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Randomly generating test problems for fuzzy relational equations

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Abstract Fuzzy relational equations play an important role in fuzzy set theory and fuzzy logic systems. To compare and evaluate the accuracy and efficiency of various solution methods proposed for solving systems of fuzzy relational equations as well as the associated optimization problems, a test problem random generator for systems of fuzzy relational equations is needed. In this paper, procedures for generating test problems of fuzzy relational equations with the sup-$\mathcal{T}$ composition are proposed for the cases of sup-$\mathcal{T}_M$, sup-$\mathcal{T}_P$, and sup-$\mathcal{T}_L$ compositions. It is shown that the test problems generated by the proposed procedures are consistent. Some properties are discussed to show that the proposed procedures randomly generate systems of fuzzy relational equations with various number of minimal solutions. Numerical examples are included to illustrate the proposed procedures.

Keywords Fuzzy relational equations · Triangular norms · Random generator

1 Introduction

The study of fuzzy relational equations based upon the max-min composition was first investigated by Sanchez (1976, 1977) in his pioneering work on the applications of

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fuzzy relations in approximate reasoning and decision making. Since then, solving various types of fuzzy relational equations has become one of the most appealing issues in fuzzy set theory. The basic type of fuzzy relational equations are those with the sup-$\mathcal{T}$ composition, or more accurately, the max-$\mathcal{T}$ composition for finite scenarios, where $\mathcal{T}$ is typically a continuous triangular norm (t-norm for short). Such a system with n variables and m constraints can be stated as follows:

$$\sup_{j \in N} \mathcal{T}(a_{ij}, x_j) = b_i, \ \forall\, i \in M, \tag{1}$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ are two index sets, a_{ij}, x_j and b_i are real numbers in the unit interval for each $i \in M$ and $j \in N$. Denote $A = (a_{ij})_{m \times n}, \mathbf{x} = (x_j)_{n \times 1}$, and $\mathbf{b} = (b_i)_{m \times 1}$. A system of fuzzy relational equations with the sup-$\mathcal{T}$ composition is also referred to as a system of sup-$\mathcal{T}$ equations for short and can be represented in the matrix form of $A \circ \mathbf{x} = \mathbf{b}$ where "$\circ$" stands for the specific sup-$\mathcal{T}$ composition. Usually, the t-norm $\mathcal{T}$ is required to be continuous, i.e., continuous as a function of two arguments. The commonly seen minimum operator $\mathcal{T}_M(x, y) = \min(x, y)$, product operator $\mathcal{T}_P(x, y) = x \cdot y$, and Łukasiewicz t-norm $\mathcal{T}_L(x, y) = \max(x + y - 1, 0)$ are all continuous. The study of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous t-norm can be reduced, in some sense, to the study of sup-$\mathcal{T}_M$ equations, sup-$\mathcal{T}_P$ equations, and sup-$\mathcal{T}_L$ equations, respectively (see, e.g., Li and Fang 2008).

The resolution of a system of sup-$\mathcal{T}$ equations is to determine the unknown vector $\mathbf{x}$ for a given coefficient matrix A and a right hand side vector $\mathbf{b}$ such that $A \circ \mathbf{x} = \mathbf{b}$. The set of all solutions, when it is non-empty, is a finitely generated root system which can be fully determined by a unique maximum solution and a finite number of minimal solutions (see, e.g., Li and Fang 2009). For a finite system of fuzzy relational equations with max-$\mathcal{T}$ composition, it is well known that its consistency can be verified by constructing and checking a potential maximum solution. However, the detection of all minimal solutions is closely related to the set covering problem and remains a challenging problem [see, e.g., Li and Fang (2009), Klir and Yuan (1995), Markovskii (2005), and Pedrycz (1991)]. Overviews of fuzzy relational equations and their applications can be found in Li and Fang (2009) and Peeva and Kyosev (2004).

The problem of minimizing a linear objective function subject to a system of fuzzy relational equations with max-min composition was first investigated in Fang and Li (1999) and later in Wu et al. (2002) and Wu and Guu (2005). Following the idea proposed in Fang and Li (1999), some variants of this type of optimization problems were discussed in Abbasi Molai and Khorram (2007), Abbasi Molai and Khorram (2008), Ghodousian and Khorram (2006), Guo and Xia (2006), Guu and Wu (2002), Khorram and Ghodousian (2006), Loetamonphong and Fang (2001) and Wu and Guu (2004). Most recently, it was shown in Li and Fang (2008) that the problem of minimizing an objective function subject to a system of fuzzy relational equations can be reduced to a 0-1 mixed integer programming problem in polynomial time. A set covering-based surrogate approach was proposed in Hu and Fang (2011) to solve the sup-$\mathcal{T}$ equation constrained optimization problems with a separable and monotone objective function of its variables.

Various solution methods have been developed for solving systems of fuzzy relational equations as well as the associated optimization problems. Examination of the accuracy and efficiency of the proposed solution methods is of theoretical and practical importance. This work intends to develop a test problem random generator for systems of fuzzy relational equations. Procedures for generating test problems of fuzzy relational equations with the sup-$\mathcal{T}_M$, sup-$\mathcal{T}_P$, and sup-$\mathcal{T}_L$ compositions are proposed. It is shown that the test problems generated by the proposed procedures are always consistent. Moreover, since the solution set of a system of fuzzy relational equations, when it is nonempty, can be characterized by one unique maximum solution and a finite number of minimal solutions, test problems with various number of minimal solutions are required for a fair assessment of a developed solution method. It is shown that the proposed procedures can generate systems of fuzzy relational equations with a unique minimal solution if $m \geq n$, and systems of fuzzy relational equations with up to $2^{\lfloor \frac{n}{2} \rfloor}$ minimal solutions if $m \geq \lceil \frac{n}{2} \rceil$.

The rest of this paper is organized as follows. In Sect. 2, some basic concepts and properties associated with fuzzy relational equations are provided. In Sect. 3, procedures for generating test problems for systems of fuzzy relational equations with the sup-$\mathcal{T}$ composition are proposed. Numerical examples are included in Sect. 4 to illustrate the proposed procedures. Conclusions are given in Sect. 5 while the pseudo codes are included in the Appendix.

2 Preliminaries

In this section, we recall some basic concepts and properties associated with the fuzzy relational equations. All proofs are omitted to keep the paper succinct and readable. Readers may refer to Klement et al. (2000) for a rather complete overview of triangular norms, and to Li and Fang (2008), Li et al. (2008), Li and Fang (2009), and Li (2009) for the detailed analysis on the resolution of a system of sup-$\mathcal{T}$ equations.

2.1 Triangular norms

A triangular norm (t-norm for short) is a binary operator $\mathcal{T} : [0, 1]^2 \rightarrow [0, 1]$, such that, for all $x, y, z \in [0, 1]$, the following four axioms are satisfied:

(T1) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ (commutativity);
(T2) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ (associativity);
(T3) $\mathcal{T}(x, y) \leq \mathcal{T}(x, z)$, whenever $y \leq z$ (monotonicity);
(T4) $\mathcal{T}(x, 1) = x$ (boundary condition).

Since t-norms are binary algebraic operators on the real unit interval [0, 1], the infix notation like $x \wedge_t y$ is also used in the literature instead of the prefix notation $\mathcal{T}(x, y)$.

A t-norm $\mathcal{T}$ is said to be continuous if it is continuous as a real function of two arguments. Due to its commutativity and monotonicity properties, a t-norm is continuous if and only if it is continuous in one of its arguments. Analogously, a t-norm is said to be left- or right-continuous if it is left- or right-continuous, respectively, in one of its arguments. The three most important continuous t-norms are the minimum t-norm

T_M, product t-norm T_P and Łukasiewicz t-norm T_L. They are defined, respectively, by

$T_M(x, y) = \min(x, y)$ (minimum, Gödel, Zadeh t-norm),
$T_P(x, y) = x \cdot y$ (probabilistic product, Goguen t-norm),
$T_L(x, y) = \max(x + y - 1, 0)$ (bounded difference, Łukasiewicz t-norm).

To characterize the solution set of a system of sup-$\mathcal{T}$ equations, two residual operators are defined with respect to a continuous t-norm $\mathcal{T}$.

Definition 2.1 Given a t-norm $\mathcal{T}$, the binary residual operators $\mathcal{I_T} : [0, 1]^2 \to [0, 1]$ and $\mathcal{J_T} : [0, 1]^2 \to [0, 1]$ are defined, respectively, by

$$\mathcal{I_T}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{T}(x, z) \leq y\}$$

and

$$\mathcal{J_T}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{T}(x, z) \geq y\}.$$

The residual operator $\mathcal{I_T}$ is known as a residual implicator or briefly an R-implicator in fuzzy logic while the residual operator $\mathcal{J_T}$ has no particular logical interpretation. In the literature, the residual implicators are also known as φ-operators which were introduced by Pedrycz (1985) in a different approach to describe the solutions of sup-$\mathcal{T}$ equations. The residual operator $\mathcal{J_T}$ was discussed in Di Nola et al. (1989) with a slightly different definition. The infix notations are used to denote these two residual operators, i.e., $x \varphi_t y = \mathcal{I_T}(x, y)$ and $x \sigma_t y = \mathcal{J_T}(x, y)$, respectively.

Theorem 2.1 (see, e.g., Li and Fang 2008) *Let $\mathcal{T}$ be a left-continuous t-norm and $\mathcal{I_T}$ its associated residual implicator. It holds for all $a, b \in [0, 1]$ that $\mathcal{T}(a, x) \leq b$ if and only if $x \leq \mathcal{I_T}(a, b)$.*

Theorem 2.2 (see, e.g., Li and Fang 2008) *Let $\mathcal{T}$ be a continuous t-norm with $\mathcal{I_T}$ and $\mathcal{J_T}$ being the associated residual operators. The equation $\mathcal{T}(a, x) = b$ has a solution for any given $a, b \in [0, 1]$ if and only if $b \leq a$, in which case the solution set of $\mathcal{T}(a, x) = b$ is given by the closed interval $[\mathcal{J_T}(a, b), \mathcal{I_T}(a, b)]$.*

Theorem 2.1 plays a crucial role in the resolution of sup-$\mathcal{T}$ equations, which is actually a special scenario of the general theory of Galois connections (see, e.g., Blyth and Janowitz 1972). The residual operators $\mathcal{I_T}$ and $\mathcal{J_T}$ of the three most important continuous t-norms are listed in Table 1.

2.2 Resolution of systems of sup-$\mathcal{T}$ equations

In this section, we focus on the resolution of a finite system of fuzzy relational equations with sup-$\mathcal{T}$ composition described in (1) or represented in the matrix form as $A \circ \mathbf{x} = \mathbf{b}$. Without loss of generality, we may assume that $b_1 \geq b_2 \geq \cdots \geq b_m > 0$ (see, e.g., Li and Fang 2008).

Table 1 Residual operators of the Gödel, Goguen, and Łukasiewicz t-norms

$\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}(x, y)$	$\mathcal{J}_{\mathcal{T}}(x, y)$
$\mathcal{T}_M$	$\begin{cases} 1, & \text{if } x \le y, \\ y, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y, \\ y, & \text{otherwise.} \end{cases}$
$\mathcal{T}_P$	$\begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y, \\ \frac{y}{x}, & \text{if } 0 < y \le x, \\ 0, & \text{otherwise.} \end{cases}$
$\mathcal{T}_L$	$\min(1 - x + y, 1)$	$\begin{cases} 1, & \text{if } x < y, \\ 1 - x + y, & \text{if } 0 < y \le x, \\ 0, & \text{otherwise.} \end{cases}$

2.2.1 Solvability and solution set

Given a system of sup-$\mathcal{T}$ equations $A \circ \mathbf{x} = \mathbf{b}$ with a continuous t-norm $\mathcal{T}$, the set of all solutions to $A \circ \mathbf{x} = \mathbf{b}$ is called its complete solution set and denoted by $S(A, \mathbf{b}) = \{\mathbf{x} \in [0, 1]^n \mid A \circ \mathbf{x} = \mathbf{b}\}$. A partial order can be defined on $S(A, \mathbf{b})$ by extending the natural order such that for any $\mathbf{x}^1, \mathbf{x}^2 \in S(A, \mathbf{b})$, $\mathbf{x}^1 \le \mathbf{x}^2$ if and only if $x_j^1 \le x_j^2$ for all $j \in N$. A system of sup-$\mathcal{T}$ equations $A \circ \mathbf{x} = \mathbf{b}$ is called consistent if $S(A, \mathbf{b}) \ne \emptyset$. Otherwise, it is inconsistent. Due to the monotonicity of the t-norm involved in the composition, if $\mathbf{x}^1, \mathbf{x}^2 \in S(A, \mathbf{b})$ and $\mathbf{x}^1 \le \mathbf{x}^2$, any $\mathbf{x}$ satisfying $\mathbf{x}^1 \le \mathbf{x} \le \mathbf{x}^2$ is also in $S(A, \mathbf{b})$. Therefore, the attention could be focused on the so-called extremal solutions as defined below.

Definition 2.2 A solution $\check{\mathbf{x}} \in S(A, \mathbf{b})$ is called a minimal or lower solution if, for any $\mathbf{x} \in S(A, \mathbf{b})$, the relation $\mathbf{x} \le \check{\mathbf{x}}$ implies $\mathbf{x} = \check{\mathbf{x}}$. A solution $\hat{\mathbf{x}} \in S(A, \mathbf{b})$ is called the maximum or greatest solution if $\mathbf{x} \le \hat{\mathbf{x}}$, $\forall\, \mathbf{x} \in S(A, \mathbf{b})$.

Theorem 2.3 (see, e.g., Li and Fang 2008) *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations. A vector $\mathbf{x} \in [0, 1]^n$ is a solution to $A \circ \mathbf{x} = \mathbf{b}$ if and only if there exists an index $j_i \in N$ for each $i \in M$ such that*

$$a_{ij_i} \wedge_t x_{j_i} = b_i \quad \text{and} \quad a_{ij} \wedge_t x_j \le b_i, \ \forall\, i \in M, j \in N.$$

Theorem 2.3 holds in a straightforward way due to the non-interactivity property of the maximum operator, i.e., $a \vee b \in \{a, b\}$. Theorems 2.2 and 2.3 lead to the next well-known solvability criterion of a system $A \circ \mathbf{x} = \mathbf{b}$ and the characterization of its solution set, both of which were first seen in Sanchez (1976, 1977).

Theorem 2.4 (see, e.g., Li and Fang 2008) *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with a continuous t-norm $\mathcal{T}$. The system is consistent if and only if the vector $A^T \varphi_t \mathbf{b}$ with its components defined by*

$$(A^T \varphi_t \mathbf{b})_j = \inf_{i \in M} \mathcal{I}_\mathcal{T}(a_{ij}, b_i), \quad \forall\, j \in N, \tag{2}$$

is a solution to $A \circ \mathbf{x} = \mathbf{b}$. *Moreover, if the system is consistent, the complete solution set* $S(A, \mathbf{b})$ *can be determined by a unique maximum solution and a finite number of minimal solutions, i.e.,*

$$S(A, \mathbf{b}) = \bigcup_{\check{\mathbf{x}} \in \check{S}(A,\mathbf{b})} \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \le \mathbf{x} \le \hat{\mathbf{x}}\}, \tag{3}$$

where $\check{S}(A, \mathbf{b})$ *is the set of all minimal solutions to* $A \circ \mathbf{x} = \mathbf{b}$ *and* $\hat{\mathbf{x}} = A^T \varphi_t \mathbf{b}$ *is the maximum solution defined in (2).*

The consistency of a system $A \circ \mathbf{x} = \mathbf{b}$ can be detected by constructing and checking the potential maximum solution $\hat{\mathbf{x}} = A^T \varphi_t \mathbf{b}$ in a time complexity of $O(mn)$. The detection of all minimal solutions is a complicated and very interesting issue for investigation. It follows from Theorems 2.2 and 2.3 that the complete solution set of the i-th equation, $i \in M$, in a consistent system $A \circ \mathbf{x} = \mathbf{b}$ is a finitely generated root system with a set of minimal solutions given by

$$\check{S}_i = \{\check{\mathbf{x}}^k \mid b_i \le a_{ik},\ k \in N\},$$

where the vector $\check{\mathbf{x}}^k$ is defined by

$$\check{x}_j^k = \begin{cases} \mathcal{J}_\mathcal{T}(a_{ik}, b_i), & \text{if } j = k, \\ 0, & \text{otherwise}, \end{cases} \quad \forall\, j \in N.$$

The complete solution set of $A \circ \mathbf{x} = \mathbf{b}$ is therefore the intersection of these root systems which remains to be a finitely generated root system. However, a minimal solution of a single equation may not necessarily be a minimal solution of the system. It has been observed that the detection of minimal solutions is closely related to the set covering problem (see, e.g., Markovskii 2005).

2.2.2 Minimal solutions and set covering problems

The close relation between the minimal solutions of a system of sup-$\mathcal{T}$ equations and the set covering problem has been noticed and described from various aspects since the structure of the complete solution set was fully understood. It provides some important information for the analysis of the number of minimal solutions and the development of algorithms to find all the minimal solutions.

With the potential maximum solution $\hat{\mathbf{x}}$ in hand, the characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$ of a system $A \circ \mathbf{x} = \mathbf{b}$ can be defined by

$$\tilde{q}_{ij} = \begin{cases} [\mathcal{J}_\mathcal{T}(a_{ij}, b_i), \hat{x}_j], & \text{if } \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, \\ \emptyset, & \text{otherwise}. \end{cases} \tag{4}$$

It was reported in Li and Fang (2008) that when $\mathcal{T}$ is a continuous Archimedean t-norm of which the product operator $\mathcal{T}_P$ and the Łukasiewicz t-norm $\mathcal{T}_L$ are typical representatives, the nonempty elements in $\tilde{Q}$ are always singletons with their values being determined by the potential maximum solution $\hat{\mathbf{x}}$. The characteristic matrix $\tilde{Q}$ in this case can be simplified as $Q = (q_{ij})_{m\times n}$ with

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Definition 2.3 Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ being its characteristic matrix. A column j of $\tilde{Q}$ is said to be in the kernel Ker($\tilde{Q}$) if there exists a row i such that $\tilde{q}_{ij}$ is the unique nonempty element in row i.

Definition 2.4 Let $Q = (q_{ij})_{m\times n} \in \{0, 1\}^{mn}$ be a binary matrix. A column j is said to cover a row i if $q_{ij} = 1$. A set of nonzero columns P forms a covering of Q if each row of Q is covered by some column of P. A column j in a covering P is called redundant if the set of columns $P\backslash\{j\}$ remains to be a covering of Q. A covering P is irredundant if it has no redundant columns. The set of all coverings of Q is denoted by $P(Q)$ while the set of all irredundant coverings of Q is denoted by $\check{P}(Q)$.

It is well-known that the set of all coverings $P(Q)$ of a binary matrix Q can be well represented by the feasible solution set of a set covering problem, i.e.,

$$\begin{aligned} &\text{Find } \mathbf{u} \in \{0, 1\}^n \\ &\text{s.t. } \ Q\mathbf{u} \geq e, \end{aligned} \tag{6}$$

where $e = (1, 1, \ldots, 1)^T_{1\times m}$. The relation between fuzzy relational equations and the set covering problem was presented by Markovskii (2005) for sup-$\mathcal{T}_P$ equations, and extended to continuous t-norms by Li and Fang (2008).

Theorem 2.5 *(see.,* e.g., Li 2009) *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous Archimedean t-norm, and Q be the corresponding simplified characteristic matrix. Then, each minimal solution of $A \circ \mathbf{x} = \mathbf{b}$ corresponds to an irredundant covering of Q, in a one-to-one manner.*

Theorem 2.5 indicates that determining all minimal solutions to a system $A \circ \mathbf{x} = \mathbf{b}$ with a continuous Archimedean t-norm $\mathcal{T}$ is equivalent to determining all irredundant coverings of its simplified characteristic matrix, which can be interpreted as a procedure of finding the minimal cover of Q.

When the system $A \circ \mathbf{x} = \mathbf{b}$ of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous non-Archimedean t-norm, the situation turns out to be a little bit complicated. For each $j \in N$, denote r_j the number of different values in $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$, set $K_j = \{1, 2, \ldots, r_j\}$ and denote the different values in $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ by $\check{v}_{jk}$ for $k \in K_j$. Let $r = \sum_{j\in N} r_j$ and $\check{v} = (\check{v}_{11}, \ldots, \check{v}_{1r_1}, \ldots, \check{v}_{n1}, \ldots, \check{v}_{nr_n})^T \in [0, 1]^r$. In this case, the characteristic matrix $\tilde{Q}$ can be converted to an augmented characteristic matrix $Q' = (q'_{ik})_{m\times r} \in \{0, 1\}^{mr}$ where

$$q'_{ik} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s \quad \text{and} \quad \check{v}_{jk} \in \tilde{q}_{ij} \quad \text{for some } j \in N, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Let $\mathbf{u} = (u_{11}, \ldots, u_{1r_1}, \ldots, u_{n1}, \ldots, u_{nr_n})^T \in \{0, 1\}^r$ such that $\sum_{k\in K_j} u_{jk} \le 1$ for each $j \in N$, i.e., at most one of u_{jk} can be 1 for $k \in K_j$. These restrictions are called the innervariable incompatibility constraints and can be represented by $G\mathbf{u} \le e^n$, where $e^n = (1, 1, \ldots, 1)^T_{1\times n}$ and $G = (g_{jk})_{n\times r}$ with

$$g_{jk} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

Definition 2.5 Let $Q' = (q'_{ik})_{m\times r} \in \{0, 1\}^{mr}$ and $G = (g_{jk})_{n\times r} \in \{0, 1\}^{nr}$ be two binary matrices. A column k of Q' is said to cover a row i of Q' if $q'_{ik} = 1$. A set of nonzero columns P forms a G-covering of Q' if each row of Q' is covered by some column in P, i.e., $Q'\mathbf{u}^P \ge e^m$, and $G\mathbf{u}^P \le e^n$ is satisfied with $\mathbf{u}^P = (u^P_k)_{r\times 1}$ where

$$u^P_k = \begin{cases} 1, & \text{if } k \in P, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

A column k in a G-covering P is called redundant if the set of columns $P\backslash\{k\}$ remains to be a G-covering of Q'. A G-covering P is irredundant if P has no redundant columns.

Theorem 2.6 (see, e.g., Li 2009) *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous non-Archimedean t-norm, Q' be the corresponding augmented characteristic matrix and G be the associated coefficient matrix of the innervariable incompatibility constraints. Then, each minimal solution to $A \circ \mathbf{x} = \mathbf{b}$ corresponds to an irredundant G-covering of Q'.*

3 Procedures for generating test problems of fuzzy relational equations with sup-$\mathcal{T}$ composition

Our aim is to generate test problems for systems of sup-$\mathcal{T}$ equations. Given a randomly generated maximum solution, the proposed procedures randomly construct the associated coefficient matrix and right hand side vector of a consistent system of fuzzy relational equations. Moreover, since test problems with various number of minimal solutions are required for a fair assessment of a developed solution method, some properties are discussed to show that the proposed procedures randomly indeed generate systems of fuzzy relational equations with various number of minimal solutions.

The test problem random generator (TPRG) for systems of fuzzy relational equations with the sup-$\mathcal{T}_M$ composition is proposed in Sect. 3.1. Since the discussion presented in Sect. 3.1 can be extended in an analogous manner to the system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous Archimedean t-norm, some of the proofs in Sects. 3.2 and 3.3 are omitted to make it succinct and readable.

3.1 System of sup-$\mathcal{T}_M$ equations

In this section, a procedure for generating test problems for systems of fuzzy relational equations with the sup-$\mathcal{T}_M$ composition is studied. We start with deriving two theorems.

Theorem 3.1.1 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}_M$ equations with $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ being the maximum solution. Then, we have the following results:*

(i) *For each $j \in N$ with $0 < \hat{x}_j < 1$, there exists an $i \in M$ such that $\hat{x}_j = b_i$.*
(ii) *$b_i \leq \max_{j \in N}\{\hat{x}_j\}$ for all $i \in M$.*
(iii) *For each $i \in M$, there exists a $j \in N$ such that*

$$\begin{cases} a_{ij} = b_i, & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [b_i, 1], & \text{if } \hat{x}_j = b_i. \end{cases}$$

Proof (i) According to Theorem 2.4, for each $j \in N$, there exists an $i \in M$ such that

$$\hat{x}_j = \mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i).$$

Moreover, if $0 < \hat{x}_j < 1$, according to Definition 2.1, we have

$$\hat{x}_j = \mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) = b_i.$$

(ii) Since $\hat{\mathbf{x}}$ is a solution to $A \circ \mathbf{x} = \mathbf{b}$, Theorem 2.3 says that, for each $i \in M$, there exists $j \in N$ such that

$$b_i = \mathcal{T}(a_{ij}, \hat{x}_j) = \min\{a_{ij}, \hat{x}_j\} \leq \hat{x}_j \leq \max_{j \in N}\{\hat{x}_j\}.$$

Therefore, $b_i \leq \max_{j \in N}\{\hat{x}_j\}$ for all $i \in M$.

(iii) According to Theorem 2.3, since $\hat{\mathbf{x}}$ is a solution to $A \circ \mathbf{x} = \mathbf{b}$, for each $i \in M$, there exists $j \in N$ such that

$$b_i = \mathcal{T}(a_{ij}, \hat{x}_j) = \min\{a_{ij}, \hat{x}_j\} = \begin{cases} a_{ij}, & \text{if } a_{ij} < \hat{x}_j, \\ \hat{x}_j, & \text{if } a_{ij} \geq \hat{x}_j, \end{cases}$$

which implies that

$$\begin{cases} a_{ij} = b_i, & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [b_i, 1], & \text{if } \hat{x}_j = b_i. \end{cases}$$

□

Theorem 3.1.2 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}_M$ equations with $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ being the maximum solution. For each $j \in N$, if $\hat{x}_j = 1$, then*

$$a_{ij} \in [0, b_i], \forall\, i \in M;$$

if $\hat{x}_j \neq 1$, then there exists an $\bar{i} \in M$ such that $\hat{x}_j = b_{\bar{i}}$. In the latter case, we have $a_{\bar{i}j} \in (b_{\bar{i}}, 1]$ and

$$\begin{cases} a_{ij} \in [0, b_i], & \text{if } b_i < b_{\bar{i}}, \\ a_{ij} \in [0, 1], & \text{if } b_i \geq b_{\bar{i}}, \end{cases} \quad \forall\, i \in M \backslash \{\bar{i}\}.$$

Proof According to Definition 2.1 and Theorem 2.4, we have

$$\mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) = \begin{cases} 1, & a_{ij} \leq b_i, \\ b_i, & a_{ij} > b_i, \end{cases} \quad \forall\, i \in M \quad \text{and} \quad j \in N,$$

and

$$\hat{x}_j = \inf_{i \in M} \mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i), \forall\, j \in N.$$

For each $j \in N$, if $\hat{x}_j = 1$, we have

$$\mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) = 1, \forall\, i \in M,$$

or equivalently,

$$a_{ij} \in [0, b_i], \forall\, i \in M.$$

If $\hat{x}_j \neq 1$, there exists an $\bar{i} \in M$, such that

$$\hat{x}_j = \mathcal{I}_{\mathcal{T}_M}(a_{\bar{i}j}, b_{\bar{i}}) = b_{\bar{i}},$$

or equivalently,

$$b_{\bar{i}} < a_{\bar{i}j} \leq 1.$$

Moreover, for each $i \in M \backslash \{\bar{i}\}$, since $\hat{x}_j = b_{\bar{i}}$, we have

$$\mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) \geq \hat{x}_j = b_{\bar{i}}.$$

If $b_i \geq b_{\bar{i}}$, we have $a_{ij} \in [0, 1]$ such that

$$\mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) \geq b_i \geq b_{\bar{i}}.$$

If $b_i < b_{\bar{i}}$, we have $0 \leq a_{ij} \leq b_i$ such that

$$\mathcal{I}_{\mathcal{T}_M}(a_{ij}, b_i) = 1 \geq \hat{x}_j.$$

□

Based on Theorems 3.1.1 and 3.1.2, we propose the following procedure for randomly generating test problems of sup-$\mathcal{T}_M$ equations:

Procedure TPRG_sup-$\mathcal{T}_M$

Step 0. [**Input**]
Input positive integers m and n.

Step 1. [**Generate a maximum solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ of the system of sup-$\mathcal{T}_M$ equations.**]

Step 1.1. If $m < n$, then go to step 1.2; otherwise, go to step 1.3.

Step 1.2. Randomly generate $\hat{x}_j \in [0, 1]$ for each $j \in M$. Let $X = \{\hat{x}_j \mid \hat{x}_j \neq 1, j \in M\}$, r be the number of different values in X, and $v_i, i = 1, 2, \ldots, r$, be the different values in X. For $j = m + 1, \ldots, n$, randomly choose $j' \in M$ and let $\hat{x}_j = \hat{x}_{j'}$.

Step 1.3. Randomly generate $\hat{x}_j \in [0, 1]$ for each $j \in N$. Let $X = \{\hat{x}_j \mid \hat{x}_j \neq 1, j \in N\}$, r be the number of different values in X, and $v_i, i = 1, 2, \ldots, r$, be the different values in X.

Step 2. [**Construct $\mathbf{b} = (b_i)_{m\times 1}$ of the system of sup-$\mathcal{T}_M$ equations.**]
For $i = 1, 2, \ldots, r$, let $b_i = v_i$. If $m > r$, randomly assign the value of $b_i \in [0, \max_{j\in N}\{\hat{x}_j\}]$, for $i = r + 1, r + 2, \ldots, m$.

Step 3. [**Construct $A = (a_{ij})_{m\times n}$ of the system of sup-$\mathcal{T}_M$ equations.**]
For each $j \in N$, if $\hat{x}_j = 1$, then randomly assign the value of $a_{ij} \in [0, b_i], \forall\, i \in M$; otherwise, find an $\bar{i} \in \{1, 2, \ldots, r\}$ such that $\hat{x}_j = b_{\bar{i}}$, randomly assign the value of $a_{\bar{i}j} \in (b_{\bar{i}}, 1]$ and

$$\begin{cases} a_{ij} \in [0, b_i], & \text{if } b_i < b_{\bar{i}}, \\ a_{ij} \in [0, 1], & \text{if } b_i \geq b_{\bar{i}}. \end{cases} \quad \forall\, i \in M\backslash\{\bar{i}\}.$$

Step 4. [**Guarantee the consistency of the system of sup-$\mathcal{T}_M$ equations.**]
For $i = r + 1, r + 2, \ldots, m$, randomly choose $j_i \in N$ such that $\hat{x}_{j_i} \geq b_i$. Update matrix A with

$$\begin{cases} a_{ij_i} = b_i, & \text{if } \hat{x}_{j_i} > b_i, \\ a_{ij_i} \in [b_i, 1], & \text{if } \hat{x}_{j_i} = b_i. \end{cases}$$

Step 5. [**Output**]
Output $A \in [0, 1]^{mn}$ and $\mathbf{b} \in [0, 1]^m$.

Proposition 3.1.1 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a system of fuzzy relational equations generated by* **TPRG_sup-**$\mathcal{T}_M$. *Then,* $A \circ \mathbf{x} = \mathbf{b}$ *is consistent.*

Proof According to Steps 2 and 3 of **TPRG_sup-**$\mathcal{T}_M$, for each $i = 1, 2, \ldots, r$, there exists an index $j_i \in N$ such that

$$\hat{x}_{j_i} = b_i < a_{ij_i},$$

which implies that

$$\mathcal{T}(a_{ij_i}, \hat{x}_{j_i}) = \min\{a_{ij_i}, \hat{x}_{j_i}\} = b_i, \quad \forall\, i = 1, 2, \ldots, r. \tag{10}$$

For each $i = r+1, r+2, \ldots, m$, according to Step 4 of **TPRG_sup-**$\mathcal{T}_M$, there exists an index $j_i \in N$ such that

$$\begin{cases} a_{ij_i} = b_i, & \text{if } \hat{x}_{j_i} > b_i, \\ a_{ij_i} \in [b_i, 1], & \text{if } \hat{x}_{j_i} = b_i, \end{cases}$$

which implies that

$$\mathcal{T}(a_{ij_i}, \hat{x}_{j_i}) = \min\{a_{ij_i}, \hat{x}_{j_i}\} = b_i, \quad \forall\, i = r+1, r+2, \ldots, m. \tag{11}$$

Moreover, according to Step 3 of **TPRG_sup-**$\mathcal{T}_M$, for each $i \in M$, we have

$$\begin{cases} a_{ij} \in [0, b_i], & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [0, 1], & \text{otherwise}, \end{cases} \quad \forall\, j \in N,$$

which implies

$$\mathcal{T}(a_{ij}, \hat{x}_j) = \min\{a_{ij}, \hat{x}_j\} \le b_i, \quad \forall\, i \in M \quad \text{and} \quad j \in N. \tag{12}$$

By Theorem 2.3, Eqs. (10)–(12) imply that the generated system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ is consistent. □

Proposition 3.1.2 *A system of sup-*$\mathcal{T}_M$ *equations* $A \circ \mathbf{x} = \mathbf{b}$ *with a unique minimal solution can be generated by* **TPRG_sup-**$\mathcal{T}_M$, *if* $m \ge n$.

Proof Consider generating a system of sup-$\mathcal{T}_M$ equations by **TPRG_sup-**$\mathcal{T}_M$. Since $m \ge n$, we can randomly generate n different numbers $x_1, x_2, \ldots, x_n \in [0, 1)$ and let $\hat{x}_j = x_j$ for each $j \in N$ in Step 1 of **TPRG_sup-**$\mathcal{T}_M$.

Let $b_j = \hat{x}_j$, for each $j \in N$, and randomly assign the value of b_i such that $0 \leq b_i \leq \max_{j \in N}\{\hat{x}_j\}$ for all $i = n+1, n+2, \ldots, m$.

According to Step 3 of **TPRG sup-$\mathcal{T}_M$**, for each $j \in N$, we have

$$a_{jj} \in (b_j, 1] \tag{13}$$

and

$$\begin{cases} a_{ij} \in [0, b_i], & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [0, 1], & \text{if } \hat{x}_j \leq b_i, \end{cases} \quad \forall\, i \in M \backslash \{j\}.$$

For each $i \in N$, let

$$\begin{cases} a_{ij} \in [0, b_i), & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [0, 1], & \text{if } \hat{x}_j < b_i, \end{cases} \quad \forall\, j \in N \backslash \{i\}. \tag{14}$$

Equations (13) and (14) imply that

$$\mathcal{T}(a_{jj}, \hat{x}_j) = b_j, \quad \forall j \in N \tag{15}$$

and, for each $i \in N$,

$$\mathcal{T}(a_{ij}, \hat{x}_j) \neq b_i, \quad \forall\, j \in N \backslash \{i\}, \tag{16}$$

respectively.

Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}_M$ equations generated by the above procedure, and $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$ be the corresponding characteristic matrix. Equations (15) and (16) imply that

$$\tilde{q}_{jj} \neq \emptyset, \ \forall\, j \in N, \tag{17}$$

and

$$\tilde{q}_{ij} = \emptyset, \ \forall\, i \in N \text{ and } j \in N \backslash \{i\}, \tag{18}$$

respectively.

Based on Definition 2.3, Eqs. (17) and (18) directly lead to the result that $j \in \mathrm{Ker}(\tilde{Q})$, for each $j \in N$. According to Theorem 3.3 in Li and Fang (2011), $A \circ \mathbf{x} = \mathbf{b}$ has a unique minimal solution. □

Proposition 3.1.3 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of fuzzy relational equations generated by* **TPRG sup-$\mathcal{T}_M$** *with $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ being the maximum solution. For each $i \in M$, let $J_i = \{j \in N \mid \hat{x}_j \geq b_i\}$ and $k_i = \mid J_i \mid$. Then, the system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ has at most $\prod_{i=1}^{m} k_i$ minimal solutions.*

Proof Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}_M$ equations generated by **TPRG_sup-**$\mathcal{T}_M$ and $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ be the corresponding characteristic matrix. According to Peeva and Kyosev (2004), the upper bound of the number of minimal solutions of $A \circ \mathbf{x} = \mathbf{b}$ is related to the number of nonempty elements in each row of $\tilde{Q}$.

For each $i \in M$, consider

$$\begin{cases} a_{ij} = b_i, & \text{if } \hat{x}_j > b_i, \\ a_{ij} \in [b_i, 1], & \text{if } \hat{x}_j = b_i, \end{cases} \quad \forall\, j \in N. \tag{19}$$

Equation (19) implies that, for each $i \in M$, if there exists a $j \in N$ such that $\hat{x}_j \geq b_i$, then

$$\mathcal{T}(a_{ij}, \hat{x}_j) = b_i,$$

or equivalently,

$$\tilde{q}_{ij} \neq \emptyset.$$

In this case, for each $i \in M$, let $J_i = \{j \in N \mid \hat{x}_j \geq b_i\}$ and $k_i =\mid J_i \mid$. According to Eq. (3.25) in Peeva and Kyosev (2004), the system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ has at most $\prod_{i=1}^{m} k_i$ minimal solutions. □

For $n \in R$, denote $\lfloor n \rfloor$ the largest integer no greater than n and $\lceil n \rceil$ the smallest integer no less than n.

Proposition 3.1.4 *A system of sup-*$\mathcal{T}_M$ *equations* $A \circ \mathbf{x} = \mathbf{b}$ *with* $2^{\lfloor \frac{n}{2} \rfloor}$ *minimal solutions can be generated by* **TPRG_sup-**$\mathcal{T}_M$*, if* $m \geq \lceil \frac{n}{2} \rceil$.

Proof Consider generating a system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ by **TPRG_sup-**$\mathcal{T}_M$. Since $m \geq \lceil \frac{n}{2} \rceil$, we can randomly generate $\lceil \frac{n}{2} \rceil$ different numbers $x_1, x_2, \ldots, x_{\lceil \frac{n}{2} \rceil} \in [0, 1)$ and let $\hat{x}_{j+\lfloor \frac{n}{2} \rfloor} = x_j, \forall\, j = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$ and $\hat{x}_j = \hat{x}_{j+\lfloor \frac{n}{2} \rfloor}, \forall\, j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ in Step 1 of **TPRG_sup-**$\mathcal{T}_M$.

For each $i = 1, 2, \ldots, \lceil \frac{n}{2} \rceil$, let $b_i = \hat{x}_{i+\lfloor \frac{n}{2} \rfloor}$ and randomly assign the value of $b_i \in [0, \max_{j\in N}\{\hat{x}_j\}], \forall\, i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \ldots, m$.

According to Step 3 of **TPRG_sup-**$\mathcal{T}_M$, for each $j \in N$, there exists an $i \in \{1, 2, \ldots, \lceil \frac{n}{2} \rceil\}$ such that $\hat{x}_j = b_i$. In this case, for each $j \in N$, we have $a_{ij} > b_i$, or equivalently,

$$\mathcal{T}(a_{ij}, \hat{x}_j) = b_i. \tag{20}$$

Let $j^* \in N$ satisfying $\hat{x}_{j^*} = \max_{j=1,2,\ldots,\lfloor\frac{n}{2}\rfloor}\{\hat{x}_j\}$. For each $j \in N\backslash\{j^*, j^* + \lfloor\frac{n}{2}\rfloor\}$, let

$$\begin{cases} a_{i'j} \in [0, b_{i'}), & \text{if } b_{i'} < b_i, \\ a_{i'j} \in [0, 1], & \text{if } b_{i'} \geq b_i, \end{cases} \quad \forall\, i' \in M\backslash\{i\},$$

which implies that, for each $j \in N\backslash\{j^*, j^* + \lfloor\frac{n}{2}\rfloor\}$,

$$\mathcal{T}(a_{i'j}, \hat{x}_j) \neq b_i', \quad \forall\, i' \in M\backslash\{i\}. \tag{21}$$

Moreover, for each $j \in \{j^*, j^* + \lfloor\frac{n}{2}\rfloor\}$, there exists an $i \in \{1, 2, \ldots, \lceil\frac{n}{2}\rceil\}$ such that $\hat{x}_{j^*} = \hat{x}_{j^*+\lfloor\frac{n}{2}\rfloor} = b_i$. Then, let

$$a_{i'j^*} \in [0, b_{i'}) \quad \text{and} \quad a_{i'j^*+\lfloor\frac{n}{2}\rfloor} \in [0, b_{i'}), \forall\, i' \in \{1, 2, \ldots, \lceil\frac{n}{2}\rceil\}\backslash\{i\}, \tag{22}$$

and

$$\begin{cases} a_{i'j^*} = a_{i'j^*+\lfloor\frac{n}{2}\rfloor} = b_{i'}, & \text{if } \hat{x}_{j^*} > b_{i'}, \\ a_{i'j^*} \in [b_{i'}, 1] \quad \text{and} \quad a_{i'j^*+\lfloor\frac{n}{2}\rfloor} \in [b_{i'}, 1], & \text{if } \hat{x}_{j^*} = b_{i'}, \end{cases}$$
$$\forall\, i' \in \left\{\left\lceil\frac{n}{2}\right\rceil + 1, \left\lceil\frac{n}{2}\right\rceil + 2, \ldots, m\right\}. \tag{23}$$

Equations (22) and (23) imply that

$$\mathcal{T}(a_{i'j^*}, \hat{x}_{j^*}) \neq b_{i'} \quad \text{and} \quad \mathcal{T}(a_{i'j^*+\lfloor\frac{n}{2}\rfloor}, \hat{x}_{j^*+\lfloor\frac{n}{2}\rfloor}) \neq b_{i'},$$
$$\forall\, i' \in \{1, 2, \ldots, \lceil\frac{n}{2}\rceil\}\backslash\{i\}, \tag{24}$$

and

$$\mathcal{T}(a_{i'j^*}, \hat{x}_{j^*}) = b_{i'} \quad \text{and} \quad \mathcal{T}(a_{i'j^*+\lfloor\frac{n}{2}\rfloor}, \hat{x}_{j^*+\lfloor\frac{n}{2}\rfloor}) = b_{i'}$$
$$\forall\, i' \in \left\{\left\lceil\frac{n}{2}\right\rceil + 1, \left\lceil\frac{n}{2}\right\rceil + 2, \ldots, m\right\}, \tag{25}$$

respectively.

According to Li and Fang (2009) and Eqs. (7)–(9), (20), (21), (24) and (25) directly lead to the result that $\check{\mathbf{x}} = \{\check{x}_1, \check{x}_2, \ldots, \check{x}_n\} = \{\overbrace{0, 0, \ldots, 0}^{\lfloor\frac{n}{2}\rfloor}, \hat{x}_{\lfloor\frac{n}{2}\rfloor+1}, \hat{x}_{\lfloor\frac{n}{2}\rfloor+2}, \ldots, \hat{x}_n\}$ is a minimal solution of the above generated system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$.

Moreover, since $\hat{x}_j = \hat{x}_{\lfloor\frac{n}{2}\rfloor+j} = b_j, \forall\, j = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor$, and

$$\mathcal{T}(a_{jj}, \hat{x}_j) = \mathcal{T}(a_{jj+\lfloor\frac{n}{2}\rfloor}, \hat{x}_{j+\lfloor\frac{n}{2}\rfloor}) = b_j, \quad \forall\, j = 1, 2, \ldots, \lfloor\frac{n}{2}\rfloor,$$

replacing the value of $\check{x}_j$ by $\hat{x}_j$ and the value of $\check{x}_{\lfloor \frac{n}{2} \rfloor + j}$ by 0 in $\check{\mathbf{x}}$, for each $j \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, results in another minimal solution of $A \circ \mathbf{x} = \mathbf{b}$. Therefore, the system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ has $2^{\lfloor \frac{n}{2} \rfloor}$ minimal solutions. □

3.2 System of sup-$\mathcal{T}_P$ equations

In this section, a procedure for generating test problems for systems of fuzzy relational equations with the sup-$\mathcal{T}_P$ composition is studied. We start with deriving the following theorem:

Theorem 3.2.1 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}_P$ equations with $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ being the maximum solution. Then, we have the following results:*

(i) *For each $i \in M$, there exists a $j \in N$ such that $b_i = a_{ij} \times \hat{x}_j$.*
(ii) *$a_{ij} \in [0, \frac{b_i}{\hat{x}_j}]$, $\forall\, i \in M$ and $j \in N$.*
(iii) *For each $j \in N$, if $\hat{x}_j \neq 1$, then there exists an $i \in M$ such that $a_{ij} = \frac{b_i}{\hat{x}_j}$.*

Proof (i) Since $\hat{\mathbf{x}}$ is a solution to $A \circ \mathbf{x} = \mathbf{b}$, Theorem 2.3 says that, for each $i \in M$, there exists a $j \in N$ such that

$$b_i = \mathcal{T}(a_{ij}, \hat{x}_j) = a_{ij} \times \hat{x}_j.$$

(ii) According to Theorem 2.4, the maximum solution $\hat{\mathbf{x}}$ is defined as

$$\hat{x}_j = \inf_{i \in M} \mathcal{I}(a_{ij}, b_i), \quad \forall\, j \in N.$$

Therefore, we have

$$\mathcal{I}(a_{ij}, b_i) \geq \hat{x}_j, \forall\, i \in M, \quad \text{and} \quad j \in N,$$

or equivalently,

$$0 \leq a_{ij} \leq \frac{b_i}{\hat{x}_j}, \forall i \in M \quad \text{and} \quad j \in N.$$

(iii) Similarly, according to Theorem 2.4, for each $j \in N$, there exists an $i \in M$ such that

$$\mathcal{I}(a_{ij}, b_i) = \hat{x}_j.$$

If $\hat{x}_j \neq 1$, we have

$$\mathcal{I}(a_{ij}, b_i) = \frac{b_i}{a_{ij}} = \hat{x}_j,$$

or equivalently,

$$a_{ij} = \frac{b_i}{\hat{x}_j}.$$

If $\hat{x}_j = 1$, we have

$$\mathcal{I}(a_{ij}, b_i) = \hat{x}_j = 1,$$

or equivalently,

$$a_{ij} \le b_i.$$

□

Based on Theorem 3.2.1, we propose the following procedure for generating test problems of sup-$\mathcal{T}_P$ equations:

Procedure TPRG_sup-$\mathcal{T}_P$

Step 0. [**Input**]
Input positive integers m and n.

Step 1. [**Generate a maximum solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ of the system of sup-$\mathcal{T}_P$ equations.**]
Randomly generate $\hat{x}_j \in [0, 1]$ for each $j \in N$.

Step 2. [**Construct $\mathbf{b} = (b_i)_{m\times 1}$ of the system of sup-$\mathcal{T}_P$ equations.**]

- **Step 2.1.** For each $i \in M$, randomly choose $j_i \in N$, assign the value of $a_{ij_i} \in (0, 1]$ and let $b_i = a_{ij_i} \times \hat{x}_{j_i}$.
- **Step 2.2.** If $\min_{i\in M}\{b_i\} \le \min_{j\in N}\{\hat{x}_j\}$, go to step 3; otherwise, go to step 2.1.

Step 3. [**Construct $A = (a_{ij})_{m\times n}$ of the system of sup-$\mathcal{T}_P$ equations.**]
For each $i \in M$, randomly assign the value of $a_{ij} \in [0, \min\{1, \frac{b_i}{\hat{x}_j}\}]$ for each $j \in N\backslash\{j_i\}$.

Step 4. [**Guarantee the consistency of the system of sup-$\mathcal{T}_P$ equations.**]
For each $j \in N\backslash\{j_1, j_2, \ldots, j_m\}$, if $\hat{x}_j \neq 1$, then randomly choose $i \in M$ such that $b_i \le \hat{x}_j$ and update matrix A with $a_{ij} = \frac{b_i}{\hat{x}_j}$.

Step 5. [**Output**]
Output $A \in [0, 1]^{mn}$ and $\mathbf{b} \in [0, 1]^m$.

Noticed that if $\min_{i \in M}\{b_i\} > \min_{j \in N}\{\hat{x}_j\}$ in Step 2.2 of **Procedure TPRG_sup-$\mathcal{T}_P$**, the procedure might have problem in randomly choose $i \in M$ such that $b_i \leq \hat{x}_j$ in Step 4. Therefore, if $\min_{i \in M}\{b_i\} \leq \min_{j \in N}\{\hat{x}_j\}$ in Step 2.2, then the procedure goes to Step 2.1 to regenerate b_i, for each $i \in M$.

To be parallel to propositions in Sect. 3.1, we have the following results for the system of sup-$\mathcal{T}_P$ equations. The proofs are omitted here.

Proposition 3.2.1 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of fuzzy relational equations generated by* **TPRG_sup-$\mathcal{T}_P$**. *Then, $A \circ \mathbf{x} = \mathbf{b}$ is consistent.*

Proposition 3.2.2 *A system of sup-$\mathcal{T}_P$ equations $A \circ \mathbf{x} = \mathbf{b}$ with a unique minimal solution can be generated by* **TPRG_sup-$\mathcal{T}_P$**, *if $m \geq n$.*

Proposition 3.2.3 *A system of sup-$\mathcal{T}_P$ equations $A \circ \mathbf{x} = \mathbf{b}$ with $2^{\lfloor \frac{n}{2} \rfloor}$ minimal solutions can be generated by* **TPRG_sup-$\mathcal{T}_P$**, *if $m \geq \lceil \frac{n}{2} \rceil$.*

3.3 System of sup-$\mathcal{T}_L$ equations

In this section, a procedure for generating test problems for systems of fuzzy relational equations with the sup-$\mathcal{T}_L$ composition is studied. We start with deriving the following theorem.

Theorem 3.3.1 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}_L$ equations with $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ being the maximum solution. Then, we have the following results:*

(i) *For each $i \in M$, there exists a $j \in N$ such that $b_i = a_{ij} + \hat{x}_j - 1$.*
(ii) *$a_{ij} \in [0, 1 - \hat{x}_j + b_i]$, $\forall\, i \in M$ and $j \in N$.*
(iii) *For each $j \in N$, if $\hat{x}_j \neq 1$, then there exists an $i \in M$ such that $a_{ij} = b_i + 1 - \hat{x}_j$.*

Proof Following the same process of Theorem 3.2.1 leads to the results. □

Based on Theorem 3.3.1, we propose the following procedure for generating test problems of sup-$\mathcal{T}_L$ equations:

Procedure TPRG_sup-$\mathcal{T}_L$

Step 0. **[Input]**
Input positive integers m and n.

Step 1. **[Generate a maximum solution $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ of the system of sup-$\mathcal{T}_L$ equations.]**
Randomly generate $\hat{x}_j \in [0, 1]$ for each $j \in N$.

Step 2. **[Construct $\mathbf{b} = (b_i)_{m \times 1}$ of the system of sup-$\mathcal{T}_L$ equations.]**
For each $i \in M$, randomly choose $j_i \in N$, assign the value of $a_{ij_i} \in (1 - \hat{x}_{j_i}, 1]$ and let $b_i = a_{ij_i} + \hat{x}_{j_i} - 1$.

Step 3. [**Construct** $A = (a_{ij})_{m\times n}$ **of the system of sup-**$\mathcal{T}_L$ **equations.**]
For each $i \in M$, randomly assign the value of $a_{ij} \in [0, \min\{1, 1 + b_i - \hat{x}_j\}]$ for each $j \in N \backslash \{j_i\}$.

Step 4. [**Guarantee the consistency of the system of sup-**$\mathcal{T}_L$ **equations.**]
For each $j \in N\backslash\{j_1, j_2, \dots, j_m\}$, if $\hat{x}_j \neq 1$, then randomly choose $i \in M$ such that $b_i \leq \hat{x}_j$ and update matrix A with $a_{ij} = b_i + 1 - \hat{x}_j$.

Step 5. [**Output**]
Output $A \in [0, 1]^{mn}$ and $\mathbf{b} \in [0, 1]^m$.

To be parallel to Propositions 3.2.1–3.2.3, we have the following results for the system of sup-$\mathcal{T}_L$ equations. The proofs are omitted here.

Proposition 3.3.1 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a system of fuzzy relational equations generated by* **TPRG_sup-**$\mathcal{T}_L$. *Then,* $A \circ \mathbf{x} = \mathbf{b}$ *is consistent.*

Proposition 3.3.2 *A system of sup-*$\mathcal{T}_L$ *equations* $A \circ \mathbf{x} = \mathbf{b}$ *with a unique minimal solution can be generated by* **TPRG_sup-**$\mathcal{T}_L$, *if* $m \geq n$.

Proposition 3.3.3 *A system of sup-*$\mathcal{T}_L$ *equations* $A \circ \mathbf{x} = \mathbf{b}$ *with* $2^{\lfloor \frac{n}{2} \rfloor}$ *minimal solutions can be generated by* **TPRG_sup-**$\mathcal{T}_L$, *if* $m \geq \lceil \frac{n}{2} \rceil$.

4 Numerical examples

In this section, numerical examples are provided to illustrate the procedures for generating test problems for systems of fuzzy relational equations with the sup-$\mathcal{T}$ composition.

Example 1 For a system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ with $n = 4$ and $m = 6$, the associated coefficient matrix and right hand side vector can be constructed by **TPRG_sup-**$\mathcal{T}_M$ as follows.

Step 1. Randomly generate a maximum solution $\hat{\mathbf{x}} = (0.4, 0.8, 0.6, 0.8)^T$ with $r = 3$. Let $\{v_1, v_2, v_3\} = \{0.8, 0.6, 0.4\}$.

Step 2. Let $b_i = v_i$, for $i = 1, 2, 3$, and randomly assign $b_4 = 0.2$, $b_5 = 0.4$ and $b_6 = 0.7$.

Step 3. Since $\hat{x}_1 = 0.4 = b_3$, we have $a_{31} \in (0.4, 1]$, and $a_{11} \in [0, 1], a_{21} \in [0, 1], a_{41} \in [0, 0.2], a_{51} \in [0, 1], a_{61} \in [0, 1]$.
Since $\hat{x}_2 = 0.8 = b_1$, we have $a_{12} \in (0.8, 1]$, and $a_{22} \in [0, 0.6], a_{32} \in [0, 0.4], a_{42} \in [0, 0.2], a_{52} \in [0, 0.4], a_{62} \in [0, 0.7]$.
Since $\hat{x}_3 = 0.6 = b_2$, we have $a_{23} \in (0.6, 1]$, and $a_{13} \in [0, 1], a_{33} \in [0, 0.4], a_{43} \in [0, 0.2], a_{53} \in [0, 0.4], a_{63} \in [0, 1]$.
Since $\hat{x}_4 = 0.8 = b_1$, we have $a_{14} \in (0.8, 1]$, and $a_{24} \in [0, 0.6], a_{34} \in [0, 0.4], a_{44} \in [0, 0.2], a_{54} \in [0, 0.4], a_{64} \in [0, 0.7]$.

Step 4. For $i = 4$, since $\hat{x}_2 \geq b_4$, choose $j = 2$ and let $a_{42} = 0.2$.
For $i = 5$, since $\hat{x}_2 \geq b_5$, choose $j = 2$ and let $a_{52} = 0.4$.
For $i = 6$, since $\hat{x}_4 \geq b_6$, choose $j = 4$ and let $a_{64} = 0.7$.

According to above procedure, the coefficient matrix A and right hand side vector $\mathbf{b}$ of the system of sup-$\mathcal{T}_M$ equations can be described as follows:

$$A = \begin{pmatrix} [0,1] & (0.8,1] & [0,1] & (0.8,1] \\ [0,1] & [0,0.6] & (0.6,1] & [0,0.6] \\ (0.4,1] & [0,0.4] & [0,0.4] & [0,0.4] \\ [0,0.2] & 0.2 & [0,0.2] & [0,0.2] \\ [0,1] & 0.4 & [0,0.4] & [0,0.4] \\ [0,1] & [0,0.7] & [0,1] & 0.7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.4 \\ 0.7 \end{pmatrix}.$$

Therefore, for example,

$$\begin{pmatrix} 0.8 & 0.9 & 0.3 & 1 \\ 0.2 & 0.4 & 0.9 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0.2 & 0.1 \\ 0.4 & 0.4 & 0.1 & 0.4 \\ 0.5 & 0.7 & 0.6 & 0.7 \end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0.4 \\ 0.7 \end{pmatrix}$$

is a system of sup-$\mathcal{T}_M$ equations with the dimensions of input data $n = 4$ and $m = 6$.

Example 2 For a system of sup-$\mathcal{T}_P$ equations $A \circ \mathbf{x} = \mathbf{b}$ with $n = 10$ and $m = 8$, the associated coefficient matrix and right hand side vector can be constructed by **TPRG_sup-$\mathcal{T}_P$** as follows.

Step 1. Randomly generate a maximum solution $\hat{\mathbf{x}} = (0.8, 0.8, 0.622, 0.6, 0.7, 0.525, 0.7, 0.8, 0.6, 0.8)^T$.

Step 2. Randomly choose $j_1 = 1$, $j_2 = 3$, $j_3 = 2$, $j_4 = 7$, $j_5 = 2$, $j_6 = 2$, $j_7 = 9$, $j_8 = 2$, and
assign $a_{1j_1} = a_{11} = 0.6$ and $b_1 = a_{1j_1} \times \hat{x}_{j_1} = 0.6 \times 0.8 = 0.48$.
assign $a_{2j_2} = a_{23} = 0.9$ and $b_2 = a_{2j_2} \times \hat{x}_{j_2} = 0.9 \times 0.622 = 0.56$.
assign $a_{3j_3} = a_{32} = 0.9$ and $b_3 = a_{3j_3} \times \hat{x}_{j_3} = 0.9 \times 0.8 = 0.72$.
assign $a_{4j_4} = a_{47} = 0.8$ and $b_4 = a_{4j_4} \times \hat{x}_{j_4} = 0.8 \times 0.7 = 0.56$.
assign $a_{5j_5} = a_{52} = 0.8$ and $b_5 = a_{5j_5} \times \hat{x}_{j_5} = 0.8 \times 0.8 = 0.64$.
assign $a_{6j_6} = a_{62} = 0.9$ and $b_6 = a_{6j_6} \times \hat{x}_{j_6} = 0.9 \times 0.8 = 0.72$.
assign $a_{7j_7} = a_{79} = 0.7$ and $b_7 = a_{7j_7} \times \hat{x}_{j_7} = 0.7 \times 0.6 = 0.42$.
assign $a_{8j_8} = a_{82} = 0.8$ and $b_8 = a_{8j_8} \times \hat{x}_{j_8} = 0.8 \times 0.8 = 0.64$.

Step 3. For each $i = 1, 2, \ldots, 8$, randomly assign $a_{ij} \in [0, \min\{1, \frac{b_i}{\hat{x}_j}\}]$ for each $j \in \{1, 2, \ldots, 10\} \backslash \{j_i\}$.

Step 4. For $j = 4$, randomly choose $i = 7$ and update $a_{74} = \frac{b_7}{\hat{x}_4} = \frac{0.42}{0.6} = 0.7$.
For $j = 5$, randomly choose $i = 2$ and update $a_{25} = \frac{b_2}{\hat{x}_5} = \frac{0.56}{0.7} = 0.8$.
For $j = 6$, randomly choose $i = 7$ and update $a_{76} = \frac{b_7}{\hat{x}_6} = \frac{0.42}{0.525} = 0.8$.

For $j = 8$, randomly choose $i = 1$ and update $a_{18} = \dfrac{b_1}{\hat{x}_8} = \dfrac{0.48}{0.8} = 0.6$.

For $j = 10$, randomly choose $i = 8$ and update $a_{810} = \dfrac{b_8}{\hat{x}_{10}} = \dfrac{0.64}{0.8} = 0.8$.

According to the above procedure, the coefficient matrix A and right hand side vector $\mathbf{b}$ of the system of sup-$\mathcal{T}_P$ equations can be described as follows:

$$A = \begin{pmatrix}
0.6 & [0, \frac{0.48}{0.8}] & [0, \frac{0.48}{0.622}] & [0, \frac{0.48}{0.6}] & [0, \frac{0.48}{0.7}] & [0, \frac{0.48}{0.525}] & [0, \frac{0.48}{0.7}] & 0.6 & [0, \frac{0.48}{0.6}] & [0, \frac{0.48}{0.8}] \\
[0, \frac{0.56}{0.8}] & [0, \frac{0.56}{0.8}] & 0.9 & [0, \frac{0.56}{0.6}] & 0.8 & [0, 1] & [0, \frac{0.56}{0.7}] & [0, \frac{0.56}{0.8}] & [0, \frac{0.56}{0.6}] & [0, \frac{0.56}{0.8}] \\
[0, \frac{0.72}{0.8}] & 0.9 & [0, 1] & [0, 1] & [0, 1] & [0, 1] & [0, 1] & [0, \frac{0.72}{0.8}] & [0, 1] & [0, \frac{0.72}{0.8}] \\
[0, \frac{0.56}{0.8}] & [0, \frac{0.56}{0.8}] & [0, \frac{0.56}{0.622}] & [0, \frac{0.56}{0.6}] & [0, \frac{0.56}{0.7}] & [0, 1] & 0.8 & [0, \frac{0.56}{0.8}] & [0, \frac{0.56}{0.6}] & [0, \frac{0.56}{0.8}] \\
[0, \frac{0.64}{0.8}] & 0.8 & [0, 1] & [0, 1] & [0, \frac{0.64}{0.7}] & [0, 1] & [0, \frac{0.64}{0.7}] & [0, \frac{0.64}{0.8}] & [0, 1] & [0, \frac{0.64}{0.8}] \\
[0, \frac{0.72}{0.8}] & 0.9 & [0, 1] & [0, 1] & [0, 1] & [0, 1] & [0, 1] & [0, \frac{0.72}{0.8}] & [0, 1] & [0, \frac{0.72}{0.8}] \\
[0, \frac{0.42}{0.8}] & [0, \frac{0.42}{0.8}] & [0, \frac{0.42}{0.622}] & 0.7 & [0, \frac{0.42}{0.7}] & 0.8 & [0, \frac{0.42}{0.7}] & [0, \frac{0.42}{0.8}] & 0.7 & [0, \frac{0.42}{0.8}] \\
[0, \frac{0.64}{0.8}] & 0.8 & [0, 1] & [0, 1] & [0, \frac{0.64}{0.7}] & [0, 1] & [0, \frac{0.64}{0.7}] & [0, \frac{0.64}{0.8}] & [0, 1] & 0.8
\end{pmatrix}$$

and

$$\mathbf{b} = (0.48, 0.56, 0.72, 0.56, 0.64, 0.72, 0.42, 0.64)^T .$$

Therefore, for example,

$$\begin{pmatrix}
0.6 & 0.5 & 0.1 & 0.1 & 0.3 & 0.8 & 0.4 & 0.6 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.9 & 0.6 & 0.8 & 0.4 & 0.5 & 0.3 & 0.5 & 0.3 \\
0.5 & 0.9 & 0.4 & 0.2 & 0.8 & 0.1 & 0.4 & 0.4 & 0.7 & 0.6 \\
0.3 & 0.5 & 0.7 & 0.5 & 0.8 & 0.1 & 0.8 & 0.3 & 0.4 & 0.6 \\
0.7 & 0.8 & 0.5 & 0.4 & 0.8 & 0.2 & 0.4 & 0.1 & 0.9 & 0.6 \\
0.5 & 0.9 & 0.7 & 0.1 & 0.5 & 0.8 & 0.7 & 0.2 & 0.9 & 0.4 \\
0.2 & 0.3 & 0.4 & 0.7 & 0.5 & 0.8 & 0.3 & 0.5 & 0.7 & 0.4 \\
0.8 & 0.8 & 0.7 & 0.5 & 0.8 & 0.3 & 0.4 & 0.7 & 0.2 & 0.8
\end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.48 \\ 0.56 \\ 0.72 \\ 0.56 \\ 0.64 \\ 0.72 \\ 0.42 \\ 0.64 \end{pmatrix}$$

is a system of sup-$\mathcal{T}_P$ equations with the dimensions of input data $n = 10$, and $m = 8$, which is consistent with the example studied in Loetamonphong and Fang (2001).

Example 3 For a system of sup-$\mathcal{T}_L$ equations $A \circ \mathbf{x} = \mathbf{b}$ with $n = 5$ and $m = 5$, the associated coefficient matrix and right hand side vector can be constructed by **TPRG_sup-$\mathcal{T}_L$** as follows.

Step 1. Randomly generate a maximum solution $\hat{\mathbf{x}} = (0.9, 0.7, 0.5, 1, 0.8)^T$.

Step 2. Randomly choose $j_1 = 2$, $j_2 = 4$, $j_3 = 2$, $j_4 = 4$, $j_5 = 4$, and randomly assign $a_{1j_1} = a_{12} = 0.7 \in (1 - 0.7, 1]$ and $b_1 = a_{1j_1} + \hat{x}_{j_1} - 1 = 0.7 + 0.7 - 1 = 0.4$.
assign $a_{2j_2} = a_{24} = 0.8 \in (1-1, 1]$ and $b_2 = a_{2j_2} + \hat{x}_{j_2} - 1 = 0.8 + 1 - 1 = 0.8$.

assign $a_{3j_3} = a_{32} = 0.5 \in (1 - 0.7, 1]$ and $b_3 = a_{3j_3} + \hat{x}_{j_3} - 1 = 0.5 + 0.7 - 1 = 0.2$.
assign $a_{4j_4} = a_{44} = 0.6 \in (1-1, 1]$ and $b_4 = a_{4j_4} + \hat{x}_{j_4} - 1 = 0.6 + 1 - 1 = 0.6$.
assign $a_{5j_5} = a_{54} = 1 \in (1-1, 1]$ and $b_5 = a_{5j_5} + \hat{x}_{j_5} - 1 = 1 + 1 - 1 = 1$.

Step 3. For each $i = 1, 2, \ldots, 5$, randomly assign the value of $a_{ij} \in [0, \min\{1, 1 + b_i - \hat{x}_j\}]$ for each $j \in \{1, 2, \ldots, 5\} \backslash \{j_i\}$.
Step 4. For $j = 1$, randomly choose $i = 2$ and update $a_{21} = b_2 + 1 - \hat{x}_1 = 0.9$.
For $j = 3$, randomly choose $i = 1$ and update $a_{13} = b_1 + 1 - \hat{x}_3 = 0.9$.
For $j = 5$, randomly choose $i = 4$ and update $a_{45} = b_4 + 1 - \hat{x}_5 = 0.8$.

According to above procedure, the coefficient matrix A and right hand side vector $\mathbf{b}$ of the system of sup-$\mathcal{T}_L$ equations can be described as follows:

$$A = \begin{pmatrix} [0, 0.5] & 0.7 & 0.9 & [0, 0.4) & [0, 0.6] \\ 0.9 & [0, 1) & [0, 1] & 0.8 & [0, 1] \\ [0, 0.3] & 0.5 & [0, 0.7] & [0, 0.2] & [0, 0.4] \\ [0, 0.7] & [0, 0.9] & [0, 1] & 0.6 & 0.8 \\ [0, 1] & [0, 1] & [0, 1] & 1 & [0, 1] \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.2 \\ 0.6 \\ 1 \end{pmatrix}.$$

Therefore, for example,

$$\begin{pmatrix} 0.2 & 0.7 & 0.9 & 0.3 & 0.5 \\ 0.9 & 0.2 & 1 & 0.8 & 0.4 \\ 0.3 & 0.5 & 0.7 & 1 & 0.3 \\ 0.7 & 0.8 & 0.3 & 0.6 & 0.8 \\ 1 & 1 & 0.5 & 1 & 0.6 \end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.2 \\ 0.6 \\ 1 \end{pmatrix}$$

is a system of sup-$\mathcal{T}_L$ equations with dimensions of input data $n = 5$, and $m = 5$, which is consistent with the example studied in Li (2009).

5 Conclusions

Computational efficiency analysis of different solution methods for solving systems of fuzzy relational equations as well as the associated optimization problems needs to use randomly generated problems with different sizes and structures. In this paper, we propose some procedures for generating test problems of systems of fuzzy relational equations with sup-$\mathcal{T}$ compositions. It is shown that the proposed procedures randomly generate consistent test problems with various number of minimal solutions for systems of fuzzy relational equations. To the best of our knowledge, such test problem generators for systems of fuzzy relational equations have not appeared in the literature. The proposed procedures are easy to use and can generate large scale systems of fuzzy relational equations. The pseudo codes of the proposed procedures are included in the Appendix.

Appendix: Pseudo codes

Main procedure TPRG_Sup_$\mathcal{T}_M$

Input: positive integers m and n.

```
If m < n
        For j = 1 to m
            Randomly generate the value of x̂_j ∈ [0, 1];
        End
        Set r = 1; v_1 = x̂_1;
        For j = 2 to m
            If x̂_j ≠ x̂_i for all i < j
                Set r = r + 1, v_r = x̂_j;
            Endif
        End
        For i = 1 to r
            If v_i = 1
                Set r = r − 1;
                For k = i to r
                    Set v_k = v_{k+1};
                End
                Break;
            Endif
        End
        For j = m + 1 to n
          Randomly choose j' ∈ {1, 2, ..., m};
          Set x̂_j = x̂_{j'};
        End
Else
        For j = 1 to n
            Randomly generate the value of x̂_j ∈ [0, 1];
        End
        Set r = 1; v_1 = x̂_1;
        For j = 2 to n
            If x̂_j ≠ x̂_i for all i < j
                Set r = r + 1, v_r = x̂_j;
            Endif
        End
        For i = 1 to r
            If v_i = 1
                Set r = r − 1;
                For k = i to r
                    Set v_k = v_{k+1};
                End
                Break;
```

```
            Endif
        End
Endif

For i = 1 to r
    Set b_i = v_i;
End

  For i = r + 1 to m
    Randomly assign b_i ∈ [0, max_{j=1,2,...,n} {x̂_j}];
End

For j = 1 to n
    If x̂_j = 1
      For i = 1 to m
        Randomly assign a_ij ∈ [0, b_i];
      End
    Else
      For i = 1 to r
        If x̂_j = b_i
          Set ī = i;
          Break;
        Endif
      End
      Randomly assign a_īj ∈ (b_ī, 1];
      For i = 1 to m
        If i ≠ ī
          If b_i < b_ī
            Randomly assign a_ij ∈ [0, b_i];
          Elseif b_i ≥ b_ī
            Randomly assign a_ij ∈ [0, 1];
          Endif
        Endif
      End
    Endif
End

For i = r + 1 to m
    For j = 1 to n
      If x̂_j ≥ b_i
        Set j̄ = j;
        Break;
      Endif
    End
    If x̂_j̄ > b_i
```

```
            Set $a_{i\bar{j}} = b_i$;
        Elseif $\hat{x}_{\bar{j}} = b_i$
            Randomly assign $a_{i\bar{j}} \in [b_i, 1]$;
        Endif
End

For $i = 1$ to $m$
        Set $b(i) = b_i$;
End

For $j = 1$ to $n$
        For $i = 1$ to $m$
            Set $A(i, j) = a_{ij}$;
        End
End

Output $A, b$;
```

Main procedure TPRG_Sup_$\mathcal{T}_P$

```
Input: positive integers $m$ and $n$.

For $j = 1$ to $n$
          Randomly generate the value of $\hat{x}_j \in [0, 1]$;
End

Do
          For $i = 1$ to $m$
                Randomly choose $j_i \in \{1, 2, \ldots, n\}$;
                Randomly assign $a_{ij_i} \in (0, 1]$;
                Set $b_i = a_{ij_i} \times \hat{x}_{j_i}$;
          End
While $\min_{i=1,2,\ldots,m} \{b_i\} > \min_{j=1,2,\ldots,n} \{\hat{x}_j\}$

For $i = 1$ to $m$
        For $j = 1$ to $n$
            If $j \neq j_i$
                Randomly assign $a_{ij} \in [0, \min\{1, \frac{b_i}{\hat{x}_j}\}]$;
            Endif
        End
End

For $j = 1$ to $n$
          If $j \notin \{j_1, j_2, \ldots, j_m\}$
```

```
                If x̂_j ≠ 1
                  For i = 1 to m
                      If b_i ≤ x̂_j
                          Set ī = i;
                          Break;
                      Endif
                  End
                  Set a_{īj} = b_ī / x̂_j;
                Endif
          Endif
End

For i = 1 to m
       Set b(i) = b_i;
End
For j = 1 to n
       For i = 1 to m
          Set A(i, j) = a_{ij};
       End
End
Output A, b;
```

Main procedure TPRG_Sup_$\mathcal{T}_L$

```
Input positive integers m and n.

For j = 1 to n
         Randomly generate the value of x̂_j ∈ [0, 1];
End

For i = 1 to m
         Randomly choose j_i ∈ {1, 2, ..., n};
         Randomly assign a_{ij_i} ∈ (1 − x̂_{j_i}, 1];
         Set b_i = a_{ij_i} + x̂_{j_i} − 1;
End

For i = 1 to m
       For j = 1 to n
          If j ≠ j_i
               Randomly assign the value of a_{ij} ∈ [0, min{1, 1 + b_i − x̂_j}];
          Endif
       End
End

For j = 1 to n
```

```
If j ∉∉ {j_1, j_2, ..., j_m}
        If x̂_j ≠ 1
           For i = 1 to m
               If b_i ≤ x̂_j
                   Set ī = i;
                   Break;
               Endif
           End
           Set a_īj = b_ī + 1 − x̂_j;
        Endif
    Endif
End

For i = 1 to m
    Set b(i) = b_i;
End

For j = 1 to n
    For i = 1 to m
       Set A(i, j) = a_ij;
    End
End

Output A, b;
```

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