Fuzzy Optim Decis Making (2012) 11:451–463
DOI 10.1007/s10700-012-9125-x

A risk index model for portfolio selection with returns subject to experts' estimations

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Published online: 25 March 2012

Abstract Portfolio selection is concerned with selecting an optimal portfolio that can strike a balance between maximizing the return and minimizing the risk among a large number of securities. Traditionally, security returns were regarded as random variables. However, there are cases that the predictions of security returns are given mainly based on experts' judgements and estimations rather than historical data. In this paper, we introduce a new type of variable to reflect the subjective estimations of the security returns. A risk index for uncertain portfolio selection is proposed and a new safe criterion for judging the portfolio investment is introduced. Based on the proposed risk index, a new mean-risk index model is developed and its crisp forms are given. In addition, to illustrate the application of the model, two numerical examples are also presented.

Keywords Portfolio selection · Uncertain programming · Mean-risk index model · Risk index

1 Introduction

Portfolio selection is concerned with selecting optimal combination of securities among a large number of candidate securities. Traditionally, security returns were regarded to be random variables and a great deal of achievements have been made in portfolio theory based on this assumption, for example, recent works Abdelaziz et al. (2007), Corazza and Favaretto (2007), Huang (2008), Lin and Liu (2008), etc. However, since the security returns, especially short term security returns are sensitive to various economic and non-economic factors, it is found in reality that sometimes the

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historical data can hardly reflect the future security returns. The predictions of security returns have to be given mainly based on experts' judgements and estimations. Therefore, many scholars argued that we should find other way other than probability theory to solve the portfolio selection problem in the situation. With the introduction and development of fuzzy set theory, scholars have tried using fuzzy set theory to help select the portfolio with returns subject to experts' evaluations since 1990's. For example, based on possibility measure, Watada (1997), Tanaka and Guo (1999), Carlsson et al. (2002), Bilbao-Terol et al. (2006), Lacagnina and Pecorella (2006) extended the mean-variance idea to solve the portfolio selection problems in different ways. Based on credibility measure, Huang (2007, 2008), Qin et al. (2009), Li et al. (2010), and Zhang et al. (2010) proposed a spectrum of credibilistic mean-variance portfolio selection models.

These researches broadened the way to handle portfolio selection problem when security returns are given mainly based on human estimations rather than historical data. However, deeper researches find that paradoxes will appear if we use fuzzy variable to describe the subjective estimations of security returns. For example, if a security return is regarded as a fuzzy variable, then we have a membership function to characterize it. Suppose it is a triangular fuzzy variable $\xi = (-0.7, 0.2, 1.1)$. Based on the membership function, it is known from possibility theory or credibility theory that *the return is exactly 0.2 with belief degree 1 in possibility measure or 0.5 in credibility measure*. However, this conclusion is hard to accept because the belief degree of *exactly 0.2* should be almost zero. In addition, it is known from possibility theory or credibility theory that the return being *exactly 0.2* and *not exactly 0.2* have the same belief degree in either possibility measure or credibility measure, which implies that the two events will happen equally likely. This conclusion is also contradictory to our judgement and is hard to accept. In 2007, Liu proposed an uncertain measure and developed an uncertainty theory which can be used to handle subjective imprecise quantity. Much research work has been done on the development of uncertainty theory and related theoretical work. For example, You (2009) proved some convergence theorems of uncertain sequences, and Gao (2009) proved some properties of continuous uncertain measure. Liu studied uncertain programming (Liu 2009). Gao et al. (2010) discussed the inference rule for uncertain systems. Peng and Iwamura (2010) gave a sufficient and necessary condition of uncertainty distribution, and Chen and Liu (2010) proved the existence and uniqueness theorem for uncertain differential equations, etc. When we use uncertain variable to describe the experts' estimations of security returns, the above mentioned paradoxes disappear immediately. Based on uncertainty theory, Zhu (2010) has solved an uncertain optimal control problem and applied it to a portfolio selection model, and Huang (2011) has defined a risk curve and has given a new selection method for uncertain portfolio selection and further proposed the uncertain mean-variance and mean-semivariance selection methods (Huang 2012). In this paper, we will also use uncertain variables to describe the experts' estimations of security returns. Different from using variance as risk measurement, We will define a new risk measurement, i.e., a risk index, and further propose a new mean-risk index selection method for portfolio selection based on the new risk measurement.

The rest of the paper is organized as follows. For better understanding of the paper, some necessary knowledge about uncertain variable will be introduced in Sect. 2.

Then the motivation for proposing an alternative risk measure will be discussed and a risk index will be proposed in Sect. 3. Based on the risk index measure, a mean-risk index model will be developed in the same section. After that, the crisp forms of the model will be presented in Sect. 4. In Sect. 5, the application of the model will be discussed by means of examples. Finally, in Sect. 6, some conclusion remarks will be given.

2 Necessary knowledge about uncertain variable

In 2007, Liu proposed an uncertain measure and an uncertain variable based on an axiomatic system of normality, self-duality, and countable subadditivity.

Definition 1 Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}\{\Lambda\}$ is called an uncertain measure if it satisfies the following three axioms (Liu 2007):

(i) (Normality) $\mathcal{M}\{\Gamma\} = 1$.
(ii) (Self-duality) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$.
(iii) (Countable subadditivity) For every countable sequence of events$\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.
It can be proven (Liu 2010) that any uncertain measure $\mathcal{M}$ is increasing. That is, for any events $\Lambda_1 \in \Lambda_2$, we have

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}. \tag{1}$$

In order to define product uncertain measure, Liu (2007) proposed the fourth axiom as follows:

(iv) (Product measure) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots, n$. The product uncertain measure is

$$\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots \wedge \mathcal{M}_n.$$

Definition 2 (Liu 2007) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

An uncertainty distribution function is used to characterize an uncertain variable and is defined as follows.

Definition 3 (Liu 2007) The uncertainty distribution $\Phi : \Re \to [0, 1]$ of an uncertain variable ξ is defined by

$$\Phi(t) = \mathcal{M}\{\xi \leq t\}.$$

For example, by a normal uncertain variable, we mean the variable that has the following normal uncertainty distribution

$$\Phi(t) = \left(1 + \exp\left(\frac{\pi(e - t)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad t \in \Re,$$

where e and σ are real numbers and $\sigma > 0$. For convenience, it is denoted in the paper by $\xi \sim \mathcal{N}(e, \sigma)$.

We call an uncertain variable the linear uncertain variable if it has the following linear uncertainty distribution

$$\Phi(t) = \begin{cases} 0, & \text{if } \; t < a \\ (t - a)/(b - a), & \text{if } \; a \leq t \leq b \\ 1, & \text{if } \; t > b. \end{cases}$$

For convenience, it is denoted in the paper by $\xi \sim \mathcal{L}(a, b)$ where $a < b$.

When the uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are represented by uncertainty distributions, the operational law is given by Liu (2010) as follows:

Theorem 1 (Liu 2010) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Let $f(t_1, t_2, \ldots, t_n)$ be strictly increasing with respect to $t_1, t_2, \ldots, t_n$. Then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

is an uncertain variable with uncertainty distribution

$$\Psi(t) = \sup_{f(t_1, t_2, \ldots, t_n) = t} \left(\min_{1 \leq i \leq n} \Phi_i(t_i)\right), \quad t \in \Re \tag{2}$$

whose inverse function is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)), \quad 0 < \alpha < 1 \tag{3}$$

if $\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)$ are unique for each $\alpha \in (0, 1)$.

To tell the size of an uncertain variable, Liu defined the expected value of uncertain variables.

Definition 4 (Liu 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_{0}^{\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r \tag{4}$$

provided that at least one of the two integrals is finite.

It can be calculated that the expected value of the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is $\mathrm{E}[\xi] = e$, and the expected value of the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is $\mathrm{E}[\xi] = (a + b)/2$.

Theorem 2 (Liu 2010) *Let ξ_1 and ξ_2 be independent uncertain variables with finite expected values. Then for any real numbers a_1 and a_2, we have*

$$\mathrm{E}[a_1\xi_1 + a_2\xi_2] = a_1\mathrm{E}[\xi_1] + a_2\mathrm{E}[\xi_2]. \tag{5}$$

3 Risk index and mean-risk index model

In portfolio selection, how to define risk is one of the most important topics. The earliest and the most popular risk definition is variance. It was given by Markowitz in 1952. He regarded the security returns as random variables and proposed that expected value of a portfolio return could be regarded as the representative of the investment return and variance the risk of the investment. The idea is that the greater the deviation from the expected value, the less likely the investors can obtain the expected return, and thus the riskier the portfolio. Therefore, for conservative investors, when making investment, they should first require that the portfolio be safe enough, i.e., the variance value of the portfolio be less than or equal to a predetermined tolerable variance level and then select among the safe portfolios the one with the maximum expected return. In mathematica way, the mean-variance model is expressed as follows:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\ \text{subject to:} \\ \qquad V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \leq c \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \ldots, n \end{cases} \tag{6}$$

where ξ_i denotes the random return of the ith security, x_i the investment proportion in the ith security, $i = 1, 2, \ldots, n$, respectively, E the expected value operator, V the variance operator, and c the predetermined maximum tolerable variance value.

Though variance is a popular risk measure, it is not so convenient to use for investors. It is seen from the model (6) that before knowing the expected return of the portfolio, the investors need to give a maximum tolerable variance level. However, it is difficult to judge if a variance level is tolerable or not when the expected value of the portfolio is unknown because with different expected value, the investors' maximum tolerable variance degree may be different. For example, suppose we have two portfolios A and B. The variance values of the two portfolio returns are both 1, but the expected return of portfolio A is 0.1 and the expected return of Portfolio B is 10. It is easy to see that though the variance values of portfolios A and B are same, Portfolio A will be regarded quite unlikely to realize the expected value 0.1, and so variance 1 is intolerable. On the contrary, variance value 1 may be tolerable for Portfolio B because Portfolio B will be likely to realize the expected value 10 with the same variance value 1. Furthermore, in reality, people usually regard the return being lower than a base target as a loss, but variance gives no idea about investors' likely loss degree. These are also true in case of uncertain portfolio selection. These difficulties motivate the author to propose an easier-to-use risk measure. As we know, people can choose to invest their money in risk free asset and gain risk free interest rate with certainty. Therefore, risk free interest rate can be set as a base target and any portfolio returns below the risk free interest rate will be regarded as losses. Since the risk free interest rate is known before investment, it is easier for investors to tell how much level below the rate they can tolerate. To obtain an average loss level, i.e., an average level of the portfolio return below the risk free interest rate, we define a risk index as follows:

Definition 5 Let ξ denote an uncertain return rate of a portfolio, and r_f the risk free interest rate. Then the risk index of the portfolio is defined by

$$RI(\xi) = E[(r_f - \xi)^+], \tag{7}$$

where E is the expected value operator of the uncertain variable and

$$(r_f - \xi)^+ = \begin{cases} r_f - \xi, & \text{if } \xi \leq r_f \\ 0, & \text{if } \xi > r_f. \end{cases}$$

Let c denote the maximum mean loss level that the investors can tolerate. Then it is clear that a portfolio is safe if

$$RI(\xi) \leq c. \tag{8}$$

For example, if the investors set $c = 0.01$, it means that the tolerable average level below risk free interest rate is 0.01.

In paper Huang (2011), Huang defined a risk curve which gives the information of each likely loss degree and the loss occurrence chance. Let ξ represent the uncertain portfolio return. The risk curve is expressed as

$$f(r) = \mathcal{M}\{r_f - \xi \geq r\}, \quad \forall r \geq 0. \tag{9}$$

Huang further defined a confidence curve which gives the investors' maximal tolerance towards the occurrence chance of each likely loss level. Then let $\alpha(r)$ denote the investors' confidence curve. A portfolio is safe if

$$f(r) \leq \alpha(r), \quad \forall r \geq 0. \tag{10}$$

Comparing Eqs. (7) with (9), we can see that risk index is an average level of risk curve. We can also see from safe criterion (10) that though risk curve provides loss degree information, it is still not very easy to use for some investors because to tell if a portfolio is safe or not, the investors have to find out their tolerance levels towards all the occurrence chances of all the loss degrees. This is not an easy task sometimes. However, taking risk index as risk measurement, the investors only need to give one level, i.e., the tolerable average level below the base profit. This is always much easier.

Usually, when making investment, the investors will require that the portfolio be safe enough and then pursue the maximum return. We use expected value as the representative of investment return and risk index the risk measurement. Let x_i denote the investment proportions in securities i, ξ_i the uncertain return rates of the ith securities, $i = 1, 2, \ldots, n$, respectively, and c the investors' tolerable average value below the risk free interest rate. Then, to pursue the maximum return among the safe portfolios, we have the model as follows:

$$\begin{cases} \max E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{subject to:} \\ \quad RI(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) \le c \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \ge 0, \quad i = 1, 2, \ldots, n \end{cases} \tag{11}$$

where RI is the risk index of the portfolio defined as

$$RI(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n) = E[(r_f - (\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n))^+].$$

It is clear that portfolios whose risk indexes are not greater than the preset level c are safe portfolios, and among them the portfolio with the maximum expected return is the optimal portfolio that the investors should select.

4 Crisp forms

Before giving the crisp form, we first give a theorem for calculating the risk index.

Theorem 3 *Let ξ be an uncertain security return with continuous uncertainty distribution Φ whose inverse function $\Phi^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$. Then the risk index of the security can be calculated via:*

$$RI(\xi) = \int_0^\beta \left(r_f - \Phi^{-1}(\alpha)\right) \mathrm{d}\alpha, \tag{12}$$

where β is defined by $\Phi^{-1}(\beta) = r_f$.

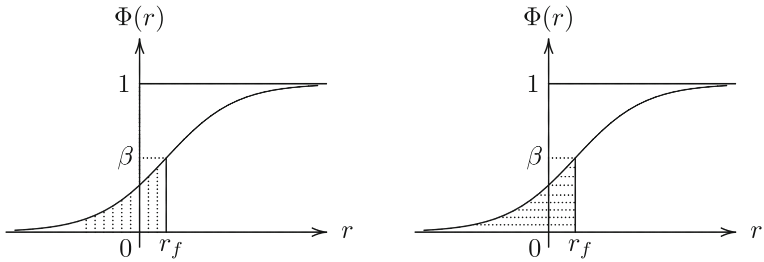

Fig. 1 Risk index via integral

Proof According to Eq. (4), we have

$$\begin{aligned} RI(\xi) &= E[(r_f - \xi)^+] \\ &= \int_0^{\infty} \mathcal{M}\{r_f - \xi \geq r\}\mathrm{d}r \\ &= \int_0^{\infty} \Phi(r_f - r)\mathrm{d}r \\ &= \int_{-\infty}^{r_f} \Phi(r)\mathrm{d}r. \end{aligned} \tag{13}$$

It is easy to see from the Eq. (13) that the theorem holds. See Fig. 1. □

Theorem 4 *Let Φ_i denote the continuous uncertainty distribution of the ith uncertain security return rate ξ_i whose inverse function $\Phi_i^{-1}(\alpha)$ exists and is unique for each $\alpha \in (0, 1)$, $i = 1, 2, \ldots, n$, respectively. Let e_i be the expected return of the ith security return rate. Then the risk index model (11) can be transformed into the following form:*

$$\begin{cases} \max x_1e_1 + x_2e_2 + \cdots + x_ne_n \\ \textit{subject to:} \\ \quad \beta r_f - \int_0^{\beta} \left(x_1\Phi_1^{-1}(\alpha) + x_2\Phi_2^{-1}(\alpha) + \cdots + x_n\Phi_n^{-1}(\alpha)\right)\mathrm{d}\alpha \leq c \\ \quad x_1\Phi_1^{-1}(\beta) + x_2\Phi_2^{-1}(\beta) + \cdots + x_n\Phi_n^{-1}(\beta) = r_f \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \geq 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{14}$$

Proof The objective function of the model (14) can be obtained directly from Theorem 2. □

Let Ψ denote the uncertainty distribution function of the portfolio return $\sum_{i=1}^{n} x_i\xi_i$. It is known from Theorem 3 that

$$RI(\xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n) = \int_0^\beta \Big(r_f - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha,$$

where β is determined by $\Psi^{-1}(\beta) = r_f$. Since $x_i \ge 0$ for $i = 1, 2, \ldots, n$, respectively, it follows from Theorem 1 that

$$\int_0^\beta \Big(r_f - \Psi^{-1}(\alpha)\Big)\mathrm{d}\alpha = \beta r_f - \int_0^\beta \Big(x_1\Phi_1^{-1}(\alpha) + x_2\Phi_2^{-1}(\alpha) + \cdots + x_n\Phi_n^{-1}(\alpha)\Big)\mathrm{d}\alpha,$$

and $x_1\Phi_1^{-1}(\beta) + x_2\Phi_2^{-1}(\beta) + \cdots + x_n\Phi_n^{-1}(\beta) = r_f$.

Thus the theorem is proven.

According to Theorem 4, we can get the crisp forms of the mean-risk index model in the situations where security returns are all normal uncertain variables or linear uncertain variables.

Theorem 5 *Suppose the return rates of the ith securities are all normal uncertain variables $\xi_i \sim \mathcal{N}(e_i, \sigma_i)$, $i = 1, 2, \ldots, n$, respectively. Then the risk index model can be transformed into the following crisp form:*

$$\begin{cases} \max \displaystyle\sum_{i=1}^{n} e_i x_i \\ \text{subject to:} \\ \quad \beta\Big(r_f - \displaystyle\sum_{i=1}^{n} e_i x_i\Big) - \displaystyle\sum_{i=1}^{n} \frac{\sqrt{3}\sigma_i x_i}{\pi}\Big[\beta \ln \beta + (1-\beta)\ln(1-\beta)\Big] \le c \\ \quad \beta = \exp\Big(\pi(r_f - \displaystyle\sum_{i=1}^{n} e_i x_i)\Big/\sqrt{3}\displaystyle\sum_{i=1}^{n}\sigma_i x_i\Big)\Big/ \\ \qquad\qquad \Big(1 + \exp\Big(\pi(r_f - \sum_{i=1}^{n} e_i x_i)\Big/\sqrt{3}\sum_{i=1}^{n}\sigma_i x_i\Big)\Big) \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{15}$$

Theorem 6 *Suppose the return rates of the ith securities are all linear uncertain variables $\xi_i \sim (a_i, b_i)$, $i = 1, 2, \ldots, n$, respectively. Then the risk index model can be transformed into the following crisp form:*

Table 1 Normal uncertain return rates of 10 securities

Security i	Uncertain return rate ξ_i	Security i	Uncertain return rate ξ_i
1	$\mathcal{N}(0.038, 0.065)$	6	$\mathcal{N}(0.028, 0.045)$
2	$\mathcal{N}(0.043, 0.06)$	7	$\mathcal{N}(0.035, 0.058)$
3	$\mathcal{N}(0.032, 0.056)$	8	$\mathcal{N}(0.033, 0.05)$
4	$\mathcal{N}(0.039, 0.067)$	9	$\mathcal{N}(0.025, 0.04)$
5	$\mathcal{N}(0.031, 0.055)$	10	$\mathcal{N}(0.05, 0.12)$

$$\begin{cases} \max \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\Big(a_i x_i + b_i x_i\Big) \\ \text{subject to:} \\ \qquad \Big(r_f - \displaystyle\sum_{i=1}^{n} a_i x_i\Big)^2 \Big/ 2\displaystyle\sum_{i=1}^{n}\Big(b_i x_i - a_i x_i\Big) \le c \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \ldots, n. \end{cases} \tag{16}$$

5 Examples

Example 1 Suppose an investor chooses from 10 securities for his/her investment. Assume that the monthly return rates of the ten securities are all normal uncertain variables. The return rate of a security is defined to be $\xi = (p' + d - p)/p$, where p' denotes the closing price of the security next month, p the closing price of the security at present, and d the dividend of the security during the month. The estimation of the candidate security return rates is given in Table 1.

Suppose the monthly risk free interest rate is 0.01. The investor sets his/her tolerable value below risk free interest rate $c = 0.01$. Then according to the model (15) in Sect. 4, we have the model as follows:

$$\begin{cases} \max \displaystyle\sum_{i=1}^{10} e_i x_i \\ \text{subject to:} \\ \qquad \beta\Big(0.01 - \displaystyle\sum_{i=1}^{10} e_i x_i\Big) - \displaystyle\sum_{i=1}^{10} \frac{\sqrt{3}\sigma_i x_i}{\pi}\Big[\beta \ln \beta + (1-\beta)\ln(1-\beta)\Big] \le 0.01 \\ \qquad \beta = \exp\Big(\pi(0.01 - \displaystyle\sum_{i=1}^{10} e_i x_i)\Big/\sqrt{3}\displaystyle\sum_{i=1}^{10}\sigma_i x_i\Big)\Big/ \\ \qquad\qquad \Big(1 + \exp\Big(\pi(0.01 - \sum_{i=1}^{10} e_i x_i)\Big/\sqrt{3}\sum_{i=1}^{10}\sigma_i x_i\Big)\Big) \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \ldots, 10. \end{cases} \tag{17}$$

Table 2 Allocation of money to 10 securities with normal uncertain returns (%)

Security i	1	2	3	4	5	6	7	8	9	10
Allocation of money	0.00	75.01	0.00	0.00	0.00	0.00	0.00	0.00	24.99	0.00

Table 3 Optimal portfolios under different risk level constraints (%)

c	Obj.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1.0	3.85	0.00	75.01	0.00	0.00	0.00	0.00	0.00	0.00	24.99	0.00
1.5	4.48	0.00	74.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	25.91
2.0	4.67	0.00	46.92	0.00	0.00	0.00	0.00	0.00	0.00	0.00	53.08
2.5	4.86	0.00	20.25	0.00	0.00	0.00	0.00	0.00	0.00	0.00	79.75

By running Solver in Excel, to obtain the maximum expected return with the risk index not greater than the preset level 0.01, the investor should allocate his/her money according to Table 2. The maximum expected return rate is 3.85 %.

To test the effect of requirement for risk index on the decision result of portfolio selection, we adjust the c value and do the experiment. The results are shown in Table 3. It is seen that when the tolerable risk level c increases, the obtained expected value of the portfolio return becomes greater, which is in consistent with the rule of "the higher the risk, the higher the return".

Example 2 Suppose in this case the candidate securities are linear uncertain variables which are given in Table 4. The risk-free interest rate is still 0.01, and the risk index level is set at $c = 0.017$. Then according to the model (16) in Sect. 4, we have the model as follows:

$$\begin{cases} \max \dfrac{1}{2}\displaystyle\sum_{i=1}^{10}\left(a_i x_i + b_i x_i\right) \\ \text{subject to:} \\ \qquad \left(0.01 - \displaystyle\sum_{i=1}^{n} a_i x_i\right)^2 \Big/ 2\displaystyle\sum_{i=1}^{n}\left(b_i x_i - a_i x_i\right) \le 0.017 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \ldots, 10. \end{cases} \tag{18}$$

By running Solver in Excel, to obtain the maximum expected return with the risk index not greater than the preset level 0.017, the investor should allocate his/her money according to Table 5. The maximum expected return rate is 4.28 %.

Again, we adjust the c values and show the portfolio selection results in Table 6. It is also seen that when the tolerable risk level c increases, the obtained expected value of the portfolio return becomes greater.

Table 4 Linear uncertain return rates of 10 securities

Security i	Uncertain return rate ξ_i	Security i	Uncertain return rate ξ_i
1	$\mathcal{L}(-0.093, 0.202)$	6	$\mathcal{L}(-0.214, 0.332)$
2	$\mathcal{L}(-0.089, 0.192)$	7	$\mathcal{L}(-0.231, 0.392)$
3	$\mathcal{L}(-0.080, 0.160)$	8	$\mathcal{L}(-0.324, 0.442)$
4	$\mathcal{L}(-0.148, 0.223)$	9	$\mathcal{L}(-0.245, 0.386)$
5	$\mathcal{L}(-0.123, 0.241)$	10	$\mathcal{L}(-0.175, 0.278)$

Table 5 Allocation of money to 10 securities with linear uncertain returns (%)

Security i	1	2	3	4	5	6	7	8	9	10
Allocation of money	0.00	24.40	75.60	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 6 Optimal portfolios under different risk level constraints (%)

c	Obj.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
1.7	4.28	0.00	24.40	75.60	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2.1	5.73	89.33	0.00	0.00	0.00	0.00	0.00	10.67	0.00	0.00	0.00
2.8	6.37	64.76	0.00	0.00	0.00	0.00	0.00	35.24	0.00	0.00	0.00
3.3	6.82	47.31	0.00	0.00	0.00	0.00	0.00	52.69	0.00	0.00	0.00

6 Conclusions

This paper has discussed the portfolio selection problem when security returns are given by experts' estimations rather than historical data. Regarding security returns as uncertain variables, the paper has introduced a risk index as an alternative risk measurement and developed a mean-risk index model. In addition, the crisp forms of the model have also been provided. The numerical examples illustrated the application of the proposed model and showed that the greater the risk level, the higher the obtained expected return.

Acknowledgments The author would like to thank the anonymous reviewers and the Associate Editor for their valuable comments. This work was supported by National Natural Science Foundation of China Grants No.70871011 and No. 71171018, Program for New Century Excellent Talents in University, and the Fundamental Research Funds for the Central Universities.

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