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Alfa-cut based linear programming methodology for constrained matrix games with payoffs of trapezoidal fuzzy numbers

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Abstract The purpose of this paper is to develop an effective methodology for solving constrained matrix games with payoffs of trapezoidal fuzzy numbers (TrFNs), which are a type of two-person non-cooperative games with payoffs expressed by TrFNs and players' strategies being constrained. In this methodology, it is proven that any Alfa-constrained matrix game has an interval-type value and hereby any constrained matrix game with payoffs of TrFNs has a TrFN-type value. The auxiliary linear programming models are derived to compute the interval-type value of any Alfa-constrained matrix game and players' optimal strategies. Thereby the TrFN-type value of any constrained matrix game with payoffs of TrFNs can be directly obtained through solving the derived four linear programming models with data taken from only 1-cut and 0-cut of TrFN-type payoffs. Validity and applicability of the models and method proposed in this paper are demonstrated with a numerical example of the market share game problem.

Keywords Fuzzy game theory · Group decision making · Interval computation · Linear programming · Algorithm

1 Introduction

There are different kinds of games to deal with antagonistic decision problems (Owen 1982). Two-person zero-sum games, which are often called matrix games for short, are an important kind of non-cooperative games. Matrix games have been extensively studied and successfully applied to many fields such as economics, finance, business,

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auctions and e-commerce as well as advertising. In some situations, however, players are not able to estimate exactly payoffs of outcomes in the game due to lack of adequate information and/or imprecision of the available information on the environments (Aggarwal et al. 2012; Bector et al. 2004; Larbani 2009; Liu and Kao 2009). This lack of precision and certainty may be appropriately modeled by using the fuzzy set (Zadeh 1965). In addition, choice of strategies for players is constrained due to some practical reason why this should be in some real-life game problems, i.e., not all mixed strategies in a game are permitted for each player (Dresher 1961). Such a matrix game with payoffs expressed by trapezoidal fuzzy numbers (TrFNs) is called a constrained matrix game with payoffs of TrFNs for short. Dresher (1961) gave a real example of the constrained matrix game with crisp payoffs. Li (1999) and Li and Cheng (2002) studied the constrained matrix games with fuzzy payoffs by using fuzzy multiobjective programming. The aim of this paper is to develop an effective methodology for solving constrained matrix games with payoffs of TrFNs. In this methodology, by using the duality theorem of linear programming and the concept of α-cuts for TrFNs, it is proven that any α-constrained matrix game has an interval-type value and hereby any constrained matrix game with payoffs of TrFNs has a TrFN-type value. The auxiliary linear programming models are derived to compute the interval-type value of any α-constrained matrix game and players' optimal strategies. Thereby the mean interval and the lower and upper limits of the TrFN-type value of any constrained matrix game with payoffs of TrFNs can be directly obtained through solving the derived four linear programming models with data taken from only 1-cut and 0-cut of TrFN-type payoffs. Thus, the fuzzy value of any constrained matrix game with payoffs of TrFNs can be easily and explicitly obtained by the representation theorem for the fuzzy set.

The rest of this paper is organized as follows. Section 2 briefly reviews some notations and definitions such as TrFNs, α-cuts and constrained matrix games as well as auxiliary linear programming models. Constrained matrix games with payoffs of TrFNs and the auxiliary linear programming models are presented in Sect. 3. In Sect. 4, validity and applicability of the proposed models and method are demonstrated with a numerical example of the market share game problem. Conclusion is made in Sect. 5.

2 Constrained matrix games and trapezoidal fuzzy numbers

2.1 Constrained matrix games and auxiliary linear programming models

Assume that $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ are sets of pure strategies for players I and II, respectively. A payoff matrix of player I is concisely expressed as $\boldsymbol{A} = (a_{ij})_{m\times n}$. Players I and II must respectively choose their mixed strategies $\boldsymbol{y} = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$ and $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$ from some convex polyhedrons, which are called constrained sets determined by some systems of linear inequalities and equations. Without loss of generality, let $Y = \{\boldsymbol{y}|\boldsymbol{B}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \boldsymbol{y} \geq 0\}$ and $Z = \{\boldsymbol{z}|\boldsymbol{E}\boldsymbol{z} \geq \boldsymbol{d}, \boldsymbol{z} \geq \boldsymbol{0}\}$ respectively represent the constrained sets of strategies for I and II, where $\boldsymbol{c} = (c_1, c_2, \ldots, c_p)^{\mathrm{T}}$, $\boldsymbol{B} = (b_{il})_{m\times p}$, $\boldsymbol{d} = (d_1, d_2, \ldots, d_q)^{\mathrm{T}}$, $\boldsymbol{E} = (e_{kj})_{q\times n}$, p and q are positive integers. Note that $\boldsymbol{B}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}$ includes $\sum_{i=1}^{m} y_i = 1$ since the latter is equivalent to both $\sum_{i=1}^{m} y_i \leq 1$ and $-\sum_{i=1}^{m} y_i \leq -1$. Likewise, $\boldsymbol{Ez} \geq \boldsymbol{d}$ includes $\sum_{j=1}^{n} z_j = 1$. Thus, a constrained matrix game $\boldsymbol{A}$ is meant that player I' payoff matrix is $\boldsymbol{A}$ (hereby player II' payoff matrix is $-\boldsymbol{A}$) and the sets of constrained strategies for I and II are Y and Z, respectively.

Suppose that players I and II are playing the constrained matrix game $\boldsymbol{A}$. If player I chooses any mixed strategy $\boldsymbol{y} \in Y$ and player II chooses any mixed strategy $\boldsymbol{z} \in Z$, then the expected payoff of player I is computed as follows:

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az} = \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} z_j. \tag{1}$$

Thus, player I should choose an optimal strategy $\boldsymbol{y}^* \in Y$ so that

$$\min_{z\in Z}\{\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{Az}\} = \max_{y\in Y}\min_{z\in Z}\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az}\} = \nu. \tag{2}$$

ν is called player I's gain-floor.

Similarly, player II should choose an optimal strategy $z^* \in Z$ so as to obtain

$$\max_{y\in Y}\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az}^*\} = \min_{z\in Z}\max_{y\in Y}\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az}\} = \omega. \tag{3}$$

ω is called player II's loss-ceiling.

Definition 1 (*Owen 1982*) Assume that there exist $\boldsymbol{y}^* \in \boldsymbol{Y}$ and $\boldsymbol{z}^* \in \boldsymbol{Z}$ so that

$$\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{Az}^* = \max_{y\in Y}\min_{z\in Z}\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az}\} = \min_{z\in Z}\max_{y\in Y}\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{Az}\}. \tag{4}$$

Then, $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ and $\nu = \boldsymbol{y}^{*T}\boldsymbol{Az}^*$ are called a saddle point (in mixed strategies) and a value of the constrained matrix game $\boldsymbol{A}$, respectively.

In a similar way to the matrix game (Owen 1982), Eq. (4) is equivalent to the linear programming models as follows:

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}\}$$
$$s.t. \begin{cases} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq 0 \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c} \\ \boldsymbol{x} \geq 0 \\ \boldsymbol{y} \geq 0 \end{cases} \tag{5}$$

and

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}\}$$
$$s.t. \begin{cases} \boldsymbol{Bs} - \boldsymbol{Az} \geq 0 \\ \boldsymbol{Ez} \geq \boldsymbol{d} \\ \boldsymbol{s} \geq 0 \\ \boldsymbol{z} \geq 0, \end{cases} \tag{6}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_q)^{\mathrm{T}}$ and $\boldsymbol{s} = (s_1, s_2, \ldots, s_p)^{\mathrm{T}}$. It is easily seen that Eqs. (5) and (6) are a pair of primal-dual linear programming problems. Therefore, if both Eqs. (5) and (6) are feasible, then the constrained matrix game $\boldsymbol{A}$ has a value and a saddle point in mixed strategies, which can be summarized as the following theorems.

Theorem 1 (Owen 1982) *If both Eqs. (5) and (6) are feasible linear programming, then they respectively have optimal solutions* $(\boldsymbol{y}^*, x^*)^{\mathrm{T}}$ *and* $(z^*, s^*)^{\mathrm{T}}$*. Moreover,* $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ *and* $v = \boldsymbol{y}^{*T} A z^*$ *are the saddle point and the value of the constrained matrix game* $\boldsymbol{A}$*, respectively.*

Theorem 2 *If* $(\boldsymbol{y}^*, x^*)^{\mathrm{T}}$ *and* $(z^*, s^*)^{\mathrm{T}}$ *respectively are feasible solutions of Eqs. (5) and (6) and* $d^{\mathrm{T}} x^* = c^{\mathrm{T}} s^*$*, then* $(\boldsymbol{y}^*, \boldsymbol{z}^*)$ *and* $v = d^{\mathrm{T}} x^* = c^{\mathrm{T}} s^*$ *are the saddle point and the value of the constrained matrix game* $\boldsymbol{A}$*, respectively.*

Theorem 2 is easily proven by using the duality theorem of linear programming. The interested reader is referred to Li (1999) and Owen (1982) for the detailed (omitted).

2.2 Trapezoidal fuzzy numbers and Alfa-cuts

A fuzzy number $\tilde{b}$ with the membership function $\mu_{\tilde{b}}(x)$ is a special fuzzy subset on the set R of real numbers, which satisfies the following conditions (Dubois and Prade 1980):

(1) There exists at least a $x_0 \in R$ so that $\mu_{\tilde{b}}(x_0) = 1$;
(2) The membership function $\mu_{\tilde{b}}(x)$ is left and right continuous.

TrFNs are a special kind of fuzzy numbers. Let $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ be a TrFN, whose membership function is defined as follows:

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - a^l)/(a^{m_1} - a^l) & \text{if } a^l \leq x < a^{m_1} \\ 1 & \text{if } a^{m_1} \leq x \leq a^{m_2} \\ (a^r - x)/(a^r - a^{m_2}) & \text{if } a^{m_2} < x \leq a^r \\ 0 & \text{else,} \end{cases} \tag{7}$$

where $[a^{m_1}, a^{m_2}]$, a^l and a^r are the mean interval and the lower and upper limits of $\tilde{a}$, respectively.

Obviously, if $a^{m_1} = a^{m_2}$ then the TrFN $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ is reduced to a triangular fuzzy number (TFN) $\tilde{a} = (a^l, a^m, a^r)$, where $a^m = a^{m_1} = a^{m_2}$. If $a^l = a^{m_1}$ and $a^{m_2} = a^r$, then $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ is reduced to an interval $a = [a^L, a^R]$, where $a^L = a^l = a^{m_1}$ and $a^R = a^{m_2} = a^r$. If $a^l = a^{m_1} = a^{m_2} = a^r$ then $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ is reduced to a real number a, where $a = a^l = a^{m_1} = a^{m_2} = a^r$. Conversely, TFNs, intervals and real numbers are easily rewritten as TrFNs. Therefore, TrFNs are an extremely congenial class of fuzzy numbers for representing imprecision and uncertainty such linguistics values and ill-quantity (Collins and Hu 2008; Zadeh 1965).

$\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ is called a non-negative TrFN if $a^l \geq 0$ and one of the values a^l, a^{m_1}, a^{m_2} and a^r is non-zero.

Let $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ and $\tilde{b} = (b^l, b^{m_1}, b^{m_2}, b^r)$ be two non-negative TrFNs. Then, their arithmetical operations are expressed as follows:

$$\tilde{a} + \tilde{b} = (a^l + b^l, a^{m_1} + b^{m_1}, a^{m_2} + b^{m_2}, a^r + b^r) \tag{8}$$

and

$$\lambda\tilde{a} = \begin{cases} (\lambda a^l, \lambda a^{m_1}, \lambda a^{m_2}, \lambda a^r) & \text{if } \lambda \geq 0 \\ (\lambda a^r, \lambda a^{m_1}, \lambda a^{m_2}, \lambda a^l) & \text{if } \lambda < 0. \end{cases} \tag{9}$$

Equations (8) and (9) mean that the sum of TrFNs and the product of a real number and a TrFN are still TrFNs.

A α-cut of a TrFN $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ is defined as $\tilde{a}(\alpha) = \{x|\mu_{\tilde{a}}(x) \geq \alpha\}$, where $\alpha \in [0, 1]$. Thus, for any $\alpha \in [0, 1]$, we can obtain a α-cut of the TrFN $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$, which is an interval, denoted by $\tilde{a}(\alpha) = [a^L(\alpha), a^R(\alpha)]$. It is easily derived from Eq. (7) that

$$a^L(\alpha) = \alpha a^{m_1} + (1 - \alpha)a^l \tag{10}$$

and

$$a^R(\alpha) = \alpha a^{m_2} + (1 - \alpha)a^r. \tag{11}$$

Obviously, 1-cut and 0-cut are just the Core and Support of $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$, respectively, i.e.,

$$\tilde{a}(1) = \text{Core}(\tilde{a}) = \{x|\mu_{\tilde{a}}(x) = 1\} = [a^{m_1}, a^{m_2}] \tag{12}$$

and

$$\tilde{a}(0) = \text{Support}(\tilde{a}) = \{x|\mu_{\tilde{a}}(x) > 0\} = [a^l, a^r]. \tag{13}$$

According to the operations over intervals (Moore 1979), it follows that

$$[\alpha a^{m_1} + (1 - \alpha)a^l, \alpha a^{m_2} + (1 - \alpha)a^r] = \alpha[a^{m_1}, a^{m_2}] + (1 - \alpha)[a^l, a^r]. \tag{14}$$

Combining with Eqs. (10)–(13), Eq. (14) is easily rewritten as follows:

$$[a^L(\alpha), a^R(\alpha)] = \alpha\tilde{a}(1) + (1 - \alpha)\tilde{a}(0) = [\alpha a^{m_1} + (1 - \alpha)a^l, \alpha a^{m_2} + (1 - \alpha)a^r], \tag{15}$$

i.e., any α-cut of a TrFN can be directly obtained from both its 1-cut and 0-cut.

According to the representation theorem for the fuzzy set (Zadeh 1965), using Eq. (15), a TrFN $\tilde{a} = (a^l, a^{m_1}, a^{m_2}, a^r)$ can be expressed as follows:

$$\tilde{a} = \bigcup_{\alpha \in [0,1]} \{\alpha \otimes \tilde{a}(\alpha)\} = \bigcup_{\alpha \in [0,1]} \{\alpha \otimes [a^L(\alpha), a^R(\alpha)]\}, \tag{16}$$

where $\alpha \otimes \tilde{a}(\alpha)$ is defined as a fuzzy set, whose membership function is defined as follows:

$$\mu_{\alpha\otimes\tilde{a}(\alpha)}(x) = \begin{cases} \alpha & \text{if } x \in \tilde{a}(\alpha) \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

i.e.,

$$\mu_{\alpha\otimes\tilde{a}(\alpha)}(x) = \begin{cases} \alpha & \text{if } a^L(\alpha) \le x \le a^R(\alpha) \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

According to Eq. (15), Eq. (16) can be further rewritten as follows:

$$\tilde{a} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes [\alpha\tilde{a}(1) + (1-\alpha)\tilde{a}(0)]\}. \tag{19}$$

Hence, it is easily derived from Eq. (19) that

$$\mu_{\tilde{a}}(x) = \max\{\alpha | x \in \alpha\tilde{a}(1) + (1-\alpha)\tilde{a}(0)\}. \tag{20}$$

More specially, combining with Eqs. (12) and (13), Eq. (20) can be written as the same as Eq. (7). Thus, it is seen from Eq. (19) that a TrFN can be directly constructed by using both its 1-cut and 0-cut.

From the aforementioned discussion, we summarize the conclusion as in the following theorem, which will be used to construct the fuzzy values of constrained matrix games with payoffs of TrFNs in Sect. 3.

Theorem 3 *A TrFN and its α-cuts have the relations (1) and (2) as follows:*

(1) Any α-cut of a TrFN can be directly obtained from both its 1-cut and 0-cut;
(2) Any TrFN can be directly constructed by using both 1-cut and 0-cut.

3 Constrained matrix games with payoffs of TrFNs and solution method

3.1 Concepts of Alfa-constrained matrix games and values

Let us consider a constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs, where the payoff matrix of player I is given as $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})_{m\times n}$, whose elements $\tilde{a}_{ij}(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are TrFNs stated as Sect. 2.2.

Definition 2 For a given $\alpha \in [0, 1]$, denote the payoff matrix of player I by $\tilde{\boldsymbol{A}}(\alpha) = (\tilde{a}_{ij}(\alpha))_{m\times n}$, whose elements $\tilde{a}_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are α-cuts of the TrFN-type payoffs $\tilde{a}_{ij}$. Then, $\tilde{\boldsymbol{A}}(\alpha)$ is called a α-constrained matrix game of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs, which is often called the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ for short.

Definition 3 For a given $\alpha \in [0, 1]$, if player I's gain-floor $\tilde{\nu}(\alpha)$ and player II's loss-ceiling $\tilde{\rho}(\alpha)$ have a common value $\tilde{V}(\alpha)$, then $\tilde{V}(\alpha)$ is called the value of the α- constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$, or the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has a value $\tilde{V}(\alpha)$, where $\tilde{V}(\alpha) = \tilde{\nu}(\alpha) = \tilde{\rho}(\alpha)$.

Definition 4 For any $\alpha \in [0, 1]$, if each α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has a value $\tilde{V}(\alpha)$, then the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs has a fuzzy value $\tilde{V}$, where $\tilde{V} = \cup_{\alpha\in[0,1]}\{\alpha \otimes \tilde{V}(\alpha)\}$.

3.2 Alfa-constrained matrix games and auxiliary linear programming models

For the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs stated as the above Sect. 3.1, according to Eqs. (8) and (9), the expected payoff for player I is computed as follows:

$$\tilde{E} = \boldsymbol{y}^{\mathrm{T}}\tilde{\boldsymbol{A}}\boldsymbol{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}\tilde{a}_{ij}y_i z_j$$
$$= \left(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{l}y_i z_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m_1}y_i z_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m_2}y_i z_j, \sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{r}y_i z_j\right), \quad (21)$$

which is a TrFN.

Due to the fact that the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs is zero-sum, the expected payoff for player II is obtained as follows:

$$-\tilde{E} = \boldsymbol{y}^{\mathrm{T}}(-\tilde{\boldsymbol{A}})\boldsymbol{z} = \sum_{i=1}^{m}\sum_{j=1}^{n}(-\tilde{a}_{ij}y_i z_j) = \left(-\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{r}y_i z_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m_2}y_i z_j,\right.$$
$$\left.-\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{m_1}y_i z_j, -\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}^{l}y_i z_j\right), \quad (22)$$

which is still a TrFN. Thus, in general, player I's gain-floor and player II's loss-ceiling should be TrFNs, denoted by $\tilde{\nu} = (\nu^l, \nu^{m_1}, \nu^{m_2}, \nu^r)$ and $\tilde{\omega} = (\omega^l, \omega^{m_1}, \omega^{m_2}, \omega^r)$, respectively. Moreover, it is easily proven that $\tilde{\nu} \leq \tilde{\omega}$ (Li 1999; Li and Cheng 2002).

In a parallel way to the crisp constrained matrix games, if $\tilde{\nu}=\tilde{\omega}$ then the common value $\tilde{V}$ is called the fuzzy value of the constrained matrix game $\tilde{A}$ with payoffs of TrFNs, where $\tilde{V} = \tilde{\nu} = \tilde{\omega}$. Obviously, $\tilde{V}$ is a TrFN, denoted by $\tilde{V}=(V^l, V^{m_1}, V^{m_2}, V^r)$.

As far as we know, unfortunately, there is no method which can always ensure that $\tilde{\nu} = \tilde{\omega}$ and hereby the constrained matrix game with payoffs of TrFNs has the fuzzy value. In this subsection, inspired by Li (2011), according to Definitions 2–4, we develop a linear programming method for solving any α-constrained matrix game. In particular, the auxiliary linear programming models are constructed to compute explicitly the TrFN-type value of the constrained matrix game $\tilde{A}$ with payoffs of TrFNs.

For any $\alpha \in [0, 1]$, let us consider a α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$, where the payoff matrix of player I is given as $\tilde{\boldsymbol{A}}(\alpha) = (\tilde{a}_{ij}(\alpha))_{m\times n}$, whose elements $\tilde{a}_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are the α-cuts of the TrFN $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^{m_1}, a_{ij}^{m_2}, a_{ij}^r)$.

Stated as earlier, the α-cuts $\tilde{a}_{ij}(\alpha)$ of $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^{m_1}, a_{ij}^{m_2}, a_{ij}^r)$ are intervals. It is easily derived from Eq. (15) that

$$\tilde{a}_{ij}(\alpha) = [a_{ij}^L(\alpha), a_{ij}^R(\alpha)] = [\alpha a_{ij}^{m_1} + (1-\alpha)a_{ij}^l, \alpha a_{ij}^{m_2} + (1-\alpha)a_{ij}^r]. \quad (23)$$

Namely, the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is the constrained matrix game with payoffs of intervals.

For any given values $a_{ij}(\alpha)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, a payoff matrix is denoted by $\boldsymbol{A}(\alpha) = (a_{ij}(\alpha))_{m\times n}$. It is easily seen from Eq. (2) that the value $\nu(\alpha)$ of the constrained matrix game $\boldsymbol{A}(\alpha)$ for player I is closely related to the values $a_{ij}(\alpha)$, i.e., entries in the payoff matrix $\boldsymbol{A}(\alpha)$. In other words, $\nu(\alpha)$ is a function of the values $a_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)$, denoted by $\nu(\alpha) = \upsilon(a_{ij}(\alpha))$ (or $\upsilon(\boldsymbol{A}(\alpha))$). Similarly, player I's optimal strategy $\boldsymbol{y}^*(\alpha) \in Y$ is also a function of the values $a_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, denoted by $\boldsymbol{y}^*(\alpha) = \boldsymbol{y}^*(a_{ij}(\alpha))$ (or $\boldsymbol{y}^*(\boldsymbol{A}(\alpha))$). Likewise, the value $\rho(\alpha)$ and player II's optimal strategy $\boldsymbol{z}^*(\alpha) \in Z$ are functions of the values $a_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)$, denoted by $\rho(\alpha) = \omega(a_{ij}(\alpha))$ (or $\omega(\boldsymbol{A}(\alpha))$) and $\boldsymbol{z}^*(\alpha) = \boldsymbol{z}^*(a_{ij}(\alpha))$ (or $\boldsymbol{z}^*(\boldsymbol{A}(\alpha))$), respectively.

According to Eqs. (1) and (2), it is easily proven that player I's gain-floor $\upsilon(\alpha) = \upsilon(a_{ij}(\alpha))$ is a monotonic and non-decreasing function of the values $a_{ij}(\alpha)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)$. In fact, for any values $a_{ij}(\alpha)$ and $a'_{ij}(\alpha)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)$, if $a_{ij}(\alpha) \le a'_{ij}(\alpha)$ then we have

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij}(\alpha) z_j \le \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a'_{ij}(\alpha) z_j \quad (24)$$

since $y_i \ge 0\ (i = 1, 2, \ldots, m)$ and $z_j \ge 0\ (j = 1, 2, \ldots, n)$, where $\boldsymbol{y} \in Y$ and $\boldsymbol{z} \in Z$. Hence,

$$\min_{z\in Z}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij}(\alpha) z_j\right\} \le \min_{z\in Z}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a'_{ij}(\alpha) z_j\right\}, \quad (25)$$

which directly implies that

$$\max_{y\in Y}\min_{z\in Z}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij}(\alpha) z_j\right\} \le \max_{y\in Y}\min_{z\in Z}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a'_{ij}(\alpha) z_j\right\}, \quad (26)$$

i.e.,

$$\nu(a_{ij}(\alpha)) \le \nu(a'_{ij}(\alpha)), \quad (27)$$

where $\boldsymbol{A}'(\alpha) = (a'_{ij}(\alpha))_{m\times n}$ is player I's payoff matrix in the constrained matrix game $\boldsymbol{A}'(\alpha)$.

According to the mini-max theorem of matrix games (Owen 1982), the constrained matrix game $\boldsymbol{A}(\alpha) = (a_{ij}(\alpha))_{m\times n}$ has a value, denoted by $V(\alpha) = V(a_{ij}(\alpha))$ (or $V(\boldsymbol{A}(\alpha))$). Obviously, $V(\alpha) = \nu(\alpha) = \rho(\alpha)$. From the above discussion, $V(\alpha) = V(a_{ij}(\alpha))$ is also a non-decreasing function of the values $a_{ij}(\alpha)$ ($i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$) in the payoff intervals $\tilde{a}_{ij}(\alpha)$.

For the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$, the expected payoffs of players are a linear combination of interval-valued payoffs. Thus, from a viewpoint of logic on interval computation, the value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ should be a closed interval as well (Moore 1979). Note that player I's value $\nu(\alpha) = \nu(a_{ij}(\alpha))$ of the constrained matrix game $\boldsymbol{A}(\alpha) = (a_{ij}(\alpha))_{m\times n}$ is a non-decreasing function of the values $a_{ij}(\alpha)$ in the payoff intervals $\tilde{a}_{ij}(\alpha)$. Hence, player I's upper bound $\nu^R(\alpha)$ of the interval-type value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ can be obtained as follows:

$$\nu^R(\alpha) = \max_{\boldsymbol{y}\in Y}\min_{\boldsymbol{z}\in Z}\left\{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}^R(\alpha)\boldsymbol{z}\right\} = \max_{\boldsymbol{y}\in Y}\min_{\boldsymbol{z}\in Z}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij}^R(\alpha) z_j\right\}, \tag{28}$$

where $\boldsymbol{A}^R(\alpha)$=$(a_{ij}^R(\alpha))_{m\times n}$. According to Eq. (5), Eq. (28) is equivalent to the linear programming model as follows:

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^R(\alpha)\}$$
$$s.t.\begin{cases}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^R(\alpha) - \boldsymbol{A}^R(\alpha)^{\mathrm{T}}\boldsymbol{y}^R(\alpha) \le 0\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^R(\alpha) \le \boldsymbol{c}\\ \boldsymbol{x}^R(\alpha) \ge 0\\ \boldsymbol{y}^R(\alpha) \ge 0.\end{cases} \tag{29}$$

If Eq. (29) is feasible linear programming, then using the Simplex method of linear programming (Owen 1982), an optimal solution of Eq. (29) is obtained, denoted by $(\boldsymbol{x}^{R^*}(\alpha), \boldsymbol{y}^{R^*}(\alpha))$. Thus, according to Theorem 2, we obtain the upper bound $\nu^R(\alpha) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{R^*}(\alpha)$ of player I's gain-floor $\tilde{\nu}(\alpha)$ and corresponding optimal strategy $\boldsymbol{y}^{R^*}(\alpha) \in Y$ for the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$.

Likewise, the lower bound $\nu^L(\alpha)$ of player I's gain-floor $\nu(\alpha)$ and the optimal strategy $\boldsymbol{y}^{L^*}(\alpha) \in Y$ for the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ are $\nu^L(\alpha) = \nu^L(a_{ij}^L(\alpha))$ and $\boldsymbol{y}^{L^*} = \boldsymbol{y}^{L^*}(a_{ij}^L(\alpha))$, respectively. According to Eq. (5), $(\nu^L(\alpha), \boldsymbol{y}^{L^*}(\alpha))$ can be obtained by solving the linear programming model as follows:

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^L(\alpha)\}$$
$$s.t.\begin{cases}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^L(\alpha) - \boldsymbol{A}^L(\alpha)^{\mathrm{T}}\boldsymbol{y}^L(\alpha) \le 0\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^L(\alpha) \le \boldsymbol{c}\\ \boldsymbol{x}^L(\alpha) \ge 0\\ \boldsymbol{y}^L(\alpha) \ge 0.\end{cases} \tag{30}$$

If Eq. (30) is feasible linear programming, then it has an optimal solution, denoted by $(\boldsymbol{y}^{L^*}(\alpha), \boldsymbol{x}^{L^*}(\alpha))^{\mathrm{T}}$. Thus, according to Theorem 2, we obtain the lower bound

$\nu^L(\alpha) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{L^*}(\alpha)$ of player I's gain-floor $\tilde{\nu}(\alpha)$ and corresponding optimal strategy $\boldsymbol{y}^{L^*}(\alpha)$ for the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$.

Thus, the lower bound $\nu^L(\alpha)$ and the upper bound $\nu^R(\alpha)$ of the player I's value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ can be obtained. Therefore, the value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is a closed interval $[\nu^L(\alpha), \nu^R(\alpha)]$. Namely, $\tilde{\nu}(\alpha) = [\nu^L(\alpha), \nu^R(\alpha)]$. It is obvious that $\tilde{\nu}(\alpha)$ is the α-cut of player I's gain-floor $\tilde{\nu}(\alpha)$ in the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs.

In the same analysis to that of player I, the upper bound $\rho^R(\alpha)$ of player II's value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and corresponding optimal strategy $\boldsymbol{z}^{R^*}(\alpha) \in Z$ are $\rho^R(\alpha) = \omega^R(a_{ij}^R(\alpha))$ and $\boldsymbol{z}^{R^*}(\alpha) = \boldsymbol{z}^{R^*}(a_{ij}^R(\alpha))$, respectively. According to Eq. (6), $(\omega^R(\alpha), \boldsymbol{z}^{R^*}(\alpha))$ can be obtained by solving the linear programming model as follows:

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^R(\alpha)\}$$
$$s.t. \begin{cases} \boldsymbol{B}\boldsymbol{s}^R(\alpha) - \boldsymbol{A}^R(\alpha)\boldsymbol{z}^R(\alpha) \geq 0 \\ \boldsymbol{E}\boldsymbol{z}^R(\alpha) \geq \boldsymbol{d} \\ \boldsymbol{s}^R(\alpha) \geq 0 \\ \boldsymbol{z}^R(\alpha) \geq 0. \end{cases} \tag{31}$$

If Eq. (31) is feasible linear programming, then using the Simplex method of linear programming, an optimal solution is obtained, denoted by $(z^{R^*}(\alpha), s^{R^*}(\alpha))$. Thus, according to Theorem 2, we obtain the upper bound $\rho^R(\alpha) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{R^*}(\alpha)$ of player II's loss-ceiling $\tilde{\omega}(\alpha)$ and corresponding optimal strategy $\boldsymbol{z}^{R^*}(\alpha)$.

Likewise, the lower bound $\rho^L(\alpha)$ of player II's loss-ceiling $\tilde{\omega}(\alpha)$ and corresponding optimal strategy $\boldsymbol{z}^{L^*}(\alpha) \in Z$ are $\rho^L(\alpha) = \omega^L(a_{ij}^L(\alpha))$ and $\boldsymbol{z}^{L^*}(\alpha) = \boldsymbol{z}^{L^*}(a_{ij}^L(\alpha))$, respectively. According to Eq. (6), $(\omega^L(\alpha), \boldsymbol{z}^{L^*}(\alpha))$ can be obtained by solving the linear programming model as follows:

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^L(\alpha)\}$$
$$s.t. \begin{cases} \boldsymbol{B}\boldsymbol{s}^L(\alpha) - \boldsymbol{A}^L(\alpha)\boldsymbol{z}^L(\alpha) \geq 0 \\ \boldsymbol{E}\boldsymbol{z}^L(\alpha) \geq \boldsymbol{d} \\ \boldsymbol{s}^L(\alpha) \geq 0 \\ \boldsymbol{z}^L(\alpha) \geq 0. \end{cases} \tag{32}$$

If Eq. (32) is feasible linear programming, then it has an optimal solution, denoted by $(z^{L^*}(\alpha), s^{L^*}(\alpha))^{\mathrm{T}}$. Thus, we obtain the lower bound $\rho^L(\alpha) = \boldsymbol{c}^{\mathrm{T}}s^{L^*}(\alpha)$ of player II's loss-ceiling $\tilde{\omega}(\alpha)$ and corresponding optimal strategy $\boldsymbol{z}^{L^*}(\alpha)$. Thus, the lower bound $\rho^L(\alpha)$ and the upper bound $\rho^R(\alpha)$ of player II's value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ can be obtained. Therefore, player II's value of the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ is a closed interval $[\rho^L(\alpha), \rho^R(\alpha)]$. Namely, $\tilde{\rho}(\alpha) = [\rho^L(\alpha), \rho^R(\alpha)]$. It is obvious that $\tilde{\rho}(\alpha)$ is the α-cut of player II's loss-ceiling $\tilde{\omega}$ in the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs.

It is easily seen that Eqs. (29) and (31) are a pair of primal-dual linear programming problems. So the maximum of $\upsilon^R(\alpha)$ is equal to the minimum of $\omega^R(\alpha)$ by the duality theorem of linear programming (Owen 1982), i.e.,

$$\nu^R(\alpha) = \rho^R(\alpha). \tag{33}$$

Likewise, Eqs. (30) and (32) are a pair of primal-dual linear programming problems. Hence,

$$\nu^L(\alpha) = \rho^L(\alpha). \tag{34}$$

Therefore, player I's gain-floor $\tilde{\nu}(\alpha) = [\nu^L(\alpha), \nu^R(\alpha)]$ is equal to player II's loss-ceiling $\tilde{\rho}(\alpha) = [\rho^L(\alpha), \rho^R(\alpha)]$. Namely, player I's gain-floor and player II's loss-ceiling have a common value. According to Definition 3, the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has a value, which is also an interval, denoted by $\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$. Obviously, $\tilde{V}(\alpha) = \tilde{\nu}(\alpha) = \tilde{\rho}(\alpha)$, i.e., $V^L(\alpha) = \nu^L(\alpha) = \rho^L(\alpha)$ and $V^R(\alpha) = \nu^R(\alpha) = \rho^R(\alpha)$. Furthermore, it is easily seen that $\tilde{V}(\alpha)$ is the α-cut of the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs, where $\tilde{V}$ is a TrFN stated as earlier. Thus, the conclusion is drawn in the following theorem.

Theorem 4 *For any $\alpha \in [0, 1]$, the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has an interval-type value $\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$, whose lower and upper bounds can be obtained through solving two linear programming problems (i.e., Eqs. (29) and (30) or Eqs. (31) and (32)), respectively.*

Theorem 5 *Any constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs always has a fuzzy value $\tilde{V}$, where $\tilde{V} = \cup_{\alpha\in[0,1]}\{\alpha \otimes \tilde{V}(\alpha)\} = \cup_{\alpha\in[0,1]}\{\alpha \otimes [V^L(\alpha), V^R(\alpha)]\}$.*

Proof For any $\alpha \in [0, 1]$, according to Theorem 4, the α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ has a value $\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$. Thus, according to Definition 4, it directly follows that the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs has a fuzzy value $\tilde{V}$. Combining with Eq. (16), we have

$$\tilde{V} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes \tilde{V}(\alpha)\} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes [V^L(\alpha), V^R(\alpha)]\}. \tag{35}$$

□

In particular, for $\alpha = 1$, according to Eqs. (29) and (30) or Eqs. (31) and (32), the linear programming problems are constructed as follows:

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^R(1)\}$$
$$s.t. \begin{cases} \boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^R(1) - \boldsymbol{A}^R(1)^{\mathrm{T}}\boldsymbol{y}^R(1) \leq 0 \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^R(1) \leq \boldsymbol{c} \\ \boldsymbol{x}^R(1) \geq 0 \\ \boldsymbol{y}^R(1) \geq 0 \end{cases} \tag{36}$$

and

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{L}(1)\}$$
$$s.t.\begin{cases}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^{L}(1)-\boldsymbol{A}^{L}(1)^{\mathrm{T}}\boldsymbol{y}^{L}(1)\leq 0\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^{L}(1)\leq \boldsymbol{c}\\ \boldsymbol{x}^{L}(1)\geq 0\\ \boldsymbol{y}^{L}(1)\geq 0,\end{cases} \tag{37}$$

or

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{R}(1)\}$$
$$s.t.\begin{cases}\boldsymbol{B}\boldsymbol{s}^{R}(1)-\boldsymbol{A}^{R}(1)\boldsymbol{z}^{R}(1)\geq 0\\ \boldsymbol{E}\boldsymbol{z}^{R}(1)\geq \boldsymbol{d}\\ \boldsymbol{s}^{R}(1)\geq 0\\ \boldsymbol{z}^{R}(1)\geq 0\end{cases} \tag{38}$$

and

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{L}(1)\}$$
$$s.t.\begin{cases}\boldsymbol{B}\boldsymbol{s}^{L}(1)-\boldsymbol{A}^{L}(1)\boldsymbol{z}^{L}(1)\geq 0\\ \boldsymbol{E}\boldsymbol{z}^{L}(1)\geq \boldsymbol{d}\\ \boldsymbol{s}^{L}(1)\geq 0\\ \boldsymbol{z}^{L}(1)\geq 0,\end{cases} \tag{39}$$

respectively.

Using the Simplex method of linear programming, we obtain the optimal solutions to the linear programming problems (i.e., Eqs. (36)–(39)), respectively, where $\nu^{R}(1)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{R}(1)$, $\nu^{L}(1)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{L}(1)$, $\rho^{R}(1)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{R}(1)$ and $\rho^{L}(1)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{L}(1)$. It easily follows from Eqs. (33) and (34) that $[V^{L}(1), V^{R}(1)]=[\nu^{L}(1),\nu^{R}(1)]=[\rho^{L}(1),\rho^{R}(1)]$. According to the notation of $\tilde{V}=(V^{l}, V^{m_1}, V^{m_2}, V^{r})$, we have

$$V^{m_1}=V^{L}(1)=\nu^{L}(1)=\rho^{L}(1),\quad V^{m_2}=V^{R}(1)=\nu^{R}(1)=\rho^{R}(1). \tag{40}$$

Namely, the mean interval of the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs can be directly obtained by solving the two linear programming problems (i.e., Eqs. (36) and (37) or Eqs. (38) and (39)). In other words, the 1-cut (or Core) of the fuzzy value is obtained as $\tilde{V}(1)=\mathrm{Core}(\tilde{V})=[V^{m_1}, V^{m_2}]=[\nu^{L}(1),\nu^{R}(1)]=[\rho^{L}(1),\rho^{R}(1)]$.

For $\alpha=0$, according to Eqs. (29) and (30) or Eqs. (31) and (32), the linear programming problems are constructed as follows:

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{R}(0)\}$$
$$s.t.\begin{cases}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^{R}(0)-\boldsymbol{A}^{R}(0)^{\mathrm{T}}\boldsymbol{y}^{R}(0)\leq 0\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^{R}(0)\leq \boldsymbol{c}\\ \boldsymbol{x}^{R}(0)\geq 0\\ \boldsymbol{y}^{R}(0)\geq 0\end{cases} \tag{41}$$

and

$$\max\{\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^L(0)\}$$
$$s.t.\begin{cases}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{x}^L(0)-\boldsymbol{A}^L(0)^{\mathrm{T}}\boldsymbol{y}^L(0)\le 0\\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{y}^L(0)\le \boldsymbol{c}\\ \boldsymbol{x}^L(0)\ge 0\\ \boldsymbol{y}^L(0)\ge 0,\end{cases} \tag{42}$$

or

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^R(0)\}$$
$$s.t.\begin{cases}\boldsymbol{B}\boldsymbol{s}^R(0)-\boldsymbol{A}^R(0)\boldsymbol{z}^R(0)\ge 0\\ \boldsymbol{E}\boldsymbol{z}^R(0)\ge \boldsymbol{d}\\ \boldsymbol{s}^R(0)\ge 0\\ \boldsymbol{z}^R(0)\ge 0\end{cases} \tag{43}$$

and

$$\min\{\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^L(0)\}$$
$$s.t.\begin{cases}\boldsymbol{B}\boldsymbol{s}^L(0)-\boldsymbol{A}^L(0)\boldsymbol{z}^L(0)\ge 0\\ \boldsymbol{E}\boldsymbol{z}^L(0)\ge \boldsymbol{d}\\ \boldsymbol{s}^L(0)\ge 0\\ \boldsymbol{z}^L(0)\ge 0,\end{cases} \tag{44}$$

respectively.

Using the Simplex method of linear programming, we obtain the optimal solutions to the linear programming problems (i.e., Eqs. (41)–(44)), respectively, where $\nu^R(0)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^R(0)$, $\nu^L(0)=\boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^L(0)$, $\rho^R(0)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^R(0)$ and $\rho^L(0)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^L(0)$. It easily follows from Eqs. (33) and (34) that $[V^L(0),V^R(0)]=[\nu^L(0),\nu^R(0)]=[\rho^L(0),\rho^R(0)]$. According to the notation of $\tilde{V}=(V^l,V^{m_1},V^{m_2},V^r)$, we have

$$V^l=V^L(0)=\nu^L(0)=\rho^L(0),\quad V^r=V^R(0)=\nu^R(0)=\rho^R(0). \tag{45}$$

Namely, the lower limit and the upper limit of the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs can be directly obtained by solving the two linear programming problems (i.e., Eqs. (41) and (41) or Eqs. (43) and (44)). In other words, the 0-cut (or Support) of the fuzzy value is obtained as $\tilde{V}(0)=\mathrm{Support}(\tilde{V})=[V^l,V^r]=[\nu^L(0),\nu^R(0)]=[\rho^L(0),\rho^R(0)]$.

Theorem 6 *The fuzzy value of the constrained matrix game* $\tilde{\boldsymbol{A}}$ *with payoffs of TrFNs can be expressed as follows:*

$$\tilde{V}=\bigcup_{\alpha\in[0,1]}\{\alpha\otimes[\alpha V^{m_1}+(1-\alpha)V^l,\alpha V^{m_2}+(1-\alpha)V^r]\}, \tag{46}$$

which is just the TrFN $\tilde{V}=(V^l,V^{m_1},V^{m_2},V^r)$*, whose mean interval and the lower and upper limits can be directly obtained through solving the above four linear* *programming problems (i.e., Eqs. (36), (37), (41) and (42) or Eqs. (38), (39), (43) and (44)), respectively.*

Proof According to Eqs. (15), (40) and (45), any α-cut $\tilde{V}(\alpha) = [V^L(\alpha), V^R(\alpha)]$ of the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs can be obtained as $\tilde{V}(\alpha) = \alpha\tilde{V}(1)+(1-\alpha)\tilde{V}(0) = [\alpha V^{m_1}+(1-\alpha)V^l, \alpha V^{m_2}+(1-\alpha)V^r]$, where $\alpha \in [0, 1]$. According to Theorem 3, the fuzzy value $\tilde{V}$ can be expressed as follows:

$$\tilde{V} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes \tilde{V}(\alpha)\} = \bigcup_{\alpha\in[0,1]} \{\alpha \otimes [\alpha V^{m_1} + (1-\alpha)V^l, \alpha V^{m_2} + (1-\alpha)V^r]\}, \tag{47}$$

which directly implies that the membership function of $\tilde{V}$ is

$$\mu_{\tilde{V}}(x) = \max\{\alpha|\alpha V^{m_1} + (1-\alpha)V^l \le x \le \alpha V^{m_2} + (1-\alpha)V^r\}$$
$$= \begin{cases} (x - V^l)/(V^{m_1} - V^l) & \text{if } V^l \le x < V^{m_1} \\ 1 & \text{if } V^{m_1} \le x \le V^{m_2} \\ (V^r - x)/(V^r - V^{m_2}) & \text{if } V^{m_2} < x \le V^r \\ 0 & \text{else.} \end{cases} \tag{48}$$

Therefore, the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs is just the TrFN $\tilde{V} = (V^l, V^{m_1}, V^{m_2}, V^r)$. □

Theorem 6 shows that the fuzzy value $\tilde{V}$ of any constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs is a TrFN, which can be directly and explicitly obtained through solving the four linear programming problems (i.e., Eqs. (36), (37), (41) and (42) or Eqs. (38), (39), (43) and (44)) with all data taken from only 1-cut and 0-cut of the TrFN-type payoffs.

3.3 Algorithm for solving constrained matrix games with payoffs of TrFNs

From the aforementioned discussion, the process and algorithm for solving constrained matrix games with payoffs of TrFNs are summarized as follows.

Step 1: Identify players, denoted by I and II;
Step 2: Identify pure strategies of players I and II, denoted the sets of pure strategies by $S_1 = \{\delta_1, \delta_2, \ldots, \delta_m\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, respectively;
Step 3: Identify constraint conditions of strategies' choices for players I and II, denoted the sets of constrained strategies by Y and Z, respectively;
Step 4: Pool opinions of outcomes for players I and II and estimate player I's payoffs expressed with TrFNs $\tilde{a}_{ij} = (a_{ij}^l, a_{ij}^{m_1}, a_{ij}^{m_2}, a_{ij}^r)$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ and hereby construct the payoff matrix $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})_{m\times n}$;
Step 5: Construct and solve the linear programming problems according to Eqs. (29) and (30) (or Eqs. (31) and (32)) and hereby obtain the value $\tilde{V}(\alpha)$ of any

α-constrained matrix game $\tilde{\boldsymbol{A}}(\alpha)$ and players' optimal strategies, where $\alpha \in [0, 1]$;

Step 6: Construct and solve the two linear programming problems according to Eqs. (36) and (37) and hereby obtain V^{m_1} and V^{m_2};

Step 7: Construct and solve the two linear programming problems according to Eqs. (41) and (42) and hereby obtain V^l and V^r;

Step 8: Construct the fuzzy value $\tilde{V}$ of the constrained matrix game $\tilde{\boldsymbol{A}}$ with payoffs of TrFNs according to the obtained values V^{m_1}, V^{m_2}, V^l and V^r. Namely, the TrFN $\tilde{V} = (V^l, V^{m_1}, V^{m_2}, V^r)$ is explicitly obtained.

4 A numerical example and computational result comparison

Suppose that there are two industrial companies p_1 and p_2 aiming to enhance the market share of an industrial product in a targeted market under the circumstance that the demand amount of the industrial product in the targeted market basically is fixed. In other words, the market share of one company increases while the market share of another company decreases. The two companies are considering about two strategies to increase the market share: strategy δ_1 (improving technology) and δ_2 (advertisement). The company p_1 needs the funds 80 and 50 (million yuan) when it takes strategies δ_1 and δ_2, respectively. However, due to the lack of the funds, the company p_1 only provides 60 (million yuan), i.e., the company p_1's mixed strategies may satisfy the constrained condition: $80y_1 + 50y_2 \leq 60$. Similarly, the company p_2 need the funds 40 and 70 (million yuan) when it takes strategies δ_1 and δ_2, respectively. However, the company p_2 only provides 50 (million yuan), i.e., the company p_2's mixed strategies may satisfy the constraint condition: $40z_1 + 70z_2 \leq 50$ (or $-40z_1 - 70z_2 \geq -50$). Due to a lack of information and imprecision of the available information, the managers of the two companies usually are not able to forecast exactly the sales amount of the companies' product. Hereby, TrFNs are suitably used to represent the sales amount of the industrial product from companies' perspectives. Thus, the above problem may be regarded as a constrained matrix game with payoffs of TrFNs. Namely, the companies p_1 and p_2 are regarded as players I and II, respectively. The sets of constrained strategies are expressed as follows:

$$Y_0 = \{y \,|80y_1 + 50y_2 \leq 60, y_1 + y_2 \leq 1, -y_1 - y_2 \leq -1, y_1 \geq 0, y_2 \geq 0\}$$

and

$$Z_0 = \{z \,|-40z_1 - 70z_2 \geq -50, z_1 + z_2 \leq 1, -z_1 - z_2 \leq -1, z_1 \geq 0, z_2 \geq 0\},$$

respectively. Let us consider the specific constrained matrix game $\tilde{\boldsymbol{A}}_0$ with payoffs of TrFNs, where the payoff matrix of the company p_1 (i.e., player I) is given as follows:

$$\tilde{\boldsymbol{A}}_0 = \begin{pmatrix} (27, 29, 32, 35) & (-25, -19, -18, -17) \\ (-11, -10, -8, -5) & (35, 40, 40.5, 41) \end{pmatrix}$$

where the element (27,29,32,35) in the matrix $\tilde{\boldsymbol{A}}_0$ is a TrFN, which indicates that the company p_1's sales amount of the industrial product is between 27 and 35 when the companies p_1 and p_2 use the strategy δ_1 (improving technology) simultaneously. Other elements (i.e., TrFNs) in the matrix $\tilde{\boldsymbol{A}}_0$ can be explained similarly.

Coefficient matrices and vectors of the constrained sets Y_0 and Z_0 for the industrial companies p_1 and p_2 are obtained as follows:

$$\boldsymbol{B} = \begin{pmatrix} 80 & 1 & -1 \\ 50 & 1 & -1 \end{pmatrix}, \quad \boldsymbol{E}^{\mathrm{T}} = \begin{pmatrix} -40 & 1 & -1 \\ -70 & 1 & -1 \end{pmatrix}$$

and

$$\boldsymbol{c} = (60, 1, -1)^{\mathrm{T}}, \quad \boldsymbol{d} = (-50, 1, -1)^{\mathrm{T}},$$

respectively.

4.1 Computational results obtained by the proposed method

According to Eq. (36), the linear programming problem can be obtained as follows:

$$\max\{-50x_1^R(1) + x_2^R(1) - x_3^R(1)\}$$
$$s.t. \begin{cases} -40x_1^R(1) + x_2^R(1) - x_3^R(1) - 32y_1^R(1) + 8y_2^R(1) \leq 0 \\ -70x_1^R(1) + x_2^R(1) - x_3^R(1) + 18y_1^R(1) - 40.5y_2^R(1) \leq 0 \\ 80y_1^R(1) + 50y_2^R(1) \leq 60 \\ y_1^R(1) + y_2^R(1) \leq 1 \\ -y_1^R(1) - y_2^R(1) \leq -1 \\ x_k^R(1) \geq 0\,(k = 1, 2, 3) \\ y_i^R(1) \geq 0\,(i = 1, 2). \end{cases} \tag{49}$$

Using the Simplex method of linear programming, the optimal solution $(\boldsymbol{x}^{R^*}(1), \boldsymbol{y}^{R^*}(1))$ to Eq. (49) can be obtained, where $\boldsymbol{y}^{R^*}(1) = (1/3, 2/3)^{\mathrm{T}}$ and $\boldsymbol{x}^{R^*}(1) = (0, 5.3333, 0)^{\mathrm{T}}$. Hence, the upper bound ν^{m_2} of the mean interval of the company p_1's gain-floor and corresponding optimal strategy $\boldsymbol{y}^{m_2^*}$ are $\nu^{m_2} = \nu^R(1) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{R^*}(1) = 5.3333$ and $\boldsymbol{y}^{m_2^*} = \boldsymbol{y}^{R^*}(1) = (1/3, 2/3)^{\mathrm{T}}$, respectively.

According to Eq. (37), the linear programming problem can be obtained as follows:

$$\max\{-50x_1^L(1) + x_2^L(1) - x_3^L(1)\}$$
$$s.t. \begin{cases} -40x_1^L(1) + x_2^L(1) - x_3^L(1) - 29y_1^L(1) + 10y_2^L(1) \leq 0 \\ -70x_1^L(1) + x_2^L(1) - x_3^L(1) + 19y_1^L(1) - 40y_2^L(1) \leq 0 \\ 80y_1^L(1) + 50y_2^L(1) \leq 60 \\ y_1^L(1) + y_2^L(1) \leq 1 \\ -y_1^L(1) - y_2^L(1) \leq -1 \\ x_k^L(1) \geq 0\,(k = 1, 2, 3) \\ y_i^L(1) \geq 0\,(i = 1, 2). \end{cases} \tag{50}$$

Using the Simplex method of linear programming, the optimal solution $(\boldsymbol{x}^{L^*}(1), \boldsymbol{y}^{L^*}(1))$ to Eq. (50) can be obtained, where $\boldsymbol{y}^{L^*}(1) = (1/3, 2/3)^{\mathrm{T}}$ and $\boldsymbol{x}^{L^*}(1) = (0, 3, 0)^{\mathrm{T}}$. Therefore, the lower bound ν^{m_1} of the mean interval of the company p_1's gain-floor and corresponding optimal strategy $\boldsymbol{y}^{m_1^*}$ are $\nu^{m_1} = \nu^L(1) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{L^*}(1) = 3$ and $\boldsymbol{y}^{m_1^*} = \boldsymbol{y}^{L^*}(1) = (1/3, 2/3)^{\mathrm{T}}$, respectively.

According to Eq. (41), the linear programming problem can be constructed as follows:

$$\max\{-50x_1^R(0) + x_2^R(0) - x_3^R(0)\}$$
$$s.t. \begin{cases} -40\,x_1^R(0) + x_2^R(0) - x_3^R(0) - 35y_1^R(0) + 5y_2^R(0) \le 0 \\ -70x_1^R(0) + x_2^R(0) - x_3^R(0) + 17y_1^R(0) - 41y_2^R \le 0 \\ 80y_1^R(0) + 50y_2^R(0) \le 60 \\ y_1^R(0) + y_2^R(0) \le 1 \\ -y_1^R(0) - y_2^R(0) \le -1 \\ x_k^R(0) \ge 0\,(k = 1, 2, 3) \\ y_i^R(0) \ge 0\,(i = 1, 2). \end{cases} \tag{51}$$

The optimal solution $(\boldsymbol{x}^{R^*}(0), \boldsymbol{y}^{R^*}(0))$ to Eq. (51) can be obtained, where $\boldsymbol{y}^{R^*}(0) = (1/3, 2/3)^{\mathrm{T}}$ and $\boldsymbol{x}^{R^*}(0) = (0, 8.3333, 0)^{\mathrm{T}}$. The upper limit ν^r of the company p_1's gain-floor and corresponding optimal strategy $\boldsymbol{y}^{r^*}$ are $\nu^r = \nu^R(0) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{R^*}(0) = 8.3333$ and $\boldsymbol{y}^{r^*} = \boldsymbol{y}^{R^*}(0) = (1/3, 2/3)^{\mathrm{T}}$, respectively.

According to Eq. (42), the linear programming problem can be obtained as follows:

$$\max\{-50x_1^L(0) + x_2^L(0) - x_3^L(0)\}$$
$$s.t. \begin{cases} 40\,x_1^L(0) + x_2^L(0) - x_3^L(0) - 27y_1^L(0) + 11y_2^L(0) \le 0 \\ -70x_1^L(0) + x_2^L(0) - x_3^L(0) + 25y_1^L(0) - 35y_2^L(0) \le 0 \\ 80y_1^L(0) + 50y_2^L(0) \le 60 \\ y_1^L(0) + y_2^L(0) \le 1 \\ -y_1^L(0) - y_2^L(0) \le -1 \\ x_k^L(0) \ge 0\,(k = 1, 2, 3) \\ y_i^L(0) \ge 0\,(i = 1, 2). \end{cases} \tag{52}$$

The optimal solution $(\boldsymbol{x}^{L^*}(0), \boldsymbol{y}^{L^*}(0))$ to Eq. (52) can be obtained, where $\boldsymbol{y}^{L^*}(0) = (1/3, 2/3)^{\mathrm{T}}$ and $\boldsymbol{x}^{L^*}(0) = (0, 1.6667, 0)^{\mathrm{T}}$. The lower limit ν^l of the company p_1's gain-floor and corresponding optimal strategy $\boldsymbol{y}^{l^*}$ are $\nu^l = \nu^L(0) = \boldsymbol{d}^{\mathrm{T}}\boldsymbol{x}^{L^*}(0) = 1.6667$ and $\boldsymbol{y}^{l^*} = \boldsymbol{y}^{L^*}(0) = (1/3, 2/3)^{\mathrm{T}}$, respectively. Thus, the company p_1's gain-floor is a TrFN $\tilde{\nu} = (\nu^l, \nu^{m_1}, \nu^{m_2}, \nu^r) = (1.6667, 3, 5.3333, 8.3333)$, whose membership function is given as follows:

$$\mu_{\tilde{\nu}}(x) = \begin{cases} (x - 1.6667)/1.3333 & \text{if } 1.6667 \le x < 3 \\ 1 & \text{if } 3 \le x \le 5.3333 \\ (8.3333 - x)/3 & \text{if } 5.3333 < x \le 8.3333 \\ 0 & \text{else.} \end{cases} \tag{53}$$

Similarly, according to Eq. (38), the linear programming problem can be obtained as follows:

$$\min\{60s_1^R(1)+s_2^R(1)-s_3^R(1)\}$$
$$s.t.\begin{cases} 80s_1^R(1)+s_2^R(1)-s_3^R(1)-32z_1^R(1)+18z_2^R(1)\geq 0 \\ 50s_1^R(1)+s_2^R(1)-s_3^R(1)+8z_1^R(1)-40.5z_2^R(1)\geq 0 \\ -40z_1^R(1)-70z_2^R(1)\geq -50 \\ z_1^R(1)+z_2^R(1)\geq 1 \\ -z_1^R(1)-z_2^R(1)\geq -1 \\ s_l^R(1)\geq 0\,(l=1,2,3) \\ z_j^R(1)\geq 0\,(j=1,2). \end{cases} \tag{54}$$

Using the Simplex method of linear programming, the optimal solution ($\boldsymbol{s}^{R^*}(1)$, $\boldsymbol{z}^{R^*}(1)$) to Eq. (54) can be obtained, where $\boldsymbol{z}^{R^*}(1)=(1,0)^{\mathrm{T}}$ and $\boldsymbol{s}^{R^*}(1)=(1.3333, 0, 74.6667)^{\mathrm{T}}$. Therefore, the upper bound ω^{m_2} of the mean interval of the company p_2's loss-ceiling and corresponding optimal strategy $\boldsymbol{z}^{m_2^*}$ are $\omega^{m_2}=\omega^R(1)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{R^*}(1)=5.3333$ and $\boldsymbol{z}^{m_2^*}=\boldsymbol{z}^{R^*}(1)=(1,0)^{\mathrm{T}}$, respectively.

According to Eq. (39), the linear programming problem can be obtained as follows:

$$\min\{60s_1^L(1)+s_2^L(1)-s_3^L(1)\}$$
$$s.t.\begin{cases} 80s_1^L(1)+s_2^L(1)-s_3^L(1)-29z_1^L(1)+19z_2^L(1)\geq 0 \\ 50s_1^L(1)+s_2^L(1)-s_3^L(1)+10z_1^L(1)-40z_2^L(1)\geq 0 \\ -40z_1^L(1)-70z_2^L(1)\geq -50 \\ z_1^L(1)+z_2^L(1)\geq 1 \\ -z_1^L(1)-z_2^L(1)\geq -1 \\ s_l^L(1)\geq 0\,(l=1,2,3) \\ z_j^L(1)\geq 0\,(j=1,2). \end{cases} \tag{55}$$

The optimal solution $(\boldsymbol{s}^{L^*}(1),\boldsymbol{z}^{L^*}(1))$ to Eq. (55) can be obtained, where $\boldsymbol{z}^{L^*}(1)=(1,0)^{\mathrm{T}}$ and $\boldsymbol{s}^{L^*}(1)=(1.30,0,75.0)^{\mathrm{T}}$. Thus, the lower bound ω^{m_1} of the mean interval of the company p_2's loss-ceiling and corresponding optimal strategy $\boldsymbol{z}^{m_1^*}$ are $\omega^{m_1}=\omega^L(1)=\boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{L^*}(1)=3$ and $\boldsymbol{z}^{m_1^*}=\boldsymbol{z}^{L^*}(1)=(1,0)^{\mathrm{T}}$, respectively.

According to Eq. (43), the linear programming problem can be obtained as follows:

$$\min\{60s_1^R(0)+s_2^R(0)-s_3^R(0)\}$$
$$s.t.\begin{cases} 80s_1^R(0)+s_2^R(0)-s_3^R(0)-35z_1^R(0)+17z_2^R(0)\geq 0 \\ 50s_1^R(0)+s_2^R(0)-s_3^R(0)+5z_1^R(0)-41z_2^R(0)\geq 0 \\ -40z_1^R(0)-70z_2^R(0)\geq -50 \\ z_1^R(0)+z_2^R(0)\geq 1 \\ -z_1^R(0)-z_2^R(0)\geq -1 \\ s_l^R(0)\geq 0\,(l=1,2,3) \\ z_j^R(0)\geq 0\,(j=1,2). \end{cases} \tag{56}$$

The optimal solution $(\boldsymbol{s}^{R^*}(0),\boldsymbol{z}^{R^*}(0))$ to Eq. (56) can be obtained, where $\boldsymbol{z}^{R^*}(0)=(1,0)^{\mathrm{T}}$ and $\boldsymbol{s}^{R^*}(0)=(1.3333,0,71.6667)^{\mathrm{T}}$, respectively. Therefore, the upper limit

ω^r of the company p_2's loss-ceiling and corresponding optimal strategy z^{r^*} are $\omega^r = \omega^R(0) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{s}^{R^*}(0) = 8.3333$ and $z^{r^*} = z^{R^*}(0) = (1, 0)^{\mathrm{T}}$, respectively.

According to Eq. (44), the linear programming problem can be obtained as follows:

$$\min\{60s_1^L(0)+s_2^L0 - s_3^L(0)\}$$
$$s.t.\begin{cases} 80s_1^L0 + s_2^L0 - s_3^L0 - 27z_1^L0 + 25z_2^L0 \geq 0 \\ 50s_1^L0 + s_2^L0 - s_3^L0 + 11z_1^L(0) - 35z_2^L(0) \geq 0 \\ -40z_1^L0 - 70z_2^L0 \geq -50 \\ z_1^L0 + z_2^L0 \geq 1 \\ -z_1^L0 - z_2^L0 \geq -1 \\ s_l^L0 \geq 0\,(l = 1, 2, 3) \\ z_j^L0 \geq 0\,(j = 1, 2). \end{cases} \tag{57}$$

The optimal solution $(\boldsymbol{s}^{L^*}(0), z^{L^*}(0))$ to Eq. (57) can be obtained, where $z^{L^*}(0) = (1, 0)^{\mathrm{T}}$ and $\boldsymbol{s}^{L^*}(0) = (1.2667, 0, 74.3333)^{\mathrm{T}}$. Therefore, the lower limit ω^l of the company p_2's loss-ceiling and corresponding optimal strategy z^{l^*} are $\omega^l = \omega^L(0) = \boldsymbol{c}^{\mathrm{T}}s^{L^*}(0) = 1.6667$ and $z^{l^*} = z^{L^*}(0) = (1, 0)^{\mathrm{T}}$, respectively. Thus, the company p_2's loss-ceiling is a TrFN $\tilde{\omega} = (\omega^l, \omega^{m_1}, \omega^{m_2}, \omega^r) = (1.6667, 3, 5.3333, 8.3333)$.

Obviously, $\tilde{\nu} = \tilde{\omega} = (1.6667, 3, 5.3333, 8.3333)$, i.e., the company p_1's gain-floor and the company p_2's loss-ceiling have the identical TrFN-type value. Therefore, the constrained matrix game $\tilde{A}_0$ with payoffs of TrFNs has the fuzzy value, which is equal to the TrFN $\tilde{V} = (1.6667, 3, 5.3333, 8.3333)$.

Furthermore, using the linear programming models (i.e., Eqs. (29)–(32)), we also can obtain the interval-type value of any α-constrained matrix game and players' optimal strategies, where $\alpha \in [0, 1]$, depicted as in Table 1.

Table 1 The interval-type values of the α-constrained matrix games and players' optimal strategies

α	Play I			Play II		
	$\boldsymbol{y}^L(\alpha)^{\mathrm{T}}$	$\boldsymbol{y}^R(\alpha)^{\mathrm{T}}$	$(\nu^L(\alpha), \nu^R(\alpha))$	$z^L(\alpha)^{\mathrm{T}}$	$z^R(\alpha)^{\mathrm{T}}$	$(\omega^L(\alpha), \omega^R(\alpha))$
0.0	(1/3, 2/3)	(1/3, 2/3)	(1.6667, 8.3333)	(1, 0)	(1, 0)	(1.6667, 8.3333)
0.1	(1/3, 2/3)	(1/3, 2/3)	(1.8000, 8.0333)	(1, 0)	(1, 0)	(1.8000, 8.0333)
0.2	(1/3, 2/3)	(1/3, 2/3)	(1.9333, 7.7333)	(1, 0)	(1, 0)	(1.9333, 7.7333)
0.3	(1/3, 2/3)	(1/3, 2/3)	(2.0667, 7.4333)	(1, 0)	(1, 0)	(2.0667, 7.4333)
0.4	(1/3, 2/3)	(1/3, 2/3)	(2.2000, 7.1333)	(1, 0)	(1, 0)	(2.2000, 7.1333)
0.5	(1/3, 2/3)	(1/3, 2/3)	(2.3333, 6.8333)	(1, 0)	(1, 0)	(2.3333, 6.8333)
0.6	(1/3, 2/3)	(1/3, 2/3)	(2.4667, 6.5333)	(1, 0)	(1, 0)	(2.4667, 6.5333)
0.7	(1/3, 2/3)	(1/3, 2/3)	(2.6000, 6.2333)	(1, 0)	(1, 0)	(2.6000, 6.2333)
0.8	(1/3, 2/3)	(1/3, 2/3)	(2.7333, 5.9333)	(1, 0)	(1, 0)	(2.7333, 5.9333)
0.9	(1/3, 2/3)	(1/3, 2/3)	(2.8667, 5.6333)	(1, 0)	(1, 0)	(2.8667, 5.6333)
1.0	(1/3, 2/3)	(1/3, 2/3)	(3.0000, 5.3333)	(1, 0)	(1, 0)	(3.0000, 5.3333)

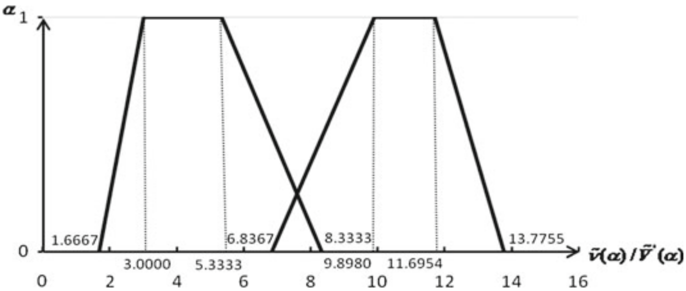

Fig. 1 The fuzzy values of the constrained matrix game with payoffs of TrFNs

It is easily seen from Table 1 that any α-constrained matrix game always has an interval-type value. Thereby, the constrained matrix game $\tilde{A}_0$ with payoffs of TrFNs has the TrFN-type value, depicted as in Fig. 1.

4.2 Computational results of the unconstrained matrix game with payoffs of TrFNs

If both companies do not take into account the constraint conditions of the strategies, then the above market share problem may be regarded as a (unconstrained) matrix game $\tilde{A}'_0$ with payoffs of TrFNs. Thus, using the methodology given by Li (2011), the linear programming problems are obtained as follows:

$$\min\{x_1'^R(1) + x_2'^R(1)\}$$
$$s.t. \begin{cases} 32x_1'^R(1) - 8x_2'^R(1) \geq 1 \\ -18x_1'^R(1) + 40.5x_2'^R(1) \geq 1 \\ x_1'^R(1) \geq 0,\ x_2'^R(1) \geq 0 \end{cases} \tag{58}$$

and

$$\min\{x_1'^L(1) + x_2'^L(1)\}$$
$$s.t. \begin{cases} 29x_1'^L(1) - 10x_2'^L(1) \geq 1 \\ -19x_1'^L(1) + 40x_2'^L(1) \geq 1 \\ x_1'^L(1) \geq 0,\ x_2'^L(1) \geq 0, \end{cases} \tag{59}$$

where $x_1'^R(1),\ x_2'^R(1),\ x_1'^L(1)$ and $x_2'^L(1)$ are decision variables.

Similarly, the two linear programming problems are constructed as follows:

$$\min\{x_1'^R(0) + x_2'^R(0)\}$$
$$s.t. \begin{cases} 35x_1'^R(0) - 5x_2'^R(0) \geq 1 \\ -17x_1'^R(0) + 41x_2'^R(0) \geq 1 \\ x_1'^R(0) \geq 0,\ x_2'^R(0) \geq 0 \end{cases} \tag{60}$$

and

$$\min\{x_1'^L(0)+x_2'^L(0)\}$$
$$s.t.\begin{cases}27x_1'^L(0)-11x_2'^L(0)\geq 1\\ -25x_1'^L(0)+35x_2'^L(0)\geq 1\\ x_1'^L(0)\geq 0,\ x_2'^L(0)\geq 0,\end{cases} \tag{61}$$

where $x_1'^R(0),\ x_2'^R(0),\ x_1'^L(0)$ and $x_2'^L(0)$ are decision variables.

Using the Simplex method of linear programming, the optimal solution $\boldsymbol{x}'^{m_1^*} = \boldsymbol{x}'^{L^*}(1) = (x_1'^{L^*}(1), x_2'^{L^*}(1))^{\mathrm{T}}$ of Eq. (59) is easily obtained, where $x_1'^{m_1^*} = 0.0515$ and $x_2'^{m_1^*} = 0.0495$. According to the methodology given by Li (2011), the lower bound v'^{m_1} of the mean interval and the company p_1's optimal strategy $\boldsymbol{y}'^{m_1^*}$ can be obtained, where

$$v'^{m_1} = v'^L(1) = 1/(x_1'^{L^*}(1)+x_2'^{L^*}(1)) = 9.8980$$

and

$$y_1'^{m_1^*} = y_1'^{L^*}(1) = x_1'^{L^*}(1)/(x_1'^{L^*}(1)+x_2'^{L^*}(1)) = 0.5102$$
$$y_2'^{m_1^*} = y_2'^{L^*}(1) = x_2'^{L^*}(1)/(x_1'^{L^*}(1)+x_2'^{L^*}(1)) = 0.4898,$$

respectively.

Likewise, solving Eqs. (58), (60) and (61), the upper bound of the mean interval, the lower and upper limits and the company p_1's optimal strategies can be obtained as follows:

$$v'^{m_2} = 11.6954,\quad \boldsymbol{y}'^{m_2^*} = (0.4924, 0.5076)^{\mathrm{T}}$$

and

$$v'^r = 13.7755,\quad \boldsymbol{y}'^{r^*} = (0.4694, 0.5306)^{\mathrm{T}}$$

and

$$v'^l = 6.8367,\quad \boldsymbol{y}'^{l^*} = (0.4694, 0.5306)^{\mathrm{T}},$$

respectively.

Thus, the company p_1's gain-floor is a TrFN $\tilde{v}' = (v'^l, v'^{m_1}, v'^{m_2}, v'^r) = (6.8367, 9.8980, 11.6954, 13.7755)$, whose membership function is given as follows:

$$\mu_{\tilde{v}'}(x) = \begin{cases}(x-6.8367)/3.0613 & \text{if } 6.8367\leq x<9.8980\\ 1 & \text{if } 9.8980\leq x\leq 11.6954\\ (13.7755-x)/2.0801 & \text{if } 11.6954<x\leq 13.7755\\ 0 & \text{else.}\end{cases}$$

In the same way, according to the methodology given by Li (2011), we have

$$\omega'^{m_1} = 9.8980, \quad z'^{m_1^*} = (0.6020, 0.3980)^{\mathrm{T}}$$
$$\omega'^{m_2} = 11.69549, \quad z'^{m_2^*} = (0.5939, 0.4061)^{\mathrm{T}}$$
$$\omega'^{r} = 13.7755, \quad z'^{r^*} = (0.5918, 0.4082)^{\mathrm{T}}$$

and

$$\omega'^{l} = 6.8367, \quad z'^{l^*} = (0.6122, 0.3878)^{\mathrm{T}},$$

respectively. Then, the company p_2's loss-celling is a TrFN $\tilde{\omega}' = (\omega'^{l}, \omega'^{m_1}, \omega'^{m_2}, \omega'^{r})$ (6.8367,9.8980,11.6954,13.7755).

Thus, the company p_1's gain-floor and the company p_2's loss-celling have the identical TrFN-type value. Hereby the (unconstrained) matrix game $\tilde{A}_0'$ with payoffs of TrFNs has the TrFN-type value $\tilde{V}' = \tilde{\nu}' = \tilde{\omega}' = (6.8367,9.8980,11.6954,13.7755)$, depicted as in Fig. 1.

It is easily seen from Fig. 1 that the fuzzy value $\tilde{V}'$ and the companies' optimal strategies in the unconstrained matrix game $\tilde{A}_0'$ with payoffs of TrFNs are different from the fuzzy value $\tilde{V}$ and the corresponding optimal strategies in the constrained matrix game $\tilde{A}_0$ with payoffs of TrFNs. Moreover, $\tilde{V}'$ is larger than $\tilde{V}$. These conclusions are rational and accordance with the actual situation as expected. On the other hand, it is shown that it is necessary to take into consideration the constraint conditions of strategies' choice for players.

5 Conclusion

Fuzzy game theory can give a basic conceptual framework for formulating and analyzing antagonistic decision problems that present some source of impreciseness and uncertainty on any of its elements. However, because of obtaining different values or defuzzification values for players, the existing methods are not rational and effective from a viewpoint of logic and the concept of zero-sum games with payoffs of TrFNs. The method proposed in this paper can always ensure that any constrained matrix game with payoffs of TrFNs has the TrFN-type value. These conclusions are in accordance with the viewpoint of logic and the zero-sum feature of matrix games with payoffs of TrFNs. The auxiliary linear programming models are derived to compute α-cuts (i.e., interval-type values) of the fuzzy value of any constrained matrix game with payoffs of TrFNs and players' optimal strategies. Particularly, the TrFN-type value of the constrained matrix game with payoffs of TrFNs can be directly and explicitly obtained through solving the derived four linear programming models with data taken from only 1-cut and 0-cut of the TrFN-type payoffs.

It is obvious that the constrained matrix games with payoffs of TrFNs is a special case of the constrained matrix games. In fact, if all the TrFN-type payoffs degenerate to real numbers, i.e., $a_{ij}^{l} = a_{ij}^{m_1} = a_{ij}^{m_2} = a_{ij}^{r}$, then the constrained matrix games with payoffs of TrFNs are reduced to the constrained matrix games. Hence, the linear programming models (i.e., Eqs. (29)–(32)) are reduced to those (i.e., Eqs. (5) and (6)) of the constrained matrix games. Furthermore, if the constraint conditions of players' strategies are not taken into account, the linear programming models are reduced to those of the (unconstrained) matrix games (Owen 1982). Therefore, the models and method proposed in this paper are extensions of those in the constrained matrix games and the matrix games.

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References

Aggarwal, A., Mehra, A., & Chandra, S. (2012). Application of linear programming with I-fuzzy sets to matrix games with I-fuzzy goals. Fuzzy Optimization and Decision Making. doi:10.1007/s10700-012-9123-z.

Bector, C.R., Chandra, S., & Vijay, V. (2004). Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optimization and Decision Making, 3*, 255–269.

Collins, W.D., & Hu, C.-Y. (2008). Studying interval valued matrix games with fuzzy logic. *Soft Computing, 12*, 147–155.

Dresher, M. (1961). *Games of strategy theory and applications*. New York: Prentice-Hall.

Dubois, D., & Prade, H. (1980). *Fuzzy sets and systems theory and applications*. New York: Academic Press.

Larbani, M. (2009). Non cooperative fuzzy games in normal form: A survey. *Fuzzy Sets and Systems, 160*, 3184–3210.

Li, D.-F. (1999). Fuzzy constrained matrix games with fuzzy payoffs. *The International Journal of Fuzzy Mathematics, 7*, 873–880.

Li, D.-F., & Cheng, C.-T. (2002). Fuzzy multiobjective programming methods for fuzzy constrained matrix games with fuzzy numbers. *Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 10*, 385–400.

Li, D.-F. (2011). Linear programming approach to solve interval-valued matrix games. *Omega, 39*, 655–666.

Liu, S.T., & Kao, C. (2009). Matrix games with interval date. *Computers and Industrial Engineering, 56*, 1697–1700.

Moore, R.E. (1979). *Method and application of interval analysis*. Philadelphia: SIAM.

Owen, G. (1982). *Game theory* (2nd ed.). New York: Academic Press.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*, 338–356.