Fuzzy Optim Decis Making (2012) 11:29–46
DOI 10.1007/s10700-011-9114-5

Exact and heuristic procedures for solving the fuzzy portfolio selection problem

J. M. Cadenas · J. V. Carrillo · M. C. Garrido · C. Ivorra · V. Liern

Received: 30 December 2010 / Accepted: 6 November 2011 / Published online: 16 November 2011

Abstract We propose a fuzzy model for the portfolio selection problem which takes into account the vagueness of the investor's preferences regarding the assumed risk. We also describe an exact method for solving it as well as a hybrid meta-heuristic procedure which is more adequate for medium and large-sized problems or in cases in which a quick solution is needed. As an application, we solve several problems based on data from the IBEX35 index and the Spanish Stock Exchange Interconnection System.

Keywords Fuzzy mathematical programming · Portfolio selection · Heuristic strategies · Genetic algorithms

1 Introduction

The modern portfolio selection problem is a classical model for determining the optimal composition of a portfolio according to an investor's preferences. The model is credited to Markowitz (1952, 1959), who is rightfully regarded as the founder

J. M. Cadenas · J. V. Carrillo · M. C. Garrido
Department of Ingeniería de la Información y Comunicaciones, Universidad de Murcia, Murcia, Spain
e-mail: jcadenas@um.es

J. V. Carrillo
e-mail: jvcarrillo@um.es

M. C. Garrido
e-mail: carmengarrido@um.es

C. Ivorra · V. Liern (✉)
Department of Matemàtiques per a l' Economia i l'Empresa, Universitat de València, València, Spain
e-mail: vicente.liern@uv.es

C. Ivorra
e-mail: carlos.ivorra@uv.es

of modern portfolio theory (Fama 1977). In a first approach, given n securities $S_1, S_2, \ldots, S_n$ in which we can invest non-negative quantities $x_1, x_2, \ldots, x_n$, a portfolio consists of a subset of securities in which all the capital should be invested, satisfying the investor's preferences. These preferences must take into account the risk that the investor is willing to assume, the expected return and, to a lesser extent, some diversification criteria.

Obviously, portfolio selection, like most financial problems, is related with uncertainty because it consists of taking a decision about future events. Therefore, we do not have at our disposal more than historical data which are usually managed with statistical methods (Markowitz 1959; Fama 1977). Moreover, it is not easy to model the investor's preferences. After the seminal work by Markowitz, attention has been given to the study of alternative models (Lai and Hwang 1992; Vercher et al. 2007) which try to deal more efficiently with the uncertainty of the data. Most of these models are based on probability distributions, which are used to characterize risk and return. For instance, in the Markowitz model, variance and mean were generally deemed satisfactory measures of risk and return, respectively (Markowitz 1959; Sharpe 1970; Konno and Wijayanayake 2001). However, according to the chosen modellization of the expected risk and return, different models coexist to select the best portfolio. Some of them propose to describe the risk by means of mean-absolute deviation (Konno and Yamazaki 1991), while other authors propose linear programming (Speranza 1993) or multiobjective (Arenas et al. 2001; Steuer et al. 2006) models.

Another way of dealing with uncertainty is to work with models based on soft computing. Watada (1997) solves this problem by using imprecise aspiration levels for an expected biobjective approach, where the membership functions of the goals are of a logistic-type. In 2000, Tanaka et al. (2000) propose using possibility distributions to model uncertainty on the expected returns and to incorporate the knowledge of financial experts by means of a possibility degree of similarity between the future state of financial markets and the state in previous periods (Inuiguchi and Tanino 2000). Multiobjective programming has also been used to design fuzzy models of portfolio selection, either for compromise solutions (Wang and Zhu 2002) or by introducing multi-indices (Arenas et al. 2001). Specific methods have even been proposed for dealing with the unfeasibility provoked by conflict between the expected return and the investor's diversification requirements (León et al. 2002; León and Liern 2001).

However, in this paper we are concerned with a very different class of vagueness related to the portfolio problem, namely the vagueness of the investor's criteria for selecting a satisfactory trade off between the risk he or she considers acceptable and the return he or she wishes to obtain. In other words, the investor must choose a point at the efficient frontier of the problem, i.e. the set of risk-return pairs (R, r) which are nondominated in the multiobjective sense (such that R is the minimum risk for a given level of return r or r is the maximum return level for a given risk R). From a theoretical point of view, the investor's preferences are usually formalized by means of utility functions, so that the final choice is that efficient portfolio maximizing a given utility function, but when we try to reflect the preferences of a real specific investor we must ask him or her directly for a point in the efficient frontier. Nevertheless, it is obvious that the investor's preferences are essentially vague, so that it is unnatural to

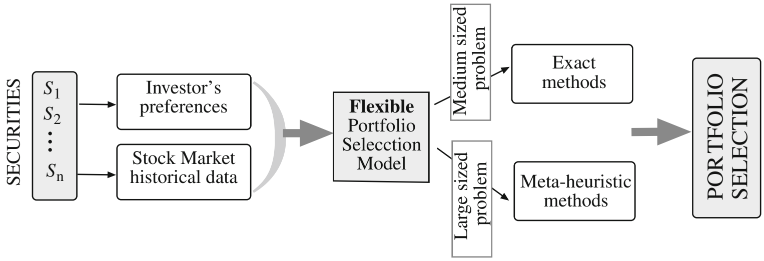

Fig. 1 Scheme of our proposal

force him or her to choose a specific point. In practice, he or she could only determine a zone or a fuzzy point.

We present several ways of dealing with the uncertainty associated with the portfolio selection problem. Namely, we are going to show that the techniques used by the authors in logistical problems (Canós et al. 2001, 2008; Cadenas et al. 2011) can be generalized to deal with risk and return fuzziness in the portfolio selection problem. These techniques allow us to include in our fuzzy model the vagueness related to the fact that the investor arbitrarily fixes the risk that he/she will assume. The uncertainty concerning the valuation of risk and returns is modeled by means of the usual variance and mean.

In this paper we propose a fuzzy model for the portfolio selection problem. To solve our model, meta-heuristic techniques are often needed because of the size of the problem or because a quick solution is required in order to reflect real-time cases. Therefore, we present a hybrid meta-heuristic method as outlined in Fig. 1. This procedure can be applied to any biobjective problem.

The paper is organized as follows: The next section introduces the framework for portfolio selection used in this paper. Section 3 examines the portfolio selection with flexible constraints, incorporating this flexibility into a fuzzy model. In Sect. 4 we present a hybrid meta-heuristic to solve the portfolio selection problem with flexible constraints. Finally, Sect. 5 outlines the most important conclusions.

2 Portfolio selection

We could consider the problem of an investor who intends to invest money in securities in such a way that the rate of return is maximized and a given level of risk cannot be surpassed. If we have n securities, r_i is the return on the ith security and x_i is the proportion of total investment funds devoted to the ith security, then we have the constraint

$$\sum_{i=1}^{n} x_i = 1.$$

The risk of investment can be measured by the variance $x^t Sx$, where $S = [s_{ij}]$ is the variance-covariance matrix. If R is the maximum risk accepted by the investor, we have the constraint

$$\sum_{i,j=1}^{n} s_{ij} x_i x_j \le R.$$

Then, one of the simplest models for the portfolio selection problem is:

$$
\begin{aligned}
(\text{MV}) \quad \text{Max.} \quad & \sum_{i=1}^{n} r_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1, && (1)\\
& \sum_{i,j=1}^{n} s_{ij} x_i x_j \leq R, && (2)\\
& x_i \geq 0, \quad i = 1, \ldots, n. && (3)
\end{aligned}
$$

Alternatively, we could have considered its economic dual model,

$$
\text{Min} \left\{ \sum_{i,j=1}^{n} s_{ij} x_i x_j \mid \sum_{i \in I_N} x_i = 1, \sum_{i=1}^{n} r_i x_i \geq r_0, \ x_i \geq 0, \ 1 \leq i \leq n \right\} \qquad (4)
$$

consisting of minimizing the risk subject to a given minimum expected return. Model (4) is the most widely used in the literature, mainly because by being a quadratic problem it is more easily handled from a mathematical point of view. However, we will deal with (MV) since it is more realistic to ask an investor what risk he or she considers acceptable rather than forcing him or her to fix a minimum return without having any reference about the risk it carries. In fact, it is the usual practice for small investors [see for instance (http://www.santander.com)].

In general, the covariance matrix S is knwon to be positive semi-definite. However, assuming that the returns are independent as random variables, S is a positive definite matrix, and we will restrict ourselves to this case. This assumptions is reasonable in practice since we can slightly modify (in a non significant way) the diagonal of S to make it a regular matrix (Steuer et al. 2006). Under this hypothesis, the solution of problem (MV) is unique.

Choosing the best investment options for a portfolio is a difficult task due to uncertainty in the economic environment and the problem of suitably reflecting decision-maker wishes in the model. Both stochastic and fuzzy programming provide different ways of handling the first kind of uncertainty (Inuiguchi and Ramík 2000). This paper presents a fuzzy approach for the second kind, in which the portfolio problem includes the subjective criteria of the decision-maker when determining the level of risk that he or she is willing to bear and the level of satisfaction to be assigned to a possible increase in return.

3 Portfolio selection with flexible constraints

The portfolio selection problem has two data concerning a decision-maker's preferences, namely the capital to be invested and the risk to be assumed. The investor can be assumed to know with certainty the capital that he/she would like to consider, and in fact in the model this quantity has been normalized to the unit. However, determining the risk to be assumed could be more flexible. As a result, it is worth incorporating this flexibility into a fuzzy model.

The main idea is to consider partially feasible solutions involving slightly greater risk than that fixed by the decision-maker, and to study the possibilities that they offer in order to improve the expected return. When compared with the logistic models [actually with the p-median case (Canós et al. 2001, 2008; Cadenas et al. 2011)], this problem happens to be more complicated because the p-median problem is linear, whereas the risk constraint in the portfolio model is quadratic. Moreover, in the p-median case, a small reduction in the demand covered affected optimal cost in a simple linear way, whereas the way in which the maximum expected return depends on the accepted risk is rather more complicated.

A fuzzy set $\tilde{S}$, of partially feasible solutions, is defined so that the membership degree of a given portfolio depends on how much its risk exceeds the risk R_0 fixed by the investor. On the other hand, a second fuzzy set $\tilde{G}$ is defined, whose membership function reflects the improvement on the return provided by a partially feasible solution with respect to the optimal crisp return z^*. In practice, we consider piecewise linear membership functions

$$\mu_{\tilde{S}}(x) = \begin{cases} 1 & \text{if } r \leq R_0, \\ 1 - \frac{r-R_0}{p_f} & \text{if } R_0 < r < R_0 + p_f, \\ 0 & \text{if } r \geq R_0 + p_f, \end{cases}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{if } z \leq z^*, \\ \frac{z-z^*}{p_g} & \text{if } z^* < z < z^* + p_g, \\ 1 & \text{if } z \geq z^* + p_g, \end{cases}$$

where r and z are the risk and the return provided by the portfolio x (which is assumed to satisfy the constraints of (MV), except the second one); the parameter p_f is the maximum increment in the risk that the decision-maker can accept, and p_g is the increment on the return that the decision-maker would consider completely satisfactory.

We have chosen piecewise linear membership functions because they are the simplest possibility in absence of a more precise criterion about the investor's preferences. But the proposed method can be applied with any reasonable alternative.

From this, we can define a global degree of satisfaction

$$\lambda(x) = \min\{\mu_{\tilde{G}}(x), \mu_{\tilde{S}}(x)\},$$

which is the membership degree for the fuzzy intersection of $\tilde{S} \cap \tilde{G}$. The fuzzy portfolio model becomes

$$\begin{aligned} \text{(FMV) Max. } & \lambda(x) \\ \text{s.t. } & \mu_{\tilde{S}}(x) \geq \lambda(x), \\ & \mu_{\tilde{G}}(x) \geq \lambda(x), \\ & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, \quad i = 1, \ldots, n. \end{aligned} \tag{5}$$

In order to solve this, the optimal solution of (MV) will be calculated for each risk level R. Therefore, we explicitly solve the Kuhn-Tucker conditions of problem (MV). To carry out the computations in a generic framework, we start by making some simplifications.

We can assume that the risk constraint (2) is active, since it will be active unless the risk level R cannot be attained by an efficient portfolio. However, it is important that the optimal portfolio that we obtain under this assumption remains optimal on the feasible set defined with (2), since this set is convex and this ensures that the Kuhn-Tucker points are optimal.

We start by solving the crisp portfolio (MV). Hence, we know the optimal crisp portfolio x, i.e. the optimal solution of the crisp corresponding the risk R_0. Call I_N to the set indexes i such that x_i is non-zero. Next we construct the auxiliary problem (AP), consisting of removing in (MV) the variables x_i with $i \notin I_N$,

$$\text{(AP)} = \text{Max}\left\{\sum_{i\in I_N} r_i x_i \;\mid\; \sum_{i\in I_N} x_i = 1,\ \sum_{i,j\in I_N} s_{ij}x_i x_j \le R,\ x_i \ge 0,\ i,j \in I_N\right\} \quad (6)$$

Let us look for solutions x_N of (AP) with $x_i > 0$. This means that the sign constraints are not active and they can be removed from (AP) and this will simplify Kuhn-Tucker conditions substantially. Later we will see that we can get the optimal solution of (MV) from them.

Calling S_N the submatrix of S resulting from removing the raws and columns corresponding to indexes $i \notin I_N$, standard linear algebra theory ensures us that we can decompose

$$S_N = A^t DA, \quad (7)$$

where the matrix D is diagonal and A is regular. Then the change of variables $y = Ax$ transforms the problem (AP) into

$$\begin{aligned} \text{Max.} & \sum_{i\in I_N} r'_i y_i \\ \text{s.t.} & \sum_{i\in I_N} b_i y_i = 1, \\ & \sum_{i\in I_N} d_i y_i^2 \le R, \\ & A^{-1} y \ge 0, \end{aligned} \quad (8)$$

where $r' = r^t A^{-1}$, $b = (1, \ldots, 1)A^{-1}$.

In order to find the Kuhn–Tucker point, we construct the Lagrangian function

$$\mathcal{L} = \sum_{i\in I_N} r'_i y_i + \delta\left(1 - \sum_{i\in I_N} b_i y_i\right) + \eta\left(R - \sum_{i\in I_N} d_i y_i^2\right) - \boldsymbol{\nu}^T A^{-1} y, \quad (9)$$

where δ, η are real numbers (the Lagrange multipliers of the capital and risk constraints respectively) and $\boldsymbol{\nu}$ is a vector.

As we are assuming that the risk constraint is active and the nonnegative conditions are inactive, their multipliers ν_i are null, and the Kuhn-Tucker conditions are

$$\sum_{i\in I_N} b_i y_i = 1, \quad \sum_{i\in I_N} d_i y_i^2 = R, \quad r_i' - b_i\delta - 2d_i y_i \eta = 0, \quad \eta \geq 0. \tag{10}$$

Moreover, we must have $\eta > 0$ since on the contrary (10) would have an infinite number of solutions which would give rise to an infinite number of Kuhn-Tucker points of (MV). They would be optimal solutions but we are assuming that the solution of (MV) is unique.

Hence:

$$y_i = \frac{r_i' - b_i\delta}{2d_i\eta}, \tag{11}$$

and, from the first constraint, we get:

$$\sum_{i\in I_N} \frac{b_i r_i'}{2d_i} - \sum_{i\in I_N} \frac{b_i^2}{2d_i}\delta = \eta.$$

Call

$$K = \sum_{i\in I_N} \frac{b_i r_i'}{2d_i}, \qquad L = \sum_{i\in I_N} \frac{b_i^2}{2d_i},$$

so that $\eta = K - L\delta$. Hence:

$$y_i(\delta) = \frac{r_i' - b_i\delta}{2d_i(K - L\delta)}.$$

The second constraint gives us:

$$\sum_{i\in I_N} \frac{(r_i' - b_i\delta)^2}{4d_i} = R(K - L\delta)^2$$

or

$$\sum_{i\in I_N} \frac{r_i'^2}{4d_i} - K\delta + \frac{L}{2}\delta^2 = R(K^2 - 2KL\delta + L^2\delta^2).$$

If

$$M = \sum_{i\in I_N} \frac{r_i'^2}{4d_i} \tag{12}$$

we get

$$(RL^2 - L/2)\delta^2 + (K - 2KLR)\delta + RK^2 - M = 0.$$

This is an algebraic second degree equation on δ, $a(R)\delta^2 + b(R)\delta + c(R) = 0$, whose discriminant, $\Delta(R) = b^2(R) - 4a(R)c(R)$, is given by

$$\Delta(R) = K^2 - 2ML + (4ML^2 - 2K^2L)R.$$

Hence, the well-known formula for the second degree equation allows us to the express the multiplier δ as a function of R:

$$\delta(R) = \frac{2KLR - K \pm \sqrt{\Delta(R)}}{2L^2R - L},$$

and the multiplier η is given by

$$\eta(R) = K - L\delta(R) = \frac{\pm\sqrt{\Delta(R)}}{1 - 2LR}.$$

The sign must be chosen so that $\eta(R) \geq 0$ (the Kuhn-Tucker sign condition). Once the right sign is chosen, this function $\delta(R)$ allows us to calculate $y_i(R)$ and, from this, we get the optimal portfolio $x_i(R)$ for each R such that $x_i(R) > 0$ and $\Delta(R) > 0$. Thus we have solved (AP) parametrically on R.

Next, we make use of the following result, in which we call x_N to the vector of variables of (AP) and x_Z to the vector of those variables of (MV) that have been removed in (AP), so that the variables of (MV) are (x_N, x_Z). Analogously, the corresponding multipliers of the sign constraints in (AP) are denoted by ν_N and those of (MV) by (ν_N, ν_Z).

Proposition 1 *Given an optimal solution x^* of (MV) providing a risk R_0, there is an interval $I = [R_-, R_+]$ containing R_0 so that for each Kuhn-Tucker point (x, δ, η, ν_N) of (AP) providing a risk $R \in I$, a vector ν_Z of multipliers can be found such that $(x, \overline{0}, \delta, \eta, \nu_N, \nu_Z)$ is a Kuhn-Tucker point of (MV).*

Proof This is obtained by comparing the Kuhn-Tucker conditions of the problems (MV) and (AP). We have obtained a parametrization $(x(R), \delta(R), \eta(R), \nu_N(R))$ of the Kuhn-Tucker point of (AP). Introducing it into the Kuhn-Tucker conditions of (MV) and setting $x_Z = 0$, all of them are automatically satisfied, except that, for indexes i such that $x_i \neq 0$, the stationary point conditions are reduced to

$$\frac{\partial \mathcal{L}}{\partial x_i} = c_i(R) - \nu_i = 0, \tag{13}$$

where $\mathcal{L}$ is the Lagrangian function of (MV) and $c_i(R)$ is a constant. And the dual feasibility conditions are $\nu_i \leq 0$. Since (13) is satisfied by $R = R_0$, we can find an interval $I = [R_-, R_+]$ containing R_0 such that for all $R \in I$, the vector $\nu_N(R)$

Table 1 Returns on five assets

Year	AmT	ATT	USS	GM	ATS
1937	−0.305	−0.173	−0.318	−0.477	−0.457
1938	0.513	0.098	0.285	0.714	0.107
1940	0.055	0.2	−0.047	0.165	−0.424
1941	−0.126	0.03	0.104	−0.043	−0.189
1942	−0.003	0.067	−0.039	0.476	0.865
1943	0.428	0.3	0.149	0.225	0.313
1944	0.192	0.103	0.26	0.29	0.637
1945	0.446	0.216	0.419	0.216	0.373
1946	−0.088	−0.046	−0.078	−0.272	−0.037

determined by (13) satisfies the dual feasibility conditions. Hence, the vector$(x(R), \overline{0}, \delta(R), \eta(R), \nu_N(R), \nu_Z(R))$ is a Kuhn-Tucker point of (MV). □

Now we have an interval I on which the efficient frontier is parametrized by

$$F(R) = \sum_{i \in I_N} r_i' y_i(R).$$

Introducing the values of δ and η in (11) , we get

$$F(R) = \mp \frac{2LR - 1}{2\sqrt{\Delta(R)}} \sum_{i \in I_N} \left(\frac{r_i'^2}{d_i} - \frac{r_i' b_i}{d_i} \frac{2KLR - K \pm \sqrt{\Delta(R)}}{2L^2 R - L} \right).$$

This expression allows us to calculate the degree of improvement of the goal $\mu_g(R)$ in the best portfolio with risk $R \in I$, whereas its degree of feasibility $\mu_f(R)$ is trivially computed.

Now we can determine the risk R^* such that $\mu_f(R^*) = \mu_g(R^*)$, which is easily shown to be the risk of the portfolio maximizing λ. The portfolio $x(R^*)$ corresponding to $y(R^*)$ by the change of variables is the optimal solution of (5). If the functions $\mu_f(R)$ and $\mu_g(R)$ did not meet in the interval I, we would need to repeat the computations from a bigger risk R_0 with a different set of non-zero variables.

3.1 A numerical example

In order to show the performance of our method in a simple example, let us use five assets from the historical data introduced by Markowitz (1959). Table 1 shows the returns of American Tobacco, AT&T, United States Steel, General Motors and Atcheson, Topeka & Santa Fe.

We have fixed a risk level $R = 0.03$. The optimal crisp portfolio is formed by assets AmT, ATT, GM, ATS and provides an optimal return $z^* = 0.103926$. For the fuzzy

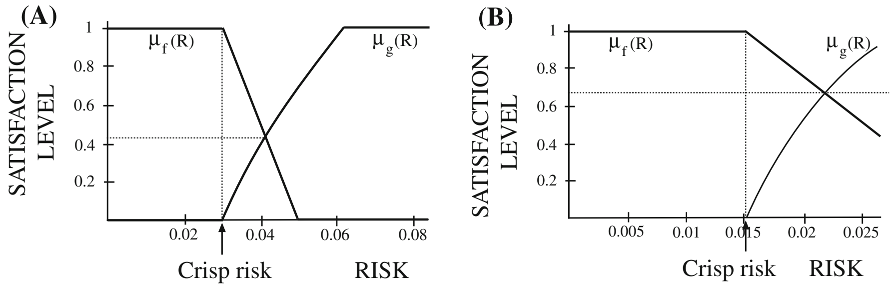

Fig. 2 Membership functions for the degree feasibility and degree of improvement on the expected return. Option (**a**) corresponds to $R = 0.03$ and option (**b**) to $R = 0.0152$

model, we have fixed $p_f = 0.02$, $p_g = 0.02$ and solved it by using MATHEMATICA. The optimal return for a given risk R, happens to be

$$F(R) = \frac{-0.02355 + 52.6832R + 2.6136\sqrt{-0.77841 + 52.6832R}}{9.09494 \times 10^{-13}R + 33.84051\sqrt{-0.77841 + 52.6832R}}$$

Computations are valid for risks in the interval $I = [0.025826, 0.083341]$. The functions $\mu_f(R)$ and $\mu_g(R)$ are shown in Fig. 2a. They intersect at $R^* = 0.041381$, corresponding to $\lambda = 0.430977$. The return on the fuzzy portfolio is 0.112545, whereas the crisp return was 0.103926.

We observe that the global degree of satisfaction is low. This means that the risk has to be increased greatly in order to obtain a not very significant increase in the return on the asset. Hence, it is not useful to replace the initial crisp portfolio by the fuzzy one. However, if the investor had chosen an initial risk $R = 0.0152$, the global degree of satisfaction would be $\lambda = 0.676198$ and the fuzzy portfolio would have a risk $R^* = 0.021676$, the expected return passing from 0.0816547 to 0.0951787 (see Fig. 2b). It has yet to be studied how solutions depend on the chosen membership functions by means of a suitable sensitivity analysis, analogous to that developed by Canós et al. (2008).

Remark 1 Notice that we have calculated the functions $\mu_f(R)$ and $\mu_g(R)$ over the interval I and we have found that both meet on it. This means that starting from a single efficient portfolio, we have parametrized a large enough piece of the efficient frontier to solve the fuzzy portfolio problem. The calculations we have made are computationally simpler than calculating many points of the frontier, since they are just algebraic operations without any iterative steps. If the intersection point had not been on the interval I, we should have calculated a second interval from a new efficient portfolio. However, for reasonable/realistic values of p_f and p_g, the intersection point will be found after a small number of iterations. For large instances of the problem, instead of using the Kuhn-Tucker conditions, it could be more efficient to work with a dotted representation of the efficient frontier calculated by means of a meta-heuristic procedure.

4 A hybrid meta-heuristic for solving the portfolio selection problem

When the size of the problem increases, classical methods (exact and deterministic) become useless as they would need too much time, due to the combinatorial explosion in the solution space. In these situations, meta-heuristics have shown their utility and effectiveness. The term meta-heuristic was coined by Glover in 1986 and refers to a master strategy that guides and modifies other heuristics to produce solutions beyond those that are normally generated in a quest for local optimality. The heuristics guided by such a meta-strategy may be high level procedures or may embody nothing more than a description of available moves for transforming one solution into another, together with an associated evaluation rule. Meta-heuristics in their modern forms are based on a variety of interpretations (Glover and Kochenberger 2003) and they have evolved rapidly. In particular, Genetic Algorithms (Mitchell 1996; Cadenas et al. 2008) are meta-heuristic methods that have already shown their effectiveness in solving optimization problems.

In this section we describe a hybrid meta-heuristic for the portfolio selection problem with flexible constraints, called `GAFUZ-PF` (Genetic algorithm + simulated Annealing + FUZzy PortFolio). This meta-heuristic uses a hybrid scheme, mixing the ideas of Simulated Annealing techniques with the classic Genetic Algorithm. Notice that in the definition of the portfolio selection problem (MV) the optimal crisp return z^* appears because $\mu_{\tilde{G}}(x)$ depends on it. Hence, `GAFUZ-PF` will calculate the crisp solution z^* as well as the fuzzy solution (this means solving two NP-hard problems). In order to realize this idea, we use three different populations on three stages of the genetic algorithm. First of all, it uses a population to achieve feasible solutions to the problem (with a risk less than or equal to R fixed) which will be part of the population of the algorithm. Once feasible solutions are found, in the second stage, the algorithm uses another population that evolves searching the best return. The third stage uses another population, which looks for the best fitness $\lambda(x)$.

Simulated Annealing (Aarts and Korst 1990; Kirkpatrick et al. 1983) is a meta-heuristic used for optimization problems. This heuristic replaces the current solution by a random "nearby" solution, chosen with a probability that depends on the difference between the corresponding function values and a global parameter T (called the temperature), which is gradually decreased during the process. The hybridization of Genetic Algorithms with other meta-heuristics has appeared in different papers in the literature (Mitchell 1996; Blum et al. 2008). We apply Simulated Annealing to define the mutation function of the Genetic Algorithm. Having a mutation function with a big mutation rate helps us to keep diversity in the population, but it introduces distortion into the population when the algorithm approaches the optimum. Therefore, it seems a good idea to use Simulated Annealing with a temperature depending on the number of generations and the fitness of the solution in order to make a big mutation rate possible at the beginning, and reduce it as the number of generations and the fitness increases. Now we specify our options and we give a general outline of our approach. Figure 3 shows the scheme of `GAFUZ-PF` hybrid meta-heuristic.

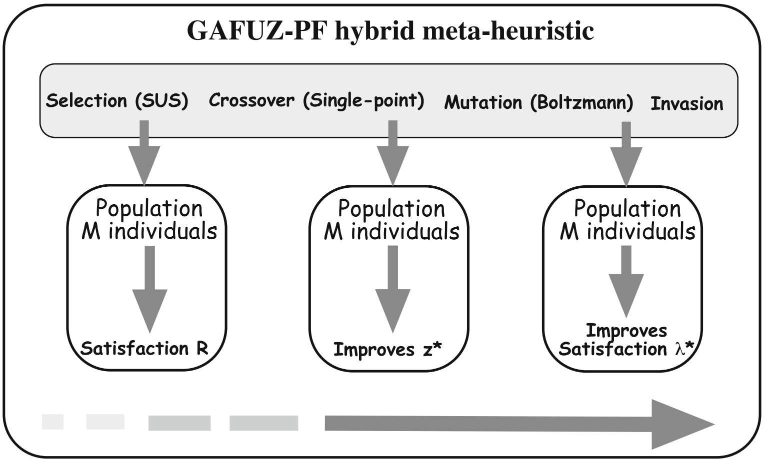

Fig. 3 Scheme of hybrid meta-heuristic

4.1 Encoding, fitness function and population size

Each element in the population of the algorithm represents a solution to the portfolio problem described. The population encoding is made having an array of n-elements, where each gene represents the investment in one asset. The size of the population is fixed at 50 individuals, as this is known to be the best average size to avoid to slow the algorithm excessively (Mitchell 1996). Furthermore, we use three different populations at the same time: the first one evolves by searching for individuals to satisfy the second constraint of the problem (MV). When it finds a suitable solution, it changes to the second stage, where it improves the return on the portfolio, taking account of the flexible constraint of R $(R + p_f)$. Once the algorithm gets a state with $\lambda(x) > 0$, it changes the population and evolves taking into account p_f and p_g. The fitness function is the objective function of the problem (5).

The algorithm scheme is the classical one, with selection, crossover, mutation, invasion and having the number of generations as stop criteria.

4.2 Selecting the individuals

In order to select individuals for the crossover operator, we use Stochastical Universal Sampling (SUS), as it has been shown to be better than Roulette Wheel (Cadenas et al. 2008). Based on their fitness, SUS places the individuals on a Roulette and spins it once with n selectors equally separated to select n individuals.

4.3 Crossover: generating new individuals

The crossover operator is the single-point crossover (Mitchell 1996) adapted to the problem constraints: there is a fixed part from one parent that goes directly to the offspring; the genes from the second part (from the second parent) are added to the son (choosing them randomly) while the sum of its genes is less than or equal to one.

4.4 Mutation

When we talk about mutation, there are certain parameters used. The mutation rate (MUR) refers to the percentage of population individuals that will be mutated. On the other hand, we have the number of genes that will change (NR). As a mutation method we use Boltzmann mutation, which is based on Simulated Annealing (Cadenas et al. 2008), with the following annealing function:

$$r = 1 - e^{-1/(n*max(\lambda(x)))}$$

It defines the mutation rate across the algorithm, reducing it as the algorithm evolves in order not to disturb the good solutions achieved when the algorithm is finishing. This rate specifies the percentage of individuals that will be mutated. As it remains near to 1, according to (Reeves 2002), it is a good choice. A mutation consists of subtracting a random value from a gene and adding this amount to another gene, both of them chosen randomly.

4.5 Invasion

Finally, invasion is another genetic operator that helps to keep diversity in the population. It introduces a number of new randomly generated individuals, which replace individuals at random in the population (except the best individual). This helps to introduce new genes into the population, as mutation does. The invasion rate (IR) selected is 1%.

4.6 Stop criteria

The detention criterion is the number of generations (NG).

4.7 General outline of the `GAFUZ-PF` meta-heuristic

Now we have shown the configuration of the algorithm, let us see its general outline in Algorithm 1.

4.8 Experiments

To test the `GAFUZ-PF` hybrid meta-heuristic we have used various test problems. The PC used for the executions has the following features: Intel Pentium IV 3.00 GHz 2,048 MB RAM.

Example 4.1 We consider five assets from the historical data introduced by Markowitz (1959) (see Example 3.1). We have fixed $p_f = 0.01$ and $p_g = 0.01$ for some values of the maximum risk R. The results obtained are shown in Table 2.

For each maximum level of risk R (first column), Table 2 shows the composition of the crisp and the fuzzy optimal portfolios (i.e. the solutions of (MV) and (FMV),

Algorithm 1 - GAFUZ-PF: Hybrid meta-heuristic for the portfolio selection

GAFUZ-PF( )
Require: r_i, s_{ij}, R, p_f, p_g
Ensure: z^*, $\overline{x}^*$, R^*, $\lambda(x)$, z, $\overline{x}$, R

```
begin
  Generate initial population with 50 individuals at random.
  Calculate the fitness of each individual.
  while Number of generations < NG do
    SELECT 50 individuals
    if ∄ individual | risk of the individual ≤ R then
      select using risk
    else
      if the sum of individual fitness is = 0 then
        select using z*
      else
        select using fitness
      end if
    end if
    CROSS the 50 individuals to obtain 50 new individuals
    Replace 49 individuals of the population with the new individuals, keeping the best (elitism)
    MUR = Boltzmann(maximum λ(x),generation_number)
    MUTATE M × MUR individuals, changing NR genes
    Replace 50 × IR individuals with random generated individuals
    Calculate the new individuals' fitness, return and risk
  end while
end
```

Table 2 Some solutions obtained by GAFUZ-PF

R	Solution	PROPORTION OF INVESTMENT					PORTFOLIO		
		AmT	ATT	USS	GM	ATS	Return	Risk	λ
0.03	Crisp	0.051	0.689	0	0.209	0.051	0.104	0.030	0
	Fuzzy	0.081	0.636	0	0.229	0.053	0.110	0.033	0.230
0.04	Crisp	0.151	0.512	0	0.279	0.057	0.112	0.040	0
	Fuzzy	0.118	0.464	0	0.298	0.058	0.114	0.041	0.213
0.05	Crisp	0.233	0.369	0	0.337	0.061	0.118	0.050	0
	Fuzzy	0.259	0.322	0	0.355	0.063	0.120	0.053	0.204

respectively) together with their corresponding return and risk. The final column shows the global satisfaction level, λ, of the fuzzy problem.

Notice that the values of λ are low, which means that increasing the risk provides a small increment on the expected return. However, there are initial risk values for which the difference between fuzzy and crisp portfolios, as well as the composition of the portfolio, can be more significant. For instance, starting from a level of risk $R = 0.0185$, the corresponding results are shown in Table 3.

We see that in this case, the composition of the portfolio changes so that the investor could prefer the fuzzy proposal for other reasons beyond its risk and return values. Moreover, the diversification of the fuzzy portfolio is clearly better from a financial point of view.

Table 3 Some efficient portfolios with risk around $R = 0.019$

R	Solution	PROPORTION OF INVESTMENT					PORTFOLIO		
		AmT	ATT	USS	GM	ATS	Return	Risk	λ
0.0185	Crisp	0	0.976	0	0	0.025	0.089	0.0185	0
	Fuzzy	0	0.907	0	0.049	0.044	0.093	0.0203	0.353
0.0190	Crisp	0	0.950	0	0.007	0.043	0.091	0.0190	0
	Fuzzy	0	0.890	0	0.065	0.045	0.094	0.0209	0.330
0.0195	Crisp	0	0.932	0	0.024	0.044	0.092	0.0195	0
	Fuzzy	0	0.875	0	0.080	0.045	0.095	0.0214	0.316

Example 4.2 We have considered the returns on 20 assets from the Spanish index IBEX35. The set of assets included in the experiment represents Acesa (ACE), Arcelor (ACR), ACS, Altadis (ALT), BBVA, Bankinter (BKT), Dragados (DRC), Endesa (ELE), FCC, Iberdrola (IBE), Metrovacesa (MVC), NH Hoteles (NHH), Banco Popular (POP), Repsol (REP), SCH (SAN), Telefónica (TEF), Unión Fenosa (UNF), Vallermoso (VAL), Acerinox (ACX), Acciona (ANA) data, respectively. We have considered the observations of the Wednesday prices as an estimate of the weekly prices. Hence, the return on the jth asset during the kth week is defined as $r_{kj} = (p_{(k+1)} - p_{kj})/p_{kj}$, where p_{kj} is the price of the jth asset on the Wednesday of the kth week. The data base used covers the period from January 1998 to March 2003. The results obtained are shown in Table 4. The time used for the algorithm to obtain a solution to this problem is 1.61 s on average.

Example 4.3 To consider a greater database, we have used the securities of the Spanish Stock Exchange Interconnection System that integrates the four existing security exchanges in Barcelona, Bilbao, Madrid and Valencia (http://www.borsabcn.es/bolsabcn/navegacion.nsf/vweb/p-eng?OpenDocument). We have selected daily returns from the period April 2001 to December 2001 because in that period it contained the greatest number of assets, specifically 144 securities.

Setting a risk level $R = 0.000015$ with tolerances $p_f = 0.000002$ and $p_g = 0.0001$, the algorithm takes 21.04 s to provide both the crisp and the fuzzy optimal portfolios that we show in Table 5 [the names of the companies can be found in (http://www.borsabcn.es/bolsabcn/navegacion.nsf/vweb/p-eng?OpenDocument)]. We can see that the fuzzy portfolio return is 5.7% greater than the crisp portfolio return and the risk that the investor is willing to assume (for the fuzzy solution) is less than or equal to $R + p_f$.

5 Conclusions

In this paper we have presented the problem of portfolio selection, which is a difficult problem to solve due to uncertainty in the economic environment and the problem of suitably reflecting a decision maker's wishes in the model. It should be remarked that

Table 4 Solutions obtained by applying GAFUZ-PF to the IBEX35 data (weekly returns from 1998 to 2003)

	$R = 0.000494$		$R = 0.000839$	
	$p_f = 0.0002$, $p_g = 0.002$		$p_f = 0.0002$, $p_g = 0.002$	
	Crisp	Fuzzy	Crisp	Fuzzy
ACE	0.316	0.200	0	0
ACR	0.011	0	0	0
ALT	0.168	0.199	0.207	0.162
DRC	0.019	0.205	0.353	0.461
IBE	0.165	0.014	0	0
MVC	0.050	0.102	0.157	0.098
NHH	0.011	0.033	0.056	0.065
POP	0.62	0	0	0
REP	0.031	0	0.001	0
UNF	0	0.077	0.0395	0.052
VAL	0.036	0	0	0
ACX	0.053	0.069	0.051	0.017
ANA	0.078	0.100	0.139	0.145
λ	0	0.454078	0	0.145928
Portfolio return	0.00202704	0.0029352	0.00353188	0.00382374
Portfolio risk	0.000491306	0.000603184	0.00083643	0.0010098

Table 5 Solutions obtained by applying GAFUZ-PF to the SIBE data (daily returns from April 2001 to December 2001)

Securities	Version	Risk	Return	λ
ALT, AZC, BDL, BYB, CAF, ESF, GAL, GCO, GUI, HKN, IBG, IBP, NMQ, OMS, PAC, SOS, SUP, UPL, ZRG	Crisp	0.0000150	0.0013220	0
ALT, ADZ, AZC, BDL, BYB, CAF, ESF, GAL, GCO, GUI, HKN, IBG, IBP, OMS, PAC, SOS, SUP, UPL, ZRG	Fuzzy	0.0000159	0.0013974	0.549034

our proposal provides efficient portfolios in the Markowitz's sense, and so we can not expect any improvement in the chances of obtaining a better return, i.e. its aim is not to improve the predictive power of the Markowitz model, but to find a trade-off between risk and expected return that fits better the investor's preferences.

We present an approach where the problem of a fuzzy portfolio includes the subjective criteria of the decision-maker when determining the level of risk that he or she is willing to bear and the level of satisfaction to be assigned to a possible increase in the return.

We have proposed an exact method to solve the fuzzy model of the portfolio problem with the main idea of finding partially-feasible solutions involving slightly greater risk than that fixed by the decision-maker, and to study the possibilities that they offer in order to improve the expected return.

We have also proposed a hybrid meta-heuristic to solve the fuzzy model for medium-sized or large problems where the traditional methods become useless as they would need too much time due to the combinatorial explosion in the solution space. The proposed meta-heuristic uses a hybrid scheme, which combines ideas from the Simulated Annealing technique and Genetic Algorithms. To test the proposed meta-heuristic we have used several test problems based on data from the IBEX35, the best-known index of Spanish Stock Markets, and the Spanish Stock Exchange Interconnection System (SIBE). The results allow us to verify that with small problems the solutions are very similar to those obtained with the exact model. Moreover, our proposal allows us to work with big problems (large time series, intra-day data, etc.) and the time taken to get the results is about 1.6 s for the IBEX35 and 21 s for the SIBE, achieving the goal we wanted with that meta-heuristic.

Acknowledgments This paper has been partially supported by the Ministerio de Ciencia e Innovación of Spain, TIN2008-06872-C04-02, TIN2008-06872-C04-03, TIN2011–27696-C02-02.

References

Aarts, E., & Korst, J. (1990). *Simulated annealing and Boltzmann machines: A stochastic approach combinatorial to optimization and neural computing*. New York: John Wiley and Sons.

Arenas Parra, M., Bilbao Terol, A., & Rodríguez Uría, M. V. (2001). A fuzzy goal programming approach to portfolio selection. *European Journal of Operational Research, 133*, 287–297.

Blum, C., Blesa, M. J., Roli, A., & Sampels, M. (Eds.). (2008). *Hybrid metaheuristics: An emerging approach to optimization. Studies in computational intelligence 114*. Berlin: Springer.

Canós, M. J., Ivorra, C., & Liern, V. (2001). The fuzzy p-median problem: A global analysis of the solutions. *European Journal of Operational Research, 130*, 430–436.

Canós, M. J., Ivorra, C., & Liern, V. (2008). Marginal analysis for the fuzzy p-median problem. *European Journal of Operational Research, 130*, 430–436.

Cadenas, M., Canós, M. J., Garrido, M. C., Ivorra, C., & Liern, V. (2011). Soft-computing based heuristics for location on networks: The p-median problem. *Applied Soft Computing, 11*, 1540–1547.

Cadenas, J. M., Canós, M. J., Carrillo, J. V., Garrido, M. C., Ivorra, C., & Liern, V. (2008). A hybrid algorithm for the fuzzy p-median problem. IADIS International conference intelligent systems and agents, Amsterdam (The Netherlands).

Fama, E. F. (1977). *Foundations of Finance*. Oxford: Basil Blackwell.

Glover, F. (1986). Future paths for integer programming and link to artificial intelligence. *Computer and Operations Research, 5*, 533–549.

Glover, F., & Kochenberger, G. (2003). *Handbook of metaheuristics*. The Netherlands: Kluwer.

Inuiguchi, M., & Ramík, J. (2000). Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. *Fuzzy Sets and Systems, 111*, 3–28.

Inuiguchi, M., & Tanino, T. (2000). Portfolio selection under independent possibilistic information. *Fuzzy Sets and Systems, 115*, 83–92.

Kirkpatrick, S., Gelat, C. D., & Vecchi, M. P., Jr. (1983). Optimization by simulated annealing. *Science, 220*, 671–680.

Konno, H., & Yamazaki, H. (1991). Mean-absolute deviation portfolio optimization model and its applications Tokyo Stock Market. *Management Science, 37*, 519–531.

Konno, H., & Wijayanayake, A. (2001). Portfolio optimization problem under concave transaction costs and minimal transactions unit constraints. *Mathematical Programming, 89*, 233–250.

Lai, Y. J., & Hwang, C. L. (1992). *Fuzzy mathematical programming: Theory and applications*. Berlin: Springer.

León, T., & Liern, V. (2001). A fuzzy method to repair infeasibility in linearly constrained problems. *Fuzzy Sets and Systems, 122*, 237–243.

León, T., Liern, V., & Vercher, E. (2002). Viability of infeasible portfolio selection problems: A fuzzy approach. *European Journal of Operational Research, 139*, 178–189.

Markowitz, H. M. (1952). Portfolio selection. *Journal of Finance, 7*, 79–91.

Markowitz, H. M. (1959). *Portfolio selection: Efficient diversification of investments*. New York: John Wiley.

Mitchell, M. (1996). *An introduction to genetic algorithms*. London: The MIT Press.

Reeves, C. R. (2002). Genetic algorithms. In F. Glover & G. A. Kochenberger (Eds.), Handbook of metaheuristics, international series in operations research & management science 57, Kluwer, pp. 62–77.

Sharpe, W. F. (1970). *Portfolio theory and capital market*. New York: McGraw-Hill.

Speranza, M. G. (1993). Linear programming model for portfolio optimization. *Finance, 14*, 107–123.

Steuer, R. E., Qi, Y., & Hirschberger, M. (2006). Portfolio optimization: New capabilities and future methods. *Zeitschrift für Betriebswirtschaft, 2*, 199–219.

Tanaka, H., Guo, P., & Türksen, I. B. (2000). Portfolio selection based on fuzzy probabilities and possibility distributions. *Fuzzy Sets and Systems, 3*, 387–397.

Vercher, E., Bermúdez, J. D., & Segura, J. V. (2007). Fuzzy portfolio optimization under downside risk measures. *Fuzzy Sets and Systems, 158*, 769–782.

Wang, S., & Zhu, S. (2002). On fuzzy portfolio selection problems. *Fuzzy Optimization and Decision Making, 1*, 361–377.

Watada, J. (1997). Fuzzy portfolio selection and its applications to decision making. *Tatra Mountains Mathematical Publications, 13*, 219–248.