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Extension of the VIKOR method for group decision making with interval-valued intuitionistic fuzzy information

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Abstract The aim of this paper is to extend the VIKOR method for multiple attribute group decision making in interval-valued intuitionistic fuzzy environment, in which all the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by interval-valued intuitionistic fuzzy number, and the information about attribute weights is partially known, which is an important research field in decision science and operation research. First, we use the interval-valued intuitionistic fuzzy hybrid geometric operator to aggregate all individual interval-valued intuitionistic fuzzy decision matrices provided by the decision-makers into the collective interval-valued intuitionistic fuzzy decision matrix, and then we use the score function to calculate the score of each attribute value and construct the score matrix of the collective interval-valued intuitionistic fuzzy decision matrix. From the score matrix and the given attribute weight information, we establish an optimization model to determine the weights of attributes, and then determine the interval-valued intuitionistic positive-ideal solution and interval-valued intuitionistic negative-ideal solution. We use the different distances to calculate the particular measure of closeness of each alternative to the interval-valued intuitionistic positive-ideal solution. According to values of the particular measure, we rank the alternatives and then select the most

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desirable one(s). Finally, a numerical example is used to illustrate the applicability of the proposed approach.

Keywords Multiple attribute group decision making (MAGDM) · Interval-valued intuitionistic fuzzy number (IVIFN) · VIKOR method

1 Introduction

Multiple attribute decision making (MADM) problems (i.e., decision making problems considering several attributes) are widely spread in real life decision situation. A MADM problem is to find a best compromise solution among all feasible alternatives assessed on the basis of multiple attributes, both quantitative and qualitative. Such problems can be dealt with using several existing methods such as VIKOR (VlseKriterijuska Optimizacija I Komoromisno Resenje) and TOPSIS (technique for order preference by similarity to ideal solution) methods, which are well-known MADM methods, developed by Opricovic (1998) and Hwang and Yoon (1992), respectively. The VIKOR and TOPSIS methods are both based on an aggregation function representing "closeness to the ideal". The VIKOR method introduces the ranking index based on the particular measure of "closeness" to the ideal solution. In contrast, the basic principle of the TOPSIS method is that the chosen alternative should have the "shortest distance" from the positive ideal solution (PIS) and the "farthest distance" from negative ideal solution (NIS). The VIKOR method is a compromise ranking approach for multiple criteria decision making (MCDM) problems. It determines a compromise solution, providing a maximum utility for the majority and a minimum regret for the opponent. There exists a large amount of literature involving VIKOR theory and application. For example, Opricovic and Tzeng (2003) suggested using fuzzy logic for the VIKOR method. Tzeng et al. (2005) used and compared the VIKOR and TOPSIS methods in solving a public transportation problem. Büyüközkan and Ruan (2008) extended the VIKOR method to effectively solve software evaluation problem under a fuzzy environment. Opricovic and Tzeng (2007) extended the VIKOR method with a stability analysis determining the weight stability intervals and with trade-offs analysis and compared the extended VIKOR method with three multicriteria decision making methods: TOPSIS, PROMETHEE, and ELECTRE. Sayadi et al. (2009) extended the VIKOR method to MADM problem with interval numbers. Chang and Hsu (2009) showed that the VIKOR method is advantageous for evaluating the relative environmental vulnerability of subdivisions in a watershed. According to a comparative analysis of VIKOR and TOPSIS written by Opricovic and Tzeng (2004), the VIKOR and TOPSIS methods, respectively, use different aggregation functions and different normalization methods. Jahanshaloo et al. (2006a) developed an algorithmic method to extend the TOPSIS for decision making problems with interval data. Yang and Hung (2007) explored the use of TOPSIS in solving a plant layout design problems. Jahanshaloo et al. (2006b) extended the TOPSIS method for decision-making problems with fuzzy data. Chen and Tsao (2008) and Park et al. (2010), respectively, extended the concept of TOPSIS to develop a method for solving MADM problems with interval-valued fuzzy data and interval-valued intuitionistic fuzzy data.

The TOPSIS method is suitable for cautious decision-maker(s), because the decision-maker(s) might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible, whereas the VIKOR method is suitable for those situations in which the decision-maker wants to have maximum profit and the risk of decisions is less important for him. In many complex decision making problems, the preference information provided by decision-maker is often imprecise or uncertain (Klir 2006) due to time pressure, lack of data, or the decision-maker's limited attention and information processing capacities (Xu and Yager 2008). In such cases, it is suitable and convenient to express the decision-maker's preference in interval-valued intuitionistic fuzzy sets (IVIFSs) (Atanassov and Gargov 1989). The fundamental characteristic of the IVIFS is that the values of its membership function and nonmembership function are intervals rather than exact numbers. Therefore, it is necessary and interesting to pay attention to the group decision making problems with interval-valued intuitionistic preference information. Recently, Xu and Yager (2009) developed new similarity measure by analyzing the results of Szmidt and Kacprzky (2004). Then they applied the developed similarity measure for consensus analysis in group decision making based on interval-valued intuitionistic fuzzy preference relations. Xu and Cai (2009) investigated how to extend an incomplete interval-valued intuitionistic fuzzy preference relation to a complete interval-valued intuitionistic fuzzy preference relation, and gave an approach to decision making based on an incomplete interval-valued intuitionistic fuzzy preference relation. Xu (2010) developed a method based on the distance measure for solving interval-valued intuitionistic fuzzy group decision making problems. They first calculated the uncertain intuitionistic fuzzy ideal solution (UIFIS) and then calculated the distance between UIFIS and each preference vector by using the Euclidean distance measure. Then they used the weighted averaging operator to fuse the individual distances into the overall distance corresponding to each alternative, from which the final result can be derived. In this paper, we extend the VIKOR method to solve multiple attribute group decision making (MAGDM) problems in interval-valued intuitionistic fuzzy environment in which all the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by interval-valued intuitionistic fuzzy number (IVIFNs), and the information about attribute weights is partially known. The remaining of this paper is organized as follows. In Sect. 2, we briefly introduce the VIKOR method. Section 3 illustrates interval-valued intuitionistic fuzzy sets (IVIFSs). Section 4 describes the developed VIKOR method to solve MAGDM problems with interval-valued intuitionistic fuzzy information. In Sect. 5, a numerical example is presented to show an application of the extended VIKOR method. Finally, a conclusion is presented in Sect. 6.

2 The VIKOR method

The VIKOR method was proposed as one applicable technique to implement within MCDM problem and developed as a multiple attribute decision making method to solve a discrete decision making problem with non-commensurable (different units) and conflicting criteria. This method focuses on ranking and selecting from a set of alternatives in the presence of conflicting criteria. In some situations, practical problems are characterized by several non-commensurable and conflicting criteria, and there may be no solution satisfying all the criteria simultaneously. So, a compromise solution for a problem with some conflicting criteria can help decision-makers to reach a final solution. The compromise solution is a feasible solution which is the closest to the ideal. The VIKOR method determines the compromise ranking list and the compromise solution by introducing the multiple criteria ranking index based on the particular measure of "closeness" to the "ideal" solution. The multiple criteria measure for compromise ranking is developed from the $L_p - metric$ used as an aggregating function in a compromise programming method.

Denote n alternatives under consideration as $O_1, O_2, \ldots, O_n$, the evaluation criteria as $C_1, C_2, \ldots, C_m$, and the rating of each alternative $O_j (j = 1, 2, \ldots, n)$ with respect to criteria $C_i (i = 1, 2, \ldots, m)$ as f_{ij}. Then the compromise ranking algorithm of the VIKOR method has the following steps:

Step 1. Determine the best rating f_i^+ and the worst rating f_i^- for all the criteria. For example, it the criterion i represents a benefit, then

$$f_i^+ = \max_j f_{ij}, \quad f_i^- = \min_j f_{ij}. \tag{1}$$

Naturally, a candidate having scores $(f_1^+, f_2^+, \ldots, f_m^+)$ would be positive ideal whereas a candidate having scores $(f_1^-, f_2^-, \ldots, f_m^-)$ would be a negative ideal candidate. It is assume that such a positive ideal candidate does not exist; otherwise, the decision would be trivial.

Step 2. Compute the values S_j and $R_j (j = 1, 2, \ldots n)$, which represent the average and the worst group scores of the alternatives O_j, respectively, with the relations

$$S_j = \sum_{i=1}^{n} w_i \frac{(f_i^+ - f_{ij})}{(f_i^+ - f_i^-)}, \quad S_j \in [0, 1], \tag{2}$$

$$R_j = \max_i \left[w_i \frac{(f_i^+ - f_{ij})}{(f_i^+ - f_i^-)} \right], \quad R_j \in [0, 1]. \tag{3}$$

Here, w_i's $(\sum_{i=1}^{m} w_i = 1, w_i \in [0, 1], i = 1, 2, \ldots, m)$ are the relative importance weights of the criteria set by the decision-maker. The smaller values of S_j and R_j correspond to the better average and the worse group scores of alternatives O_j, respectively.

Step 3. Compute the Q_j values for $j = 1, 2, \ldots, m$ with the relation

$$Q_j = \frac{v(S_j - S^+)}{(S^- - S^+)} + \frac{(1 - v)(R_j - R^+)}{(R^- - R^+)}, \tag{4}$$

where

$$S^{+} = \min_{j} S_j, \quad S^{-} = \max_{j} S_j, \tag{5}$$

$$R^{+} = \min_{j} R_j, \quad R^{-} = \max_{j} R_j, \tag{6}$$

and v is the weight of decision making strategy "the majority of attribute" (or "the maximum group utility"). The compromise can be selected with "voting by majority" ($v > 0.5$), with "consensus" ($v = 0.5$), with "veto" ($v < 0.5$).

Step 4. Rank the alternatives by sorting each S, R and Q values in an decreasing order. The result is a set of three ranking lists denoted as $S_{[\cdot]}$, $R_{[\cdot]}$ and $Q_{[\cdot]}$.

Step 5. Propose the alternative O_{j_1} corresponding to $Q_{[1]}$ (the smallest among Q_j values) as compromise solution if

C1. The alternative O_{j_1} has an *acceptable advantage* ; in other words, $Q_{[2]} - Q_{[1]} \geq DQ$ where $DQ = \frac{1}{(m-1)}$, and m is the number of alternatives.

C2. The alternative O_{j_1} is *stable within the decision making process*; in other words, it is also the best ranked in $S_{[\cdot]}$ or $R_{[\cdot]}$.

If one of the above conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives O_{j_1} and O_{j_2} where $Q_{j_2} = Q_{[2]}$ if only the condition C2 is not satisfied, or
- Alternatives $O_{j_1}, O_{j_2}, \ldots, O_{j_k}$ if the condition C1 is not satisfied; and O_{j_k} is determined by the relation $Q_{[k]} - Q_{[1]} < DQ$ for the maximum k where $Q_{j_k} = Q_{[k]}$ (the positions of these alternatives are in closeness).

3 Interval-valued intuitionistic fuzzy sets

Let X be a non-empty and finite set with $\text{Card}(X) = n$. Let $D[0, 1]$ be the set of all closed subintervals of the unit interval $[0, 1]$. An interval-valued intuitionistic fuzzy set (Atanassov and Gargov 1989) (IVIFS) A in X is an object having the form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{7}$$

where $\mu_A : X \to D[0, 1]$, $\nu_A : X \to D[0, 1]$ with the condition $\sup \mu_A(x) + \sup \nu_A(x) \leq 1$ for any $x \in X$.

The intervals $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x to A. Then for each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are denoted by $\mu_{AL}(x)$, $\mu_{AU}(x)$, $\nu_{AL}(x)$ and $\nu_{AU}(x)$, respectively, and thus we can replace Eq. (7) with

$$A = \{\langle x, [\mu_{AL}(x), \mu_{AU}(x)], [\nu_{AL}(x), \nu_{AU}(x)]\rangle : x \in X\}, \tag{8}$$

where $0 \leq \mu_{AU}(x) + \nu_{AU}(x) \leq 1$ for any $x \in X$.

For each IVIFS A in X, Park et al. (2008) called

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \mu_{AU}(x) - \nu_{AU}(x), 1 - \mu_{AL}(x) - \nu_{AL}(x)] \tag{9}$$

an intuitionistic fuzzy interval of X in A.

For convenience, Xu (2007a) called $\tilde{a} = \langle [a, b], [c, d] \rangle$ an interval-valued intuitionistic fuzzy number (IVIFN), where $[a, b] \subset [0, 1]$, $[c, d] \subset [0, 1]$ and $b + d \leq 1$.

Atanassov and Gargov (1989) and Atanassov (1994) introduced some basic operations on IVIFSs, which not only can ensure that the operational results are IVIFSs but also are useful in the calculus of variables under interval-valued intuitionistic fuzzy environment. Motivated by the operations in Atanassov (1994), Atanassov and Gargov (1989)), Xu (2007a) and Xu and Chen (2007a,b) defined three operational laws of IVIFNs, which are useful in the remainder of this paper, as follows:

Let $\tilde{a}_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$, $\tilde{a}_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$ and $\tilde{a} = \langle [a, b], [c, d] \rangle$ be three IVIFNs; then

(1) $\tilde{a}_1 \otimes \tilde{a}_2 = \langle [a_1 a_2, b_1 b_2], [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2] \rangle$;
(2) $\tilde{a}^\lambda = \langle [a^\lambda, b^\lambda], [1 - (1 - c)^\lambda, 1 - (1 - d)^\lambda] \rangle, \ \lambda > 0$;
(3) $\lambda \tilde{a} = \langle [1 - (1 - a)^\lambda, 1 - (1 - b)^\lambda], [c^\lambda, d^\lambda] \rangle, \ \lambda > 0$;

which can ensure the operational results are also IVIFNs. Moreover, Xu (2007a) defined a score function s to measure a IVIFN $\tilde{a}$ as follows:

$$s(\tilde{a}) = \frac{1}{2}(a - c + b - d), \tag{10}$$

where $s(\tilde{a}) \in [-1, 1]$. The larger the value of $s(\tilde{a})$, the higher the IVIFN $\tilde{a}$. Especially, if $s(\tilde{a}) = 1$, then $\tilde{a} = \langle [1, 1], [0, 0] \rangle$, which is the largest IVIFN; if $s(\tilde{a}) = -1$, then $\tilde{a} = \langle [0, 0], [1, 1] \rangle$, which is the smallest IVIFN.

Xu (2007a) also defined an accuracy function h to evaluate the accuracy degree of a IVIFN $\tilde{a}$ as follows:

$$h(\tilde{a}) = \frac{1}{2}(a + b + c + d), \tag{11}$$

where $h(\tilde{a}) \in [0, 1]$. The larger the value of $h(\tilde{a})$, the higher the accuracy degree of the IVIFN $\tilde{a}$.

From Eq. (9), we define the hesitancy degree of the IVIFN $\tilde{a} = \langle [a, b], [c, d] \rangle$ as the midpoint of intuitionistic fuzzy interval of $\tilde{a}$, i.e.,

$$\pi(\tilde{a}) = \frac{1}{2}((1 - a - c) + (1 - b - d)). \tag{12}$$

Then we get the relation between the hesitancy degree and the accuracy degree of the IVIFN $\tilde{a}$

$$\pi(\tilde{a}) = \frac{1}{2}((1 - a - c) + (1 - b - d)) = 1 - h(\tilde{a}),$$

i.e.,

$$\pi(\tilde{a}) + h(\tilde{a}) = 1. \tag{13}$$

From Eq. (13), we know that the higher the accuracy degree $h(\tilde{a})$, the lower the hesitancy degree $\pi(\tilde{a})$.

Based on the score and the accuracy functions, Xu (2007a) defined a method to compare two IVIFNs as follows:

Definition 1 Let $\tilde{a}_1 = \langle[a_1, b_1], [c_1, d_1]\rangle$ and $\tilde{a}_2 = \langle[a_2, b_2], [c_2, d_2]\rangle$ be two IVIFNs, $s(\tilde{a}_1) = \frac{1}{2}(a_1 - c_1 + b_1 - d_1)$ and $s(\tilde{a}_2) = \frac{1}{2}(a_2 - c_2 + b_2 - d_2)$ be the score of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and $h(\tilde{a}_1) = \frac{1}{2}(a_1 + b_1 + c_1 + d_1)$ and $h(\tilde{a}_2) = \frac{1}{2}(a_2 + b_2 + c_2 + d_2)$ be the accuracy degree of $\tilde{a}_1$ and $\tilde{a}_2$, respectively; then:

- if $s(\tilde{a}_1) < s(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
- if $s(\tilde{a}_1) = s(\tilde{a}_2)$, then
  - (1) if $h(\tilde{a}_1) = h(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, which indicates indifference between $\tilde{a}_1$ and $\tilde{a}_2$, denotes by $\tilde{a}_1 \sim \tilde{a}_2$;
  - (2) if $h(\tilde{a}_1) < h(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.

4 The VIKOR method with interval-valued intuitionistic fuzzy data

In this section, we extend the VIKOR method to solve MAGDM problems in which all preference information provided by decision-makers is expressed as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by IVIFN, and the information about attribute weights is partially known.

For MAGDM problem, let $O = \{O_1, O_2, \ldots, O_n\}$ be the set of n alternatives, $D = \{d_1, d_2, \ldots, d_l\}$ be the set of l decision-makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ be the weight vector of decision-makers, where $\lambda_k \geq 0, k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} \lambda_k = 1$. Let $U = \{u_1, u_2, \ldots, u_m\}$ be the set of m attributes. In general, the decision-makers need to determine the importance degrees of a set U of m attributes. Thus we suppose that the decision-makers provide the attribute weight information may be presented in the following forms (Kim and Ahn 1999; Kim et al. 1999), for $i \neq j$:

1. A weak ranking: $\{w_i \geq w_j\}$;
2. A strict ranking: $\{w_i - w_j \geq \delta_i (> 0)\}$;
3. A ranking with multiples: $\{w_i \geq \delta_i w_j\}, 0 \leq \delta_i \leq 1$;
4. An interval form: $\{\delta_i \leq w_i \leq \delta_i + \epsilon_i\}, 0 \leq \delta_i \leq \delta_i + \epsilon_i \leq 1$;
5. A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$, for $j \neq k \neq l$.

For convenience, we denote by H the set of the known information about attribute weights provided by the decision-makers. Let $R^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n}$ be an interval-valued intuitionistic fuzzy decision matrix, provided by decision-maker $d_k (k = 1, 2, \ldots, l)$, as the following form:

	O_1	O_2	$\cdots$	O_n
u_1	$\tilde{r}_{11}^{(k)}$	$\tilde{r}_{12}^{(k)}$	$\cdots$	$\tilde{r}_{1n}^{(k)}$
u_2	$\tilde{r}_{21}^{(k)}$	$\tilde{r}_{22}^{(k)}$	$\cdots$	$\tilde{r}_{2n}^{(k)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	$\tilde{r}_{m1}^{(k)}$	$\tilde{r}_{m2}^{(k)}$	$\cdots$	$\tilde{r}_{mn}^{(k)}$

where $\tilde{r}_{ij}^{(k)} = \langle [a_{ij}^{(k)}, b_{ij}^{(k)}], [c_{ij}^{(k)}, d_{ij}^{(k)}] \rangle$ is an IVIFN representing the performance rating of the alternative O_j with respect to the attribute $u_i \in U$, provided by the decision-maker $d_k \in D$ (i.e., $[a_{ij}^{(k)}, b_{ij}^{(k)}]$ indicates the degree that the alternative $O_j \in O$ satisfy the attribute u_i, expressed by the decision-maker d_k, while $[c_{ij}^{(k)}, d_{ij}^{(k)}]$ indicates the degree that the alternative $O_j \in O$ does not satisfy the attribute u_i, expressed by the decision-maker d_k) and

$$\begin{aligned} &[a_{ij}^{(k)}, b_{ij}^{(k)}] \subset [0, 1], [c_{ij}^{(k)}, d_{ij}^{(k)}] \subset [0, 1], b_{ij}^{(k)} + d_{ij}^{(k)} \le 1, \\ &i = 1, 2, \ldots, m; \;\; j = 1, 2, \ldots, n. \end{aligned} \tag{14}$$

To extend the VIKOR method in the process of group decision making, we first need to fuse all individual decision opinion into group opinion. To do this, we use the IIFHG operator (Wei and Wang 2007; Xu and Chen 2007a) to aggregate all individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (\tilde{r}_{ij}^{(k)})_{m\times n} (k = 1, 2, \ldots, l)$ into the collective interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{r}_{ij})_{m\times n}$,

	O_1	O_2	$\cdots$	O_n
u_1	$\tilde{r}_{11}$	$\tilde{r}_{12}$	$\cdots$	$\tilde{r}_{1n}$
u_2	$\tilde{r}_{21}$	$\tilde{r}_{22}$	$\cdots$	$\tilde{r}_{2n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
u_m	$\tilde{r}_{m1}$	$\tilde{r}_{m2}$	$\cdots$	$\tilde{r}_{mn}$

where

$$\begin{aligned} \tilde{r}_{ij} &= \mathrm{IIFHG}_{\alpha,\lambda}(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(l)}) \\ &= (\dot{\tilde{r}}_{ij}^{(\sigma(1))})^{\alpha_1} \otimes (\dot{\tilde{r}}_{ij}^{(\sigma(2))})^{\alpha_2} \otimes \cdots \otimes (\dot{\tilde{r}}_{ij}^{(\sigma(l))})^{\alpha_l} \\ &= \left\langle \left[\prod_{k=1}^{l} (\dot{a}_{ij}^{(\sigma(k))})^{\alpha_k}, \prod_{k=1}^{l} (\dot{b}_{ij}^{(\sigma(k))})^{\alpha_k} \right], \right. \\ &\qquad \left. \left[1 - \prod_{k=1}^{l} (1 - \dot{c}_{ij}^{(\sigma(k))})^{\alpha_k}, 1 - \prod_{k=1}^{l} (1 - \dot{d}_{ij}^{(\sigma(k))})^{\alpha_k} \right] \right\rangle, \end{aligned} \tag{15}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)^T$ is weight vector of IIFHG operator with $\alpha_k > 0 (k = 1, 2, \ldots, l)$ and $\sum_{k=1}^{l} \alpha_k = 1$, and $\dot{\tilde{r}}_{ij}^{(\sigma(k))} = \langle[\dot{a}_{ij}^{\sigma((k))}, \dot{b}_{ij}^{\sigma((k))}], [\dot{c}_{ij}^{\sigma((k))}, \dot{d}_{ij}^{\sigma((k))}]\rangle$ is the kth largest of the weighted IVIFNs $\dot{\tilde{r}}_{ij}^{(k)} (\dot{\tilde{r}}_{ij}^{(k)} = (\tilde{r}_{ij}^{(k)})^{l\lambda_k}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Here, we denote by $\tilde{r}_{ij} = \langle[a_{ij}, b_{ij}], [c_{ij}, d_{ij}]\rangle, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

Let J_1 be a collection of benefit attributes (i.e., the larger u_i, the greater preference) and J_2 be a collection of cost attributes (i.e., the smaller u_i, the greater preference). The interval-valued intuitionistic PIS, denoted by O^*, and the interval-valued intuitionistic NIS, denoted by O^-, are defined as follows:

$$
\begin{aligned}
O^* &= \left\{\left\langle u_i, \left(\max_j \tilde{r}_{ij} | i \in J_1\right), \left(\min_j \tilde{r}_{ij} | i \in J_2\right)\right\rangle | i = 1, 2, \ldots, m\right\}^T \\
&= \{\tilde{r}_1^+, \tilde{r}_2^+, \ldots, \tilde{r}_m^+\}^T, \qquad (16) \\
O^- &= \left\{\left\langle u_i, \left(\min_j \tilde{r}_{ij} | i \in J_1\right), \left(\max_j \tilde{r}_{ij} | i \in J_2\right)\right\rangle | i = 1, 2, \ldots, m\right\}^T \\
&= \{\tilde{r}_1^-, \tilde{r}_2^-, \ldots, \tilde{r}_m^-\}^T, \qquad (17)
\end{aligned}
$$

where $\max_j \tilde{r}_{ij} = \langle[\max_j a_{ij}, \max_j b_{ij}], [\min_j c_{ij}, \min_j d_{ij}]\rangle$ and $\min_j \tilde{r}_{ij} = \langle[\min_j a_{ij}, \min_j b_{ij}], [\max_j c_{ij}, \max_j d_{ij}]\rangle$. Here we denote $\tilde{r}_i^+ = \langle[a_i^+, b_i^+], [c_i^+, d_i^+]\rangle$ and $\tilde{r}_i^- = \langle[a_i^-, b_i^-], [c_i^-, d_i^-]\rangle, i = 1, 2, \ldots, m$. It is assumed that such a interval-valued intuitionistic PIS does not exists; otherwise, the decision would be trivial.

The average score and the worst group score of alternative can be measured by the Hamming distance or Euclidean distance. So, we adopt several definitions proposed by Park et al. (2008) for measuring distances between IVIFNs, which are, respectively, extension of Burillo and Bustince (1996) method and Grzegorzewski (2004) method for intuitionistic fuzzy sets. The average scores $\tilde{S}_j$ and the worst group scores $\tilde{R}_j$ of the alternatives $O_j (j = 1, 2, \ldots, n)$, respectively, are derived from:

(1) The average score $\tilde{S}_j$:

- The extension of Burillo and Bustince's method, d_1:

$$
\begin{aligned}
\tilde{S}_j^{d_1} &= \sum_{i=1}^{m} \frac{w_i \times d_1(\tilde{r}_i^+, \tilde{r}_{ij})}{d_1(\tilde{r}_i^+, \tilde{r}_i^-)} \\
&= \sum_{i=1}^{m} \frac{w_i \times (|a_i^+ - a_{ij}| + |b_i^+ - b_{ij}| + |c_i^+ - c_{ij}| + |d_i^+ - d_{ij}|)}{|a_i^+ - a_i^-| + |b_i^+ - b_i^-| + |c_i^+ - c_i^-| + |d_i^+ - d_i^-|}. \qquad (18)
\end{aligned}
$$

- The extension of modified Burillo and Bustince's method, d_2:

$$
\begin{aligned}
\tilde{S}_j^{d_2} &= \sum_{i=1}^{m} \frac{w_i \times d_2(\tilde{r}_i^+, \tilde{r}_{ij})}{d_2(\tilde{r}_i^+, \tilde{r}_i^-)} \\
&= \sum_{i=1}^{m} \Big[w_i \times \big(|a_i^+ - a_{ij}| + |b_i^+ - b_{ij}| + |c_i^+ - c_{ij}| + |d_i^+ - d_{ij}|
\end{aligned}
$$

$$+\ \left||a_i^+ - b_i^+| - |a_{ij} - b_{ij}|\right| + \left||c_i^+ - d_i^+| - |c_{ij} - d_{ij}|\right|\big)$$
$$\times \big(|a_i^+ - a_i^-| + |b_i^+ - b_i^-| + |c_i^+ - c_i^-| + |d_i^+ - d_i^-|$$
$$+\ \left||a_i^+ - b_i^+| - |a_i^- - b_i^-|\right| + \left||c_i^+ - d_i^+| - |c_i^- - d_i^-|\right|\big)^{-1}\Big]. \quad (19)$$

- The extension of Grzegorzewski's method, d_H:

$$\tilde{S}_j^{d_H} = \sum_{i=1}^{m} \frac{w_i \times d_H(\tilde{r}_i^+, \tilde{r}_{ij})}{d_H(\tilde{r}_i^+, \tilde{r}_i^-)}$$
$$= \sum_{i=1}^{m} \frac{w_i \times \left[\max(|a_i^+ - a_{ij}|, |b_i^+ - b_{ij}|) + \max(|c_i^+ - c_{ij}|, |d_i^+ - d_{ij}|)\right]}{\max(|a_i^+ - a_i^-|, |b_i^+ - b_i^-|) + \max(|c_i^+ - c_i^-|, |d_i^+ - d_i^-|)}. \quad (20)$$

(2) The worst group score $\tilde{R}_j$

- The extension of Burillo and Bustince's method, d_1:

$$\tilde{R}_j^{d_1} = \max_{1\le i\le m}\left[\frac{w_i \times d_1(\tilde{r}_i^+, \tilde{r}_{ij})}{d_1(\tilde{r}_i^+, \tilde{r}_i^-)}\right]$$
$$= \max_{1\le i\le m}\left[\frac{w_i \times (|a_i^+ - a_{ij}| + |b_i^+ - b_{ij}| + |c_i^+ - c_{ij}| + |d_i^+ - d_{ij}|)}{|a_i^+ - a_i^-| + |b_i^+ - b_i^-| + |c_i^+ - c_i^-| + |d_i^+ - d_i^-|}\right]. \quad (21)$$

- The extension of modified Burillo and Bustince's method, d_2:

$$\tilde{R}_j^{d_2} = \max_{1\le i\le m}\left[\frac{w_i \times d_2(\tilde{r}_i^+, \tilde{r}_{ij})}{d_2(\tilde{r}_i^+, \tilde{r}_i^-)}\right]$$
$$= \max_{1\le i\le m}\Big[w_i \times \big(|a_i^+ - a_{ij}| + |b_i^+ - b_{ij}| + |c_i^+ - c_{ij}| + |d_i^+ - d_{ij}|$$
$$+\ \left||a_i^+ - b_i^+| - |a_{ij} - b_{ij}|\right| + \left||c_i^+ - d_i^+| - |c_{ij} - d_{ij}|\right|\big)$$
$$\times \big(|a_i^+ - a_i^-| + |b_i^+ - b_i^-| + |c_i^+ - c_i^-| + |d_i^+ - d_i^-|$$
$$+\ \left||a_i^+ - b_i^+| - |a_i^- - b_i^-|\right| + \left||c_i^+ - d_i^+| - |c_i^- - d_i^-|\right|\big)^{-1}\Big]. \quad (22)$$

- The extension of Grzegorzewski's method, d_H:

$$\tilde{R}_j^{d_H} = \max_{1\le i\le m}\left[\frac{w_i \times d_H(\tilde{r}_i^+, \tilde{r}_{ij})}{d_H(\tilde{r}_i^+, \tilde{r}_i^-)}\right]$$
$$= \max_{1\le i\le m}\left[\frac{w_i \times \left[\max(|a_i^+ - a_{ij}|, |b_i^+ - b_{ij}|) + \max(|c_i^+ - c_{ij}|, |d_i^+ - d_{ij}|)\right]}{\max(|a_i^+ - a_i^-|, |b_i^+ - b_i^-|) + \max(|c_i^+ - c_i^-|, |d_i^+ - d_i^-|)}\right]. \quad (23)$$

Here, w_i's ($\sum_{i=1}^{m} w_i = 1, w_i \in [0, 1], i = 1, 2, \ldots, m$) are the weights of attribute, expressing their relative importance, which are provided by decision-makers. The smaller values of $\tilde{S}_j$ and $\tilde{R}_j$ correspond to the better average and the worse group scores for the alternatives $O_j (j = 1, 2, \ldots, n)$, respectively.

The values $\tilde{Q}_j (j = 1, 2, \ldots, n)$ of the alternative O_j is defined as the following:

$$\tilde{Q}_j = \frac{v(\tilde{S}_j - \tilde{S}^*)}{\tilde{S}^- - \tilde{S}^*} + \frac{(1 - v)(\tilde{R}_j - \tilde{R}^*)}{\tilde{R}^- - \tilde{R}^*}, \quad j = 1, 2, \ldots, n. \tag{24}$$

where

$$\tilde{S}^* = \min_j \tilde{S}_j, \quad \tilde{S}^- = \max_j \tilde{S}_j, \tag{25}$$

$$\tilde{R}^* = \min_j \tilde{R}_j, \quad \tilde{R}^- = \max_j \tilde{R}_j, \tag{26}$$

and v is the weight of decision making strategy "the majority of attribute" (or "the maximum group utility"). The compromise can be selected with "voting by majority" ($v > 0.5$), with "consensus" ($v = 0.5$), with "veto" ($v < 0.5$).

Based on the VIKOR method, we rank the alternatives by sorting each $\tilde{S}$, $\tilde{R}$ and $\tilde{Q}$ values in decreasing order. The result is a set of three ranking lists denoted as $\tilde{S}_{[\cdot]}$, $\tilde{R}_{[\cdot]}$ and $\tilde{Q}_{[\cdot]}$. We propose the alternative O_{j_1} corresponding to $\tilde{Q}_{[1]}$ (that has minimum $\tilde{Q}_j$) the best alternative and it is chosen as compromise solution if

C1. The alternative O_{j_1} has an *acceptable advantage*; in other words, $\tilde{Q}_{[2]} - \tilde{Q}_{[1]} \geq D\tilde{Q}$ where $D\tilde{Q} = \frac{1}{(m-1)}$, and m is the number of alternatives.

C2. The alternative O_{j_1} is *stable within the decision making process*; in other words, it is also the best ranked in $\tilde{S}_{[\cdot]}$ or $\tilde{R}_{[\cdot]}$.

If one of the above conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

– Alternatives O_{j_1} and O_{j_2} where $\tilde{Q}_{j_2} = \tilde{Q}_{[2]}$ if only the condition C2 is not satisfied, or
– Alternatives $O_{j_1}, O_{j_2}, \ldots, O_{j_k}$ if the condition C1 is not satisfied; and O_{j_k} is determined by the relation $\tilde{Q}_{[k]} - \tilde{Q}_{[1]} < D\tilde{Q}$ for the maximum k where $\tilde{Q}_{j_k} = \tilde{Q}_{[k]}$ (the positions of these alternatives are in closeness).

4.1 A model for determining attribute weights

In some situations, however, the information about attribute weights provided by the decision-makers is usually incomplete (see, Kim and Ahn 1999; Kim et al. 1999) because of time pressure, lack of knowledge, or data, and their limited expertise related to the problem domain. So an interesting and important issue is how to utilize the collective interval-valued intuitionistic fuzzy decision matrix and the known weight information to find the most desirable alternative(s).

Xu (2007b) presented an approach to determining the weight of attributes as follows.

Definition 2 Let $R = (\tilde{r}_{ij})_{m\times n}$ be the collective interval-valued intuitionistic fuzzy decision matrix. Then we call $S = (s_{ij})_{m\times n}$ the score matrix of $R = (\tilde{r}_{ij})_{m\times n}$, where

$$s_{ij} = s(\tilde{r}_{ij}) = \frac{1}{2}(a_{ij} - c_{ij} + b_{ij} - d_{ij}),\ i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \quad (27)$$

and $s(\tilde{r}_{ij})$ is called the score of $\tilde{r}_{ij}$.

Based on the score matrix, we present the overall score values of each alternatives $O_j (j = 1, 2, \ldots, n)$:

$$s_j(w) = \sum_{i=1}^{m} w_i s_{ij}, \quad j = 1, 2, \ldots, n. \quad (28)$$

Obviously, the greater the value $s_j(w)$, the better the alternative O_j. When we only consider the alternative O_j, then a reasonable vector of attribute weights $w = (w_1, w_2, \ldots, w_m)^T$ should be determined. Thus, we establish the following optimization model to maximize $s_j(w)$:

$$(\mathbf{M-1}) \quad \text{Maximize } s_j(w) = \sum_{i=1}^{m} w_i s_{ij}$$

$$\text{Subject to}: w = (w_1, \ldots, w_m)^T \in H, w_i \geq 0,\ i = 1, \ldots, m, \sum_{i=1}^{m} w_i = 1.$$

By solving the model $(\mathbf{M-1})$, we get the optimal solution $w^{(j)} = (w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)})^T$ corresponding to the alternative O_j. However, in the process of determining the weight vector $w = (w_1, w_2, \ldots, w_m)^T$, we need to consider all the alternatives $O_j (j = 1, 2, \ldots, n)$ as a whole. Thus, we construct a combined weight vector as follows:

$$w = \omega_1 w^{(1)} + \omega_2 w^{(2)} + \cdots + \omega_n w^{(n)} = \begin{pmatrix} w_1^{(1)} & w_1^{(2)} & \cdots & w_1^{(n)} \\ w_2^{(1)} & w_2^{(2)} & \cdots & w_2^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ w_m^{(1)} & w_m^{(2)} & \cdots & w_m^{(n)} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = W\omega \quad (29)$$

where

$$W = \begin{pmatrix} w_1^{(1)} & w_1^{(2)} & \cdots & w_1^{(n)} \\ w_2^{(1)} & w_2^{(2)} & \cdots & w_2^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ w_m^{(1)} & w_m^{(2)} & \cdots & w_m^{(n)} \end{pmatrix}$$

and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is undetermined nonnegative vector satisfying the condition:

$$\omega^T \omega = 1. \tag{30}$$

Let $\bar{s}_j = (s_{ij}, s_{2j}, \ldots, s_{mj})^T (j = 1, 2, \ldots, n)$, then the score matrix S can be expressed as $S = (\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n)^T$. By Eqs. (28) and (29), we have

$$s_j(w) = \sum_{i=1}^{m} w_i s_{ij} = w^T \bar{s}_j = (W\omega)^T \bar{s}_j. \tag{31}$$

To determine the combined weight vector $w = (w_1, w_2, \ldots, w_m)^T$, we should make all the overall score values $s_j(w)(j = 1, 2, \ldots, n)$ as greater as possible, which means to maximize the following vector $s(w) = (s_1(w), s_2(w), \ldots, s_n(w))$ under the condition Eq. (30). In order to do that, we establish the following multiple objective optimization model:

$$(\mathbf{M-2}) \quad \text{Maximize } s(w) = (s_1(w), s_2(w), \ldots, s_n(w))$$
$$\text{Subject to} : \omega^T \omega = 1.$$

By the equal weighted summation method, the model $(\mathbf{M-2})$ can be transformed into a single objective optimization model:

$$(\mathbf{M-3}) \quad \text{Maximize } s(w)^T s(w)$$
$$\text{Subject to} : \omega^T \omega = 1.$$

Let $f(\omega) = s(w)^T s(w$, then by Eq. (31), we have

$$f(\omega) = s(w)^T s(w) = \omega^T (S^T W)^T (S^T W)\omega. \tag{32}$$

Let $C = (S^T W)^T (S^T W)$, then $C^T = (S^T W)^T (S^T W) = C$, i.e., C is real symmetrical matrix. Moreover, $C \geq$). Therefore, C is nonnegative definite matrix.

Here, we introduce two useful theorems:

Theorem 1 (Horn and Johnson 1990). *Let $B = (r_{ij})_{n \times n}$ be a real symmetrical matrix, i.e., $B^T = B$, then*

$$\max \frac{x^T B x}{x^T x} = \lambda_{\max} \tag{33}$$

where $\lambda_{\max}$ is the largest eigenvalue of B, and x is a nonzero vector.

Theorem 2 (Horn and Johnson 1990). *Let $B = (r_{ij})_{n\times n}$ be a real irreducible non-negative matrix, then*

(1) *B has a largest eigenvalue $\lambda_{\max}$, which is also a unique eigenvalue of B.*
(2) *Let $v = (v_1, v_2, \ldots, v_n)^T$ be the eigenvector of $\lambda_{\max}$, then all $v_j > 0 (j = 1, 2, \ldots, n)$, i.e., w is positive eigenvector.*

By Theorems 1 and 2, we know that $f(\omega)$ has a largest value $\max f(\omega)$, which is also the largest eigenvalue $\bar{\lambda}_{\max}$ of C. $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the eigenvector of $\bar{\lambda}_{\max}$, where $\bar{\lambda}_{\max}$ is unique, and all $\omega_j > 0 (j = 1, 2, \ldots, n)$. After normalizing the eigenvector ω, we can utilize Eq. (29) to derive the weight vector $w = (w_1, w_2, \ldots, w_m)^T$.

4.2 An approach to MAGDM with incomplete attribute weight information

Based on the analysis above, in the following we present an approach to multiple attribute interval-valued intuitionistic fuzzy group decision making with incomplete attribute weight information:

Step 1. Utilize the IIFHG operator to aggregate all individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (\tilde{r}_{ij}^{(k)})_{m\times n} (k = 1, 2, \ldots, l)$ into a collective interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{r}_{ij})_{m\times n}$.
Step 2. Calculate the score matrix $S = (s_{ij})_{m\times n}$ of the collective interval-valued intuitionistic fuzzy decision matrix R.
Step 3. Utilize the model $(\mathbf{M-1})$ to obtain the optimal weight vectors $w^{(j)} = (w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)})^T (j = 1, 2, \ldots, n)$ corresponding to the alternatives $O_j (j = 1, 2, \ldots, n)$, and then construct the weight matrix W.
Step 4. Calculate the normalized eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ of the matrix $(S^T W)^T (S^T W)$.
Step 5. Utilize Eq. (29) to derive the weight vector $w = (w_1, w_2, \ldots, w_m)^T$.
Step 6. Utilize Eqs. (16) and (17) to determine the interval-valued intuitionistic PIS O^* and interval-valued intuitionistic NIS O^-.
Step 7. Utilize Eqs. (18)–(23) to calculate the average scores $\tilde{S}_j$ and the worst group scores $\tilde{R}_j$ of each alternative $O_j (j = 1, 2, \ldots, n)$, respectively.
Step 8. Utilize Eq. (24) to calculate the value $\tilde{Q}_j$ of each alternative $O_j (j = 1, 2, \ldots, n)$.
Step 9. Rank the alternatives $O_j (j = 1, 2, \ldots, n)$, by sorting each $\tilde{S}$, $\tilde{R}$ and $\tilde{Q}$ values in decreasing order and then select the alternative that minimum $\tilde{Q}_j$ as the best alternative (i.e., it is chosen as compromise solution).

5 Numerical example

Let $O = \{O_1, O_2, O_3, O_4\}$ be the set of four alternatives, $D = \{d_1, d_2, d_3, d_4\}$ be the set of four experts whose weight vector is $\lambda = (0.3, 0.2, 0.3, 0.2)^T$ from each strategic decision area. Let $J = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of five attributes and

Table 1 Interval-valued intuitionistic fuzzy decision matrix $R^{(1)}$

	O_1	O_2	O_3	O_4
u_1	⟨[0.5, 0.6], [0.2, 0.3]⟩	⟨[0.3, 0.4], [0.4, 0.6]⟩	⟨[0.4, 0.5], [0.3, 0.5]⟩	⟨[0.3, 0.5], [0.4, 0.5]⟩
u_2	⟨[0.3, 0.5], [0.4, 0.5]⟩	⟨[0.1, 0.3], [0.2, 0.4]⟩	⟨[0.7, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.7, 0.8]⟩
u_3	⟨[0.6, 0.7], [0.2, 0.3]⟩	⟨[0.3, 0.4], [0.4, 0.5]⟩	⟨[0.5, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.5, 0.8]⟩
u_4	⟨[0.5, 0.7], [0.1, 0.2]⟩	⟨[0.2, 0.4], [0.5, 0.6]⟩	⟨[0.4, 0.6], [0.2, 0.3]⟩	⟨[0.2, 0.3], [0.4, 0.6]⟩
u_5	⟨[0.1, 0.4], [0.3, 0.5]⟩	⟨[0.7, 0.8], [0.1, 0.2]⟩	⟨[0.5, 0.6], [0.2, 0.3]⟩	⟨[0.2, 0.3], [0.5, 0.6]⟩

Table 2 Interval-valued intuitionistic fuzzy decision matrix $R^{(2)}$

	O_1	O_2	O_3	O_4
u_1	⟨[0.4, 0.5], [0.2, 0.4]⟩	⟨[0.3, 0.5], [0.4, 0.5]⟩	⟨[0.4, 0.6], [0.3, 0.4]⟩	⟨[0.3, 0.4], [0.4, 0.6]⟩
u_2	⟨[0.3, 0.4], [0.4, 0.6]⟩	⟨[0.1, 0.3], [0.3, 0.7]⟩	⟨[0.6, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.6, 0.8]⟩
u_3	⟨[0.6, 0.7], [0.1, 0.2]⟩	⟨[0.3, 0.4], [0.4, 0.5]⟩	⟨[0.7, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.7, 0.8]⟩
u_4	⟨[0.5, 0.6], [0.1, 0.3]⟩	⟨[0.2, 0.3], [0.6, 0.7]⟩	⟨[0.4, 0.6], [0.3, 0.4]⟩	⟨[0.3, 0.4], [0.4, 0.6]⟩
u_5	⟨[0.1, 0.3], [0.3, 0.5]⟩	⟨[0.6, 0.8], [0.1, 0.2]⟩	⟨[0.5, 0.6], [0.2, 0.4]⟩	⟨[0.2, 0.4], [0.5, 0.6]⟩

Table 3 Interval-valued intuitionistic fuzzy decision matrix $R^{(3)}$

	O_1	O_2	O_3	O_4
u_1	⟨[0.4, 0.7], [0.1, 0.2]⟩	⟨[0.4, 0.5], [0.2, 0.4]⟩	⟨[0.2, 0.4], [0.3, 0.4]⟩	⟨[0.3, 0.4], [0.2, 0.4]⟩
u_2	⟨[0.3, 0.5], [0.3, 0.4]⟩	⟨[0.2, 0.4], [0.4, 0.5]⟩	⟨[0.6, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.6, 0.8]⟩
u_3	⟨[0.6, 0.7], [0.1, 0.2]⟩	⟨[0.4, 0.5], [0.3, 0.4]⟩	⟨[0.5, 0.7], [0.1, 0.3]⟩	⟨[0.1, 0.3], [0.5, 0.7]⟩
u_4	⟨[0.5, 0.6], [0.1, 0.3]⟩	⟨[0.1, 0.2], [0.7, 0.8]⟩	⟨[0.5, 0.7], [0.2, 0.3]⟩	⟨[0.2, 0.3], [0.5, 0.7]⟩
u_5	⟨[0.3, 0.5], [0.4, 0.5]⟩	⟨[0.6, 0.7], [0.2, 0.3]⟩	⟨[0.6, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.6, 0.8]⟩

Table 4 Interval-valued intuitionistic fuzzy decision matrix $R^{(4)}$

	O_1	O_2	O_3	O_4
u_1	⟨[0.6, 0.7], [0.2, 0.3]⟩	⟨[0.4, 0.5], [0.4, 0.5]⟩	⟨[0.4, 0.5], [0.3, 0.4]⟩	⟨[0.3, 0.4], [0.4, 0.5]⟩
u_2	⟨[0.3, 0.4], [0.3, 0.4]⟩	⟨[0.1, 0.2], [0.2, 0.3]⟩	⟨[0.6, 0.7], [0.1, 0.3]⟩	⟨[0.1, 0.3], [0.6, 0.7]⟩
u_3	⟨[0.7, 0.8], [0.1, 0.2]⟩	⟨[0.3, 0.4], [0.5, 0.6]⟩	⟨[0.5, 0.8], [0.1, 0.2]⟩	⟨[0.1, 0.2], [0.5, 0.8]⟩
u_4	⟨[0.5, 0.6], [0.1, 0.3]⟩	⟨[0.2, 0.3], [0.4, 0.6]⟩	⟨[0.4, 0.5], [0.2, 0.3]⟩	⟨[0.2, 0.3], [0.4, 0.5]⟩
u_5	⟨[0.1, 0.2], [0.5, 0.7]⟩	⟨[0.6, 0.7], [0.1, 0.2]⟩	⟨[0.5, 0.6], [0.3, 0.4]⟩	⟨[0.3, 0.4], [0.5, 0.6]⟩

suppose that u_1, u_2, u_3 and u_5 are benefit attributes and u_4 is cost attribute. That is, $J_1 = \{u_1, u_2, u_3, u_5\}$ and $J_2 = \{u_4\}$. The experts $d_k(k = 1, 2, 3, 4)$ represent, respectively, the characteristics of the alternatives $O_j(j = 1, 2, 3, 4)$ by the IVIFNs $r_{ij}^{(k)}(i = 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4)$ with respect to the attributes $u_i(i = 1, 2, 3, 4, 5)$, listed in Tables 1, 2, 3 and 4 (i.e., interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (r_{ij}^{(k)})_{5\times 4}(k = 1, 2, 3, 4)$).

Assume that the information about attribute weights, given by decision-makers, is shown as follows, respectively:

$$\begin{aligned} &d_1 : w_1 \leq 0.3, \quad 0.2 \leq w_3 \leq 0.5; \\ &d_2 : 0.1 \leq w_2 \leq 0.2, \quad w_5 \leq 0.4; \\ &d_3 : w_3 - w_2 \geq w_5 - w_4, \quad w_4 \geq w_1; \\ &d_4 : w_3 - w_1 \leq 0.1, \quad 0.1 \leq w_4 \leq 0.3. \end{aligned}$$

Then the set H of the known information about attribute weights provided by the decision-makers is

$$\begin{aligned} H = \{&w_1 \leq 0.3,\ 0.2 \leq w_3 \leq 0.5,\ 0.1 \leq w_2 \leq 0.2, w_5 \leq 0.4, \\ &w_3 - w_2 \geq w_5 - w_4,\ w_4 \geq w_1, w_3 - w_1 \leq 0.1,\ 0.1 \leq w_4 \leq 0.3\}. \end{aligned}$$

Step 1. Utilize the IIFHG operator (let $\alpha = (0.155, 0.345, 0.345, 0.155)^T$ be its weight vector derived by the normal distribution based method Xu 2005) to aggregate the individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (r_{ij}^{(k)})_{5\times4} (k = 1, 2, 3, 4)$ into the collective interval-valued intuitionistic fuzzy decision matrix $R = (r_{ij})_{5\times4}$ (Table 5).

Step 2. Calculate the score matrix $S = (s_{ij})_{5\times4}$ of the collective interval-valued intuitionistic fuzzy decision matrix R (Table 6):

Step 3. Use the method $(\mathbf{M-1})$ to obtain the optimal weight vectors $w^{(j)} = (w_1^{(j)}, w_2^{(j)}, w_3^{(j)}, w_4^{(j)}, w_5^{(j)})^T (j = 1, 2, 3, 4)$ corresponding to the alternatives $O_j (j = 1, 2, 3, 4)$:

$$w^{(1)} = (0.2667, 0.1, 0.3667, 0.2667, 0)^T,$$
$$w^{(2)} = (0.16, 0.1, 0.26, 0.16, 0.32)^T,$$

Table 5 Collective interval-valued intuitionistic fuzzy decision matrix R

	O_1	O_2
u_1	⟨[0.4385, 0.6199], [0.1549, 0.2848]⟩	⟨[0.3502, 0.4797], [0.3114, 0.4681]⟩
u_2	⟨[0.3000, 0.4573], [0.3404, 0.4710]⟩	⟨[0.1138, 0.3010], [0.2511, 0.4773]⟩
u_3	⟨[0.6116, 0.7117], [0.1089, 0.2083]⟩	⟨[0.3379, 0.4387], [0.3872, 0.4887]⟩
u_4	⟨[0.5000, 0.6395], [0.0980, 0.2567]⟩	⟨[0.1758, 0.3134], [0.5305, 0.6496]⟩
u_5	⟨[0.1323, 0.3623], [0.3747, 0.5482]⟩	⟨[0.6395, 0.7521], [0.1089, 0.2083]⟩
	O_3	O_4
u_1	⟨[0.3516, 0.4906], [0.2940, 0.4214]⟩	⟨[0.3000, 0.4170], [0.3114, 0.4887]⟩
u_2	⟨[0.6395, 0.7711], [0.0980, 0.2263]⟩	⟨[0.1000, 0.2103], [0.6012, 0.7678]⟩
u_3	⟨[0.5213, 0.7804], [0.0980, 0.2083]⟩	⟨[0.1000, 0.2366], [0.5577, 0.7569]⟩
u_4	⟨[0.4387, 0.6252], [0.2263, 0.3262]⟩	⟨[0.2103, 0.3109], [0.4050, 0.5613]⟩
u_5	⟨[0.5452, 0.6502], [0.1770, 0.3005]⟩	⟨[0.1849, 0.3121], [0.5031, 0.6118]⟩

Table 6 Collective score matrix S

	O_1	O_2	O_3	O_4
u_1	0.3093	0.0252	0.0634	−0.0415
u_2	−0.0270	−0.1568	0.5431	−0.5294
u_3	0.5030	−0.0496	0.4977	−0.4890
u_4	0.3924	−0.3454	0.2557	−0.2225
u_5	−0.2141	0.5372	0.3589	−0.3089

$$w^{(3)} = (0.1, 0.2, 0.2, 0.25, 0.25)^T,$$
$$w^{(4)} = (0.3, 0.1, 0.2, 0.3, 0.1)^T$$

and construct the weight matrix

$$W = \begin{pmatrix} 0.2667 & 0.16 & 0.1 & 0.3 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.3667 & 0.26 & 0.2 & 0.2 \\ 0.2667 & 0.16 & 0.25 & 0.3 \\ 0 & 0.32 & 0.25 & 0.1 \end{pmatrix}$$

then

$$(S^T W)^T (S^T W) = \begin{pmatrix} 0.3455 & 0.2621 & 0.2835 & 0.2848 \\ 0.2621 & 0.2634 & 0.2683 & 0.2266 \\ 0.2835 & 0.2683 & 0.2808 & 0.2423 \\ 0.2848 & 0.2266 & 0.2423 & 0.2365 \end{pmatrix}.$$

Step 4. Calculate the normalized eigenvectors ω of the matrix $(S^T W)^T (S^T W)$:

$$\omega = (0.2764, 0.2390, 0.2519, 0.2326)^T.$$

Step 5. Use Eq. (29) to derive the weight vector w:

$$w = W\omega = \begin{pmatrix} 0.2667 & 0.16 & 0.1 & 0.3 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.3667 & 0.26 & 0.2 & 0.2 \\ 0.2667 & 0.16 & 0.25 & 0.3 \\ 0 & 0.32 & 0.25 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2764 \\ 0.2390 \\ 0.2519 \\ 0.2326 \end{pmatrix}$$
$$= (0.2069, 0.1252, 0.2604, 0.2447, 0.1627)^T.$$

Step 6. Utilize Eqs. (16) and (17) to determine the interval-valued intuitionistic PIS O^* and interval-valued intuitionistic NIS O^-:

$$O^* = \{\langle[0.4385, 0.6199], [0.1549, 0.2848]\rangle, \langle[0.6395, 0.7711], [0.0980, 0.2263]\rangle, \langle[0.6116, 0.7117], [0.0980, 0.2083]\rangle, \langle[0.1758, 0.3109], [0.5305, 0.6496]\rangle, \langle[0.6395, 0.7521], [0.1089, 0.2083]\rangle\}^T$$

and

$$O^- = \{\langle[0.3000, 0.4170], [0.3114, 0.4887]\rangle, \langle[0.1000, 0.2103], [0.6012, 0.7678]\rangle, \langle[0.1000, 0.2366], [0.5577, 0.7569]\rangle, \langle[0.5000, 0.6395], [0.0980, 0.2567]\rangle, \langle[0.1323, 0.3121], [0.5031, 0.6118]\rangle\}^T.$$

Step 7. Utilize Eqs. (18)–(24) to compute the values $\tilde{S}_j$, $\tilde{R}_j$ and $\tilde{Q}_j$ for each alternative $O_j (j = 1, 2, 3, 4)$, respectively, and rank the alternatives $O_j (i = 1, 2, 3, 4)$ by sorting $\tilde{S}_j$, $\tilde{R}_j$ and $\tilde{Q}_j$ in an decreasing order (see Table 7).

Using the extended VIKOR method, the decision results as follows:

(a) In the case of d_1, the best candidate (alternative) is O_2 and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_4 \succ O_1$ if $0 \le v < 0.423$; the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_4 \sim O_1$ if $v = 0.423$; the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_1 \succ O_4$ if $0.423 < v \le 1$.

(b) In the case of d_2, the best candidate is O_2 and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_4 \succ O_1$ if $0 \le v < 0.364$; the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_4 \sim O_1$ if $v = 0.364$; the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_1 \succ O_4$ if $0.364 < v \le 1$.

(c) In the case of d_H, the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_1 \succ O_4$ if $0 \le v < 0.856$; the best candidate is O_2 and O_3, and the ranked ordered of all candidates is $O_2 \sim O_3 \succ O_1 \succ O_4$ if $v = 0.856$; the best candidate is O_1, O_2 and O_3, and the ranked ordered of all candidates is $O_2 \succ O_3 \succ O_1 \succ O_4$ if $0.856 < v \le 1$.

Therefore, the decision results obtained by the extended VIKOR method depend on the weight v and distance measures d_1, d_2 and d_H.

As mentioned in introduction, Park et al. (2010) extended the TOPSIS method to solve MAGDM problems with interval-valued intuitionistic fuzzy data. They solved this example using the extended TOPSIS method and determined the ranking of the alternatives as follows:

$$O_3 \succ O_2 \succ O_1 \succ O_4.$$

The compromise solution O_3 obtained by the extended TOPSIS method is different with the compromise solution O_2 of the extended VIKOR method. These different solutions derive from differences in aggregation functions.

Table 7 Decision results obtained from the extended VIKOR method

	$\tilde{S}_j^{d_1}$	$\tilde{R}_j^{d_1}$	$\tilde{Q}_j^{d_1}$
(a) The values $\tilde{S}_j^{d_1}$, $\tilde{R}_j^{d_1}$ and $\tilde{O}_j^{d_1}$ and preference ranking order obtained from d_1			
O_1	0.4528	0.2447	$1 - 0.8410v$
O_2	0.3954	0.1675	0
O_3	0.3985	0.1994	$0.4132 - 0.4046v$
O_4	0.7564	0.2254	$0.7500 - 0.2500v$
Ranking order	$O_2 \succ O_3 \succ O_1 \succ O_4$	$O_2 \succ O_3 \succ O_4 \succ O_1$	$O_2 \succ O_3 \succ O_4 \succ O_1$ $(0 \leq v < 0.423)$ $O_2 \succ O_3 \succ O_4 \sim O_1$ $(v = 0.423)$ $O_2 \succ O_3 \succ O_1 \succ O_4$ $(0.432 < v \leq 1)$
(b) The values $\tilde{S}_j^{d_2}$, $\tilde{R}_j^{d_2}$ and $\tilde{O}_j^{d_2}$ and preference ranking order obtained from d_2			
O_1	0.4651	0.2447	$1 - 0.8032v$
O_2	0.3918	0.1645	0
O_3	0.4153	0.2050	$0.5050 - 0.4419v$
O_4	0.7643	0.2275	$0.7855 + 0.2145v$
Ranking order	$O_2 \succ O_3 \succ O_1 \succ O_4$	$O_2 \succ O_3 \succ O_4 \succ O_1$	$O_2 \succ O_3 \succ O_4 \succ O_1$ $(0 \leq v < 0.364)$ $O_2 \succ O_3 \succ O_4 \sim O_1$ $(v = 0.364)$ $O_2 \succ O_3 \succ O_1 \succ O_4$ $(0.364 < v \leq 1)$
(c) The values $\tilde{S}_j^{d_H}$, $\tilde{R}_j^{d_H}$ and $\tilde{O}_j^{d_H}$ and preference ranking order obtained from d_H			
O_1	0.4793	0.2447	$0.7934 - 0.5896v$
O_2	0.4160	0.1844	$0.0456v$
O_3	0.3979	0.2050	$0.2711(1 - v)$
O_4	0.7973	0.2604	1
Ranking order	$O_3 \succ O_2 \succ O_1 \succ O_4$	$O_2 \succ O_3 \succ O_1 \succ O_4$	$O_2 \succ O_3 \succ O_1 \succ O_4$ $(0 \leq v < 0.856)$ $O_2 \sim O_3 \succ O_1 \succ O_4$ $(v = 0.856)$ $O_3 \succ O_2 \succ O_1 \succ O_4$ $(0.856 < v \leq 1)$

6 Conclusion

In this paper, we have forced on the MAGDM problems, which occur in many decision areas, such as multi-response processes and hydropower systems. The IIFHG operator have been used to aggregate interval-valued intuitionistic information. We have utilized the optimization model to determine the weights associated with attributes. In the process of aggregation information, the IIFHG operator can avoid losing the original intuitionistic information and then ensure the accuracy and validity of the aggregated results. Moreover, based on IIFHG operator, we have developed a procedure for solving the MAGDM problems where all attribute values are expressed in IVIFNs and the the information about attribute weights is partially known. In the procedure, we have extended the VIKOR method to interval-valued intuitionistic fuzzy environment, used the different distances to calculate the particular measure of closeness of each alternative to the interval-valued intuitionistic PIS, and used the extended VIKOR method to rank and select the optimal alternative. Finally, a numerical example is given to verify the developed procedure and to demonstrate its effectiveness.

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