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Optimizing fuzzy portfolio selection problems by parametric quadratic programming

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Abstract This paper develops a robust method to describe fuzzy returns by employing parametric possibility distributions. The parametric possibility distributions are obtained by equivalent value (EV) reduction methods. For common type-2 triangular and trapezoidal fuzzy variables, their reduced fuzzy variables are studied in the current development. The parametric possibility distributions of reduced fuzzy variables are first derived, then the second moment formulas for the reduced fuzzy variables are established. Taking the second moment as a new risk measure, the reward-risk and risk-reward models are developed to optimize fuzzy portfolio selection problems. The mathematical properties of the proposed optimization models are analyzed, including the analytical representations for the second moments of linear combinations of reduced fuzzy variables as well as the convexity of second moments with respect to decision vectors. On the basis of the analytical representations for the second moments, the reward-risk and risk-reward models can be turned into their equivalent parametric quadratic convex programming problems, which can be solved by conventional solution methods or general-purpose software. Finally, some numerical experiments are performed to demonstrate the new modeling ideas and the efficiency of solution method.

Keywords Portfolio selection · Reduced fuzzy variable · Parametric possibility distribution · Moment · Parametric programming

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1 Introduction

Portfolio selection problem was first proposed by Markowitz (1952), and mean-variance method has been widely accepted as a practical tool for portfolio optimization. The use of the semivariance rather than variance as the risk measure was also proposed by Markowitz (1959). Since then, several other risk measures have been documented in portfolio literature. For example, Konno and Yamakazi (1991) measured investment risk by absolute deviation and developed mean-absolute-deviation models; Jorion (1996) studied Value-at-Risk (VaR) as a risk measure, and applied the mean-VaR model in finance industry, and Rockafellar and Uryaser (2000) reduced investment risk by minimizing conditional Value-at-Risk (CVaR), and established mean-CVaR model.

The conventional portfolio methods require security returns are random variables, and probability theory is the main research tool. However, the observed values of security returns in real-world problems are sometimes imprecise or vague. Imprecise evaluations may result from unquantifiable, incomplete and non obtainable information. On the basis of fuzzy theory (Zadeh 1965, 1978; Liu 2004), some researchers have proposed various fuzzy methods for dealing with this impreciseness and ambiguity in portfolio selection. For instance, Arenas-Parra et al. (2001) discussed the optimal portfolio for a private investor by taking into account three criteria: return, risk and liquidity; Chen et al. (2009) proposed a possibilistic mean variance portfolio selection model, and solved it by a cutting plane algorithm; Huang (2009) gave a detailed survey about fuzzy portfolio selection based on credibility measure; Duan and Stahlecker (2011) studied a static portfolio selection problem, in which future returns of securities are given as fuzzy sets, and Wu and Liu (2011) proposed mean-spread method for optimal portfolio selection problems to avoid the difficulty of computing the variance of fuzzy variable. The interested reader may also refer to the book (Huang, 2010) about recent development of portfolio analysis under uncertainty.

In fuzzy community, Zadeh (1975) first proposed the concept of a type-2 fuzzy set as an extension of ordinary fuzzy set. A type-2 fuzzy set is characterized by a fuzzy membership function, where the degree of membership for any element in this set is a fuzzy number in the interval [0, 1]. Since then, type-2 fuzzy theory has been well developed in the literature. For example, Mizumoto and Tanaka (1976) investigated the algebraic structures that the fuzzy grades of type-2 fuzzy sets form under the operations of join, meet and negation, and showed that normal convex fuzzy grades form a distributive lattice under join and meet; Dubois and Prade (1979) developed the operations in a fuzzy-valued logic; Karnik and Mendel (2001) proposed a defuzzification method by using the concept of centroid for a type-2 fuzzy set; Mendel and John (2002) showed us that a type-2 fuzzy set represents the uncertainty in terms of secondary membership function and footprint of uncertainty; Mitchell (2005) introduced a similarity measure for measuring the similarity or compatibility between two type-2 fuzzy sets; Mendel (2007) described the important advances for both general and interval type-2 fuzzy sets and systems; Liu and Liu (2010) studied type-2 fuzziness in an axiomatic framework referred to as fuzzy possibility theory; Chen and Zhang (2011) gave some new results about arithmetic of type-2 fuzzy variables, and Qin et al. (2011a,b) proposed the critical value and mean value reduction methods for secondary possibility distribution functions. In the current development, a new EV reduction method will be discussed for type-2 triangular and trapezoidal fuzzy variables.

On the other hand, we will study the portfolio selection problems from a new viewpoint, in which fuzzy returns are characterized by parametric possibility distributions. More precisely, to deal with type-2 fuzziness encountered in portfolio selection problems, the security returns can be represented by parametric possibility distributions, which are obtained by using the EV reduction methods. On the basis of the second moments of the reduced fuzzy variables, the reward-risk and risk-reward models are developed to deal with fuzzy portfolio optimization problems. In our models, the investment return is quantified by the expected value, while the investment risk is gauged by the second moment of a portfolio. The mathematical properties of the proposed optimization models are analyzed, including the analytical representations for the second moments of linear combinations of reduced fuzzy variables as well as the convexity of second moments with respect to decision vectors. Using the analytical representations for the second moments, the reward-risk and risk-reward models can be turned into their equivalent parametric quadratic convex programming problems, which can be solved by conventional numerical algorithms or general-purpose software. Some numerical experiments are performed to demonstrate the above modeling ideas and the efficiency of solution method.

The rest of this paper is organized as follows. For common type-2 triangular and trapezoidal fuzzy variables, Sect. 2 derives the parametric possibility distributions of reduced fuzzy variables, and Sect. 3 establishes the second moment formulas of reduced fuzzy variables by L–S integral and discusses their convexity with respect to fuzzy parameters. In Sect. 4, we develop the reward-risk and risk-reward models for fuzzy portfolio selection problems. Section 5 deals with the equivalent parametric programming problems of the proposed optimization models as well as their solution methods. Section 6 provides two numerical examples to demonstrate the new modeling ideas and the effectiveness of the solution method. Section 7 concludes the paper.

2 Parametric possibility distributions of reduced fuzzy variables

If a fuzzy variable ξ takes its values in the unit interval $[0, 1]$, then it is called a *regular fuzzy variable*. Suppose $\mu_\xi(t)$ is a generalized possibility distribution (not necessarily normalized) of regular fuzzy variable ξ. Then for any $t \in [0, 1]$, the possibility, necessity and credibility of fuzzy event $\{\xi \le t\}$ are computed by

$$\mathrm{Pos}\{\xi \le t\} = \sup_{0\le u\le t} \mu_\xi(u), \tag{1}$$

$$\mathrm{Nec}\{\xi \le t\} = \sup_{0\le u\le 1} \mu_\xi(u) - \sup_{t<u\le 1} \mu_\xi(u), \tag{2}$$

and

$$\mathrm{Cr}\{\xi \le t\} = \frac{1}{2}\left(\sup_{0\le u\le 1} \mu_\xi(u) + \sup_{0\le u\le t} \mu_\xi(u) - \sup_{t<u\le 1} \mu_\xi(u)\right). \tag{3}$$

If the possibility distribution $\mu_\xi(t)$ is normalized, then the credibility defined above coincides with the concept defined in Liu and Liu (2002).

Let ξ be a regular fuzzy variable with a generalized possibility distribution μ_ξ. Then the pessimistic equivalent value (PEV) of ξ is calculated by the following integral

$$\mathrm{EV}_*[\xi] = \int_{[0,1]} t\mathrm{d}\left(\mathrm{Pos}\{\xi \leq t\}\right), \tag{4}$$

while the optimistic equivalent value (OEV) of ξ is calculated by the following integral

$$\mathrm{EV}^*[\xi] = \int_{[0,1]} t\mathrm{d}\left(\mathrm{Nec}\{\xi \leq t\}\right). \tag{5}$$

The equivalent value (EV) of ξ is computed by the following integral

$$\mathrm{EV}[\xi] = \int_{[0,1]} t\mathrm{d}\left(\mathrm{Cr}\{\xi \leq t\}\right). \tag{6}$$

The three integrals in Eqs. (4), (5) and (6) are all L–S integrals, and the properties and calculation methods about L–S integral can be found in the book (Carter and Van Brunt, 2000).

A type-2 fuzzy variable represents the uncertainty in terms of second possibility distribution and support of uncertainty. In this section, we give a new method to reduce the uncertainty in second possibility distribution. Different from the critical value and mean value reduction methods defined by Qin et al. (2011a,b), the EV reduction method considered in the current development is based on classical L–S integral instead of fuzzy integrals.

Let $\tilde{\xi}$ be a type-2 fuzzy variable. In order to reduce the uncertainty in secondary possibility distribution, we now define the PEV, OEV and EV of regular fuzzy variable $\tilde{\mathrm{P}}\mathrm{os}\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) = \mathrm{t}\}$ as its representing values. The methods are referred to as the PEV, OEV and EV reduction method for $\tilde{\xi}$, respectively. In the following sections, the variables obtained by the three EV reduction methods are called reduced fuzzy variables.

A type-2 fuzzy variable $\tilde{\xi}$ is called trapezoidal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(t)$ is the regular fuzzy variable

$$\left(\frac{t-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{t-r_1}{r_2-r_1}, \frac{r_2-t}{r_2-r_1}\right\}, \frac{t-r_1}{r_2-r_1}, \frac{t-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{t-r_1}{r_2-r_1}, \frac{r_2-t}{r_2-r_1}\right\}\right) \tag{7}$$

for $t \in [r_1, r_2]$, the regular fuzzy variable $\tilde{1}$ for $t \in [r_2, r_3]$, and the regular fuzzy variable

$$\left(\frac{r_4-t}{r_4-r_3} - \theta_l \min\left\{\frac{r_4-t}{r_4-r_3}, \frac{t-r_3}{r_4-r_3}\right\}, \frac{r_4-t}{r_4-r_3}, \frac{r_4-t}{r_4-r_3} + \theta_r \min\left\{\frac{r_4-t}{r_4-r_3}, \frac{t-r_3}{r_4-r_3}\right\}\right) \tag{8}$$

for $t \in [r_3, r_4]$, where $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty that $\tilde{\xi}$ takes the value t. We denote the type-2 trapezoidal fuzzy variable $\tilde{\xi}$ with the above secondary possibility distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$.

For a type-2 trapezoidal fuzzy variable $\tilde{\xi}$, its reduced fuzzy variables have the following parametric possibility distributions:

Proposition 1 *Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$ be a type-2 trapezoidal fuzzy variable, and ξ_*, ξ^* and ξ its reduced fuzzy variables obtained by the PEV, OEV and EV methods, respectively. Then we have:*

(i) The parametric possibility distribution of ξ_* is

$$\mu_{\xi_*}(t;\theta_l) = \begin{cases} \frac{(2-\theta_l)(t-r_1)}{2(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2} \\ \frac{(2+\theta_l)t-\theta_l r_2-2r_1}{2(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{(-2-\theta_l)t+\theta_l r_3+2r_4}{2(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2} \\ \frac{(2-\theta_l)(r_4-t)}{2(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4, \end{cases}$$

and denote $\xi_* = (r_1, r_2, r_3, r_4; h_*(\theta_l))$, where $h_*(\theta_l) = \mu_{\xi_*}\left(\frac{r_1+r_2}{2};\theta_l\right) = \mu_{\xi_*}\left(\frac{r_3+r_4}{2};\theta_l\right) = \frac{2-\theta_l}{4}$.

(ii) The parametric possibility distribution of ξ^* is

$$\mu_{\xi^*}(t;\theta_r) = \begin{cases} \frac{(2+\theta_r)(t-r_1)}{2(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2} \\ \frac{(2-\theta_r)t+\theta_r r_2-2r_1}{2(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{(\theta_r-2)t-\theta_r r_3+2r_4}{2(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2} \\ \frac{(2+\theta_r)(r_4-t)}{2(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4, \end{cases}$$

and denote $\xi^* = (r_1, r_2, r_3, r_4; h^*(\theta_r))$, where $h^*(\theta_r) = \mu_{\xi^*}\left(\frac{r_1+r_2}{2};\theta_r\right) = \mu_{\xi^*}\left(\frac{r_3+r_4}{2};\theta_r\right) = \frac{2+\theta_r}{4}$.

(iii) The parametric possibility distribution of ξ is

$$\mu_\xi(t;\theta_l,\theta_r) = \begin{cases} \frac{(4+\theta_r-\theta_l)(t-r_1)}{4(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2} \\ \frac{(4-\theta_r+\theta_l)t+(\theta_r-\theta_l)r_2-4r_1}{4(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{(-4+\theta_r-\theta_l)t-(\theta_r-\theta_l)r_3+4r_4}{4(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2} \\ \frac{(4+\theta_r-\theta_l)(r_4-t)}{4(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4, \end{cases}$$

and denote $\xi = (r_1, r_2, r_3, r_4; h(\theta_l, \theta_r))$, where $h(\theta_l,\theta_r) = \mu_\xi\left(\frac{r_1+r_2}{2};\theta_l,\theta_r\right) = \mu_\xi\left(\frac{r_3+r_4}{2};\theta_l,\theta_r\right) = \frac{4+\theta_r-\theta_l}{8}$.

Proof We only prove assertion (i), and the rest can be proved similarly.

Since ξ_* is the reduced fuzzy variable of $\tilde{\xi}$ by the PEV method, it follows from Eqs. (7) and (8) that the parametric possibility distribution $\mu_{\xi_*}(t;\theta_l)$ is computed as follows

$$\begin{aligned} \mu_{\xi_*}(t;\theta_l) &= \mathrm{Pos}\{\xi_* = t\} \\ &= \begin{cases} \frac{1}{2}\left(2\frac{t-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{t-r_1}{r_2-r_1}, \frac{r_2-t}{r_2-r_1}\right\}\right), & r_1 \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{1}{2}\left(2\frac{r_4-t}{r_4-r_3} - \theta_l \min\left\{\frac{r_4-t}{r_4-r_3}, \frac{t-r_3}{r_4-r_3}\right\}\right), & r_3 \le t \le r_4 \end{cases} \\ &= \begin{cases} \frac{(2-\theta_l)(t-r_1)}{2(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2} \\ \frac{(2+\theta_l)t-\theta_l r_2-2r_1}{2(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{(-2-\theta_l)t+\theta_l r_3+2r_4}{2(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2} \\ \frac{(2-\theta_l)(r_4-t)}{2(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4, \end{cases} \end{aligned}$$

which completes the proof of assertion (i).

Corollary 1 *Let $\tilde{\xi}$ be a type-2 trapezoidal fuzzy variable. Then the parametric possibility distributions $\mu_{\xi_*}(t;\theta_l)$, $\mu_{\xi^*}(t;\theta_r)$ and $\mu_\xi(t;\theta_l,\theta_r)$ satisfy the following conditions as shown in Fig. 1:*

$$\mu_{\xi^*}(t;\theta_r) \ge \mu_\xi(t;\theta_l,\theta_r) \ge \mu_{\xi_*}(t;\theta_l). \tag{9}$$

As a consequence of Proposition 1, the reduced fuzzy variables of type-2 triangular fuzzy variable have the following parametric possibility distributions:

Corollary 2 *Let $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$ be a type-2 triangular fuzzy variable, and ξ_*, ξ^* and ξ its reduced fuzzy variables obtained by the PEV, OEV and EV methods, respectively. Then we have:*

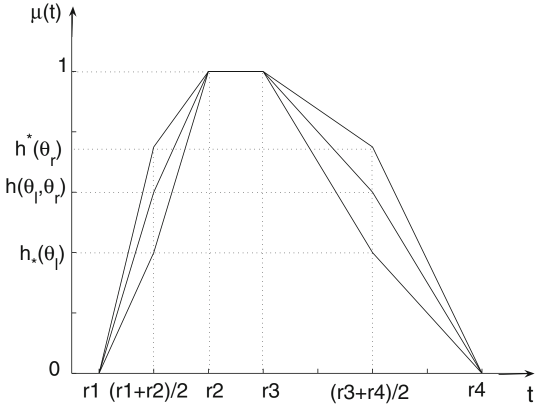

Fig. 1 The relationships among parametric possibility distributions of ξ_*, ξ^* and ξ

(i) The parametric possibility distribution of ξ_* *is*

$$\mu_{\xi_*}(t;\theta_l)=\begin{cases}\frac{(2-\theta_l)(t-r_1)}{2(r_2-r_1)}, & r_1\le t\le\frac{r_1+r_2}{2}\\ \frac{(2+\theta_l)t-\theta_l r_2-2r_1}{2(r_2-r_1)}, & \frac{r_1+r_2}{2}\le t\le r_2\\ \frac{(-2-\theta_l)t+\theta_l r_2+2r_3}{2(r_3-r_2)}, & r_2\le t\le\frac{r_2+r_3}{2}\\ \frac{(2-\theta_l)(r_3-t)}{2(r_3-r_2)}, & \frac{r_2+r_3}{2}\le t\le r_3,\end{cases}$$

and denote $\xi_*=(r_1,r_2,r_3;h_*(\theta_l))$*, where* $h_*(\theta_l)=\mu_{\xi_*}\left(\frac{r_1+r_2}{2};\theta_l\right)=\mu_{\xi_*}\left(\frac{r_2+r_3}{2};\theta_l\right)=\frac{2-\theta_l}{4}$.

(ii) The parametric possibility distribution of ξ^* *is*

$$\mu_{\xi^*}(t;\theta_r)=\begin{cases}\frac{(2+\theta_r)(t-r_1)}{2(r_2-r_1)}, & r_1\le t\le\frac{r_1+r_2}{2}\\ \frac{(2-\theta_r)t+\theta_r r_2-2r_1}{2(r_2-r_1)}, & \frac{r_1+r_2}{2}\le t\le r_2\\ \frac{(\theta_r-2)t-\theta_r r_2+2r_3}{2(r_3-r_2)}, & r_2\le t\le\frac{r_2+r_3}{2}\\ \frac{(2+\theta_r)(r_3-t)}{2(r_3-r_2)}, & \frac{r_2+r_3}{2}\le t\le r_3,\end{cases}$$

and denote $\xi^*=(r_1,r_2,r_3;h^*(\theta_r))$*, where* $h^*(\theta_r)=\mu_{\xi^*}\left(\frac{r_1+r_2}{2};\theta_r\right)=\mu_{\xi^*}\left(\frac{r_2+r_3}{2};\theta_r\right)=\frac{2+\theta_r}{4}$.

(iii) The parametric possibility distribution of ξ *is*

$$\mu_{\xi}(t;\theta_l,\theta_r)=\begin{cases}\frac{(4+\theta_r-\theta_l)(t-r_1)}{4(r_2-r_1)}, & r_1\le t\le\frac{r_1+r_2}{2}\\ \frac{(4-\theta_r+\theta_l)t+(\theta_r-\theta_l)r_2-4r_1}{4(r_2-r_1)}, & \frac{r_1+r_2}{2}\le t\le r_2\\ \frac{(-4+\theta_r-\theta_l)t-(\theta_r-\theta_l)r_2+4r_3}{4(r_3-r_2)}, & r_2\le t\le\frac{r_2+r_3}{2}\\ \frac{(4+\theta_r-\theta_l)(r_3-t)}{4(r_3-r_2)}, & \frac{r_2+r_3}{2}\le t\le r_3,\end{cases}$$

and denote $\xi = (r_1, r_2, r_3; h(\theta_l, \theta_r))$, *where* $h(\theta_l, \theta_r) = \mu_\xi\left(\frac{r_1+r_2}{2}; \theta_l, \theta_r\right) = \mu_\xi\left(\frac{r_2+r_3}{2}; \theta_l, \theta_r\right) = \frac{4+\theta_r-\theta_l}{8}$.

For type-2 trapezoidal and triangular fuzzy variables, Proposition 1 and Corollary 2 show that their reduced fuzzy variables ξ_*, ξ^* and ξ are characterized by parametric possibility distributions, i.e., their possibility distributions depend on parameters θ_l and θ_r. As a consequence, a reduced fuzzy variable is more flexible than an ordinary fuzzy variable in the applications of the real-world decision making problems under uncertainty.

3 Moments of reduced fuzzy variables

Let ξ be a reduced fuzzy variable with a parametric possibility distribution $\mu_\xi(t;\theta)$. To measure the variation of the parametric possibility distribution about its expected value $\mathrm{E}[\xi]$, we adopt the following index about the nth moment of ξ,

$$\mathrm{M}_n[\xi] = \int\limits_{(-\infty,+\infty)} (t - \mathrm{E}[\xi])^n \mathrm{d}\left(\mathrm{Cr}\{\xi \le t\}\right), \tag{10}$$

where the credibility distribution is defined by the parametric possibility distribution $\mu_\xi(u;\theta)$ of ξ as follows

$$\mathrm{Cr}\{\xi \le t\} = \frac{1}{2}\left(\sup_{u\in\Re}\mu_\xi(u;\theta) + \sup_{u\le t}\mu_\xi(u;\theta) - \sup_{u>t}\mu_\xi(u;\theta)\right), \quad t \in \Re.$$

In addition, the integral in Eq. (10) is L–S integral. When $n = 2$, $\mathrm{M}_2[\xi]$ is called the second moment of ξ. In the following, we will establish the second moment formulas for reduced fuzzy variables. First, we have the following results about type-2 trapezoidal fuzzy variable.

Theorem 1 *Let $\tilde{\xi}$ be a type-2 trapezoidal fuzzy variable defined as $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4; \theta_l, \theta_r)$, and ξ_*, ξ^* and ξ its reduced fuzzy variables obtained by the PEV, OEV and EV methods, respectively. Then we have:*

(i) The second moment of ξ_* is

$$\mathrm{M}_2[\xi_*] = \frac{1}{48}\left(5r_1^2+5r_2^2+5r_3^2+5r_4^2+2r_1r_2+2r_3r_4-6r_1r_3-6r_1r_4-6r_2r_3-6r_2r_4\right)$$
$$-\frac{1}{256}\theta_l^2(r_1-r_2-r_3+r_4)^2 - \frac{1}{32}\theta_l\left(r_1^2-r_2^2-r_3^2+r_4^2+2r_2r_3-2r_1r_4\right),$$

which is equivalent to the following parametric matrix form

$$\mathrm{M}_2[\xi_*] = \frac{1}{2}r^T Q_* r,$$

where $r = (r_1, r_2, r_3, r_4)^T$, *and the matrix*

$$Q_* = \begin{bmatrix} -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{128}\theta_l^2 + \frac{1}{24} & \frac{1}{128}\theta_l^2 - \frac{1}{8} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} \\ \frac{1}{128}\theta_l^2 + \frac{1}{24} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l + \frac{5}{24} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{128}\theta_l^2 - \frac{1}{8} \\ \frac{1}{128}\theta_l^2 - \frac{1}{8} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l - \frac{1}{8} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{128}\theta_l^2 + \frac{1}{24} \\ -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{128}\theta_l^2 - \frac{1}{8} & \frac{1}{128}\theta_l^2 + \frac{1}{24} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} \end{bmatrix}.$$

(ii) *The second moment of* ξ^* *is*

$$\begin{aligned} \mathrm{M}_2[\xi^*] = \tfrac{1}{48}\left(5r_1^2+5r_2^2+5r_3^2+5r_4^2+2r_1r_2+2r_3r_4-6r_1r_3-6r_1r_4-6r_2r_3-6r_2r_4\right) \\ -\tfrac{1}{256}\theta_r^2(r_1 - r_2 - r_3 + r_4)^2 + \tfrac{1}{32}\theta_r\left(r_1^2 - r_2^2 - r_3^2 + r_4^2 + 2r_2r_3 - 2r_1r_4\right), \end{aligned}$$

which is equivalent to the following parametric matrix form

$$\mathrm{M}_2[\xi^*] = \frac{1}{2}r^T Q^* r,$$

where $r = (r_1, r_2, r_3, r_4)^T$, *and the matrix*

$$Q^* = \begin{bmatrix} -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{128}\theta_r^2 + \frac{1}{24} & \frac{1}{128}\theta_r^2 - \frac{1}{8} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} \\ \frac{1}{128}\theta_r^2 + \frac{1}{24} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r + \frac{5}{24} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{128}\theta_r^2 - \frac{1}{8} \\ \frac{1}{128}\theta_r^2 - \frac{1}{8} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r - \frac{1}{8} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{128}\theta_r^2 + \frac{1}{24} \\ -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{128}\theta_r^2 - \frac{1}{8} & \frac{1}{128}\theta_r^2 + \frac{1}{24} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} \end{bmatrix}.$$

(iii) *The second moment of* ξ *is*

$$\begin{aligned} \mathrm{M}_2[\xi] = \tfrac{1}{48}\left(5r_1^2+5r_2^2+5r_3^2+5r_4^2+2r_1r_2+2r_3r_4-6r_1r_3-6r_1r_4-6r_2r_3-6r_2r_4\right) \\ -\tfrac{1}{1{,}024}(\theta_r-\theta_l)^2(r_1-r_2-r_3+r_4)^2+\tfrac{1}{64}(\theta_r-\theta_l)\left(r_1^2-r_2^2-r_3^2+r_4^2+2r_2r_3-2r_1r_4\right), \end{aligned}$$

which is equivalent to the following parametric matrix form

$$\mathrm{M}_2[\xi] = \frac{1}{2}r^T Q r,$$

where $r = (r_1, r_2, r_3, r_4)^T$, and the elements of the symmetric matrix Q include

$$
\begin{aligned}
Q_{11} &= Q_{44} = -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24},\\
Q_{12} &= Q_{34} = \frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{24},\\
Q_{13} &= Q_{24} = \frac{1}{512}(\theta_r - \theta_l)_r^2 - \frac{1}{8},\\
Q_{14} &= -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8},\\
Q_{22} &= Q_{33} = -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24},\\
Q_{23} &= -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8}.
\end{aligned}
$$

Moreover, the second moments $\mathrm{M}_2[\xi_*]$, $\mathrm{M}_2[\xi^*]$ *and* $\mathrm{M}_2[\xi]$ *are all parametric quadratic convex functions with respect to vector $r \in \Re^4$.*

Proof We only prove assertion (iii), and the rest can be proved similarly.

Since ξ is the reduced fuzzy variable by the EV method, its parametric possibility distribution $\mu_\xi(t; \theta_l, \theta_r)$ is given by assertion (iii) of Proposition 1. As a consequence, the credibility distribution of ξ is

$$
\mathrm{Cr}\{\xi \le t\} = \begin{cases}
0, & t < r_1\\
\frac{(4+\theta_r-\theta_l)(t-r_1)}{8(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2}\\
\frac{(4-\theta_r+\theta_l)t+(\theta_r-\theta_l)r_2-4r_1}{8(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2\\
\frac{1}{2}, & r_2 \le t \le r_3\\
1 - \frac{(-4+\theta_r-\theta_l)t-(\theta_r-\theta_l)r_3+4r_4}{8(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2}\\
1 - \frac{(4+\theta_r-\theta_l)(r_4-t)}{8(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4\\
1, & t > r_4,
\end{cases}
$$

and the expected value of ξ is

$$
\mathrm{E}[\xi] = \frac{1}{4}(r_1 + r_2 + r_3 + r_4) + \frac{1}{32}(\theta_r - \theta_l)(r_1 - r_2 - r_3 + r_4),
$$

which is denoted by m. Therefore, the second moment of ξ is calculated by

$$
\begin{aligned}
\mathrm{M}_2[\xi] &= \int_{(-\infty,+\infty)} (t-m)^2 \mathrm{d}(\mathrm{Cr}\{\xi \le t\})\\
&= \int_{\left(r_1, \frac{r_1+r_2}{2}\right)} (t-m)^2 \mathrm{d}\left(\frac{(4+\theta_r-\theta_l)(t-r_1)}{8(r_2-r_1)}\right) + \int_{\left(\frac{r_1+r_2}{2}, r_2\right)} (t-m)^2 \mathrm{d}\left(\frac{(4-\theta_r+\theta_l)t+(\theta_r-\theta_l)r_2-4r_1}{8(r_2-r_1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int\limits_{\left(r_3, \frac{r_3+r_4}{2}\right)} (t-m)^2 \mathrm{d}\left(1 - \tfrac{(-4+\theta_r-\theta_l)t-(\theta_r-\theta_l)r_3+4r_4}{8(r_4-r_3)}\right) \\
&+ \int\limits_{\left(\frac{r_3+r_4}{2}, r_4\right)} (t-m)^2 \mathrm{d}\left(1 - \tfrac{(4+\theta_r-\theta_l)(r_4-t)}{8(r_4-r_3)}\right) \\
&= \tfrac{4+\theta_r-\theta_l}{8(r_2-r_1)} \int_{r_1}^{\frac{r_1+r_2}{2}} (t-m)^2 \mathrm{d}t + \tfrac{4-\theta_r+\theta_l}{8(r_2-r_1)} \int_{\frac{r_1+r_2}{2}}^{r_2} (t-m)^2 \mathrm{d}t - \tfrac{-4+\theta_r-\theta_l}{8(r_4-r_3)} \int_{r_3}^{\frac{r_3+r_4}{2}} (t-m)^2 \mathrm{d}t \\
&+ \tfrac{4+\theta_r-\theta_l}{8(r_4-r_3)} \int_{\frac{r_3+r_4}{2}}^{r_4} (t-m)^2 \mathrm{d}t \\
&= \tfrac{1}{48}\left(5r_1^2 + 5r_2^2 + 5r_3^2 + 5r_4^2 + 2r_1r_2 + 2r_3r_4 - 6r_1r_3 - 6r_1r_4 - 6r_2r_3 - 6r_2r_4\right) \\
&- \tfrac{1}{1{,}024}(\theta_r-\theta_l)^2(r_1-r_2-r_3+r_4)^2 + \tfrac{1}{64}(\theta_r-\theta_l)(r_1^2-r_2^2-r_3^2+r_4^2+2r_2r_3-2r_1r_4) \\
&= \tfrac{1}{2} r^T Q r.
\end{aligned}
$$

On the other hand, the integrand $(t-m)^2$ and the credibility distribution $\mathrm{Cr}\{\xi \le t\}$ are both nonnegative, so $\mathrm{M}_2[\xi] \ge 0$ holds for any vector $r \in \Re^4$. In addition, Q is a 4×4 symmetric parametric matrix. Therefore, $\mathrm{M}_2[\xi]$ is a positive semidefinite quadratic form. In other words, for any parameters θ_l and θ_r, the second moment $\mathrm{M}_2[\xi]$ is a parametric quadratic convex function with respect to vector $r \in \Re^4$. The proof of the theorem is complete.

As a corollary of Theorem 1, the second moment formulas for the reduced fuzzy variables of type-2 triangular fuzzy variable are as follows.

Corollary 3 *Let $\tilde{\xi}$ be a type-2 triangular fuzzy variable defined as $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$, and ξ_*, ξ^* and ξ its reduced fuzzy variables obtained by the PEV, OEV and EV methods, respectively. Then we have:*

(i) The second moment of ξ_* is

$$
\begin{aligned}
\mathrm{M}_2[\xi_*] = &\tfrac{1}{48}\left(5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 4r_2r_3 - 6r_1r_3\right) \\
&- \tfrac{1}{256}\theta_l^2(r_1 - 2r_2 + r_3)^2 - \tfrac{1}{32}\theta_l(r_3 - r_1)^2,
\end{aligned}
$$

which is equivalent to the following parametric matrix form

$$
\mathrm{M}_2[\xi_*] = \frac{1}{2} r^T R_* r,
$$

where $r = (r_1, r_2, r_3)^T$, and the matrix

$$
R_* = \begin{bmatrix}
-\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} \\
\frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{32}\theta_l^2 + \frac{1}{6} & \frac{1}{64}\theta_l^2 - \frac{1}{12} \\
-\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24}
\end{bmatrix}.
$$

(ii) The second moment of ξ^* *is*

$$\begin{aligned}\mathrm{M}_2[\xi^*] = \tfrac{1}{48}\left(5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 4r_2r_3 - 6r_1r_3\right)\\ -\tfrac{1}{256}\theta_r^2(r_1 - 2r_2 + r_3)^2 + \tfrac{1}{32}\theta_r(r_3 - r_1)^2,\end{aligned}$$

which is equivalent to the following parametric matrix form

$$\mathrm{M}_2[\xi^*] = \frac{1}{2}r^T R^* r,$$

where $r = (r_1, r_2, r_3)^T$, *and the matrix*

$$R^* = \begin{bmatrix} -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} \\ \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{32}\theta_r^2 + \frac{1}{6} & \frac{1}{64}\theta_r^2 - \frac{1}{12} \\ -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} \end{bmatrix}.$$

(iii) The second moment of ξ *is*

$$\begin{aligned}\mathrm{M}_2[\xi] = \tfrac{1}{48}(5r_1^2 + 4r_2^2 + 5r_3^2 - 4r_1r_2 - 4r_2r_3 - 6r_1r_3) - \tfrac{1}{1{,}024}\\ (\theta_r - \theta_l)^2(r_1 - 2r_2 + r_3)^2 + \tfrac{1}{64}(\theta_r - \theta_l)(r_3 - r_1)^2,\end{aligned}$$

which is equivalent to the following parametric matrix form

$$\mathrm{M}_2[\xi] = \frac{1}{2}r^T R r,$$

where $r = (r_1, r_2, r_3)^T$, *and the matrix*

$$R = \begin{bmatrix} -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24} & \frac{1}{256}(\theta_r - \theta_l)^2 - \frac{1}{12} & -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8} \\ \frac{1}{256}(\theta_r - \theta_l)^2 - \frac{1}{12} & -\frac{1}{128}(\theta_r - \theta_l)^2 + \frac{1}{6} & \frac{1}{256}(\theta_r - \theta_l)^2 - \frac{1}{12} \\ -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8} & \frac{1}{256}(\theta_r - \theta_l)^2 - \frac{1}{12} & -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24} \end{bmatrix}.$$

Moreover, the second moments $\mathrm{M}_2[\xi_*]$, $\mathrm{M}_2[\xi^*]$ *and* $\mathrm{M}_2[\xi]$ *are all parametric quadratic convex functions with respect to vector* $r \in \Re^3$.

4 Moment risk criteria in portfolio optimization

In this section, we adopt the second moment as a risk measure and develop reward-risk and risk-reward models to optimize fuzzy portfolio selection problems in which the fuzzy returns are characterized by parametric possibility distributions.

4.1 Reward-risk and risk-reward models

Portfolio selection was first proposed by Markowitz (1952) to study the problem of how to allocate one's capital to a number of potential securities so that the investment can bring a profitable return. On the basis of probability theory, the key principle of mean-variance model is to use the expected return of a portfolio as the investment return and to use the variance of a portfolio return as the investment risk.

Due to the complexity of security market under uncertainty, the observed values of security returns in real-world problems are sometimes imprecise or vague. To deal with this situation, fuzzy portfolio selection problems were developed in the literature, in which fixed possibility distribution functions or membership functions of security returns are usually supposed to be available (Huang 2010; Wu and Liu 2011). In this section, we will develop a robust approach to dealing with fuzzy portfolio selection problem. In our method, we will employ parametric possibility distribution functions instead of fixed possibility distribution functions to describe the reduced security returns, and the parametric possibility distributions are obtained by using the EV reduction method for type-2 fuzzy returns. In other words, the reduced fuzzy returns have parametric possibility distributions, so they can serve as the representatives of type-2 fuzzy returns. When the parameters vary in the unit interval [0, 1], the distribution functions run over the entire footprints of type-2 fuzzy returns. In the following, we will adopt this modeling idea to construct fuzzy portfolio selection problems.

More precisely, given a collection of potential securities indexed from 1 to n, let $\tilde{\xi}_i$ be the type-2 fuzzy returns of security i in the next time period, $i = 1, 2, \ldots, n$. According to type-2 possibility distributions of returns $\tilde{\xi}_i$, we employ the EV reduction method to get their reduced fuzzy returns ξ_i for $i = 1, 2, \ldots, n$, which are characterized by parametric possibility distributions. Using the second moment of reduced fuzzy returns as a new risk criterion, we can build meaningful portfolio selection models. Assume that an investor invests all his fund in n potential securities, and nonnegative numbers x_i is the investment proportion to security i such that $\sum_{i=1}^{n} x_i = 1$. As a consequence, the return investor would obtain by using this portfolio is represented by

$$R(x, \xi) = \sum_{i=1}^{n} x_i \xi_i = \xi^T x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$. The reward associated with such a portfolio is defined as the expected return

$$\mathrm{E}[R(x, \xi)] = \mathrm{E}\left[\xi^T x\right].$$

Note that a security with a high return usually results in a high level of risk. Thus, it is necessary to define a risk measure for the return $R(x, \xi)$. In this paper, we will gauge the risk associated with return $R(x, \xi)$ by its second moment

$$\mathrm{M}_2[R(x, \xi)] = \mathrm{M}_2\left[\xi^T x\right].$$

Using $\mathrm{E}[R(x, \xi)]$ and $M_2[R(x, \xi)]$ as optimization indices, we have two formulations about portfolio selection problem according to an investor's altitude towards risk.

On the one hand, if an investor desires to maximize expected return under the condition that the maximum acceptable risk is ψ, then he may employ the following mathematical model

$$\begin{cases} \max & \mathrm{E}[\xi^T x] \\ \text{subject to:} & \mathrm{M}_2[\xi^T x] \leq \phi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{cases} \tag{11}$$

where $\phi \geq 0$ plays the role of a parameter. This formulation of the portfolio selection problem is called a *reward-risk* model, with $\mathrm{E}[\xi^T x]$ representing reward and $\mathrm{M}_2[\xi^T x]$ standing for risk.

On the other hand, if an investor is looking for a portfolio with minimum risk under prescribing a minimum acceptable level ψ of expected portfolio return, then he may consider the following optimization model

$$\begin{cases} \min & \mathrm{M}_2[\xi^T x] \\ \text{subject to:} & \mathrm{E}[\xi^T x] \geq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{cases} \tag{12}$$

where $\psi \geq 0$ plays the role of a parameter. This formulation of the portfolio selection problem is called a *risk-reward* model, which is also a parametric optimization problem with parameter ψ. In finance, the optimal objective value as a function of ψ plays an important role. Its graph is called the *efficient frontier* with the horizontal axis corresponding to risk and the vertical one corresponding to return.

4.2 Linear combinations of reduced fuzzy returns and their moment formulas

Since $\mathrm{M}_2[\xi^T x]$ in problems (11) and (12) is the second moment of linear combination of reduced fuzzy returns, we can deduce its second moment formulas in the following case.

First, for type-2 trapezoidal fuzzy variables, we have the following calculation formulas:

Theorem 2 *Let $\tilde{\xi}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}, \tilde{r}_{i4}; \theta_l, \theta_r)$, $i = 1, 2, \ldots, n$ be mutually independent type-2 trapezoidal fuzzy returns, and $\xi_{i,*}$, ξ_i^* and ξ_i the reduced fuzzy returns of $\tilde{\xi}_i$ obtained by the PEV, OEV and EV methods, respectively. Then for any $x_i \in \Re$, $i = 1, 2, \ldots, n$, we have:*

(i) The second moment of $\xi_*^T x$ is

$$\mathrm{M}_2\left[\xi_*^T x\right] = \frac{1}{2} x^T D_* x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi_*^T = (\xi_{1,*}, \xi_{2,*}, \ldots, \xi_{n,*})^T$ *and* $D_* = F^T Q_* F$ *with the matrix*

$$F = \begin{bmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \\ r_{13} & r_{23} & \cdots & r_{n3} \\ r_{14} & r_{24} & \cdots & r_{n4} \end{bmatrix} \tag{13}$$

and the matrix

$$Q_* = \begin{bmatrix} -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{128}\theta_l^2 + \frac{1}{24} & \frac{1}{128}\theta_l^2 - \frac{1}{8} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} \\ \frac{1}{128}\theta_l^2 + \frac{1}{24} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l + \frac{5}{24} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{128}\theta_l^2 - \frac{1}{8} \\ \frac{1}{128}\theta_l^2 - \frac{1}{8} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l - \frac{1}{8} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{128}\theta_l^2 + \frac{1}{24} \\ -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{128}\theta_l^2 - \frac{1}{8} & \frac{1}{128}\theta_l^2 + \frac{1}{24} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} \end{bmatrix}.$$

(ii) *The second moment of* $\xi^{*T} x$ *is*

$$\mathrm{M}_2\left[\xi^{*T} x\right] = \frac{1}{2} x^T D^* x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_n^*)^T$ *and* $D^* = F^T Q^* F$, F *is the matrix defined by (13) and the matrix*

$$Q^* = \begin{bmatrix} -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{128}\theta_r^2 + \frac{1}{24} & \frac{1}{128}\theta_r^2 - \frac{1}{8} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} \\ \frac{1}{128}\theta_r^2 + \frac{1}{24} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r + \frac{5}{24} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{128}\theta_r^2 - \frac{1}{8} \\ \frac{1}{128}\theta_r^2 - \frac{1}{8} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r - \frac{1}{8} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{128}\theta_r^2 + \frac{1}{24} \\ -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{128}\theta_r^2 - \frac{1}{8} & \frac{1}{128}\theta_r^2 + \frac{1}{24} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} \end{bmatrix}.$$

(iii) *The second moment of* $\xi^T x$ *is*

$$\mathrm{M}_2\left[\xi^T x\right] = \frac{1}{2} x^T D x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ *and* $D = F^T Q F$, F *is the matrix defined by (13) and the elements of the symmetric matrix* Q *include*

$$\begin{aligned} Q_{11} = Q_{44} &= -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24}, \\ Q_{12} = Q_{34} &= \frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{24}, \\ Q_{13} = Q_{24} &= \frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{8}, \\ Q_{14} &= -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8}, \end{aligned}$$

$$Q_{22} = Q_{33} = -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24},$$
$$Q_{23} = -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8}.$$

Moreover, the second moments $\mathrm{M}_2[\xi_*^T x]$, $\mathrm{M}_2[\xi^{*T} x]$ *and* $\mathrm{M}_2[\xi^T x]$ *are all parametric quadratic convex functions with respect to decision vector* $x \in \Re^n$.

Proof We only prove assertion (iii), and the rest can be proved similarly.

Note that ξ_i's are mutually independent reduced fuzzy returns. If we denote $R(x, \xi) = \xi^T x$, then its α-cut is $R_\alpha(x, \xi) = \sum_{i=1}^n x_i \xi_{i,\alpha}$. As a consequence, the parametric possibility distribution of $R(x, \xi)$ is as follows

$$\mu_R(t; \theta_l, \theta_r) = \begin{cases} \frac{(4+\theta_r-\theta_l)(t-r_1)}{4(r_2-r_1)}, & r_1 \le t \le \frac{r_1+r_2}{2} \\ \frac{(4-\theta_r+\theta_l)t+(\theta_r-\theta_l)r_2-4r_1}{4(r_2-r_1)}, & \frac{r_1+r_2}{2} \le t \le r_2 \\ 1, & r_2 \le t \le r_3 \\ \frac{(-4+\theta_r-\theta_l)t-(\theta_r-\theta_l)r_3+4r_4}{4(r_4-r_3)}, & r_3 \le t \le \frac{r_3+r_4}{2} \\ \frac{(4+\theta_r-\theta_l)(r_4-t)}{4(r_4-r_3)}, & \frac{r_3+r_4}{2} \le t \le r_4, \end{cases}$$

where $r_1 = \sum_{i=1}^n x_i r_{i1}$, $r_2 = \sum_{i=1}^n x_i r_{i2}$, $r_3 = \sum_{i=1}^n x_i r_{i3}$ and $r_4 = \sum_{i=1}^n x_i r_{i4}$.

According to assertion (iii) in Theorem 1, the second moment of $\xi^T x$ can be represented as

$$\mathrm{M}_2[\xi^T x] = \frac{1}{2}[r_1, r_2, r_3, r_4] Q \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} \ge 0.$$

In addition, we have the next equation

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{21} & \dots & r_{n1} \\ r_{12} & r_{22} & \dots & r_{n2} \\ r_{13} & r_{23} & \dots & r_{n3} \\ r_{14} & r_{24} & \dots & r_{n4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If we denote $x = (x_1, x_2, \dots, x_n)^T$, and the matrix

$$F = \begin{bmatrix} r_{11} & r_{21} & \dots & r_{n1} \\ r_{12} & r_{22} & \dots & r_{n2} \\ r_{13} & r_{23} & \dots & r_{n3} \\ r_{14} & r_{24} & \dots & r_{n4} \end{bmatrix},$$

then $(r_1, r_2, r_3, r_4)^T = Fx$. Therefore, the second moment of fuzzy return $\xi^T x$ is

$$\mathrm{M}_2\left[\xi^T x\right] = \frac{1}{2}x^T F^T QFx = \frac{1}{2}x^T Dx \geq 0,$$

where $D = F^T QF$. Note that $\mathrm{M}_2[\xi^T x] \geq 0$ holds for any $x \in \Re^n$, and D is a symmetric parametric matrix. Therefore, for any parameters θ_l and θ_r, the second moment $\mathrm{M}_2[\xi^T x]$ is a parametric quadratic convex function with respect to decision vector $x \in \Re^n$. The proof of the theorem is complete.

As a corollary of Theorem 2, for type-2 triangular fuzzy returns, we have the following results:

Corollary 4 *Let $\tilde{\xi}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \tilde{r}_{i3}; \theta_l, \theta_r)$, $i = 1, 2, \ldots, n$ be mutually independent type-2 triangular fuzzy returns, and $\xi_{i,*}$, ξ_i^* and ξ_i its reduced fuzzy returns obtained by the PEV, OEV and EV methods, respectively. Then for any $x_i \in \Re$, $i = 1, 2, \ldots, n$, we have:*

(i) The second moment of $\xi_*^T x$ is

$$\mathrm{M}_2\left[\xi_*^T x\right] = \frac{1}{2}x^T H_* x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi_*^T = (\xi_{1,*}, \xi_{2,*}, \ldots, \xi_{n,*})^T$ and $H_* = S^T R_* S$ with the matrix

$$S = \begin{bmatrix} r_{11} & r_{21} & r_{31} & \cdots & r_{n1} \\ r_{12} & r_{22} & r_{32} & \cdots & r_{n2} \\ r_{13} & r_{23} & r_{33} & \cdots & r_{n3} \end{bmatrix}, \tag{14}$$

and the matrix

$$R_* = \begin{bmatrix} -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} & \frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} \\ \frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{32}\theta_l^2 + \frac{1}{6} & \frac{1}{64}\theta_l^2 - \frac{1}{12} \\ -\frac{1}{128}\theta_l^2 + \frac{1}{16}\theta_l - \frac{1}{8} & \frac{1}{64}\theta_l^2 - \frac{1}{12} & -\frac{1}{128}\theta_l^2 - \frac{1}{16}\theta_l + \frac{5}{24} \end{bmatrix}.$$

(ii) The second moment of $\xi^{*T} x$ is

$$\mathrm{M}_2\left[\xi^{*T} x\right] = \frac{1}{2}x^T H^* x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi^* = (\xi_1^*, \xi_2^*, \ldots, \xi_n^*)^T$ and $H^* = S^T R^* S$, S is the matrix defined by (14) and the matrix

$$R^* = \begin{bmatrix} -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} & \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} \\ \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{32}\theta_r^2 + \frac{1}{6} & \frac{1}{64}\theta_r^2 - \frac{1}{12} \\ -\frac{1}{128}\theta_r^2 - \frac{1}{16}\theta_r - \frac{1}{8} & \frac{1}{64}\theta_r^2 - \frac{1}{12} & -\frac{1}{128}\theta_r^2 + \frac{1}{16}\theta_r + \frac{5}{24} \end{bmatrix}.$$

(iii) The second moment of $\xi^T x$ is

$$\mathrm{M}_2\left[\xi^T x\right] = \frac{1}{2}x^T Hx,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ and $H = S^T RS$, S is the matrix defined by (14) and the matrix

$$R=\begin{bmatrix} -\frac{1}{512}(\theta_r-\theta_l)^2+\frac{1}{32}(\theta_r-\theta_l)+\frac{5}{24} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{512}(\theta_r-\theta_l)^2-\frac{1}{32}(\theta_r-\theta_l)-\frac{1}{8} \\ \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{128}(\theta_r-\theta_l)^2+\frac{1}{6} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} \\ -\frac{1}{512}(\theta_r-\theta_l)^2-\frac{1}{32}(\theta_r-\theta_l)-\frac{1}{8} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{512}(\theta_r-\theta_l)^2+\frac{1}{32}(\theta_r-\theta_l)+\frac{5}{24} \end{bmatrix}.$$

Moreover, the second moments $\mathrm{M}_2[\xi_*^T x]$, $\mathrm{M}_2[\xi^{*T} x]$ and $\mathrm{M}_2[\xi^T x]$ are all parametric quadratic convex functions with respect to decision vector $x \in \Re^n$.

5 Equivalent parametric programming and solution methods

5.1 Equivalent parametric programming

Assume $\tilde{\xi}_i$'s are mutually independent type-2 trapezoidal fuzzy returns. We now discuss the equivalent parametric programming problems of problems (11) and (12). According to Theorem 2, the second moment of fuzzy return is represented by

$$\mathrm{M}_2\left[\xi^T x\right] = \frac{1}{2}x^T Dx,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $D = F^T QF$, and the information matrix

$$F = \begin{bmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \\ r_{13} & r_{23} & \cdots & r_{n3} \\ r_{14} & r_{24} & \cdots & r_{n4} \end{bmatrix}$$

is the knowledge about security returns. In addition, the elements in the symmetric matrix Q are determined by

$$
\begin{aligned}
Q_{11} = Q_{44} &= -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24},\\
Q_{12} = Q_{34} &= \frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{24},\\
Q_{13} = Q_{24} &= \frac{1}{512}(\theta_r - \theta_l)_r^2 - \frac{1}{8},\\
Q_{14} &= -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8},\\
Q_{22} = Q_{33} &= -\frac{1}{512}(\theta_r - \theta_l)^2 - \frac{1}{32}(\theta_r - \theta_l) + \frac{5}{24},\\
Q_{23} &= -\frac{1}{512}(\theta_r - \theta_l)^2 + \frac{1}{32}(\theta_r - \theta_l) - \frac{1}{8},
\end{aligned}
$$

where θ_l and θ_r characterize the degree of fuzziness.

On the other hand, we discuss the equivalent parametric form of $\mathrm{E}[\xi^T x]$. Owing to the independence of security returns, we have

$$\mathrm{E}\left[\xi^T x\right] = \sum_{i=1}^{n} x_i \mathrm{E}[\xi_i],$$

which can be expressed as

$$\mathrm{E}\left[\xi^T x\right] = c^T x,$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $c = (\mathrm{E}[\xi_1], \mathrm{E}[\xi_2], \ldots, \mathrm{E}[\xi_n])^T$ and

$$\mathrm{E}[\xi_i] = \frac{1}{4}(r_{i1} + r_{i2} + r_{i3} + r_{i4}) + \frac{1}{32}(\theta_r - \theta_l)(r_{i1} - r_{i2} - r_{i3} + r_{i4}).$$

From the discussions in Sect. 4, when the security returns are characterized by type-2 trapezoidal fuzzy variables, the reward-risk problem (11) can be turned into the following equivalent parametric quadratic programming one

$$
\begin{cases}
\max & c^T x\\
\text{subject to:} & \frac{1}{2} x^T D x \le \phi\\
& \sum_{i=1}^{n} x_i = 1\\
& x_i \ge 0, i = 1, 2, \ldots, n.
\end{cases}
\tag{15}
$$

Similarly, the risk-reward problem (12) can be represented as the following equivalent parametric quadratic programming one

$$\begin{cases} \min & \frac{1}{2}x^T Dx \\ \text{subject to:} & c^T x \geq \psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n. \end{cases} \tag{16}$$

In problems (15) and (16), the second moment $1/2x^T Dx$ is a quadratic convex function with respect to decision vector x (Theorem 2). Expected value $c^T x$ and other constraints are linear functions about decision vector x. Therefore, problems (15) and (16) can be converted into the following parametric quadratic convex programming ones with parameters θ_l and θ_r:

$$\begin{cases} \min & -c^T x \\ \text{subject to:} & \frac{1}{2}x^T Dx \leq \phi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n, \end{cases} \tag{17}$$

and

$$\begin{cases} \min & \frac{1}{2}x^T Dx \\ \text{subject to:} & -c^T x \leq -\psi \\ & \sum_{i=1}^n x_i = 1 \\ & x_i \geq 0, i = 1, 2, \ldots, n. \end{cases} \tag{18}$$

5.2 Solution methods

Since problems (17) and (18) are quadratic convex programming ones, we may use conventional solution methods or general-purpose software to solve them. For example, problem (17) has a linear objective function with a quadratic convex constraint and linear equality and inequality constraints, so we can solve it by cutting plane method (Kelley 1960).

For the sake of presentation, we denote

$$f(x) = -c^T x, \quad h(x) = \frac{1}{2}x^T Dx, \quad g(x) = \phi - h(x),$$

and

$$T = \left\{ x \in \Re^n \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, 2, \ldots, n \right\}.$$

The procedure of cutting plane method for problem (17) is described as follows:

Step 1. Set $k = 1$. Solve the following linear programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & x \in T, \end{cases} \tag{19}$$

and get the solution x_k.

Step 2. If $g(x_k) \geq 0$, then x_k is the global optimal solution to problem (17); Otherwise, add the following inequality to problem (19) as a cutting plane

$$g(x_k) + \nabla g(x_k)(x - x_k) \geq 0.$$

Step 3. Solve the following programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & g(x_k) + \nabla g(x_k)(x - x_k) \geq 0 \\ & x \in T, \end{cases} \tag{20}$$

and get the solution x_{k+1}. Set $k = k + 1$, and go to Step 2.

Owing to the convexity of problem (17), the obtained optimal solution by cutting plane method is a global optimal solution. In addition to the cutting plane method, the equivalent parametric quadratic convex programming problems (17) and (18) can also be solved by general-purpose software such as Lingo, which will be considered in the next section.

6 Numerical experiments

In this section, we demonstrate the developed modeling ideas by two numerical examples. The first is solved by cutting plane method, while the second is solved by Lingo software.

Example 1 Consider an investor intends to invest his fund in three securities. Let x_i denote the investment proportion to security i, and $\tilde{\xi}_i$'s mutually independent type-2 triangular fuzzy returns for $i = 1, 2, 3$. The parametric distribution types of $\tilde{\xi}_i$, $i = 1, 2, 3$ are as follows.

$$\tilde{\xi}_1 = (\widetilde{1.002}, \widetilde{1.033}, \widetilde{1.045}\ ;\ \theta_l, \theta_r), \tilde{\xi}_2 = (\widetilde{1.009}, \widetilde{1.027}, \widetilde{1.059}\ ;\ \theta_l, \theta_r),$$
$$\tilde{\xi}_3 = (\widetilde{1.012}, \widetilde{1.038}, \widetilde{1.073}\ ;\ \theta_l, \theta_r).$$

For this portfolio selection problem, we build it as problem (17). In this case, the portfolio selection problem is equivalent to the following parametric quadratic convex programming problem

$$\begin{cases} \min & -c^T x \\ \text{subject to:} & \frac{1}{2}x^T Hx \le \phi \\ & \sum_{i=1}^{3} x_i = 1 \\ & x_i \ge 0, i = 1, 2, 3, \end{cases} \tag{21}$$

where $H = S^T RS$ with the matrix

$$S = \begin{bmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{bmatrix}$$

and the matrix

$$R = \begin{bmatrix} -\frac{1}{512}(\theta_r-\theta_l)^2+\frac{1}{32}(\theta_r-\theta_l)+\frac{5}{24} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{512}(\theta_r-\theta_l)^2-\frac{1}{32}(\theta_r-\theta_l)-\frac{1}{8} \\ \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{128}(\theta_r-\theta_l)^2+\frac{1}{6} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} \\ -\frac{1}{512}(\theta_r-\theta_l)^2-\frac{1}{32}(\theta_r-\theta_l)-\frac{1}{8} & \frac{1}{256}(\theta_r-\theta_l)^2-\frac{1}{12} & -\frac{1}{512}(\theta_r-\theta_l)^2+\frac{1}{32}(\theta_r-\theta_l)+\frac{5}{24} \end{bmatrix}.$$

We now solve it by cutting plane method with the following parametric values: $\theta_l = 0.4$, $\theta_r = 0.8$ and $\phi = 0.28 \times 10^{-3}$. As a consequence, the matrix

$$H = \begin{bmatrix} 0.3462 & 0.3742 & 0.4628 \\ 0.3742 & 0.4560 & 0.5516 \\ 0.4628 & 0.5516 & 0.6700 \end{bmatrix} \times 10^{-3},$$

and the vector $c = (1.0280, 1.0307, 1.0404)^T$. So, the convex programming problem reads

$$\begin{cases} \min & f(x) \\ \text{subject to:} & h(x) \le 0.28 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \ge 0, \end{cases} \tag{22}$$

where

$$f(x) = -(1.0280x_1 + 1.0307x_2 + 1.0404x_3),$$
$$h(x) = 0.1731x_1^2+0.2280x_2^2+0.3350x_3^2+0.3742x_1x_2+0.4628x_1x_3+0.5516x_2x_3.$$

Let $g(x) = 0.28 - h(x)$, $T = \{x \in \Re^3 : x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \ge 0\}$, and $Y = \{x \in \Re^3 : g(x) \ge 0\}$. We now apply the cutting plane method to solve problem (22). The solution procedure is as follows.

Iteration 1: Solve the following linear programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & x \in T. \end{cases} \tag{23}$$

We get the solution $x_1 = (0, 0, 1)^T$, and $g(x_1) = -0.0550 < 0$ that means $x_1 \notin Y$. Hence, x_1 is not the optimal solution to problem (22), and we add the following inequality to problem (23) as a cutting plane

$$g(x_1) + \bigtriangledown g(x_1)(x - x_1) \geq 0,$$

that is, $0.4628x_1 + 0.5516x_2 + 0.6700x_3 \leq 0.6150$.

Iteration 2: Solve the following programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & 0.4628x_1 + 0.5516x_2 + 0.6700x_3 \leq 0.6150 \\ & x \in T. \end{cases}$$

We get the solution $x_2 = (0.2654440, 0, 0.7345560)^T$, and $g(x_2) = -0.0032 < 0$ that means $x_2 \notin Y$. Hence, x_2 is not the optimal solution to problem (22), and we add the following inequality to problem (23) as a cutting plane

$$g(x_2) + \bigtriangledown g(x_2)(x - x_2) \geq 0,$$

that is, $0.4318x_1 + 0.5045x_2 + 0.6150x_3 \leq 0.5632$.

Iteration 3: Solve the following programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & 0.4318x_1 + 0.5045x_2 + 0.6150x_3 \leq 0.5632 \\ & x \in T. \end{cases}$$

We get the solution $x_3 = (0.2827511, 0, 0.7172489)^T$, and $g(x_3) = -3.5625 \times 10^{-5} < 0$ that means $x_3 \notin Y$. Hence, x_3 is not the optimal solution to problem (22), and we add the following inequality to problem (23) as a cutting plane

$$g(x_3) + \bigtriangledown g(x_3)(x - x_3) \geq 0,$$

that is, $0.4298x_1 + 0.5014x_2 + 0.6114x_3 \leq 0.5600$.

Iteration 4: Solve the following programming problem

$$\begin{cases} \min & f(x) \\ \text{subject to:} & 0.4298x_1 + 0.5014x_2 + 0.6114x_3 \leq 0.5600 \\ & x \in T. \end{cases}$$

We get the solution $x_4 = (0.2830396, 0, 0.7169604)^T$, and $g(x_4) = 1.6758 \times 10^{-5} > 0$ that means $x_4 \in Y$.

As a consequence, the cutting plane method produces the following global optimal solution to problem (22)

$$x^* = (0.2830396, 0, 0.7169604)^T.$$

Example 2 Consider an investor intends to invest his fund in twenty-two securities. Let x_i be the investment proportion to security i, and $\tilde{\xi}_i$'s mutually independent type-2 trapezoidal fuzzy returns for $i = 1, 2, \ldots, 22$. The parametric distribution types of $\tilde{\xi}_i, i = 1, 2, \ldots, 22$ are as follows.

$$\begin{aligned}
&\tilde{\xi}_1 = (\widetilde{0.9946}, \widetilde{0.9967}, \widetilde{1.0012}, \widetilde{1.0016}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_2 = (\widetilde{1.0011}, \widetilde{1.0020}, \widetilde{1.0061}, \widetilde{1.0092}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_3 = (\widetilde{0.9986}, \widetilde{1.0073}, \widetilde{1.0081}, \widetilde{1.0094}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_4 = (\widetilde{0.9983}, \widetilde{1.0096}, \widetilde{1.0122}, \widetilde{1.0263}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_5 = (\widetilde{1.0033}, \widetilde{1.0122}, \widetilde{1.0262}, \widetilde{1.0310}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_6 = (\widetilde{1.0146}, \widetilde{1.0159}, \widetilde{1.0248}, \widetilde{1.0499}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_7 = (\widetilde{1.0209}, \widetilde{1.0225}, \widetilde{1.0416}, \widetilde{1.0553}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_8 = (\widetilde{1.0291}, \widetilde{1.0299}, \widetilde{1.0468}, \widetilde{1.0679}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_9 = (\widetilde{1.0259}, \widetilde{1.0468}, \widetilde{1.0618}, \widetilde{1.0709}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{10} = (\widetilde{1.0350}, \widetilde{1.0514}, \widetilde{1.0671}, \widetilde{1.0830}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{11} = (\widetilde{1.0388}, \widetilde{1.0469}, \widetilde{1.0702}, \widetilde{1.0851}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{12} = (\widetilde{1.0385}, \widetilde{1.0629}, \widetilde{1.0758}, \widetilde{1.0986}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{13} = (\widetilde{1.0414}, \widetilde{1.0569}, \widetilde{1.0770}, \widetilde{1.1024}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{14} = (\widetilde{1.0511}, \widetilde{1.0529}, \widetilde{1.0769}, \widetilde{1.1116}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{15} = (\widetilde{1.0422}, \widetilde{1.0766}, \widetilde{1.0877}, \widetilde{1.1168}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{16} = (\widetilde{1.0373}, \widetilde{1.0914}, \widetilde{1.0972}, \widetilde{1.1171}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{17} = (\widetilde{1.0460}, \widetilde{1.0932}, \widetilde{1.1048}, \widetilde{1.1269}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{18} = (\widetilde{1.0640}, \widetilde{1.0760}, \widetilde{1.1130}, \widetilde{1.1300}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{19} = (\widetilde{1.0615}, \widetilde{1.0785}, \widetilde{1.1155}, \widetilde{1.1275}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{20} = (\widetilde{1.0456}, \widetilde{1.0986}, \widetilde{1.1221}, \widetilde{1.1257}\ ;\ \theta_l, \theta_r)\\
&\tilde{\xi}_{21} = (\widetilde{1.0549}, \widetilde{1.0896}, \widetilde{1.1279}, \widetilde{1.1293}\ ;\ \theta_l, \theta_r), && \tilde{\xi}_{22} = (\widetilde{1.0619}, \widetilde{1.0992}, \widetilde{1.1257}, \widetilde{1.1533}\ ;\ \theta_l, \theta_r).
\end{aligned}$$

We build this portfolio selection problem as the risk-reward model (18). In this case, the portfolio selection problem is equivalent to the following parametric quadratic convex programming problem

$$\begin{cases}
\min & \frac{1}{2}x^T Dx\\
\text{subject to:} & -c^T x \le -\psi\\
& \sum_{i=1}^{22} x_i = 1\\
& x_i \ge 0, i = 1, 2, \ldots, 22,
\end{cases} \tag{24}$$

where $D = F^T QF$, the information matrix F defined by (13) is the knowledge about security returns, and the matrix Q defined in Theorem 2 is a parametric matrix about θ_l and θ_r. We next solve the convex programming problem (24) by Lingo software.

In the case when $\theta_l = \theta_r = 0$, the reduced returns $\xi_i, i = 1, 2, \ldots, 22$ are trapezoidal fuzzy variables shown in Table 1, and the possibility $h(\theta_l, \theta_r)$, the expected values $\mathrm{E}[\xi_i]$ and second moments $\mathrm{M}_2[\xi_i]$ are also computed and provided in Table 1. For this portfolio optimization problem, F^T is the 22×4 return matrix consisting of the second, third, fourth and fifth columns in Table 1. The vector c and matrix Q are different for various values of θ_l and θ_r. In the following, we solve problem (24) according to six cases.

Table 1 The values of $h(\theta_l, \theta_r)$, $\mathrm{E}[\xi_i]$ and $\mathrm{M}_2[\xi_i]$ for case 1

Reduced returns	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$h(\theta_l, \theta_r)$	$\mathrm{E}[\xi_i]$	$\mathrm{M}_2[\xi_i]$
ξ_1	0.9946	0.9967	1.0012	1.0016	0.5	0.9985	8.4560×10^{-6}
ξ_2	1.0011	1.0020	1.0061	1.0092	0.5	1.0046	9.7368×10^{-6}
ξ_3	0.9986	1.0073	1.0081	1.0094	0.5	1.0059	11.6342×10^{-6}
ξ_4	0.9983	1.0096	1.0122	1.0263	0.5	1.0116	72.1262×10^{-6}
ξ_5	1.0033	1.0122	1.0262	1.0310	0.5	1.0182	112.9417×10^{-6}
ξ_6	1.0146	1.0159	1.0248	1.0499	0.5	1.0263	148.4233×10^{-6}
ξ_7	1.0209	1.0225	1.0416	1.0553	0.5	1.0351	186.8173×10^{-6}
ξ_8	1.0291	1.0299	1.0468	1.0679	0.5	1.0434	212.4830×10^{-6}
ξ_9	1.0259	1.0468	1.0618	1.0709	0.5	1.0514	246.6504×10^{-6}
ξ_{10}	1.0350	1.0514	1.0671	1.0830	0.5	1.0591	275.3465×10^{-6}
ξ_{11}	1.0388	1.0469	1.0702	1.0851	0.5	1.0602	314.7442×10^{-6}
ξ_{12}	1.0385	1.0629	1.0758	1.0986	0.5	1.0690	379.5300×10^{-6}
ξ_{13}	1.0414	1.0569	1.0770	1.1024	0.5	1.0694	447.9675×10^{-6}
ξ_{14}	1.0511	1.0529	1.0769	1.1116	0.5	1.0731	496.5719×10^{-6}
ξ_{15}	1.0422	1.0766	1.0877	1.1168	0.5	1.0808	543.6212×10^{-6}
ξ_{16}	1.0373	1.0914	1.0972	1.1171	0.5	1.0858	596.4111×10^{-6}
ξ_{17}	1.0460	1.0932	1.1048	1.1269	0.5	1.0927	647.9422×10^{-6}
ξ_{18}	1.0640	1.0760	1.1130	1.1300	0.5	1.0958	681.1047×10^{-6}
ξ_{19}	1.0615	1.0785	1.1155	1.1275	0.5	1.0958	681.1049×10^{-6}
ξ_{20}	1.0456	1.0986	1.1221	1.1257	0.5	1.0980	788.3903×10^{-6}
ξ_{21}	1.0549	1.0896	1.1279	1.1293	0.5	1.1004	844.0823×10^{-6}
ξ_{22}	1.0619	1.0992	1.1257	1.1533	0.5	1.1100	958.4863×10^{-6}

Case 1: If $\theta_l = \theta_r = 0$, then the reduced fuzzy returns are collected in Table 1, the vector c is

$$c = (0.9985, 1.0046, 1.0059, 1.0116, 1.0182, 1.0263, 1.0351, 1.0434, 1.0514, 1.0591, 1.0602, 1.0690, 1.0694, 1.0731, 1.0808, 1.0858, 1.0927, 1.0958, 1.0958, 1.0980, 1.1004, 1.1100)^T,$$

and the symmetric matrix Q is

$$Q = \begin{bmatrix} \frac{5}{24} & \frac{1}{24} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{24} & \frac{5}{24} & -\frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{5}{24} & \frac{1}{24} \\ -\frac{1}{8} & -\frac{1}{8} & \frac{1}{24} & \frac{5}{24} \end{bmatrix}.$$

Thus, the matrix $D{=}F^T QF$ can be calculated accordingly. For different values of ψ, the obtained optimal allocation proportions to the 22 securities are reported in Table 2.

Table 2 The allocation proportions to the 22 securities with $\theta_l = \theta_r = 0$ (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1100
x_1	100	0	0	0	0	0	0	0
x_3	0	89.25370	63.82854	41.84918	16.48889	0	0	0
x_{10}	0	0	11.02331	27.20371	41.47247	14.53177	0	0
x_{12}	0	5.402724	21.07221	30.94711	24.27382	0	0	0
x_{14}	0	5.343573	4.075949	0	0	0	0	0
x_{17}	0	0	0	0	17.76482	60.22061	37.95650	0
x_{18}	0	0	0	0	0	14.24227	1.018994	0
x_{19}	0	0	0	0	0	11.00535	52.73823	0
x_{22}	0	0	0	0	0	0	8.286278	100

If $\theta_l = \theta_r$, then the optimal allocation proportions are the same as those in the case $\theta_l = \theta_r = 0$, because the parameter $h(\theta_l, \theta_r) = 0.5$ is a constant.

Case 2: If $\theta_l = 0.5, \theta_r = 0.3$, then the reduced fuzzy returns are provided in Table 3, the vector c is

$$\begin{aligned} c = (&0.9985, 1.0046, 1.0059, 1.0116, 1.0182, 1.0262, 1.0350, 1.0433, 1.0514,\\ &1.0591, 1.0602, 1.0690, 1.0694, 1.0729, 1.0809, 1.0860, 1.0929,\\ &1.0957, 1.0958, 1.0983, 1.1006, 1.1101)^T, \end{aligned}$$

and the symmetric matrix Q is

$$Q = \begin{bmatrix} 0.2020 & 0.0417 & -0.1249 & -0.1188 \\ 0.0417 & 0.2145 & -0.1313 & -0.1249 \\ -0.1249 & -0.1313 & 0.2145 & 0.0417 \\ -0.1188 & -0.1249 & 0.0417 & 0.2020 \end{bmatrix}.$$

As a consequence, for different values of ψ, the obtained optimal allocation proportions to the 22 securities are reported in Table 4.

Case 3: If $\theta_l = 0.3, \theta_r = 0.5$, then the reduced fuzzy returns are shown in Table 5, the vector c is

$$\begin{aligned} c = (&0.9985, 1.0046, 1.0058, 1.0116, 1.0181, 1.0264, 1.0352, 1.0436, 1.0513,\\ &1.0591, 1.0603, 1.0689, 1.0695, 1.0733, 1.0808, 1.0855,\\ &1.0926, 1.0958, 1.0957, 1.0977, 1.1002, 1.1100)^T, \end{aligned}$$

and the symmetric matrix Q is

$$Q = \begin{bmatrix} 0.2145 & 0.0417 & -0.1249 & -0.1313 \\ 0.0417 & 0.2020 & -0.1188 & -0.1249 \\ -0.1249 & -0.1188 & 0.2020 & 0.0417 \\ -0.1313 & -0.1249 & 0.0417 & 0.2145 \end{bmatrix}.$$

Therefore, for different values of ψ, the corresponding optimal allocation proportions to the 22 securities are reported in Table 6.

Case 4: If $\theta_l = 0.8, \theta_r = 0.2$, then the reduced fuzzy returns are shown in Table 7, the vector c is

$$\begin{aligned} c = (&0.9986, 1.0046, 1.0060, 1.0115, 1.0183, 1.0259, 1.0348, 1.0430, 1.0516,\\ &1.0591, 1.0601, 1.0690, 1.0692, 1.0725, 1.0809, 1.0864, 1.0932, 1.0957,\\ &1.0958, 1.0989, 1.1010, 1.1102)^T, \end{aligned}$$

Table 3 The values of $h(\theta_l, \theta_r)$, $\mathrm{E}[\xi_i]$ and $\mathrm{M}_2[\xi_i]$ for case 2

Reduced returns	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$h(\theta_l, \theta_r)$	$\mathrm{E}[\xi_i]$	$\mathrm{M}_2[\xi_i]$
ξ_1	0.9946	0.9967	1.0012	1.0016	0.475	0.9985	8.3661×10^{-6}
ξ_2	1.0011	1.0020	1.0061	1.0092	0.475	1.0046	9.5841×10^{-6}
ξ_3	0.9986	1.0073	1.0081	1.0094	0.475	1.0059	11.2696×10^{-6}
ξ_4	0.9983	1.0096	1.0122	1.0263	0.475	1.0116	69.6971×10^{-6}
ξ_5	1.0033	1.0122	1.0262	1.0310	0.475	1.0182	111.1558×10^{-6}
ξ_6	1.0146	1.0159	1.0248	1.0499	0.475	1.0262	144.7546×10^{-6}
ξ_7	1.0209	1.0225	1.0416	1.0553	0.475	1.0350	184.2536×10^{-6}
ξ_8	1.0291	1.0299	1.0468	1.0679	0.475	1.0433	208.6550×10^{-6}
ξ_9	1.0259	1.0468	1.0618	1.0709	0.475	1.0514	241.0200×10^{-6}
ξ_{10}	1.0350	1.0514	1.0671	1.0830	0.475	1.0591	268.9168×10^{-6}
ξ_{11}	1.0388	1.0469	1.0702	1.0851	0.475	1.0602	309.7398×10^{-6}
ξ_{12}	1.0385	1.0629	1.0758	1.0986	0.475	1.0690	368.7624×10^{-6}
ξ_{13}	1.0414	1.0569	1.0770	1.1024	0.475	1.0694	437.5981×10^{-6}
ξ_{14}	1.0511	1.0529	1.0769	1.1116	0.475	1.0729	486.8914×10^{-6}
ξ_{15}	1.0422	1.0766	1.0877	1.1168	0.475	1.0809	526.6141×10^{-6}
ξ_{16}	1.0373	1.0914	1.0972	1.1171	0.475	1.0860	576.5704×10^{-6}
ξ_{17}	1.0460	1.0932	1.1048	1.1269	0.475	1.0929	627.8856×10^{-6}
ξ_{18}	1.0640	1.0760	1.1130	1.1300	0.475	1.0957	671.7693×10^{-6}
ξ_{19}	1.0615	1.0785	1.1155	1.1275	0.475	1.0958	671.7695×10^{-6}
ξ_{20}	1.0456	1.0986	1.1221	1.1257	0.475	1.0983	769.9708×10^{-6}
ξ_{21}	1.0549	1.0896	1.1279	1.1293	0.475	1.1006	831.3250×10^{-6}
ξ_{22}	1.0619	1.0992	1.1257	1.1533	0.475	1.1101	934.5710×10^{-6}

and the symmetric matrix Q is

$$Q = \begin{bmatrix} 0.1889 & 0.0424 & -0.1243 & -0.1070 \\ 0.0424 & 0.2264 & -0.1445 & -0.1243 \\ -0.1243 & -0.1445 & 0.2264 & 0.0424 \\ -0.1070 & -0.1243 & 0.0424 & 0.1889 \end{bmatrix}.$$

Hence, for different values of ψ, the obtained optimal allocation proportions to the 22 securities are reported in Table 8.

Case 5: If $\theta_l = 0.2$, $\theta_r = 0.8$, then the reduced fuzzy returns are collected in Table 9, the vector c is

$$\begin{aligned} c = (&0.9985, 1.0046, 1.0057, 1.0117, 1.0181, 1.0267, 1.0353, 1.0438, 1.0511, \\ &1.0591, 1.0604, 1.0689, 1.0696, 1.0737, 1.0807, 1.0851, 1.0923, 1.0958, \\ &1.0957, 1.0971, 1.0998, 1.1098)^T, \end{aligned}$$

Table 4 The allocation proportions to the 22 securities with $\theta_l = 0.5$, $\theta_r = 0.3$ (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1101
x_1	100	0	0	0	0	0	0	0
x_2	0	90.60023	73.72593	59.48196	0	0	0	0
x_3	0	0	0	0	0.7518797	0	0	0
x_{10}	0	0	0	0	99.24812	17.74691	0	0
x_{15}	0	0	0	3.978558	0	0	0	0
x_{17}	0	9.39977	26.27407	36.53948	0	23.68560	19.58042	0
x_{19}	0	0	0	0	0	58.56749	76.44873	0
x_{22}	0	0	0	0	0	0	3.970854	100

Table 5 The values of $h(\theta_l, \theta_r)$, $\mathrm{E}[\xi_i]$ and $\mathrm{M}_2[\xi_i]$ for case 3

Reduced returns	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$h(\theta_l, \theta_r)$	$\mathrm{E}[\xi_i]$	$\mathrm{M}_2[\xi_i]$
ξ_1	0.9946	0.9967	1.0012	1.0016	0.525	0.9985	8.5458×10^{-6}
ξ_2	1.0011	1.0020	1.0061	1.0092	0.525	1.0046	9.8891×10^{-6}
ξ_3	0.9986	1.0073	1.0081	1.0094	0.525	1.0058	11.9946×10^{-6}
ξ_4	0.9983	1.0096	1.0122	1.0263	0.525	1.0116	74.5548×10^{-6}
ξ_5	1.0033	1.0122	1.0262	1.0310	0.525	1.0181	114.7263×10^{-6}
ξ_6	1.0146	1.0159	1.0248	1.0499	0.525	1.0264	152.0476×10^{-6}
ξ_7	1.0209	1.0225	1.0416	1.0553	0.525	1.0352	189.3696×10^{-6}
ξ_8	1.0291	1.0299	1.0468	1.0679	0.525	1.0436	216.2789×10^{-6}
ξ_9	1.0259	1.0468	1.0618	1.0709	0.525	1.0513	252.2699×10^{-6}
ξ_{10}	1.0350	1.0514	1.0671	1.0830	0.525	1.0591	281.7762×10^{-6}
ξ_{11}	1.0388	1.0469	1.0702	1.0851	0.525	1.0603	319.7449×10^{-6}
ξ_{12}	1.0385	1.0629	1.0758	1.0986	0.525	1.0689	390.2974×10^{-6}
ξ_{13}	1.0414	1.0569	1.0770	1.1024	0.525	1.0695	458.3293×10^{-6}
ξ_{14}	1.0511	1.0529	1.0769	1.1116	0.525	1.0733	506.1679×10^{-6}
ξ_{15}	1.0422	1.0766	1.0877	1.1168	0.525	1.0808	560.6262×10^{-6}
ξ_{16}	1.0373	1.0914	1.0972	1.1171	0.525	1.0855	616.1605×10^{-6}
ξ_{17}	1.0460	1.0932	1.1048	1.1269	0.525	1.0926	667.9496×10^{-6}
ξ_{18}	1.0640	1.0760	1.1130	1.1300	0.525	1.0958	690.4380×10^{-6}
ξ_{19}	1.0615	1.0785	1.1155	1.1275	0.525	1.0957	690.4384×10^{-6}
ξ_{20}	1.0456	1.0986	1.1221	1.1257	0.525	1.0977	806.6192×10^{-6}
ξ_{21}	1.0549	1.0896	1.1279	1.1293	0.525	1.1002	856.7529×10^{-6}
ξ_{22}	1.0619	1.0992	1.1257	1.1533	0.525	1.1100	982.3943×10^{-6}

and the symmetric matrix Q is

$$Q = \begin{bmatrix} 0.2264 & 0.0424 & -0.1243 & -0.1445 \\ 0.0424 & 0.1889 & -0.1070 & -0.1243 \\ -0.1243 & -0.1070 & 0.1889 & 0.0424 \\ -0.1445 & -0.1243 & 0.0424 & 0.2264 \end{bmatrix}.$$

So, for different values of ψ, the obtained optimal allocation proportions to the 22 securities are reported in Table 10.

Case 6: In this case, we adopt two methods to treat the parameters θ_l and θ_r that describe the uncertainty in the secondary possibility distributions of security returns.

Firstly, the values of parameters θ_l and θ_r are not fixed in advance but generated randomly from the unite interval [0, 1]. After the values of θ_l and θ_r have been generated randomly, the corresponding values of vector c and matrix Q are determined by their definitions. With various values of ψ, the obtained optimal allocation proportions to the 22 securities are reported in Table 11.

Secondly, the parameters θ_l and θ_r are treated as variables in problem (24). That is, we add the constraints $0 \le \theta_l \le 1$ and $0 \le \theta_r \le 1$ to (24). With various values of ψ, the obtained optimal allocation proportions to the 22 securities with optimal values of θ_l and θ_r are reported in Table 12, from which we can see that the optimal values $\theta_l^* = 1$ and $\theta_r^* = 0$ hold for every acceptable return level ψ, which imply that the

Table 6 The allocation proportions to the 22 securities with $\theta_l = 0.3$, $\theta_r = 0.5$ (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1100
x_1	100	0	0	0	0	0	0	0
x_3	0	89.48148	65.16265	51.06778	31.59490	4.893689	0	0
x_{12}	0	0	31.89893	36.94796	31.07405	4.169525	0	0
x_{13}	0	0	2.938423	0	0	0	0	0
x_{14}	0	10.51852	0	0	0	0	0	0
x_{17}	0	0	0	0	9.552336	52.31492	35.38658	0
x_{18}	0	0	0	11.98427	27.77872	38.62187	56.63898	0
x_{22}	0	0	0	0	0	0	7.974441	100

Table 7 The values of $h(\theta_l, \theta_r)$, $\mathrm{E}[\xi_i]$ and $\mathrm{M}_2[\xi_i]$ for case 4

Reduced returns	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$h(\theta_l, \theta_r)$	$\mathrm{E}[\xi_i]$	$\mathrm{M}_2[\xi_i]$
ξ_1	0.9946	0.9967	1.0012	1.0016	0.425	0.9986	8.1855×10^{-6}
ξ_2	1.0011	1.0020	1.0061	1.0092	0.425	1.0046	9.2776×10^{-6}
ξ_3	0.9986	1.0073	1.0081	1.0094	0.425	1.0060	10.5274×10^{-6}
ξ_4	0.9983	1.0096	1.0122	1.0263	0.425	1.0115	64.8369×10^{-6}
ξ_5	1.0033	1.0122	1.0262	1.0310	0.425	1.0183	107.5799×10^{-6}
ξ_6	1.0146	1.0159	1.0248	1.0499	0.425	1.0259	137.2846×10^{-6}
ξ_7	1.0209	1.0225	1.0416	1.0553	0.425	1.0348	179.0919×10^{-6}
ξ_8	1.0291	1.0299	1.0468	1.0679	0.425	1.0430	200.9023×10^{-6}
ξ_9	1.0259	1.0468	1.0618	1.0709	0.425	1.0516	229.7265×10^{-6}
ξ_{10}	1.0350	1.0514	1.0671	1.0830	0.425	1.0591	256.0573×10^{-6}
ξ_{11}	1.0388	1.0469	1.0702	1.0851	0.425	1.0601	299.7203×10^{-6}
ξ_{12}	1.0385	1.0629	1.0758	1.0986	0.425	1.0690	347.2265×10^{-6}
ξ_{13}	1.0414	1.0569	1.0770	1.1024	0.425	1.0692	416.8363×10^{-6}
ξ_{14}	1.0511	1.0529	1.0769	1.1116	0.425	1.0725	467.2765×10^{-6}
ξ_{15}	1.0422	1.0766	1.0877	1.1168	0.425	1.0809	492.5931×10^{-6}
ξ_{16}	1.0373	1.0914	1.0972	1.1171	0.425	1.0864	536.6149×10^{-6}
ξ_{17}	1.0460	1.0932	1.1048	1.1269	0.425	1.0932	587.6247×10^{-6}
ξ_{18}	1.0640	1.0760	1.1130	1.1300	0.425	1.0957	653.0928×10^{-6}
ξ_{19}	1.0615	1.0785	1.1155	1.1275	0.425	1.0958	653.0929×10^{-6}
ξ_{20}	1.0456	1.0986	1.1221	1.1257	0.425	1.0989	732.5597×10^{-6}
ξ_{21}	1.0549	1.0896	1.1279	1.1293	0.425	1.1010	805.5506×10^{-6}
ξ_{22}	1.0619	1.0992	1.1257	1.1533	0.425	1.1102	886.7184×10^{-6}

minimum risk in terms of second moment is attained in the most conservative situation for the values of parameters θ_l and θ_r, and the computational results coincide with the theoretical analysis for the second moments of reduced fuzzy returns.

7 Conclusions

This paper studied the EV reduction methods for bounded type-2 fuzzy variables in fuzzy possibility theory, and applied the EV reduction method to fuzzy portfolio problem, in which the fuzzy returns are characterized by parametric possibility distributions. The major new results are as follows.

(i) Based on the PEV, OEV and EV of regular fuzzy variable, we got the reduced fuzzy variables of type-2 triangular and trapezoidal fuzzy variables, and derived their parametric possibility distributions, where the parameters characterize the degree of uncertainty in secondary possibility distributions.

Table 8 The allocation proportions to the 22 securities with $\theta_l = 0.8$, $\theta_r = 0.2$ (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1102
x_1	100	0	0	0	0	0	0	0
x_3	0	92.30769	75.05641	60.86134	36.12650	0	0	0
x_{12}	0	0	0	0	9.652744	16.56257	0	0
x_{16}	0	0	6.176309	12.98253	21.69078	20.34938	0.2297486	0
x_{17}	0	0	0	0	0	32.63032	61.63691	0
x_{18}	0	7.692308	18.76728	26.15613	32.52997	0	0	0
x_{19}	0	0	0	0	0	30.45773	26.85449	0
x_{22}	0	0	0	0	0	0	11.27886	100

Table 9 The values of $h(\theta_l, \theta_r)$, $\mathrm{E}[\xi_i]$ and $\mathrm{M}_2[\xi_i]$ for case 5

Reduced returns	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$h(\theta_l, \theta_r)$	$\mathrm{E}[\xi_i]$	$\mathrm{M}_2[\xi_i]$
ξ_1	0.9946	0.9967	1.0012	1.0016	0.575	0.9985	8.7246×10^{-6}
ξ_2	1.0011	1.0020	1.0061	1.0092	0.575	1.0046	10.1926×10^{-6}
ξ_3	0.9986	1.0073	1.0081	1.0094	0.575	1.0057	12.7024×10^{-6}
ξ_4	0.9983	1.0096	1.0122	1.0263	0.575	1.0117	79.4101×10^{-6}
ξ_5	1.0033	1.0122	1.0262	1.0310	0.575	1.0181	118.2917×10^{-6}
ξ_6	1.0146	1.0159	1.0248	1.0499	0.575	1.0267	159.1636×10^{-6}
ξ_7	1.0209	1.0225	1.0416	1.0553	0.575	1.0353	194.4397×10^{-6}
ξ_8	1.0291	1.0299	1.0468	1.0679	0.575	1.0438	223.7740×10^{-6}
ξ_9	1.0259	1.0468	1.0618	1.0709	0.575	1.0511	263.4764×10^{-6}
ξ_{10}	1.0350	1.0514	1.0671	1.0830	0.575	1.0591	294.6355×10^{-6}
ξ_{11}	1.0388	1.0469	1.0702	1.0851	0.575	1.0604	329.7355×10^{-6}
ξ_{12}	1.0385	1.0629	1.0758	1.0986	0.575	1.0689	411.8317×10^{-6}
ξ_{13}	1.0414	1.0569	1.0770	1.1024	0.575	1.0696	479.0298×10^{-6}
ξ_{14}	1.0511	1.0529	1.0769	1.1116	0.575	1.0737	525.1062×10^{-6}
ξ_{15}	1.0422	1.0766	1.0877	1.1168	0.575	1.0807	594.6296×10^{-6}
ξ_{16}	1.0373	1.0914	1.0972	1.1171	0.575	1.0851	655.3849×10^{-6}
ξ_{17}	1.0460	1.0932	1.1048	1.1269	0.575	1.0923	707.8167×10^{-6}
ξ_{18}	1.0640	1.0760	1.1130	1.1300	0.575	1.0958	709.0989×10^{-6}
ξ_{19}	1.0615	1.0785	1.1155	1.1275	0.575	1.0957	709.0994×10^{-6}
ξ_{20}	1.0456	1.0986	1.1221	1.1257	0.575	1.0971	842.5051×10^{-6}
ξ_{21}	1.0549	1.0896	1.1279	1.1293	0.575	1.0998	881.8342×10^{-6}
ξ_{22}	1.0619	1.0992	1.1257	1.1533	0.575	1.1098	1030.1881×10^{-6}

(ii) For the reduced fuzzy variables of type-2 triangular and trapezoidal fuzzy variables, their second moment formulas were established, and the convexity of second moments with respect to fuzzy parameters was discussed.

(iii) Taking the second moment as a new risk measure, the reward-risk and risk-reward models were developed to optimize fuzzy portfolio selection problems. The mathematical properties of the proposed optimization models were analyzed, including the analytical representations for the second moments of linear combinations of reduced fuzzy variables as well as the convexity of second moments with respect to decision vectors.

(vi) Using the analytical representations of second moments, the reward-risk and risk-reward models can be turned into their equivalent parametric quadratic convex programming problems, which can be solved by conventional solution methods or general-purpose software. The solution results reported in the numerical experiments demonstrated the credibility of the proposed parametric methods.

Finally, we want to emphasize that the parametric method developed in this paper is a robust approach to optimizing fuzzy portfolio selection problems, and it requires

Table 10 The allocation proportions to the 22 securities with $\theta_l = 0.2$, $\theta_r = 0.8$ (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1098
x_1	100	0	0	0	0	0	0	0
x_3	0	88.72754	65.03165	46.02068	26.70933	0	0	0
x_{10}	0	0	0	39.33343	35.51741	19.61853	0	0
x_{12}	0	9.693229	34.96835	0	0	0	0	0
x_{14}	0	1.579234	0	0	0	0	0	0
x_{18}	0	0	0	14.64589	37.77325	80.38147	100	0
x_{22}	0	0	0	0	0	0	0	100

Table 11 The allocation proportions to the 22 securities with (θ_l, θ_r) generated randomly (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1100
x_1	100	0	0	0	0	0	0	0
x_3	0	87.05899	58.84898	40.79798	13.97035	0	0	0
x_{10}	0	12.94101	41.15102	51.93352	65.39866	14.24975	0	0
x_{17}	0	0	0	7.26850	20.63099	79.73565	70.63985	0.88382
x_{18}	0	0	0	0	0	6.01460	16.85598	0
x_{22}	0	0	0	0	0	0	12.50417	99.11618

Table 12 The allocation proportions to the 22 securities with optimal θ_l^*, θ_r^* (%)

ψ	0.9985	1.0129	1.0278	1.0399	1.0587	1.0886	1.0958	1.1100
x_1	100	0	0	0	0	0	0	0
x_3	0	87.14882	66.70455	52.54180	30.53685	0	0	0
x_{10}	0	12.85118	21.50448	22.32564	23.60149	14.28422	0	0
x_{17}	0	0	11.79096	25.13256	45.86166	85.71578	86.38012	1.95079
x_{22}	0	0	0	0	0	0	13.61988	98.04921
θ_l^*	1	1	1	1	1	1	1	1
θ_r^*	0	0	0	0	0	0	0	0

less information about the security returns. In practical modeling process, the parametric method only requires the information about the distribution types about security returns such as triangular and normal distributions, but not requires the knowledge about the concrete values of parameters in the distributions. The decision makers may select various values of parameters according to their preference or attitudes towards risk. From this viewpoint, our parametric method has advantages over some existing fuzzy methods in which the security returns are assumed to be characterized by fixed possibility distribution functions or membership functions.

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