Fuzzy Optim Decis Making (2013) 12:89–98
DOI 10.1007/s10700-012-9143-8

On comonotonic functions of uncertain variables

Xiaohu Yang

Published online: 5 October 2012

Abstract Uncertain variable is used to model quantities in uncertainty. This paper considers comonotonic functions of an uncertain variable, and gives their uncertainty distributions. Besides, it proves the linearity of expected value operator on comonotonic functions of an uncertain variable.

Keywords Uncertainty theory · Uncertain variable · Expected value · Comonotonicity

1 Introduction

Probability theory is a branch of axiomatic mathematics to deal with random phenomena and is applicable only when the estimated probability is closed enough to the real frequency. However, we are usually lack of historical data to estimate the probability in practice. In that case, some experts are invited to evaluate their belief degree. Since human beings tend to overweight unlikely events (Kahneman and Tversky 1979), the belief degree may have a much larger range than the real frequency. If we insist on treating the belief degree as probability, some counterintuitive results will happen. Interested readers may refer to Liu (2012) for an example. In order to deal with this situation, an uncertainty theory was founded by Liu (2007) and refined by Liu (2011) based on normality, duality, subadditivity and product axioms.

X. Yang (✉)
Department of Statistics, Xi'an University of Finance and Economics,
Xi'an, 710061 Shaanxi, China
e-mail: yangxiaohu@live.xaufe.edu.cn

X. Yang
College of Mathematics and Information Science, Weinan Normal University,
Weinan, 714000 Shaanxi, China

In the framework of uncertainty theory, many researchers have contributed a lot. Gao (2009) provided some properties of continuous uncertain measure, and Zhang (2011) proposed some inequalities on uncertain measure. Based on uncertain measure, a concept of uncertain variable was proposed as a measurable function from an uncertainty space to a real number set. You (2009) proposed some convergence theorems on a sequence of uncertain variables. Then Tian (2011) and Yang (2011) proposed some moment inequalities of uncertain variables. In order to describe an uncertain variable, a concept of uncertainty distribution was defined. Peng and Iwamura (2010) gave a sufficient and necessary condition for a function being an uncertainty distribution of an uncertain variable. Expected value is an important index to compare uncertain variables. Liu and Ha (2010) gave some theorems on expected values of monotone functions of uncertain variables.

This paper will consider the comonotonic functions of an uncertain variable. The rest of this paper is organized as follows. In Sect. 2, we review some basic results of uncertainty theory which will be used throughout the paper. Then the concept of comonotonic functions is given in Sect. 3. After that, the uncertainty distributions and expected values of comonotonic functions of uncertain variables are given in Sects. 4 and 5, respectively. At last, some remarks are made in Sect. 6.

2 Preliminary

Uncertainty theory is a branch of axiomatic mathematics to model human uncertainty. The fundamental concept in uncertainty theory is uncertain measure.

Definition 1 (Liu 2007) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M} : \mathcal{L} \to [0, 1]$ is called an uncertain measure if it satisfies the following axioms:

Axiom 1: (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2: (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3: (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots,$ we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Besides, the product uncertain measure on the product σ-algebra $\mathcal{L}$ was defined via Liu (2009) by the following axiom.

Axiom 4: (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Definition 2 (Liu 2007) An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \mid \xi(\gamma) \in B\}$$

is an event.

In order to describe an uncertain variable, a concept of uncertainty distribution is defined.

Definition 3 (Liu 2007) The uncertainty distribution of an uncertain variable ξ is defined by

$$\Phi(x) = \mathcal{M}\{\xi \le x\}$$

for any $x \in \Re$.

The inverse function $\Phi^{-1}(\alpha)$ is called an inverse uncertainty distribution of ξ. It plays an important role in the operation of uncertain variables.

Theorem 1 (Liu 2011) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(r) = f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)\right).$$

Definition 4 (Liu 2007) The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x$$

provided that at least one of the two integrals exists.

Assume that ξ has a regular uncertainty distribution Φ. Then Liu (2011) proved that

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x$$

and

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha$$

provided that the expected value $E[\xi]$ exists. Following that, Liu and Ha (2010) proved that

$$E[f(\xi_1, \xi_2, \ldots, \xi_n)] \\ = \int_0^1 f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1-r), \ldots, \Phi_n^{-1}(1-r)\right) \mathrm{d}r$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables, $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. In addition, Liu (2011) proved the linearity of expected value operator on independent uncertain variables, that is $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ for independent uncertain variables ξ and η and any real numbers a and b.

3 Comonotonic functions

This section will introduce the concept of comonotonic functions, and give a theorem about them.

Definition 5 Two functions f and g are said to be comonotonic if

$$(f(x_1) - f(x_2))(g(x_1) - g(x_2)) \geq 0$$

for any real numbers x_1 and x_2.

Example 1 Consider the functions $f(x) = |x|$ and $g(x) = x^2$. For two real numbers x_1 and x_2, if $f(x_1) = |x_1| \leq |x_2| = f(x_2)$, then $g(x_1) = x_1^2 \leq x_2^2 = g(x_2)$. As a result, we have

$$(f(x_1) - f(x_2))(g(x_1) - g(x_2)) \geq 0$$

for any real numbers x_1 and x_2. Thus f and g are comonotonic functions.

Theorem 2 *Let f and g be two monotone increasing functions on $\Re$. Then they are comonotonic functions.*

Proof Let x_1 and x_2 be two real numbers with $x_1 < x_2$. Since f and g are monotone increasing functions, we have $f(x_1) \leq f(x_2)$, $g(x_1) \leq g(x_2)$. Then

$$(f(x_1) - f(x_2))(g(x_1) - g(x_2)) \geq 0.$$

As a result, f and g are comonotonic functions. □

Example 2 Consider the functions $f(x) = x$ and $g(x) = \exp(x)$. Since both of them are increasing functions, they are comonotonic functions.

Theorem 3 *Let f and g be two monotone decreasing functions on $\Re$. Then they are comonotonic functions.*

Proof Let x_1 and x_2 be two real numbers with $x_1 < x_2$. Since f and g are monotone decreasing functions, we have $f(x_1) \geq f(x_2)$, $g(x_1) \geq g(x_2)$. Then

$$(f(x_1) - f(x_2))(g(x_1) - g(x_2)) \geq 0.$$

As a result, f and g are comonotonic functions. □

Example 3 Consider the functions $f(x) = -x$ and $g(x) = \exp(-x)$. Since both of them are decreasing functions, they are comonotonic functions.

Theorem 4 *Let f_1 and f_2 be two comonotonic functions. Then for any real numbers y_1 and y_2, either $\{f_1(x) \leq y_1\} \subset \{f_2(x) \leq y_2\}$ or $\{f_2(x) \leq y_2\} \subset \{f_1(x) \leq y_1\}$ holds.*

Proof The theorem will be proved by a contradiction method. Write $A_1 = \{x \mid f_1(x) \leq y_1\}$ and $A_2 = \{x \mid f_2(x) \leq y_2\}$. Assume that $A_1 \not\subset A_2$ and $A_2 \not\subset A_1$. Then taking $x_1 \in A_1 \backslash A_2$, and $x_2 \in A_2 \backslash A_1$, we have

$$f_1(x_1) \leq y_1 < f_1(x_2),$$
$$f_2(x_2) \leq y_2 < f_2(x_1).$$

Thus

$$(f_1(x_1) - f_1(x_2))(f_2(x_1) - f_2(x_2)) < 0$$

which contradicts to the definition of comonotonic functions. Thus either $A_1 \subset A_2$ or $A_2 \subset A_1$. The theorem is proved. □

4 Comonotonic functions of an uncertain variable

In this section, we first consider the uncertain measures of two events generated from comonotonic functions of an uncertain variable. Then we give the uncertainty distribution of comonotonic functions of an uncertain variable.

Theorem 5 *Let ξ be an uncertain variable, and f_1 and f_2 be two comonotonic functions. Then for any real numbers y_1 and y_2, we have*

$$\mathcal{M}\{(f_1(\xi) \leq y_1) \cap (f_2(\xi) \leq y_2)\} = \mathcal{M}\{f_1(\xi) \leq y_1\} \wedge \mathcal{M}\{f_2(\xi) \leq y_2\}.$$

Proof Write $A_1 = \{x \mid f_1(x) \le y_1\}$ and $A_2 = \{x \mid f_2(x) \le y_2\}$. Then we have $A_1 \subset A_2$ or $A_2 \subset A_1$ by Theorem 4. Besides, we have $\{f_1(\xi) \le y_1\} = \{\xi \in A_1\}$ and $\{f_2(\xi) \le y_2\} = \{\xi \in A_2\}$ Thus either $\{f_1(\xi) \le y_1\} \subset \{f_2(\xi) \le y_2\}$ or $\{f_2(\xi) \le y_2\} \subset \{f_1(\xi) \le y_1\}$ holds. By the property of uncertain measure, we have

$$\mathcal{M}\{(f_1(\xi) \le y_1) \cap (f_2(\xi) \le y_2)\} = \mathcal{M}\{f_1(\xi) \le y_1\} \wedge \mathcal{M}\{f_2(\xi) \le y_2\}.$$

□

Theorem 6 *Let ξ be an uncertain variable, and f_1 and f_2 be two comonotonic functions. Then for any real numbers y_1 and y_2, we have*

$$\mathcal{M}\{(f_1(\xi) \ge y_1) \cap (f_2(\xi) \ge y_2)\} = \mathcal{M}\{f_1(\xi) \ge y_1\} \wedge \mathcal{M}\{f_2(\xi) \ge y_2\}.$$

Proof Since f_1 and f_2 are comonotonic functions, $-f_1$ and $-f_2$ are also comonotonic functions. Then for any real numbers y_1 and y_2, by Theorem 5 we have

$$\begin{aligned}\mathcal{M}\{(f_1(\xi) \ge y_1) \cap (f_2(\xi) \ge y_2)\} &= \mathcal{M}\{(-f_1(\xi) \le -y_1) \cap (-f_2(\xi) \le -y_2)\} \\ &= \mathcal{M}\{-f_1(\xi) \le -y_1\} \wedge \mathcal{M}\{-f_2(\xi) \le -y_2\} \\ &= \mathcal{M}\{f_1(\xi) \ge y_1\} \wedge \mathcal{M}\{f_2(\xi) \ge y_2\}.\end{aligned}$$

The theorem is thus proved. □

Before proving the theorem on the inverse uncertainty distributions of comonotonic functions of an uncertain variable, we first introduce the definition of monotonicity of a binary function.

Definition 6 A binary function $H(x, y)$ is said to be increasing if $f(x_1, y_1) \le f(x_2, y_2)$ whenever $x_1 \le x_2$ and $y_1 \le y_2$, and $f(x_1, y_1) < f(x_2, y_2)$ whenever $x_1 < x_2$ and $y_1 < y_2$.

Theorem 7 *Let $H(x, y)$ be an increasing function, and f and g be two comonotonic functions. Assume that $f(\xi)$ and $g(\xi)$ are two regular uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then $H(f(\xi), g(\xi))$ has an inverse uncertainty distribution $H\left(\Phi^{-1}(\alpha), \Psi^{-1}(\alpha)\right)$.*

Proof Since $H(x, y)$ is an increasing function, we have

$$\begin{aligned}&\mathcal{M}\left\{H(f(\xi), g(\xi)) \le H\left(\Phi^{-1}(\alpha), \Psi^{-1}(\alpha)\right)\right\} \\ \ge\ &\mathcal{M}\left\{\left(f(\xi) \le \Phi^{-1}(\alpha)\right) \cap \left(g(\xi) \le \Psi^{-1}(\alpha)\right)\right\} \\ =\ &\mathcal{M}\left\{f(\xi) \le \Phi^{-1}(\alpha)\right\} \wedge \left\{g(\xi) \le \Psi^{-1}(\alpha)\right\} \\ =\ &\alpha \wedge \alpha = \alpha.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{M}\left\{H(f(\xi), g(\xi)) \le H\left(\Phi^{-1}(\alpha), \Psi^{-1}(\alpha)\right)\right\} \\
\le\ &\mathcal{M}\left\{\left(f(\xi) \le \Phi^{-1}(\alpha)\right) \cup \left(g(\xi) \le \Psi^{-1}(\alpha)\right)\right\} \\
=\ &\mathcal{M}\left\{f(\xi) \le \Phi^{-1}(\alpha)\right\} \vee \left\{g(\xi) \le \Psi^{-1}(\alpha)\right\} \\
=\ &\alpha \vee \alpha = \alpha.
\end{aligned}$$

Thus

$$\mathcal{M}\left\{H(f(\xi), g(\xi)) \le H\left(\Phi^{-1}(\alpha), \Psi^{-1}(\alpha)\right)\right\} = \alpha$$

which means $H(f(\xi), g(\xi))$ has an inverse uncertainty distribution $H\left(\Phi^{-1}(\alpha), \Psi^{-1}(\alpha)\right)$. □

Example 4 Let f and g be two comonotonic functions. Assume that $f(\xi)$ and $g(\xi)$ are two regular uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then $f(\xi) + g(\xi)$ has an inverse uncertainty distribution $\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)$.

Example 5 Let f and g be two comonotonic functions. Assume that $f(\xi)$ and $g(\xi)$ are two nonnegative regular uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then $f(\xi) \times g(\xi)$ has an inverse uncertainty distribution $\Phi^{-1}(\alpha) \times \Psi^{-1}(\alpha)$.

Example 6 Let f and g be two comonotonic functions. Assume that $f(\xi)$ and $g(\xi)$ are two regular uncertain variables with uncertainty distributions Φ and Ψ, respectively. Then $f(\xi) \wedge g(\xi)$ has an inverse uncertainty distribution $\Phi^{-1}(\alpha) \wedge \Psi^{-1}(\alpha)$, and $f(\xi) \vee g(\xi)$ has an inverse uncertainty distribution $\Phi^{-1}(\alpha) \vee \Psi^{-1}(\alpha)$.

5 Expected value

In this section, we give a theorem on the expected values of comonotonic functions of uncertain variables.

Theorem 8 *Let f and g be two comonotonic functions. Then for any uncertain variable ξ, we have*

$$E[f(\xi) + g(\xi)] = E[f(\xi)] + E[g(\xi)].$$

Proof Assume that $f(\xi)$ and $g(\xi)$ have uncertainty distributions Φ and Ψ, respectively. Then $f(\xi) + g(\xi)$ has an inverse uncertainty distribution $\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)$.

Thus we have

$$E[f(\xi)+g(\xi)] = \int_0^1 \left(\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\right)\mathrm{d}\alpha$$
$$= \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha$$
$$= E[f(\xi)] + E[g(\xi)].$$

□

Remark 1 The theorem does not hold if f and g are not assumed comonotonic. Here, we give an example. Let the universal set $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with

$$\mathcal{M}\{\gamma_1\} = 0.5, \quad \mathcal{M}\{\gamma_2\} = 0.4, \quad \mathcal{M}\{\gamma_3\} = 0.3.$$

Consider an uncertain variable ξ defined by

$$\xi(\gamma_1) = -1, \quad \xi(\gamma_2) = 1, \quad \xi(\gamma_3) = 2.$$

Then

$$\xi^{-1}(\gamma_1) = -1, \quad \xi^{-1}(\gamma_2) = 1, \quad \xi^{-1}(\gamma_3) = 1/2,$$
$$(\xi+\xi^{-1})(\gamma_1) = -2, \quad (\xi+\xi^{-1})(\gamma_2) = 2, \quad (\xi+\xi^{-1})(\gamma_3) = 5/2.$$

By the definition of expected value, we have

$$E[\xi] = 0.3, \quad E[\xi^{-1}] = -0.05, \quad E[\xi+\xi^{-1}] = 0.15.$$

So $E[\xi+\xi^{-1}] \neq E[\xi]+E[\xi^{-1}]$. However, for monotone functions f and g, we have the following corollaries.

Corollary 1 *Let f and g be increasing functions, and ξ be an uncertain variable. Then*

$$E[f(\xi)+g(\xi)] = E[f(\xi)] + E[g(\xi)].$$

Proof Note that increasing functions are comonotonic with each other. The corollary follows immediately. □

Corollary 2 *Let f and g be decreasing functions, and ξ be an uncertain variable. Then*

$$E[f(\xi)+g(\xi)] = E[f(\xi)] + E[g(\xi)].$$

Proof Note that decreasing functions are comonotonic with each other. The corollary follows immediately. □

Example 7 Both $f(x) = x$ and $g(x) = \exp(x)$ are increasing functions. Thus for uncertain variable ξ, we have

$$E[\xi + \exp(\xi)] = E[\xi] + E[\exp(\xi)].$$

Example 8 Let ξ be a nonnegative uncertain variable. Note that $f(x) = x$ and $g(x) = x^2$ are increasing functions on $[0, +\infty)$. Then we have

$$E\left[\xi + \xi^2\right] = E[\xi] + E\left[\xi^2\right].$$

Example 9 Let ξ be an uncertain variable taking values in $[-\pi, \pi]$. Note that $f(x) = \cos x$ and $g(x) = -x^2$ are comonotonic functions on $[-\pi, \pi]$. Then we have

$$E\left[\cos\xi - \xi^2\right] = E[\cos\xi] + E\left[-\xi^2\right] = E[\cos\xi] - E\left[\xi^2\right].$$

6 Conclusions

This paper investigated comonotonic functions of uncertain variables, and gave some properties of their uncertainty distributions. In addition, it proved that the expected value of comonotonic functions of an uncertain variable is additive.

Acknowledgments This work was supported by National Bureau of Statistics Foundation (Program No.2011LY092) and Shaanxi Featuring Military and Civilian Integration (Program No.11JMR09).

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