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A bilevel fuzzy principal-agent model for optimal nonlinear taxation problems

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Abstract This paper presents a bilevel fuzzy principal-agent model for optimal nonlinear taxation problems with asymmetric information, in which the government and the monopolist are the principals, the consumer is their agent. Since the assessment of the government and the monopolist about the consumer's taste is subjective, therefore, it is reasonable to characterize this assessment as a fuzzy variable. What's more, a bilevel fuzzy optimal nonlinear taxation model is developed with the purpose of maximizing the expected social welfare and the monopolist's expected welfare under the incentive feasible mechanism. The equivalent model for the bilevel fuzzy optimal nonlinear taxation model is presented and Pontryagin maximum principle is adopted to obtain the necessary conditions of the solutions for the fuzzy optimal nonlinear taxation problems. Finally, one numerical example is given to illustrate the effectiveness of the proposed model, the results demonstrate that the consumer's purchased quantity not only relates with the consumer's taste, but also depends on the structure of the social welfare.

Keywords Fuzzy programming · Game theory · Principal-agent · Optimal taxation · Asymmetric information

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1 Introduction

Since the pioneering work of Mirrlees (1971), a large body of research on optimal taxation problems has been developed, covering many different aspects of the tax system. For example, Laffont (1987) studied the optimal control of a discriminating monopolist and found that the monopolist had a tendency to underproduce, with individual consumption levels smaller than the first best and derived the optimal tax for the government with asymmetric information. Bierbrauer (2009) extended the model of optimal income taxation due to Mirrlees (1971) and applied a mechanism design approach to analyze the implementation of the tax system. Hamilton and Slutsky (2007) suggested that tax schedules should balance the government budget for every possible vector of revelations. Boadway and Gahvari (2006) studied the optimal commodity taxation problem when the time taken in consumption was a perfect substitute for either labor or leisure. Moresi (1998) analyzed the nonlinear optimal taxation problems under asymmetric information in a general equilibrium model which showed that output and input taxes were not necessary if profits were optimally taxed under a weak separability assumption on technologies and described the corresponding effects on the structure of the production sector relative to the first-best solution. Finally, the paper investigated conditions under which the provision of public goods can be decentralized. Parker (1999) modeled an optimal linear taxation problem when occupational choice and labor supply were endogenous. Workers chose among paid employment, self-employment, and unemployment. The implications of making unemployment benefits as well as tax rates endogenous were also analyzed, and robustness checks were performed. However, among all the literature associated with optimal taxation problems under asymmetric information, they characterized the unknown tastes of the consumers as random variables, seldom did this work involve fuzzy variables to denote the unknown tastes. Since the consumer's taste is often fuzziness by nature and cannot be exactly assessed by the government and the monopolist, it is reasonable to denote the unknown taste by a fuzzy variable.

The notion of fuzzy sets was originally introduced by Zadeh (1965). From then on, fuzzy sets theory was being perfected and became an effective method to deal with fuzziness. Owing to its strong effectiveness, many researchers such as Zimmermann (1985), Costantino and Di Gravio (2009), Chen (2009), Tanaka et al. (2000), Luhandjula (2007) and Pedrycz (2007) applied fuzzy sets theory successfully to optimization problems. Recently, Liu and Liu (2002) laid a new foundation for optimization theory in uncertain environments, in which fuzzy programming models (Huang 2009; Ke et al. 2010; Lan et al. 2009, 2010; Sun et al. 2010, and Wang et al. 2007) were proposed to deal with the optimization problems. Recently, Cui et al. (2007) studied a fuzzy principal-agent problem which was the pioneering work in fuzzy principal-agent fields. One of main contributions of their work is that the authors proved that the objective function of the model, i.e., the expected utility of the principal, can be transformed into an integral form which made it possible to study fuzzy principal-agent problems by using the optimal control methods. The incentive compatible constraint and participation constraint, as the main contents of the principal-agent problems, were not considered in that paper. The purpose of this paper is to study an optimal nonlinear taxation problem in which the consumer's taste is his private information, but can be assessed by the government and the monopolist as a fuzzy variable. To maximize the expected welfare of the government and the monopolist, a bilevel fuzzy optimal nonlinear taxation model is formulated. The equivalent model for the bilevel fuzzy optimal nonlinear taxation model is presented and the necessary conditions of the solutions for the bilevel fuzzy optimal nonlinear taxation model are obtained by the Pontryagin maximum principle. Moreover, one example is given to illustrate the effectiveness of the proposed model and the results demonstrate the consumer's purchased quantity not only relates with the consumer's taste, but also depends on the structure of the social welfare.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts about fuzzy variables. Section 3 establishes a bilevel fuzzy optimal nonlinear taxation model under incentive compatibility constraints and participation constraints. After that, Sect. 4 presents the equivalent model for the bilevel fuzzy optimal nonlinear taxation model and the necessary conditions of the solutions for the fuzzy optimal nonlinear taxation problem are obtained by the Pontryagin maximum principle. One example is given in Sect. 5 to illustrate the effectiveness of the proposed model, the results demonstrate the consumer's purchased quantity not only relates with the consumer's taste, but also depends on the structure of the social welfare. Section 6 summarizes the main results in this paper.

2 Preliminaries

A credibility space is defined as a triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$, where Θ is a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ, and Cr a credibility measure defined on $\mathcal{P}(\Theta)$ (Liu 2006).

Definition 1 (Nahmias 1978) Let $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ be a credibility space. If ξ is a function from Θ to $\Re$, then it is called a fuzzy variable.

The credibility of a fuzzy event $A \in \mathcal{P}(\Theta)$ can be defined by

$$\mathrm{Cr}(A) = \frac{1}{2}(\mathrm{Pos}(A) + 1 - \mathrm{Pos}(A^c)). \tag{1}$$

The membership function of ξ can be derived from the credibility measure by

$$\mu_\xi(x) = (2\mathrm{Cr}(\{\xi = x\})) \wedge 1, \forall x \in \Re. \tag{2}$$

Definition 2 (Liu 2002) Let ξ be a fuzzy variable defined on a credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then the credibility distribution $\Phi : \Re \to [0, 1]$ of ξ is defined by

$$\Phi(x) = \mathrm{Cr}(\{\theta \in \Theta \mid \xi(\theta) \le x\}), \forall x \in \Re. \tag{3}$$

or equivalently

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\mathrm{d}y, \forall x \in \Re, \tag{4}$$

where $\phi : \Re \rightarrow [0, +\infty)$ is the credibility density function, and satisfies $\int_{-\infty}^{+\infty} \phi(y)\mathrm{d}y = 1$.

Definition 3 (Liu and Liu 2002) Suppose that ξ is a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then its expected value $E[\xi]$ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathrm{Cr}(\{\xi \geq x\})\mathrm{d}x - \int_{-\infty}^{0} \mathrm{Cr}(\{\xi \leq x\})\mathrm{d}x, \tag{5}$$

where at least one of the two integrals is finite.

Lemma 1 (Xue et al. 2008) *If $f : \Re \rightarrow \Re$ is an increasing function and ξ is a fuzzy variable with continuous membership function and finite expected value, then*

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)\phi(x)\mathrm{d}x \tag{6}$$

provided that the integral is finite.

3 A bilevel fuzzy optimal nonlinear taxation model

In this section, an optimal nonlinear taxation problem is considered, in which there are three participants, the government, the monopolist and the consumer, among them, the government and the monopolist are the principals, the consumer is their agent. Thus, there exist two principal-agent relationships: the relationship between the government and the consumer as well as the relationship between the monopolist and the consumer. The consumer should give payments for the goods to the monopolist as the transfer, meanwhile pay the corresponding taxation to the government. The government designs a nonlinear taxation intended for the consumer, while the monopolist also makes a pricing of the goods to attract the consumer to purchase the goods. In the optimal nonlinear taxation problem, the consumer gets access to the information which is not available to the government and the monopolist. This extra information is referred to as the consumer's characteristic, which denotes the consumer's taste to the price. The government's nonlinear taxation mechanism can be specified by considering a vector, denoted by $(q(\cdot), \tau(q(\cdot)), r(\cdot))$, in which $q(\cdot)$ denotes the consumer's purchased quantity; $\tau(q(\cdot))$ is the corresponding nonlinear taxation the consumer pays to the government when his purchased quantity is $q(\cdot)$; $r(\cdot)$ is the transfer that the consumer pays to the monopolist. Thus, the welfare of the consumer with a taste x, when he pays τ and r for the purchased quantity q is denoted by $V(q, \tau, r, x)$. While the monopolist's welfare is denoted by $U(q, r)$ which depends on the consumer's purchased quantity and the payment from the consumer.

To be more specific, the following assumptions are made throughout the paper.

(i) Let Ω denote the set of the consumer's possible taste. Assume that the government and the monopolist have the same subjective assessment of the consumer's taste, and this subjective assessment is denoted by a fuzzy variable ξ. The taste ξ has a continuous membership function with a support $\Omega = [0, b]$, where $b \in (0, +\infty)$ and denotes the consumer's highest taste towards the price. The credibility density function and the credibility distribution function of ξ are $\phi(x)$ and $\Phi(x)$, respectively, both of them denote the structure of the government's and the monopolist's subjective assessment.

(ii) The consumer's welfare $V(q, \tau, r, x) = \pi(q, x) - \tau(q) - r$, where $\pi(q, x)$ is the consumer's utility to purchase q with the taste x. Moreover, assume that $q(0) = 0$ and $\tau(0) = 0$. For each $x \in \Omega$, assume that

$$\frac{\partial \pi(q,x)}{\partial q} \geq 0, \; \frac{\partial^2 \pi(q,x)}{\partial q^2} \leq 0, \; \frac{\partial \pi(q,x)}{\partial x} \geq 0, \; \frac{\partial^2 \pi(q,x)}{\partial q \partial x} > 0,$$
$$\frac{\partial^3 \pi(q,x)}{\partial q^2 \partial x} \geq 0, \; \frac{\partial^3 \pi(q,x)}{\partial q \partial x^2} \leq 0, \; \frac{\mathrm{d}^2 \tau(q)}{\mathrm{d}q^2} \geq 0, \; \frac{\mathrm{d}^2 q}{\mathrm{d}x^2} \leq 0, \; \frac{\mathrm{d}^2 r}{\mathrm{d}x^2} \geq 0.$$

(iii) The monopolist's welfare $U(q, r) = r - cq$, where c is the marginal cost of the monopolist.

(iv) The social welfare is

$$S(q, \tau(q), r, x) = V(q, \tau(q), r, x) + \alpha U(q, r) + \beta \tau(q),$$

where $\alpha \leq 1$ is the weight put on the monopolist's welfare and $\beta \geq 1$ is the social cost of funds.

(v) For each mechanism $(q(\cdot), \tau(q(\cdot)), r(\cdot))$,

$$\frac{\partial \pi(q(x), x)}{\partial q} - c - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} \geq 0, \forall x \in \Omega.$$

(vi) For each mechanism $(q(\cdot), \tau(q(\cdot)), r(\cdot))$,

$$\alpha \left(\frac{\partial \pi(q(x), x)}{\partial q} - c - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} \right) + \beta \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} \geq 0, \forall x \in \Omega.$$

(vii) $\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{1 - \Phi(x)}{\phi(x)} \right) \leq 1, \forall x \in \Omega.$

Remark 1 Let us give a total explanation about the above assumptions. It is obvious that the Assumptions i-iv are quite reasonable. Assumption v and Assumption vi can be illustrated as follows. It follows from the sum of the consumer's and the monopolist's welfare

$$\begin{aligned} &V(q(x), \tau(q(x)), r(x), x) + U(q(x), r(x)) \\ &\quad = \pi(q(x), x) - \tau(q(x)) - cq(x), \forall x \in \Omega \end{aligned}$$

that

$$\frac{\partial(V(q(x),\tau(q(x)),r(x),x)+U(q(x),r(x)))}{\partial q} = \frac{\partial\pi(q(x),x)}{\partial q} - c - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}, \forall x \in \Omega.$$

Thus, Assumption v is equivalent to $\frac{\partial(V(q(x),\tau(q(x)),r(x),x)+U(q(x),r(x)))}{\partial q} \geq 0$ for all x and for each mechanism, which means that the consumer's marginal revenue should not be less than his marginal cost including the goods' price per unit and the marginal taxation. Similarly, Assumption vi implies that the consumer's marginal revenue should not be less than his marginal cost under the social view. Note that Assumption vii is not too strict since some common fuzzy variables such as triangular fuzzy variable satisfies it.

In the optimal nonlinear taxation problem, the monopolist is subordinate to the government, the government aims to maximize the expected social welfare by designing a nonlinear taxation $\tau(q(\cdot))$ for the consumer under the incentive compatibility constraint and participation constraint, while the monopolist aims to maximize his expected welfare by designing the mechanism $(q(\cdot), r(\cdot))$ to consult the consumer with a taste $x \in \Omega$ to choose $(q(x), r(x))$, i.e., to pay the transfer $r(x)$ for quantity $q(x)$.

We consider a three-step game described as follows. In Step 1, the government designs a mechanism. A mechanism is a game where the government offers a contract specified by $(q(\cdot), \tau(q(\cdot)), r(\cdot))$. The message space could be the set of the consumer's tastes of the goods. Nonetheless, the consumer needs not reveal its true taste. In Step 2, the consumer accepts or rejects the contract. A consumer who rejects the contract is assumed to receive a payoff of 0 (his reservation utility). In Step 3, the consumer who accepts the contract plays the game according to the specifications of the accepted contract. Note that the consumer plays a completely passive role in the design of contract here. We can justify this by assuming that there are many competitive consumers in the market while there is only one government and one monopolist. Therefore, it is reasonable to assume that the government has the sole authority in the design of contract while the consumer merely responds to the offer. In general, it is difficult to obtain the optimal contract parameters in a three-step game like this one. However, a simple but fundamental result called the Revelation Principle Gibbard (1973) states that to obtain the highest expected payoff in such three-step games, the government can restrict his attention to direct mechanisms, which is a game where the consumer's only action is to submit a claim about his or her taste. The Revelation Principle further states that the government can content himself with direct mechanisms in which the consumer, regardless of his taste, accepts the contract and truthfully announces his taste.

Truthful announcement of tastes implies that the incentive compatible constraint of the consumer must be satisfied. The incentive compatible constraint of the consumer is that if for each $x \in \Omega$, the consumer with the taste x will choose $(q(x), \tau(q(x)), r(x))$ to maximize his own welfare, i.e.,

$$V(q(x), \tau(q(x)), r(x), x) \geq V(q(y), \tau(q(y)), r(y), x), \forall x, y \in \Omega. \tag{7}$$

In addition, the contracts must be individually rational so that the consumer wants to participate in the trade, i.e.,

$$V(q(x), \tau(q(x)), r(x), x) \geq 0, \quad \forall x \in \Omega. \tag{8}$$

As a consequence, the bilevel fuzzy optimal nonlinear taxation model can be formulated as follows:

$$\begin{cases} \max_{\tau(q(\cdot))} E\left[S(q(\xi), \tau(q(\xi)), r(\xi), \xi)\right] \\ \text{subject to:} \\ \quad \begin{cases} \max_{(q(\cdot), r(\cdot))} E\left[U(q(\xi), r(\xi))\right] \\ \text{subject to:} \\ \quad V(q(x), \tau(q(x)), r(x), x) \geq V(q(y), \tau(q(y)), r(y), x), \quad \forall x, y \in \Omega \\ \quad V(q(x), \tau(q(x)), r(x), x) \geq 0, \quad \forall x \in \Omega. \end{cases} \end{cases} \tag{9}$$

It is obvious that the constraint in Model (9) is also an optimization problem, we call it the lower level optimization problem of Model (9), so Model (9) is a bilevel model.

Remark 2 Cui et al. (2007) studied a fuzzy principal-agent problem which was the pioneering work in fuzzy principal-agent fields. Our work is the further study of Cui et al.'s work, but different from it. Cui et al.'s model is

$$\begin{cases} \max_{l(\cdot)} E\left[W(l(\xi), \xi)\right] \\ \text{subject to:} \\ 0 \leq \frac{\mathrm{d}l(x)}{\mathrm{d}x} \leq M, \quad \forall x \in \Omega. \end{cases} \tag{10}$$

It is easy for us to list the differences between Model (9) and Model (10). A common principal-agent problem was studied in Cui et al.'s work, but we present a novel bilevel fuzzy principal-agent model for optimal nonlinear taxation problems; Cui et al.'s Model (10) contained only one decision variable $l(x)$, but three decision variables $q(x)$, $r(x)$ and $\tau(q(x))$ are refereed in Model (9); Only one monotonic constraint was considered in Cui et al.'s Model (10), but the incentive compatible constraint and the participation constraint were not considered, in fact, considering the incentive compatible constraint and participation constraint is more practical, thus the above two constraints are discussed in our paper; Cui et al.'s work presented a single level principal-agent model, while a bilevel one is established in our paper.

4 Model analyzing

4.1 Equivalent model for the optimal nonlinear taxation problem

In this subsection, we consider one equivalent model of (9). First, we provide an equivalent model for the lower level optimization problem of (9).

Proposition 1 *The incentive compatibility constraint (7) can be written as*

$$\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\frac{\mathrm{d}q(x)}{\mathrm{d}x}+\frac{\mathrm{d}r(x)}{\mathrm{d}x}=\frac{\partial\pi(q(x),x)}{\partial q}\frac{\mathrm{d}q(x)}{\mathrm{d}x},\quad \forall x\in\Omega \tag{11}$$

and

$$\frac{\mathrm{d}q(x)}{\mathrm{d}x}\geq 0,\quad \forall x\in\Omega. \tag{12}$$

Proof By Assumption ii, $V(q(x),\tau(q(x)),r(x),x)=\pi(q(x),x)-\tau(q(x))-r(x)$. Let $L(x,y)=\pi(q(y),x)-\tau(q(y))-r(y)$, which denotes the welfare of the consumer with the taste x but choosing the mechanism $(q(y),r(y))$, where $x,y\in\Omega$ and $x\neq y$. Thus, for any given x, Inequality (7) can be written as

$$L(x,x)\geq L(x,y),\quad \forall y\in\Omega,$$

which means that $L(x,y)$ obtains its maximal value at (x,x), i.e., a consumer with taste x has no incentive to pretend to be a that with taste y, $y\neq x$. Thus, $L(x,y)$ satisfies the first-order condition $\frac{\partial L(x,y)}{\partial y}|_{y=x}=0$ and the second-order condition $\frac{\partial^2 L(x,y)}{\partial y^2}|_{y=x}\leq 0$. It follows from the first-order condition that

$$\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\frac{\mathrm{d}q(x)}{\mathrm{d}x}+\frac{\mathrm{d}r(x)}{\mathrm{d}x}=\frac{\partial\pi(q(x),x)}{\partial q}\frac{\mathrm{d}q(x)}{\mathrm{d}x},\quad \forall x\in\Omega. \tag{13}$$

By differentiating (13) with respect to x,

$$\begin{aligned}&\frac{\mathrm{d}^2\tau(q(x))}{\mathrm{d}q^2}\left(\frac{\mathrm{d}q(x)}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\frac{\mathrm{d}^2q(x)}{\mathrm{d}x^2}+\frac{\mathrm{d}^2r(x)}{\mathrm{d}x^2}\\&=\frac{\partial^2\pi(q(x),x)}{\partial q^2}\left(\frac{\mathrm{d}q(x)}{\mathrm{d}x}\right)^2+\frac{\partial\pi(q(x),x)}{\partial q}\frac{\mathrm{d}^2q(x)}{\mathrm{d}x^2}\\&\quad+\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}\frac{\mathrm{d}q(x)}{\mathrm{d}x},\quad \forall x\in\Omega.\end{aligned} \tag{14}$$

By the second-order condition and Assumptions ii and v, we can obtain

$$\begin{aligned}&\frac{\partial^2\pi(q(x),x)}{\partial q^2}\left(\frac{\mathrm{d}q(x)}{\mathrm{d}x}\right)^2+\frac{\partial\pi(q(x),x)}{\partial q}\frac{\mathrm{d}^2q(x)}{\mathrm{d}x^2}\\&\quad-\frac{\mathrm{d}^2\tau(q(x))}{\mathrm{d}q^2}\left(\frac{\mathrm{d}q(x)}{\mathrm{d}x}\right)^2-\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\frac{\mathrm{d}^2q(x)}{\mathrm{d}x^2}-\frac{\mathrm{d}^2r(x)}{\mathrm{d}x^2}\leq 0,\quad \forall x\in\Omega.\end{aligned} \tag{15}$$

Applying (14) to (15) yields

$$-\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}\frac{\mathrm{d}q(x)}{\mathrm{d}x}\leq 0,\quad \forall x\in\Omega. \tag{16}$$

Note that $\frac{\partial^2 \pi(q(x),x)}{\partial q \partial x} > 0$ by Assumption ii, thus, $\frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0, \forall x \in \Omega$. That is, (7) $\Rightarrow$ (11) and (12).

On the other hand, by $\frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0$ and $\frac{\partial^2 \pi(q,x)}{\partial q \partial x} > 0$ and integrating (11) yields

$$\begin{aligned}\tau(q(x)) - \tau(q(y)) + r(x) - r(y) &= -\int_x^y \frac{\partial \pi(q(s), s)}{\partial q} \frac{\mathrm{d}q(s)}{\mathrm{d}s} \mathrm{d}s \\ &\leq -\int_x^y \frac{\partial \pi(q(s), x)}{\partial q} \frac{\mathrm{d}q(s)}{\mathrm{d}s} \mathrm{d}s = \pi(q(x), x) \\ &\quad - \pi(q(y), x)\end{aligned}$$

when $y > x$; and

$$\begin{aligned}\tau(q(x)) - \tau(q(y)) + r(x) - r(y) &= \int_y^x \frac{\partial \pi(q(s), s)}{\partial q} \frac{\mathrm{d}q(s)}{\mathrm{d}s} \mathrm{d}s \\ &\leq \int_y^x \frac{\partial \pi(q(s), x)}{\partial q} \frac{\mathrm{d}q(s)}{\mathrm{d}s} \mathrm{d}s = \pi(q(x), x) \\ &\quad - \pi(q(y), x)\end{aligned}$$

when $y < x$. Therefore, the incentive constraint (7) is satisfied. That is, (11) and (12) $\Rightarrow$ (7). Therefore, the proof of the proposition is complete. □

Remark 3 The incentive constraint can be written as its first-order condition because, actually, the incentive constraint is an optimization problem with respect to the consumer's taste; moreover, the inequality $\frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0$ is a control constraint inequality in the equivalent model of (9) in the following subsection.

Proposition 2 *The participation constraint* (8) *can be written as*

$$r(0) = \pi(q(0), 0) - \tau(q(0)). \tag{17}$$

Proof By Assumption ii, $V(q(x), \tau(q(x)), r(x), x) = \pi(q(x), x) - \tau(q(x)) - r(x)$, hence,

$$\begin{aligned}\frac{\mathrm{d}V(q(x), \tau(q(x)), r(x), x)}{\mathrm{d}x} &= \frac{\partial \pi(q(x), x)}{\partial q} \frac{\mathrm{d}q(x)}{\mathrm{d}x} + \frac{\partial \pi(q(x), x)}{\partial x} \\ &\quad - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} \frac{\mathrm{d}q(x)}{\mathrm{d}x} - \frac{\mathrm{d}r(x)}{\mathrm{d}x}.\end{aligned} \tag{18}$$

It follows from (11) that

$$\frac{\mathrm{d}V(q(x), \tau(q(x)), r(x), x)}{\mathrm{d}x} = \frac{\partial \pi(q(x), x)}{\partial x} \geq 0. \tag{19}$$

That means that $V(q(x), \tau(q(x)), r(x), x)$ is increasing with respect to x. Consequently, the participation constraint (8) is equivalent to

$$V(q(0), \tau(q(0)), r(0), 0) = \pi(q(0), 0) - \tau(q(0)) - r(0) \geq 0. \tag{20}$$

In fact, the constraint (20) is binding under the optimal mechanism. Since for any feasible mechanism $(q(\cdot), r(\cdot))$ of Model (9), a new mechanism $(q(\cdot), r^*(\cdot))$ can be established, where $r^*(0) = \pi(q(0), 0) - \tau(q(0))$ and

$$\frac{\mathrm{d}r^*(x)}{\mathrm{d}x} = \frac{\mathrm{d}r(x)}{\mathrm{d}x}.$$

It is easy to testify that $(q(\cdot), r^*(\cdot))$ is also feasible for Model (9) and $r^*(x) \geq r(x)$ for all $x \in \Omega$. Since $\frac{\mathrm{d}U}{\mathrm{d}r} = 1 > 0$, i.e., the monopolist's welfare is strictly increasing with respect to r, hence,

$$U(q(x), r^*(x)) \geq U(q(x), r(x)),$$

which means that the monopolist will choose the biggest transfer satisfying the participation constraint. Thus, an optimal mechanism should satisfy

$$r(0) = \pi(q(0), 0) - \tau(q(0)).$$

□

Remark 4 Equation (17) can be viewed as a boundary condition in the equivalent model of (9) in the following subsection.

Proposition 3 *The objective function of the lower level problem of Model* (9) *can be written as*

$$E[U(q(\xi), r(\xi))] = \int_0^b \left[\pi(q(x), x) - \tau(q(x)) - cq(x) - \left(\frac{1 - \Phi(x)}{\phi(x)}\right) \frac{\partial \pi(q(x), x)}{\partial x}\right] \phi(x)\mathrm{d}x. \tag{21}$$

Proof By Assumption iii, the monopolist's welfare $U(q(x), r(x)) = r(x) - cq(x), \forall x \in \Omega$. Moreover, integrating (11) yields

$$\tau(q(x)) + r(x) = \pi(q(x), x) - \pi(q(0), 0) - \int_0^x \frac{\partial \pi(q(s), s)}{\partial s}\mathrm{d}s + \tau(q(0)) + r(0). \tag{22}$$

Therefore,

$$r(x) = \pi(q(x), x) - \tau(q(x)) - \int_0^x \frac{\partial \pi}{\partial s}(q(s), s)\mathrm{d}s - \pi(q(0), 0) + \tau(q(0)) + r(0). \tag{23}$$

By Eq. (17), $r(0) = \pi(q(0), 0) - \tau(q(0))$, the transfer

$$r(x) = \pi(q(x), x) - \tau(q(x)) - \int_0^x \frac{\partial \pi}{\partial s}(q(s), s)\mathrm{d}s. \tag{24}$$

Substituting (24) into the monopolist's welfare $U(q(x), r(x)) = r(x) - cq(x)$ yields

$$U(q(x), r(x)) = \pi(q(x), x) - \tau(q(x)) - cq(x) - \int_0^x \frac{\partial \pi}{\partial s}(q(s), s)\mathrm{d}s, \quad \forall x \in \Omega. \tag{25}$$

Since

$$\frac{\mathrm{d}U(q(x), r(x))}{\mathrm{d}x} = \left[\frac{\partial \pi(q(x), x)}{\partial q} - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} - c\right]\frac{\mathrm{d}q(x)}{\mathrm{d}x},$$

it follows from $\frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0$ and $\frac{\partial \pi(q(x),x)}{\partial q} - c - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} \geq 0$ in Assumption v that $\frac{\mathrm{d}U(q(x),r(x))}{\mathrm{d}x} \geq 0$, i.e., the monopolist's welfare is increasing with respect to x. It follows from Lemma 1 that the monopolist's expected welfare can be written as

$$E[U(q(\xi), r(\xi))] = \int_0^b \left[\pi(q(x), x) - \tau(q(x)) - cq(x) - \int_0^x \frac{\partial \pi}{\partial s}(q(s), s)\mathrm{d}s\right]\phi(x)\mathrm{d}x. \tag{26}$$

Integrating (26) by parts yields

$$E[U(q(\xi), r(\xi))] = \int_0^b \left[\pi(q(x), x) - \tau(q(x)) - cq(x) - \left(\frac{1 - \Phi(x)}{\phi(x)}\right)\frac{\partial \pi(q(x), x)}{\partial x}\right]\phi(x)\mathrm{d}x. \tag{27}$$

Therefore, the proof of the proposition is complete. □

Proposition 4 *For any given function $\tau(q)$, the lower level problem of Model (9) can be written as*

$$\begin{cases} \max_{q(\cdot)} \int_0^b \left[\pi(q(x),x) - \tau(q(x)) - cq(x) - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial\pi(q(x),x)}{\partial x}\right]\phi(x)\mathrm{d}x \\ \text{subject to:} \\ \quad \frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0, \quad \forall x \in \Omega. \end{cases} \tag{28}$$

Proof Note that Eq. (21) has nothing to do with $r(\cdot)$, thus, the decision vector $(q(\cdot), r(\cdot))$ is simplified as $q(\cdot)$. According to Propositions 1–3, it is obvious that the lower level problem of Model (9) is equivalent to Model (28). □

Proposition 5 *The function $q(x)$ is the optimal solution of Model (28) if and only if*

$$\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} = \frac{\partial\pi(q(x),x)}{\partial q} - c - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}. \tag{29}$$

Proof First, the concavity of the objective function in Model (28) is proved. Since the second variation of $E[U(q(\xi), r(\xi))]$ with respect to q, i.e., $\delta^2 E[U(q(\xi), r(\xi))]$ is

$$\frac{1}{2}\int_0^b \left\{\left[\frac{\partial^2\pi(q(x),x)}{\partial q^2} - \frac{\mathrm{d}^2\tau(q(x))}{\mathrm{d}q^2} - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q^2\partial x}\right]\phi(x)\right\}(\delta q)^2\mathrm{d}x,$$

according to Assumption ii, $\delta^2 E[U(q(\xi), r(\xi))] \leq 0$, i.e., $E[U(q(\xi), r(\xi))]$ is concave with respect to q. In order to maximize the monopolist's welfare, it follows from the first-order condition, i.e., $\delta E[U(q(\xi), r(\xi))] = 0$ that

$$\int_0^b \left\{\left[\frac{\partial\pi(q(x),x)}{\partial q} - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} - c - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}\right]\phi(x)\right\}\delta q\mathrm{d}x = 0, \tag{30}$$

since $\phi(x) > 0$, Eq. (29) $\frac{\partial\pi(q(x),x)}{\partial q} - \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q} - c - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x} = 0$ obviously holds.

On the other hand, Eq. (29) also means that the constraint $\frac{\mathrm{d}q(x)}{\mathrm{d}x} \geq 0$ holds. Since the derivation of (29) with respect to x is

$$\begin{aligned} &\frac{\partial\pi^2(q(x),x)}{\partial q^2}\frac{\mathrm{d}q(x)}{\mathrm{d}x} + \frac{\partial^2\pi(q(x),x)}{\partial q\partial x} - \frac{\mathrm{d}^2\tau(q(x))}{\mathrm{d}q^2}\frac{\mathrm{d}q(x)}{\mathrm{d}x} \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x} \\ &\quad - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q^2\partial x}\frac{\mathrm{d}q(x)}{\mathrm{d}x} - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q\partial x^2} = 0, \end{aligned}$$

thus,

$$\left[\frac{\partial\pi^2(q(x),x)}{\partial q^2}-\frac{\mathrm{d}^2\tau(q(x))}{\mathrm{d}q^2}-\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q^2\partial x}\right]\frac{\mathrm{d}q(x)}{\mathrm{d}x}$$
$$=-\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x}$$
$$+\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q\partial x^2},$$

it follows from Assumptions ii and vii that $\frac{\mathrm{d}q(x)}{\mathrm{d}x}\geq 0, \forall x\in\Omega$. Therefore, the proof of the proposition is complete. □

Proposition 6 *The objective function of Model (9) can be written as*

$$E[S(q(\xi),\tau(q(\xi)),r(\xi),\xi)]$$
$$=E[V(q(\xi),\tau(q(\xi)),r(\xi),\xi)+\alpha U(q(\xi),r(\xi))+\beta\tau(q(\xi))]$$
$$=\int_0^b\Bigg[(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial\pi(q(x),x)}{\partial x}+\alpha[\pi(q(x),x)-cq(x)$$
$$-\tau(q(x))]+\beta\tau(q(x))\Bigg]\phi(x)\mathrm{d}x. \tag{31}$$

Proof *It follows from Eqs. (22) and (17) that*

$$V(q(x),\tau(q(x)),r(x),x)=\pi(q(x),x)-\tau(q(x))-r(x)$$
$$=\int_0^x\frac{\partial\pi}{\partial s}(q(s),s)\mathrm{d}s,\quad\forall x\in\Omega. \tag{32}$$

According to Eqs. (25) and (32), the social welfare $S(q(x),\tau(q(x)),r(x),x)$ can be written as

$$S(q(x),\tau(q(x)),r(x),x)=V(q(x),\tau(q(x)),r(x),x)+\alpha U(q(x),r(x))$$
$$+\beta\tau(q(x))=\Bigg[(1-\alpha)\int_0^x\frac{\partial\pi}{\partial s}(q(s),s)\mathrm{d}s+\alpha[\pi(q(x),x)$$
$$-cq(x)-\tau(q(x))]+\beta\tau(q(x))\Bigg]. \tag{33}$$

Differentiating Eq. (33) with respect to x yields

$$\begin{aligned}
&\frac{\mathrm{d}S(q(x),\tau(q(x)),r(x),x)}{\mathrm{d}x}\\
&=\frac{\mathrm{d}\left[(1-\alpha)\int_0^x \frac{\partial\pi}{\partial s}(q(s),s)\mathrm{d}s+\alpha[\pi(q(x),x)-cq(x)-\tau(q(x))]+\beta\tau(q(x))\right]}{\mathrm{d}x}\\
&=\left[\alpha\left(\frac{\partial\pi(q(x),x)}{\partial q}-c-\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\right)+\beta\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\right]\frac{\mathrm{d}q(x)}{\mathrm{d}x}+\frac{\partial\pi(q(x),x)}{\partial x}.
\end{aligned} \tag{34}$$

Since $\frac{\mathrm{d}q(x)}{\mathrm{d}x}\geq 0$ *from Proposition 1,* $\alpha\left(\frac{\partial\pi(q(x),x)}{\partial q}-c-\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\right)+\beta\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}\geq 0$ *in Assumption vi, and* $\frac{\partial\pi(q(x),x)}{\partial x}\geq 0$ *in Assumption ii, then* $\frac{\mathrm{d}S(q(x),\tau(q(x)),r(x),x)}{\mathrm{d}x}\geq 0$*, i.e., the social welfare is increasing with respect to x. It follows from Lemma 1 and integrates by parts that Eq. (31) holds.* □

Theorem 1 *Model (9) is equivalent to*

$$\begin{cases}
\max_{\tau(q(\cdot))}\displaystyle\int_0^b\left[(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial\pi(q(x),x)}{\partial x}\right.\\
\quad\left.+\alpha[\pi(q(x),x)-cq(x)-\tau(q(x))]+\beta\tau(q(x))\right]\phi(x)\mathrm{d}x\\
\text{subject to:}\\
\frac{\mathrm{d}\tau(q(x))}{\mathrm{d}q}=\frac{\partial\pi(q(x),x)}{\partial q}-c-\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q\partial x},\quad\forall x\in\Omega.
\end{cases} \tag{35}$$

Proof It is easy to verify the theorem according to Propositions 1–6. □

4.2 Necessary conditions for the optimal taxation problem

Let us rewrite this problem as a control problem as follows. It follows from Proposition 5 that Eq. (29) implies that the inequality $\mathrm{d}q(x)/\mathrm{d}x\geq 0$ holds, thus, it is reasonable to add $\mathrm{d}q(x)/\mathrm{d}x\geq 0$ into the control problem. We set q and τ as the state variables. Furthermore, let $u(x)=\mathrm{d}q(x)/\mathrm{d}x$ be the control variable and $\mathrm{d}\tau(q(x))/\mathrm{d}x=\mathrm{d}\tau(q(x))/\mathrm{d}q\times\mathrm{d}q(x)/\mathrm{d}x$. Note that it is quite reasonable to add an upper bound to $\mathrm{d}q(x)/\mathrm{d}x$, i.e., $0\leq\mathrm{d}q(x)/\mathrm{d}x\leq M$, which means the rate of the goods' quantity that the consumer purchases increases with his taste should be under a certain level. Thus, Model (35) can be written as:

$$\begin{cases} \max_{u(\cdot)} \int\limits_0^b \left[(1-\alpha) \left(\dfrac{1-\Phi(x)}{\phi(x)} \right) \dfrac{\partial \pi(q(x),x)}{\partial x} \right. \\ \quad \left. +\alpha[\pi(q(x),x) - cq(x) - \tau(q(x))] + \beta\tau(q(x)) \right] \phi(x) \mathrm{d}x \\ \text{subject to:} \\ \quad \frac{\mathrm{d}q(x)}{\mathrm{d}x} = u(x), \quad \forall x \in \Omega \\ \quad \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}x} = u(x) \left[\frac{\partial \pi(q(x),x)}{\partial q} - c - \left(\frac{1-\Phi(x)}{\phi(x)} \right) \frac{\partial^2 \pi(q(x),x)}{\partial q \partial x} \right], \quad \forall x \in \Omega \\ \quad 0 \le u(x) \le M, \quad \forall x \in \Omega \\ \quad q(0) = 0 \\ \quad \tau(0) = 0. \end{cases} \tag{36}$$

Theorem 2 *If $q^*(x)$, $\tau^*(q^*(x))$ and $u^*(x)$ are the optimal solutions to Model (36), then there exist adjoint states $\lambda(x)$ and $\gamma(x)$ such that*

$$\begin{cases} H(q^*(x), \tau^*(q^*(x)), u^*(x), \lambda(x), \gamma(x), x) \\ \quad = \max\limits_{0 \le u(x) \le M} H(q^*(x), \tau^*(q^*(x)), u(x), \lambda(x), \gamma(x), x) \\ \frac{\mathrm{d}q(x)}{\mathrm{d}x} = u(x), \quad \forall x \in \Omega \\ \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}x} = u(x) \left[\frac{\partial \pi(q(x),x)}{\partial q} - c - \left(\frac{1-\Phi(x)}{\phi(x)} \right) \frac{\partial^2 \pi(q(x),x)}{\partial q \partial x} \right], \quad \forall x \in \Omega \\ \frac{\mathrm{d}\lambda}{\mathrm{d}x} = -\left[(1-\alpha) \left(\frac{1-\Phi(x)}{\phi(x)} \right) \frac{\partial^2 \pi(q(x),x)}{\partial q \partial x} + \alpha \left[\frac{\partial \pi(q(x),x)}{\partial q} - c \right] \right. \\ \qquad \left. +(\beta-\alpha) \left[\frac{\partial \pi(q(x),x)}{\partial q} - c - \left(\frac{1-\Phi(x)}{\phi(x)} \right) \frac{\partial^2 \pi(q(x),x)}{\partial q \partial x} \right] \right] \phi(x) \\ \qquad -\gamma(x) u(x) \left[\frac{\partial^2 \pi(q(x),x)}{\partial q^2} - \left(\frac{1-\Phi(x)}{\phi(x)} \right) \frac{\partial^3 \pi(q(x),x)}{\partial q^2 \partial x} \right] \\ \gamma(x) = (\beta-\alpha)(1-\Phi(x)) \\ \lambda(b) = 0 \\ \gamma(b) = 0 \\ \tau(0) = 0 \\ q(0) = 0. \end{cases} \tag{37}$$

Proof The Hamiltonian

$$\begin{aligned} H(q, \tau(q), u, \lambda, \gamma, x) = & \left\{ \frac{1}{2}(1-\alpha) \left(\frac{1-\Phi(x)}{\phi(x)} \right) q(x) \right. \\ & \left. +\alpha \left(x + \frac{1}{2} x q(x) - cq(x) - \tau(q(x)) \right) + \beta\tau(q(x)) \right\} \phi(x) \\ & +\lambda(x) u(x) + \gamma(x) u(x) \left(\frac{1}{2} x - c - \frac{1-\Phi(x)}{2\phi(x)} \right) \end{aligned} \tag{38}$$

where $\lambda(x)$ and $\gamma(x)$ are adjoint states.

According to Pontryagin maximum principle Casas et al. (2001), if $u^*(x)$ and $\tau^*(q^*(x))$ are the optimal solutions to the problem (36), then there exist adjoint states $\lambda(x)$ and $\gamma(x)$ such that the following conditions hold.

(1) The canonical differential equations of the system are:

$$\begin{cases} \frac{\mathrm{d}\lambda}{\mathrm{d}x} = -\frac{\partial H}{\partial q} \\ \frac{\mathrm{d}\gamma}{\mathrm{d}x} = -\frac{\partial H}{\partial \tau}. \end{cases} \tag{39}$$

(2) The boundary conditions of the system are:

$$\begin{cases} q(0) = 0 \\ \tau(0) = 0 \\ \lambda(b) = 0 \\ \gamma(b) = 0. \end{cases} \tag{40}$$

(3) $u^*(x)$ maximizes the Hamiltonian (38) over $0 \leq u(x) \leq M$ for all x, i.e.,

$$\begin{aligned} &H(q^*(x), \tau^*(q^*(x)), u^*(x), \lambda(x), \gamma(x), x) \\ &\quad = \max_{0 \leq u(x) \leq M} H(q^*(x), \tau^*(q^*(x)), u(x), \lambda(x), \gamma(x), x). \end{aligned} \tag{41}$$

According to the canonical differential equations (39), we can obtain

$$\begin{aligned} \frac{\mathrm{d}\lambda}{\mathrm{d}x} = &-\left[(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q \partial x} + \alpha\left[\frac{\partial \pi(q(x),x)}{\partial q} - c\right]\right. \\ &\left. +(\beta-\alpha)\left[\frac{\partial \pi(q(x),x)}{\partial q} - c - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^2\pi(q(x),x)}{\partial q \partial x}\right]\right]\phi(x) \\ &-\gamma(x)u(x)\left[\frac{\partial^2\pi(q(x),x)}{\partial q^2} - \left(\frac{1-\Phi(x)}{\phi(x)}\right)\frac{\partial^3\pi(q(x),x)}{\partial q^2 \partial x}\right]. \end{aligned} \tag{42}$$

It is similar that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}x} = -\frac{\partial H}{\partial \tau} = (\alpha - \beta)\phi(x). \tag{43}$$

Considering the boundary condition $\gamma(b) = 0$, we get

$$\gamma(x) = (\beta - \alpha)(1 - \Phi(x)). \tag{44}$$

Thus, the proof of the theorem is complete. □

5 Numerical example

In this section, we present a numerical example to obtain the necessary conditions of the solutions for the fuzzy optimal nonlinear taxation model by analytical methods. Suppose that the consumer's utility function $\pi(q, x) = x + \frac{1}{2}xq$ and the welfare of the monopolist $U(q, r) = r - cq$, thus, the welfare of the consumer $V(q, \tau, r, x) =$

$x+\frac{1}{2}xq-\tau(q)-r$. Since the government and the monopolist have the same subjective assessment of the consumer's taste and consider it as a fuzzy variable ξ with a support $[0, b]$, where b denotes the consumer's highest taste to the price.

Without loss of generality, using a triangular fuzzy variable $\xi = (0, a, b)$ to denote the government's and the monopolist's subjective assessment of the consumer's taste with the credibility density function

$$\phi(x) = \begin{cases} \frac{1}{2a}, & \text{if } \ 0 \le x < a \\ \frac{1}{2(b-a)}, & \text{if } \ a \le x \le b \\ 0, & \text{otherwise,} \end{cases}$$

and the distribution function

$$\Phi(x) = \begin{cases} \frac{x}{2a}, & \text{if } \ 0 \le x < a \\ \frac{x+b-2a}{2(b-a)}, & \text{if } \ a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

Note that $a \to b$ means that the consumer's taste is very high and he has strong desire to purchase the goods; similarly, $a \to 0$ means that the consumer's taste is very low and he hardly has any desire to purchase the goods. $a \to \frac{b}{2}$ means that the consumer's taste is moderate and he hesitates to purchase the goods, he may purchase many or few.

Thus, the government's optimal taxation problem can be formulated as follows:

$$\begin{cases} \max_{u(\cdot)} \int_0^b \left[\frac{1}{2}(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right) q(x) \right. \\ \left. +\alpha\left(x+\frac{1}{2}xq(x)-cq(x)-\tau(q(x))\right)+\beta\tau(q(x))\right]\phi(x)\mathrm{d}x \\ \text{subject to:} \\ \quad \frac{\mathrm{d}q(x)}{\mathrm{d}x} = u(x), \quad \forall x \in \Omega \\ \quad \frac{\mathrm{d}\tau(q(x))}{\mathrm{d}x} = u(x)\left(\frac{1}{2}x-c-\frac{1-\Phi(x)}{2\phi(x)}\right), \quad \forall x \in \Omega \\ \quad 0 \le u(x) \le M, \quad \forall x \in \Omega \\ \quad q(0) = 0 \\ \quad \tau(0) = 0. \end{cases} \tag{45}$$

The Hamiltonian

$$\begin{aligned} H(q, \tau(q), u, \lambda, \gamma, x) = &\left\{\frac{1}{2}(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right)q(x)\right. \\ &+\alpha\left(x+\frac{1}{2}xq(x)-cq(x)-\tau(q(x))\right) \\ &\left.+\beta\tau(q(x))\right\}\phi(x)+\lambda(x)u(x)+\gamma(x)u(x)\left(\frac{1}{2}x-c-\frac{1-\Phi(x)}{2\phi(x)}\right) \end{aligned} \tag{46}$$

where $\lambda(x)$ and $\gamma(x)$ are adjoint states.

From the necessary conditions derived from the Pontryagin maximum principle, it is known that the canonical system of differential equations is as follows:

$$\begin{cases} \dfrac{d\lambda}{dx} = -\dfrac{\partial H}{\partial q} \\ \dfrac{d\gamma}{dx} = -\dfrac{\partial H}{\partial \tau}. \end{cases} \tag{47}$$

It follows from (46) and (47) that

$$\frac{d\lambda}{dx} = -\left[\frac{1}{2}(1-\alpha)\left(\frac{1-\Phi(x)}{\phi(x)}\right) + \alpha\left(\frac{1}{2}x - c\right) + (\beta-\alpha)\left(\frac{1}{2}x - c - \frac{1-\Phi(x)}{2\phi(x)}\right)\right]\phi(x).$$

Since $\lambda(b) = 0$, we can obtain

$$\begin{aligned} \lambda(x) &= \lambda(b) - \int_x^b \frac{d\lambda}{dx} dx \\ &= \begin{cases} \int_x^b \left\{\left[\frac{1}{2}(1-\alpha)(2a-y) + \alpha\left(\frac{1}{2}y - c\right) + (\beta-\alpha)\left(\frac{1}{2}y - c - \frac{1}{2}(2a-y)\right)\right] \times \frac{1}{2a}\right\} dy, & \text{if } 0 \le x < a \\ \int_x^b \left\{\left[\frac{1}{2}(1-\alpha)(b-y) + \alpha\left(\frac{1}{2}y - c\right) + (\beta-\alpha)\left(\frac{1}{2}y - c - \frac{1}{2}(b-y)\right)\right] \times \frac{1}{2(b-a)}\right\} dy, & \text{if } a \le x \le b \end{cases} \\ &= \begin{cases} \frac{1}{2a}\left[\left(\frac{\beta}{2} - \frac{1}{4}\right)(b^2 - x^2) + ((1-\beta)a - \beta c)(b-x)\right], & \text{if } 0 \le x < a \\ \frac{1}{2(b-a)}\left[\left(\frac{\alpha}{2} - \frac{1}{4}\right)(b^2 - x^2) + \left(\frac{1}{2}(1-\beta)b - \beta c\right)(b-x)\right], & \text{if } a \le x \le b. \end{cases} \end{aligned} \tag{48}$$

$$\frac{d\gamma}{dx} = (\alpha-\beta)\phi(x) = \begin{cases} \dfrac{\alpha-\beta}{2a}, & \text{if } 0 \le x < a \\ \dfrac{\alpha-\beta}{2(b-a)}, & \text{if } a \le x \le b. \end{cases} \tag{49}$$

According to the boundary condition $\gamma(b) = 0$, we get

$$\gamma(x) = (\beta-\alpha)(1-\Phi(x)) = \begin{cases} (\beta-\alpha)\left(1 - \frac{x}{2a}\right), & \text{if } 0 \le x < a \\ (\beta-\alpha)\left(1 - \frac{x+b-2a}{2(b-a)}\right), & \text{if } a \le x \le b. \end{cases} \tag{50}$$

According to (48) and (50), when $0 \leq x < a$,

$$\begin{aligned}
&\lambda(x) + \gamma(x)\left(\frac{1}{2}x - c - \frac{1-\Phi(x)}{2\phi(x)}\right) \\
&= -\frac{1}{2a}\left(\frac{\beta}{2} - \frac{1}{4}\right)x^2 - \frac{1}{2a}[(1-\beta)a - \beta c + (\beta - \alpha)]x \\
&+\beta - \alpha + \frac{1}{2a}\left[\left(\frac{\beta}{2} - \frac{1}{4}\right)b^2 + [(1-\beta)a - \beta c + (\beta - \alpha)]b\right] \\
&\triangleq D_1(x),
\end{aligned} \tag{51}$$

and when $a \leq x \leq b$,

$$\begin{aligned}
&\lambda(x) + \gamma(x)\left(\frac{1}{2}x - c - \frac{1-\Phi(x)}{2\phi(x)}\right) = -\frac{1}{2(b-a)}\left(\frac{\alpha}{2} - \frac{1}{4}\right)x^2 \\
&-\frac{1}{2(b-a)}\left[\left(\frac{1}{2}(1-\beta)b - \beta c\right) + (\beta - \alpha)\right]x + (\beta - \alpha)\left(1 - \frac{b-2a}{2(b-a)}\right) \\
&+\frac{1}{2(b-a)}\left[\left(\frac{\alpha}{2} - \frac{1}{4}\right)b^2 + \left[\left(\frac{1}{2}(1-\beta)b - \beta c\right) + (\beta - \alpha)\right]b\right] \triangleq D_2(x).
\end{aligned} \tag{52}$$

It follows from the symbols of the quadratic functions $D_1(x)$ and $D_2(x)$ and Theorem 2 that the optimal mechanism satisfies:

$$\begin{cases}
q^*(x) = 0,\ \tau^*(q(x)) = 0, & \text{if } \alpha \in [0,1],\ x = 0 \\
q^*(x) = 0,\ \tau^*(q(x)) = 0, & \text{if } \alpha \in [0,1], 0 < x < a, \\
 & (1-\beta)a - \beta c + \beta - \alpha + \beta b = \frac{b}{2} \\
q^*(x) = Mx, & \\
\tau^*(q(x)) = Mx^2 - \left(c + \frac{b}{2}\right)Mx, & \text{if } \alpha \in [0, \frac{1}{2}),\ a \leq x < b, \\
 & -\frac{1}{2}\beta b - \beta c + \beta - \alpha + \alpha b = 0 \\
q^*(x) = 0,\ \tau^*(q(x)) = 0, & \text{if } \alpha \in (\frac{1}{2}, 1],\ a \leq x < b, \\
 & -\frac{1}{2}\beta b - \beta c + \beta - \alpha + \alpha b = 0 \\
q^*(x) = 0,\ \tau^*(q(x)) = 0, & \text{if } \alpha = \frac{1}{2},\ a \leq x < b, \\
 & n_2 a + z_2 < 0,\ n_2 b + z_2 < 0 \\
q^*(x) = Mx, & \\
\tau^*(q(x)) = Mx^2 - \left(c + \frac{b}{2}\right)Mx, & \text{if } \alpha = \frac{1}{2},\ a \leq x < b, \\
 & n_2 a + z_2 > 0,\ n_2 b + z_2 > 0 \\
q^*(x) = Mx, & \\
\tau^*(q(x)) = Mx^2 - \left(c + \frac{b}{2}\right)Mx, & \text{if } x = b,\ 2a - b > 0 \\
q^*(x) = 0,\ \tau^*(q(x)) = 0, & \text{if } x = b,\ 2a - b < 0 \\
q^*(x) = 0,\ \tau^*(q(x)) = 0, \quad \text{or} & \\
q^*(x) = Mx, & \\
\tau^*(q(x)) = Mx^2 - \left(c + \frac{b}{2}\right)Mx, & \text{if } x = b,\ 2a - b = 0.
\end{cases}$$

The obtained conclusions can be explained as follows:

(1) It is obvious that if $x = 0$, i.e., the consumer's has no taste about the goods, the optimal purchased quantity $q^*(x) = 0$ in this case, which means that the consumer is totally unwilling to purchase the goods.
(2) If $a \to 0$, i.e., the consumer's taste is very low, the optimal purchased quantity $q^*(x) = 0$, which means the consumer hardly has any desire to purchase the goods.
(3) If $\alpha \in [0, 1/2)$ and $a \to b$, i.e., the consumer's taste is very high and the weight of the monopolist's welfare in the social welfare is not high, which means the consumer may have more surplus, the optimal purchased quantity $q^*(x) = Mx$, he is happy to buy the goods as many as possible; conversely, If $\alpha \in (1/2, 1]$ and $a \to b$, the consumer's surplus is few, the optimal purchased quantity $q^*(x) = 0$, hence, the consumer is reluctant to purchase the goods; naturally, when $\alpha = 1/2$, the optimal purchased quantity $q^*(x) = Mx$ or 0, the consumer hesitates to purchase the goods, he may purchase many or none.
(4) If $x = b$, i.e., the consumer's true taste is greatly high, if $a > b/2$, which means the consumer's taste which is assessed by the government and the monopolist is higher than the average taste, in other words, the government's and the monopolist's subjective assessment of the consumer's taste is consistent with the consumer's true taste, hence, the consumer is willing to purchase as many as goods, the optimal purchased quantity $q^*(x) = Mx$; similarly, if the consumer's taste is lower than the average taste, i.e., $a < b/2$, the government's and the monopolist's subjective assessment of the consumer's taste is not consistent with the consumer's true taste, the consumer is unwilling to purchase any goods, the optimal purchased quantity $q^*(x) = 0$; if $a = b/2$, the optimal purchased quantity $q^*(x) = Mx$ or 0, the consumer hesitates to purchase the goods, he may purchase many or none.

In a word, the results illustrate that the consumer's purchased quantity not only relates with the consumer's taste, but also depends on the structure of the social welfare.

6 Conclusions

This paper interpreted a novel bilevel fuzzy principal-agent model for an optimal nonlinear taxation problem in which the government and the monopolist face the consumer with unknown taste denoted by a fuzzy variable. A fuzzy bilevel optimal nonlinear taxation model was set up with the purpose of maximizing the expected social welfare and the monopolist's expected welfare. The equivalent model for the bilevel fuzzy optimal nonlinear taxation model was presented and the Pontryagin maximum principle was adopted to obtain the necessary conditions of the solutions for the fuzzy bilevel optimal nonlinear taxation model, and one numerical example was given to illustrate the effectiveness of the proposed model. The results demonstrate that the consumer's purchased quantity not only relates with the consumer's taste, but also depends on the structure of the social welfare. The future work includes that the study of the multi-consumer optimal taxation problems.

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