frac{23}{11}, \frac{28}{11}, \frac{28}{11}\right)$
$\tilde{x}_1$	1	0	$-\frac{6}{11}$	0	0	$-\frac{4}{11}$	$\frac{1}{11}$	$\left(\frac{9}{11}, \frac{28}{11}, \frac{19}{11}, \frac{19}{11}\right)$
$\tilde{x}_2$	0	1	$-\frac{20}{11}$	0	0	$-\frac{6}{11}$	$-\frac{4}{11}$	$\left(-\frac{9}{11}, \frac{26}{11}, \frac{35}{11}, \frac{35}{11}\right)$

Therefore, the new optimal solution is:

$$\tilde{x}_1 \simeq \left(\frac{9}{11}, \frac{28}{11}, \frac{19}{11}, \frac{19}{11}\right), \tilde{x}_2 \simeq \left(-\frac{9}{11}, \frac{26}{11}, \frac{35}{11}, \frac{35}{11}\right), \tilde{x}_3 \simeq \tilde{0} \text{ and}$$
$$\tilde{x}_4 \simeq \left(-\frac{5}{11}, \frac{23}{11}, \frac{28}{11}, \frac{28}{11}\right).$$

5 Concluding remarks

In this paper, we introduced a new approach based on dual simplex method to obtain the fuzzy optimal solution of a kind of fuzzy linear programming problems with symmetric trapezoidal fuzzy numbers, without converting them to crisp linear programming problems. Then, we studied the sensitivity analysis for these problems when the data are perturbed, while the fuzzy optimal solution remains invariant.

A topic for further research that would be interesting is the extension of these results to unsymmetrical cases.

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