Fuzzy Optim Decis Making (2011) 10:125–152
DOI 10.1007/s10700-011-9099-0

Set covering-based surrogate approach for solving sup-$\mathcal{T}$ equation constrained optimization problems

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Published online: 15 March 2011

Abstract This work considers solving the sup-$\mathcal{T}$ equation constrained optimization problems from the integer programming viewpoint. A set covering-based surrogate approach is proposed to solve the sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables. This is our first trial of developing integer programming-based techniques to solve sup-$\mathcal{T}$ equation constrained optimization problems. Our computational results confirm the efficiency of the proposed method and show its potential for solving large scale sup-$\mathcal{T}$ equation constrained optimization problems.

Keywords Fuzzy relational equations · Triangular norms · Fuzzy optimization · Set covering problems

1 Introduction

Consider a sup-$\mathcal{T}$ equation constrained optimization problem in the following form:

This work was supported by the National Science Council (NSC) of R.O.C., under Grant NSC 99-2918-I-214-001.

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$$\text{(OP-T)}\quad \begin{aligned} &\min\ z = f(\mathbf{x}) \\ &\text{s.t.}\quad A \circ \mathbf{x} = \mathbf{b}, \\ &\qquad\ \ \mathbf{x} \in [0,1]^n, \end{aligned}$$

where $f : [0,1]^n \to \mathbf{R}$ is a real-valued function, $A = (a_{ij})_{m\times n} \in [0,1]^{m\times n}$, $\mathbf{x} = (x_j)_{n\times 1} \in [0,1]^n$, $\mathbf{b} = (b_i)_{m\times 1} \in [0,1]^m$ and $\circ$ stands for the specific sup-$\mathcal{T}$ composition with $\mathcal{T}$ being a continuous t-norm. In this way, $A \circ \mathbf{x} = \mathbf{b}$ represents a system of fuzzy relational equations with sup-$\mathcal{T}$ composition (or a system of sup-$\mathcal{T}$ equations for short).

The resolution of a system of fuzzy relational equations with max- min composition was first investigated by Sanchez (1976, 1977). It was shown in the literature that a system of fuzzy relational equations are well defined with respect to the max-$\mathcal{T}$ (or sup-$\mathcal{T}$) composite operation, with $\mathcal{T}$ being a continuous triangular norm. The minimum operator is the most frequently used triangular norm (see, e.g., Pedrycz (1985)). For a finite system of fuzzy relational equations with max-$\mathcal{T}$ composition, it is well known that its consistency can be verified in polynomial time by constructing and checking a potential maximum solution. Moreover, the solution set of a consistent system of fuzzy relational equations with max-$\mathcal{T}$ composition can be characterized by one maximum solution and a finite number of minimal solutions. However, as shown in Elbassioni (2008), Li and Fang (2008), and Markovskii (2005), the detection of all minimal solutions is a hard problem that is closely related to the hypergraph transversal problem, for which the best known algorithm runs in incremental quasi-polynomial time (see, e.g., Khachiyan et al. (2006)). Overviews of fuzzy relational equations and their applications can be found in Li and Fang (2009) and Peeva and Kyosev (2004).

The problem of minimizing a linear objective function subject to a consistent system of sup-$\mathcal{T}_M$ equations was first investigated by Fang and Li (1999) and later by Wu et al. (2002) and Wu and Guu (2005). Following the idea of Fang and Li (1999), the linear optimization problem subject to a system of sup-$\mathcal{T}_P$ equations was discussed by Loetamonphong and Fang (2001), Guu and Wu (2002), and Ghodousian and Khorram (2006), where $\mathcal{T}_P$ is the product operator. Furthermore, this problem was investigated under various composite operations by Wu and Guu (2005), Khorram and Ghodousian (2006), and Abbasi Molai and Khorram (2007). However, it was pointed out by Zimmermann (2007) and Shieh (2010) that the algorithms proposed by Khorram and Ghodousian (2006) and Abbasi Molai and Khorram (2007), respectively, may not lead to the optimal solution in some cases. Some other generalizations on this issue can be found in Abbasi Molai and Khorram (2008). When the problem of minimizing a nonlinear objective function is concerned, the situation could be very complicated. Lu and Fang (2001) designed a genetic algorithm to solve nonlinear optimization problems subject to a system of sup-$\mathcal{T}_M$ equations. Yang and Cao (2007) considered a special subclass of the problems of this type where the objective functions are posynomials. Fuzzy relational equation constrained geometric programming was investigated by Yang and Cao (2005), which is a generalization of the so-called latticized linear programming problem considered in Wang et al. (1991). The multi-objective optimization problem was discussed in Wang (1995) and Loetamonphong et al. (2002). These papers considered solving the sup-$\mathcal{T}$ equation constrained optimization problems with some specific compositions as well as the relatively small size instances.

When larger scale sup-$\mathcal{T}$ equation constrained optimization problems are faced, a systematic method is needed. Most recently, it was shown by Li and Fang (2008) that the problem of minimizing an objective function subject to a system of fuzzy relational equations can be reduced to a $0-1$ mixed integer programming problem in polynomial time. If the objective function is linear, or more generally, separable and monotone in each of the variables, then it can be further reduced to a set covering problem.

The set covering problem (SCP) is known to be one of Karp's 21 NP-complete problems and has been extensively studied. See, for instance, Caprara et al. (2000) and Golumbic and Hartman (2005). Many algorithms have been proposed in the literature for finding the exact solution to the problem (see Balas and Ho (1980), Beasley (1987), Fisher and Kedia (1990), and Balas and Carrera (1996).) Since the computational time is long for any exact algorithm, large set covering problems are usually solved by means of a greedy type heuristic. Classical greedy algorithms are very fast in practice, but typically do not provide high quality solutions, as reported in Balas and Ho (1980) and Balas and Carrera (1996). The most effective heuristic approaches to SCP are those based on Lagrangian relaxation with subgradient optimization, following the work of Balas and Ho (1980), and then the improvements of Beasley (1990), Fisher and Kedia (1990), and Balas and Carrera (1996). An alternative for Lagrangian relaxation is the surrogate relaxation. Lopes and Lorena (1994) proposed an effective heuristic approach based on continuous surrogate relaxation and subgradient optimization. The experimental results reported in Lopes and Lorena (1994) suggested that the use of surrogate instead of Lagrangian relaxation with subgradient optimization allows one to obtain near optimal multipliers in a shorter computing time. Beasley and Chu proposed a genetic algorithm-based heuristic approach in Beasley and Chu (1996). Their computational results showed that the genetic algorithm is capable of producing high-quality solutions for the set covering problems.

This paper follows the idea of Li and Fang (2008) and considers the solutions to the sup-$\mathcal{T}$ equation constrained optimization problems. Taking advantage of the well developed techniques and clarity of exposition in the theory of integer programming, a set covering-based surrogate approach is proposed to solve the sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables. This approach essentially provides a systematic method for solving the sup-$\mathcal{T}$ equation constrained optimization problems from an integer programming viewpoint. Computational results of our experiments confirm the efficiency of the proposed method and show its potential for solving large scale sup-$\mathcal{T}$ equation constrained optimization problems.

The rest of this paper is organized as follows. Some basic concepts and important properties associated with fuzzy relational equations are provided in Sect. 2. In Sect. 3, we summarize the related results in Li and Fang (2008), which shows that the sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables can be polynomially reduced to a set covering problem. To solve the resulting set covering problem, a surrogate heuristic algorithm is presented in Sect. 4. Numerical examples are included in Sect. 5 to illustrate the set covering-based surrogate approach for solving the sup-$\mathcal{T}$ equation constrained optimization problems. Conclusions are provided in Sect. 6.

2 Preliminaries

In this section, we recall some basic concepts and important properties associated with fuzzy relational equations, which are indispensable for the introduction of the sup-$\mathcal{T}$ equation constrained optimization problems in this context. All proofs are omitted to keep the paper succinct and readable. Readers may refer to Klement et al. (2000) for a rather complete overview of triangular norms, and to Li and Fang (2008), Li et al. (2008), and Li and Fang (2009) for the detailed analysis on the resolution and optimization of a system of sup-$\mathcal{T}$ equations.

2.1 Triangular norms

It is well known that a system of fuzzy relational equations can be well defined with sup-$\mathcal{T}$ composition, where $\mathcal{T}$ is a continuous triangular norm. A triangular norm (t-norm for short) is a binary operator $\mathcal{T} : [0, 1]^2 \to [0, 1]$, such that for all $x, y, z \in [0, 1]$, the following four axioms are satisfied:

(T1) $\mathcal{T}(x, y) = \mathcal{T}(y, x)$, (commutativity);
(T2) $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$, (associativity);
(T3) $\mathcal{T}(x, y) \leq \mathcal{T}(x, z)$, whenever $y \leq z$,(monotonicity);
(T4) $\mathcal{T}(x, 1) = x$, (boundary condition).

Since t-norms are just binary algebraic operators on the real unit interval [0, 1], the infix notation like $x \wedge_t y$ is usually used in the literature instead of the prefix notation $\mathcal{T}(x, y)$.

A t-norm $\mathcal{T}$ is said to be continuous if it is continuous as a real function of two arguments. Due to its commutativity and monotonicity properties, a t-norm is continuous if and only if it is continuous in one of its arguments. Analogously, a t-norm is said to be left- or right-continuous if it is left- or right-continuous, respectively, in one of its arguments. The three most important continuous t-norms are the minimum $\mathcal{T}_M$, the product $\mathcal{T}_P$, and the Łukasiewicz t-norm $\mathcal{T}_L$ defined, respectively, by

$\mathcal{T}_M(x, y) = \min(x, y)$, (minimum, Gödel t-norm, Zadeh t-norm),
$\mathcal{T}_P(x, y) = x \cdot y$, (probabilistic product, Goguen t-norm),
$\mathcal{T}_L(x, y) = \max(x + y - 1, 0)$, (bounded difference, Łukasiewicz t-norm).

To characterize the solution set of the system of sup-$\mathcal{T}$ equations, two residual operators can be defined with respect to a continuous t-norm $\mathcal{T}$.

Definition 2.1 The binary residual operators $\mathcal{I}_\mathcal{T} : [0, 1]^2 \to [0, 1]$ and $\mathcal{J}_\mathcal{T} : [0, 1]^2 \to [0, 1]$ with respect to a t-norm $\mathcal{T}$ are defined, respectively, by

$$\mathcal{I}_\mathcal{T}(x, y) = \sup\{z \in [0, 1] \mid \mathcal{T}(x, z) \leq y\},$$
$$\mathcal{J}_\mathcal{T}(x, y) = \inf\{z \in [0, 1] \mid \mathcal{T}(x, z) \geq y\}.$$

The residual operator $\mathcal{I}_\mathcal{T}$ is known as a residual implicator or briefly an R-implicator in fuzzy logic while the residual operator $\mathcal{J}_\mathcal{T}$ has no particular logical interpretation. The infix notations are usually used to denote these two residual operators, i.e.,

Table 1 Residual operators of the Gödel, Goguen, and Łukasiewicz t-norms

$\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}(x, y)$	$\mathcal{J}_{\mathcal{T}}(x, y)$
$\mathcal{T}_M$	$\begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y, \\ y, & \text{otherwise.} \end{cases}$
$\mathcal{T}_P$	$\begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y, \\ \frac{y}{x}, & \text{if } 0 < y \leq x, \\ 0, & \text{otherwise.} \end{cases}$
$\mathcal{T}_L$	$\min(1 - x + y, 1)$	$\begin{cases} 1, & \text{if } x < y, \\ 1 - x + y, & \text{if } 0 < y \leq x, \\ 0, & \text{otherwise.} \end{cases}$

$\mathcal{I}_{\mathcal{T}}(x, y) = x\varphi_t y$ and $\mathcal{J}_{\mathcal{T}}(x, y) = x\sigma_t y$, respectively. In the literature of fuzzy relational equations, the residual implicators are also known as φ-operators which were introduced by Pedrycz (1985) in a different approach to describing the solutions of sup-$\mathcal{T}$ equations. The residual operator $\mathcal{J}_{\mathcal{T}}$ was also discussed in Di Nola et al. (1989) with a slightly different definition.

Theorem 2.1 *Let $\mathcal{T}$ be a left-continuous t-norm and $\mathcal{I}_{\mathcal{T}}$ its associated residual implicator. It holds for all $a, b \in [0, 1]$ that $\mathcal{T}(a, x) \leq b$ if and only if $x \leq \mathcal{I}_{\mathcal{T}}(a, b)$.*

Theorem 2.2 *Let $\mathcal{T}$ be a continuous t-norm and $\mathcal{I}_{\mathcal{T}}$ and $\mathcal{J}_{\mathcal{T}}$ its associated residual operators. The equation $\mathcal{T}(a, x) = b$ has a solution for given $a, b \in [0, 1]$ if and only if $b \leq a$, in which case the solution set of $\mathcal{T}(a, x) = b$ is given by the closed interval $[\mathcal{J}_{\mathcal{T}}(a, b), \mathcal{I}_{\mathcal{T}}(a, b)]$.*

Theorem 2.1 plays a crucial role in the resolution of sup-$\mathcal{T}$ equations, which is actually a special scenario of the general theory of Galois connections Blyth and Janowitz (1972). The residual operators $\mathcal{I}_{\mathcal{T}}$ and $\mathcal{J}_{\mathcal{T}}$ of the three most important continuous t-norms are listed in Table 1.

2.2 Resolution of systems of sup-$\mathcal{T}$ equations

2.2.1 Solvability and the solution set

In this section, we focus on the resolution of a finite system of fuzzy relational equations $A \circ \mathbf{x} = \mathbf{b}$ with sup-$\mathcal{T}$ composition where $\mathcal{T}$ is a continuous t-norm and the coefficient matrix $A = (a_{ij})_{m \times n} \in [0, 1]^{mn}$, the unknown vector $\mathbf{x} = (x_j)_{n \times 1} \in [0, 1]^n$ and the right hand side constants $\mathbf{b} = (b_i)_{m \times 1} \in [0, 1]^m$. For the convenience of description, two index sets are defined by $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. The set of all solutions to $A \circ \mathbf{x} = \mathbf{b}$ is called its complete solution set and denoted by $S(A, \mathbf{b}) = \{\mathbf{x} \in [0, 1]^n \mid A \circ \mathbf{x} = \mathbf{b}\}$. A partial order can be defined on $S(A, \mathbf{b})$ by extending the natural order such that for any $\mathbf{x}^1, \mathbf{x}^2 \in S(A, \mathbf{b})$, $\mathbf{x}^1 \leq \mathbf{x}^2$ if and only if $x_j^1 \leq x_j^2$ for all $j \in N$. A system of sup-$\mathcal{T}$ equations $A \circ \mathbf{x} = \mathbf{b}$ is called consistent if $S(A, \mathbf{b}) \neq \emptyset$, otherwise, it is inconsistent. Due to the monotonicity of the t-norm involved in the composition, if $\mathbf{x}^1, \mathbf{x}^2 \in S(A, \mathbf{b})$ and $\mathbf{x}^1 \leq \mathbf{x}^2$, any $\mathbf{x}$ such that $\mathbf{x}^1 \leq \mathbf{x} \leq \mathbf{x}^2$ is also in $S(A, \mathbf{b})$. Therefore, the attention could be focused on the so-called extremal solutions as defined below.

Definition 2.2 A solution $\check{\mathbf{x}} \in S(A, \mathbf{b})$ is called a minimal or lower solution if for any $\mathbf{x} \in S(A, \mathbf{b})$, the relation $\mathbf{x} \leq \check{\mathbf{x}}$ implies $\mathbf{x} = \check{\mathbf{x}}$. A solution $\hat{\mathbf{x}} \in S(A, \mathbf{b})$ is called the maximum or greatest solution if $\mathbf{x} \leq \hat{\mathbf{x}}, \forall\, \mathbf{x} \in S(A, \mathbf{b})$.

Theorem 2.3 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations. A vector $\mathbf{x} \in [0, 1]^n$ is a solution to $A \circ \mathbf{x} = \mathbf{b}$ if and only if there exists an index $j_i \in N$ for each $i \in M$ such that*

$$a_{ij_i} \wedge_t x_{j_i} = b_i \quad and \quad a_{ij} \wedge_t x_j \leq b_i, \quad i \in M, j \in N.$$

Theorem 2.3 holds in a straightforward way due to the non-interactivity property of the maximum operator, i.e., $a \vee b \in \{a, b\}$. Theorems 2.2 and 2.3 lead to the following well-known solvability criteria of a system $A \circ \mathbf{x} = \mathbf{b}$ and the characterization of its solution set, both of which were first seen in Sanchez (1976, 1977).

Theorem 2.4 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with a continuous t-norm $\mathcal{T}$. The system is consistent if and only if the vector $A^T \varphi_t \mathbf{b}$ with its components defined by*

$$(A^T \varphi_t \mathbf{b})_j = \inf_{i \in M} \mathcal{I}_{\mathcal{T}}(a_{ij}, b_i), \quad j \in N,$$

is a solution to $A \circ \mathbf{x} = \mathbf{b}$. Moreover, if the system is consistent, the complete solution set $S(A, \mathbf{b})$ can be determined by a unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A, \mathbf{b}) = \bigcup_{\check{\mathbf{x}} \in \check{S}(A, \mathbf{b})} \{\mathbf{x} \in [0, 1]^n \mid \check{\mathbf{x}} \leq \mathbf{x} \leq \hat{\mathbf{x}}\},$$

where $\check{S}(A, \mathbf{b})$ is the set of all minimal solutions to $A \circ \mathbf{x} = \mathbf{b}$ and $\hat{\mathbf{x}} = A^T \varphi_t \mathbf{b}$.

Clearly, the consistency of a system $A \circ \mathbf{x} = \mathbf{b}$ can be detected by constructing and checking the potential maximum solution in a time complexity of $O(mn)$. The detection of all minimal solutions is rather complicated and a very interesting issue for investigation. It has been observed for a long time that the detection of minimal solutions is closely related with the set covering problem. See, e.g., Markovskii (2005).

2.2.2 Minimal solutions and set covering problems

The close relation between minimal solutions of a system of sup-$\mathcal{T}$ equations and some set covering problems has been noticed and described from various aspects since the structure of the complete solution set was fully understood. It provides some important information for the analysis of the number of minimal solutions and the development of algorithms to obtain all the minimal solutions.

With the potential maximum solution $\hat{\mathbf{x}}$, the characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ of a system $A \circ \mathbf{x} = \mathbf{b}$ can be defined by

$$\tilde{q}_{ij} = \begin{cases} [\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i), \hat{x}_j], & \text{if } \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, \\ \emptyset, & \text{otherwise,} \end{cases}$$

and obtained in a time complexity of $O(mn)$. It was reported in Li and Fang (2008) that when $\mathcal{T}$ is a continuous Archimedean t-norm of which the product operator $\mathcal{T}_P$ and the Łukasiewicz t-norm $\mathcal{T}_L$ are typical representatives, the nonempty elements in $\tilde{Q}$ are always singletons with their values determined by the potential maximum solution. The characteristic matrix $\tilde{Q}$ in this case can be further simplified as $Q = (q_{ij})_{m\times n}$ with

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3 Let $Q = (q_{ij})_{m\times n} \in \{0, 1\}^{mn}$ be a binary matrix. A column j is said to cover a row i if $q_{ij} = 1$. A set of nonzero columns P forms a covering of Q if each row of Q is covered by some column from P. A column j in a covering P is called redundant if the set of columns $P\backslash\{j\}$ remains to be a covering of Q. A covering P is irredundant if it has no redundant columns. The set of all coverings of Q is denoted by $P(Q)$ while the set of all irredundant coverings of Q is denoted by $\check{P}(Q)$.

It is well-known that the set of all coverings $P(Q)$ of a binary matrix Q can be well represented by the feasible solution set of a set covering problem, i.e.,$\{\mathbf{u} \in \{0, 1\}^n \mid Q\mathbf{u} \geq e\}$ where $e = (1, 1, \ldots, 1)^T \in \{0, 1\}^m$, while the irredundant coverings of Q correspond to the minimal elements in $\{\mathbf{u} \in \{0, 1\}^n \mid Q\mathbf{u} \geq e\}$.

Theorem 2.5 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous Archimedean t-norm. The set of all minimal solutions $\check{S}(A, \mathbf{b})$ of the system $A \circ \mathbf{x} = \mathbf{b}$ is one-to-one corresponding to the set of all irredundant coverings $\check{P}(Q)$ of its simplified characteristic matrix Q.*

Theorem 2.5 indicates that determining all minimal solutions to a system $A \circ \mathbf{x} = \mathbf{b}$ with a continuous Archimedean t-norm $\mathcal{T}$ is equivalent to determining all irredundant coverings of its simplified characteristic matrix, which can be interpreted as a procedure of finding the minimal cover of Q.

When the system of sup-$\mathcal{T}$ equations $A \circ \mathbf{x} = \mathbf{b}$ with $\mathcal{T}$ being a continuous non-Archimedean t-norm, the situation turns out to be a little bit complicated. Denote r_j the numbers of different values in $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ for each $j \in N$, $r = \sum_{j\in N} r_j$, $K_j = \{1, 2, \ldots, r_j\}$, and $\check{v}_{jk}$, for $k \in K_j$, the different values in $\{\mathcal{J}_{\mathcal{T}}(a_{ij}, b_i) \mid \mathcal{T}(a_{ij}, \hat{x}_j) = b_i, i \in M\}$ for every $j \in N$. Let $\check{v} = (\check{v}_{11}, \ldots, \check{v}_{1r_1}, \ldots, \check{v}_{n1}, \ldots, \check{v}_{nr_n})^T \in [0, 1]^r$ and

$$x_j = \sum_{k\in K_j} \check{v}_{jk} u_{jk}, \quad j \in N$$

where $u_{jk} \in \{0, 1\}, \forall\, k \in K_j, j \in N$. Obviously, for each $j \in N$, at most one of $u_{jk}, k \in K_j$ can be 1, i.e., $\sum_{k \in K_j} u_{jk} \leq 1, j \in N$. These restrictions are called the innervariable incompatibility constraints and can be represented by $G\mathbf{u} \leq e^n$, where $e^n = (1, 1, \cdots, 1)^T \in \{0, 1\}^n$, $\mathbf{u} = (u_{11}, \cdots, u_{1r_1}, \cdots, u_{n1}, \ldots, u_{nr_n})^T \in \{0, 1\}^r$ and $G = (g_{jk})_{n \times r}$ with

$$g_{jk} = \begin{cases} 1, \text{ if } \sum_{s=1}^{j-1} r_s < k \leq \sum_{s=1}^{j} r_s, \\ 0 \ \text{ otherwise.} \end{cases}$$

Actually, the incompatibility constraints with $r_j = 1$ are redundant and hence can be removed. Clearly, if $r_j = 1$ for all $j \in N$, all the incompatibility constraints are redundant and no additional difficulties will be imposed. In this case, the values of the nonzero elements in a minimal solution are uniquely determined although they may be different from those in the maximum solution.

As a consequence of this transformation, the characteristic matrix $\tilde{Q}$ can be converted to its augmented characteristic matrix $Q = (q_{ik})_{m \times r} \in \{0, 1\}^{mr}$ where

$$q_{ik} = \begin{cases} 1, \text{ if } \sum_{s=1}^{j-1} r_s < k \leq \sum_{s=1}^{j} r_s, \ \check{v}_k \in \tilde{q}_{ij}, \ j \in N, \\ 0. \text{ otherwise.} \end{cases}$$

Definition 2.4 Let $Q = (q_{ik})_{m \times r} \in \{0, 1\}^{mr}$ and $G = (q_{jk})_{n \times r} \in \{0, 1\}^{nr}$ be two binary matrices. A column k of Q is said to cover a row i of Q if $q_{ik} = 1$. A set of nonzero columns P forms a G-covering of Q if each row of Q is covered by some column in P, i.e., $Q\mathbf{u}^P \geq e^m$, and also satisfies $G\mathbf{u}^P \leq e^n$ where $\mathbf{u}^P = (u_k^P)_{r \times 1}$ and

$$u_k^P = \begin{cases} 1, \text{ if } k \in P, \\ 0, \text{ otherwise.} \end{cases}$$

A column k in a G-covering P is called redundant if the set of columns $P \backslash \{k\}$ remains to be a G-covering of Q. A G-covering P is irredundant if P has no redundant columns. The set of all G-coverings of Q is denoted by $P_G(Q)$ while the set of all irredundant G-coverings of Q is denoted by $\check{P}_G(Q)$.

Theorem 2.6 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous non-Archimedean t-norm. Each minimal solution to $A \circ \mathbf{x} = \mathbf{b}$ corresponds to an irredundant G-covering of Q, where Q and G are the augmented characteristic matrix and the coefficient matrix of the innervariable incompatibility constraints, respectively.*

3 The sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables

In this section, we discuss the sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables. The material presented in this section is essentially based on Li and Fang (2008) with some modification.

When the problem (OP-T) has a linear objective function, it is of the form

$$\text{(LO-T)}\quad \begin{aligned} \min\ & z = \mathbf{c}^T\mathbf{x} \\ \text{s.t.}\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n, \end{aligned}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbf{R}^n$ is the weight (or cost) vector, and c_j represents the weight associated with the variable x_j, $j = 1, 2, \ldots, n$.

Theorem 3.1 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous t-norm. The maximum solution $\hat{\mathbf{x}}$ is an optimal solution that minimizes the objective function $z = \mathbf{c}^T\mathbf{x}$ over $S(A, \mathbf{b})$ if $c_j \leq 0$ for all $j \in N$. One of the minimal solutions is an optimal solution that minimizes the objective function $z = \mathbf{c}^T\mathbf{x}$ over $S(A, \mathbf{b})$ if $c_j \geq 0$ for all $j \in N$.*

Theorem 3.1 was first stated by Fang and Li (1999) for sup-$\mathcal{T}_M$ equations, which is valid for general continuous t-norms since it only depends on the structure of the complete solution set $S(A, \mathbf{b})$.

With the aid of Theorem 3.1, any given weight vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbf{R}^n$ can be separated into two parts, i.e., $\mathbf{c}^+ = (c_1^+, c_2^+, \ldots, c_n^+)^T$ and $\mathbf{c}^- = (c_1^-, c_2^-, \ldots, c_n^-)^T$ such that, for every $j \in N$,

$$c_j^+ = \begin{cases} c_j, & \text{if } c_j > 0, \\ 0, & \text{if } c_j \leq 0, \end{cases} \quad \text{and} \quad c_j^- = \begin{cases} 0, & \text{if } c_j > 0, \\ c_j, & \text{if } c_j \leq 0. \end{cases}$$

Hence, $\mathbf{c} = c^+ + c^-$ with $c^+ \geq 0$ and $c^- \leq 0$. Two subproblems can be defined, respectively, as

$$\begin{aligned} \min\ & z^+ = \sum_{j \in N} c_j^+ x_j \\ \text{s.t.}\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n, \end{aligned}$$

and

$$\begin{aligned} \min\ & z^- = \sum_{j \in N} c_j^- x_j \\ \text{s.t.}\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n. \end{aligned}$$

Theorem 3.2 *Let $A \circ \mathbf{x} = \mathbf{b}$ be a consistent system of sup-$\mathcal{T}$ equations with $\mathcal{T}$ being a continuous t-norm. For any weight vector $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbf{R}^n$, the vector $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ with*

$$x_j^* = \begin{cases} x_j^{+*}, & \text{if } c_j > 0, \\ \hat{x}_j, & \text{if } c_j \le 0, \end{cases}$$

is an optimal solution that minimizes the objective function $\mathbf{c}^T\mathbf{x}$ *over* $S(A, \mathbf{b})$, *where* $\mathbf{x}^{+*} = (x_1^{+*}, x_2^{+*}, \ldots, x_n^{+*})^T$ *is a solution that minimizes* $\sum_{j\in N} c_j^+ x_j$ *over* $S(A, \mathbf{b})$, *and* $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$ *is the maximum solution that minimizes* $\sum_{j\in N} c_j^- x_j$ *over* $S(A, \mathbf{b})$.

By Theorem 3.2, for an arbitrary weight vector, solving the linear optimization problem subject to a system of sup-$\mathcal{T}$ equations can be decomposed into two subproblems, one of which can be solved analytically while another is not easy to handle. The subproblem with nonnegative weight vector is inevitably an NP-hard problem since the classical set covering problem can be regarded as a special scenario of this problem. On the other hand, this problem can be polynomially reduced to a set covering problem or a constrained set covering problem, where the existence of additional constraints depends on whether the involved continuous triangular norm is Archimedean or not.

Theorem 3.3 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a consistent system of sup-*$\mathcal{T}$ *equations with* $\mathcal{T}$ *being a continuous Archimedean t-norm. Denote* $\hat{\mathbf{x}}$ *its maximum solution and* Q *its associated simplified characteristic matrix. For any given weight vector* $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$, *the following problem*

$$(\mathrm{LO-Ar}) \quad \begin{aligned} \min\ & z_{\mathbf{x}}^+ = \sum_{j\in N} c_j^+ x_j \\ \text{s.t.}\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned}$$

is equivalent to the set covering problem

$$(\mathrm{SCP}) \quad \begin{aligned} \min\ & z_{\mathbf{u}}^+ = \sum_{j\in N} (c_j^+ \hat{x}_j) u_j \\ \text{s.t.}\ & Q\mathbf{u} \ge \mathbf{e}^m, \\ & \mathbf{u} \in \{0, 1\}^n, \end{aligned}$$

in the sense that any optimal solution $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ *to problem (SCP) defines an optimal solution* $\mathbf{x}^{+*} = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \ldots, \hat{x}_n u_n^*)^T$ *to problem (LO-Ar).*

Theorem 3.4 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a consistent system of sup-*$\mathcal{T}$ *equations with* $\mathcal{T}$ *being a continuous non-Archimedean t-norm. Denote* $\hat{\mathbf{x}}$ *its maximum solution,* Q *its augmented characteristic matrix, and* G *the associated coefficient matrix of the innervariable incompatibility constraints. For any given weight vector* $\mathbf{c} = (c_1, c_2, \ldots, c_n)^T$, *the following problem*

$$(\mathrm{LO-nAr}) \quad \begin{aligned} \min\ & z_{\mathbf{x}}^+ = \sum_{j\in N} c_j^+ x_j \\ \text{s.t.}\ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned}$$

is equivalent to the constrained set covering problem

$$(\text{CSCP}) \quad \begin{aligned} \min\ z_{\mathbf{u}}^{+} &= \sum_{j\in N}\sum_{k\in K_j}(c_j^{+}\check{v}_{jk})u_{jk} \\ \text{s.t.}\ \ & Q\mathbf{u} \geq \mathbf{e}^m, \\ & G\mathbf{u} \leq \mathbf{e}^n, \\ & \mathbf{u} \in \{0,1\}^r, \end{aligned}$$

in the sense that any optimal solution $\mathbf{u}^* = (u_{11}^*, \ldots, u_{1r_1}^*, \ldots, u_{n1}^*, \cdots, u_{nr_n}^*)^T$ *to problem (CSCP) defines an optimal solution*

$$\mathbf{x}^{+*} = \left(\sum_{k\in K_1}\check{v}_{1k}u_{1k}^*, \sum_{k\in K_2}\check{v}_{2k}u_{2k}^*, \ldots, \sum_{k\in K_n}\check{v}_{nk}u_{nk}^*\right)^T$$

to problem (LO-nAr).

Once the optimal solutions $\mathbf{x}^{+*}$ to problems (LO-Ar) and (LO-nAr) are obtained for any given weight vector $\mathbf{c}$, the corresponding optimal solution that minimizes the objective function $\mathbf{c}^T\mathbf{x}$ over $S(A, \mathbf{b})$ can be obtained according to Theorem 3.2. Moreover, note that the weight $c_j^{+}\check{v}_{jk}$ in problem (CSCP) is nonnegative for each $k \in K_j$ and $j \in N$, and hence, the constraint $G\mathbf{u} \leq \mathbf{e}^n$ can be further removed and the problem (CSCP) consequently becomes a set covering problem.

The procedure for solving sup-$\mathcal{T}$ equation constrained linear optimization problems can be directly extended to the case where the objective function is separable and monotone in each of the variables, i.e., $z = \sum_{j\in N} f_j(x_j)$ with $f_j : [0, 1] \to \mathbf{R}$ being a monotone function for every $j \in N$. Without loss of generality, we may assume that $f_j(0) = 0$ for every $j \in N$.

Theorem 3.5 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a consistent system of sup-*$\mathcal{T}$ *equations with* $\mathcal{T}$ *being a continuous Archimedean t-norm. Denote* $\hat{\mathbf{x}}$ *its maximum solution and* Q *its associated simplified characteristic matrix. Given the objective function* $z = \sum_{j\in N} f_j(x_j)$ *with* $f_j : [0, 1] \to \mathbf{R}$ *being a monotone function for every* $j \in N$*. Let* $N^- \triangleq \{j \in N \mid f_j\ is a decreasing function\}$, $N^+ \triangleq N \setminus N^-$*. The following problem*

$$(\text{SMO}-\text{Ar}) \quad \begin{aligned} \min\ z_{\mathbf{x}}^{+} &= \sum_{j\in N^+} f_j(x_j) \\ \text{s.t.}\ \ & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0,1]^n, \end{aligned}$$

is equivalent to the set covering problem

$$(\text{SMSCP}) \quad \begin{aligned} \min\ z_{\mathbf{u}}^{+} &= \sum_{j\in N^+} f_j(\hat{x}_j)u_j \\ \text{s.t.}\ \ & Q\mathbf{u} \geq \mathbf{e}^m, \\ & \mathbf{u} \in \{0,1\}^n, \end{aligned}$$

in the sense that any optimal solution $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ *to problem (SMSCP) defines an optimal solution* $\mathbf{x}^{+*} = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \ldots, \hat{x}_n u_n^*)^T$ *to problem (SMO-Ar).*

Theorem 3.6 *Let* $A \circ \mathbf{x} = \mathbf{b}$ *be a consistent system of sup-*$\mathcal{T}$ *equations with* $\mathcal{T}$ *being a continuous non-Archimedean t-norm. Denote* $\hat{\mathbf{x}}$ *its maximum solution,* Q *its augmented characteristic matrix, and* G *the associated coefficient matrix of the inner-variable incompatibility constraints. Given the objective function* $z = \sum_{j \in N} f_j(x_j)$ *with* $f_j : [0, 1] \to \mathbf{R}$ *being a monotone function for every* $j \in N$*. The following problem*

$$(\mathrm{SMO-nAr}) \quad \begin{aligned} \min\ z_{\mathbf{x}}^+ = & \sum_{j \in N^+} f_j(x_j) \\ \text{s.t.} \quad & A \circ \mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \in [0, 1]^n, \end{aligned}$$

is equivalent to the constrained set covering problem

$$(\mathrm{SMCSCP}) \quad \begin{aligned} \min\ z_{\mathbf{u}}^+ = & \sum_{j \in N^+} \sum_{k \in K_j} f_j(\check{v}_{jk}) u_{jk} \\ \text{s.t.} \quad & Q\mathbf{u} \geq \mathbf{e}^m, \\ & G\mathbf{u} \leq \mathbf{e}^n, \\ & \mathbf{u} \in \{0, 1\}^r, \end{aligned}$$

in the sense that any optimal solution $\mathbf{u}^* = (u_{11}^*, \ldots, u_{1r_1}^*, \ldots, u_{n1}^*, \ldots, u_{nr_n}^*)^T$ *to problem (SMCSCP) defines an optimal solution*

$$\mathbf{x}^{+*} = \left(\sum_{k \in K_1} \check{v}_{1k} u_{1k}^*, \sum_{k \in K_2} \check{v}_{2k} u_{2k}^*, \ldots, \sum_{k \in K_n} \check{v}_{nk} u_{nk}^* \right)^T$$

to problem (SMO-nAr).

Once the optimal solutions $\mathbf{x}^{+*} = (x_1^{+*}, x_2^{+*}, \ldots, x_n^{+*})^T$ to problems (SMO-Ar) and (SMO-nAr) are obtained, the corresponding optimal solution that minimizes the objective function $\sum_{j \in N} f_j(x_j)$ over $S(A, \mathbf{b})$, i.e. $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$, can be obtained with

$$x_j^* = \begin{cases} x_j^{+*}, & \text{if } j \in N^+, \\ \hat{x}_j, & \text{if } j \in N^-. \end{cases}$$

Similarly, the weight $f_j(\check{v}_{jk})$ in problem (SMCSCP) is nonnegative for each $k \in K_j$ and $j \in N^+$, and hence, the constraint $G\mathbf{u} \leq \mathbf{e}^n$ can be further removed. Consequently, the problem (SMCSCP) becomes a set covering problem.

According to above theorems, the problem of minimizing a separable and monotone objective function in each of the variables subject to a system of sup-$\mathcal{T}$ equations, with $\mathcal{T}$ being a continuous t-norm, can be polynomially reduced to a set covering problem.

This reduction can be completed in a time complexity of $O(mn)$ if the t-norm $\mathcal{T}$ is Archimedean, or in a time complexity of $O(m^2 n)$ if it is non-Archimedean.

The set covering problem is a well-known NP-hard combinatorial optimization problem and, therefore, difficult to solve, i.e., computational time is high for any exact algorithm. Hence, large set covering problems are usually solved by means of a greedy type heuristic for quickly find optimal or near optimal solutions. However, classical greedy algorithms typically do not provide high quality solutions, an effective heuristic approach based upon continuous surrogate relaxations and subgradient optimization has been proposed for solving set covering problems. The computational results in Lopes and Lorena (1994) show that the surrogate heuristic approach is faster and more stable than the well considered heuristic algorithms based on Lagrangian relaxations.

4 A surrogate heuristic for set covering problems

In this section, a surrogate heuristic for solving the resulting set covering problem of the sup-$\mathcal{T}$ equation constrained optimization problem is presented.

Consider the set covering problem

$$\begin{aligned} \min\ z &= \sum_{j\in J} c'_j u_j \\ \text{s.t.}\ \ & Q\mathbf{u} \geq \mathbf{e}^m, \\ & u_j \in \{0,1\}, \quad j \in J, \end{aligned} \tag{1}$$

where $c'_j \geq 0$ represents the weight associated with the variable u_j, $j \in J \triangleq \{1, 2, \dots, n'\}$, Q is a matrix ($m \times n'$) of zeros and ones, $\mathbf{e}^m$ is the m-vector of 1's, and $u_j = 1$ if column j is in the solution, $u_j = 0$ otherwise.

The continuous surrogate relaxation of the set covering problem (1) can be defined as follows:

$$(S_\mathbf{w})\quad \begin{aligned} \min\ z &= \sum_{j\in J} c'_j u_j \\ \text{s.t.}\ \ & \mathbf{w}^T Q\mathbf{u} \geq \mathbf{w}^T\mathbf{e}^m, \\ & u_j \in [0,1], \quad j \in J, \end{aligned}$$

where $\mathbf{w} = (w_1, w_2, \dots, w_m)^T \in \mathbf{R}_+^m$ is the surrogate multiplier vector. It is easy to observe that the continuous surrogate relaxation, $(S_\mathbf{w})$, corresponds to a very particular case of a classical knapsack problem; in this case, its optimal solution can be achieved resorting to the well known properties established for such problems. To find the optimal solution of $(S_\mathbf{w})$, for a given $\mathbf{w} \in \mathbf{R}_+^m$, the efficiency of the jth variable of $(S_\mathbf{w})$ is defined as follows:

$$d_j \triangleq \frac{c'_j}{\mathbf{w}^T q_j}, \quad j \in J,$$

where q_j is the jth column of matrix Q. If the optimal solution $\mathbf{u_w} = (u_{w1}, u_{w2}, \ldots, u_{wn'})^T$ of $(S_\mathbf{w})$ is assumed to be ordered according to their efficiency, i.e., $d_{w1} \leq d_{w2} \leq \ldots \leq d_{wn'}$, then

$$\mathbf{u_w} = \left(1, 1, 1, \ldots, 1, \frac{\mathbf{w}^T \mathbf{e}^m - \sum_{j=1}^{j^*-1} \mathbf{w}^T q_{wj}}{\mathbf{w}^T q_{wj^*}}, 0, 0, 0, \ldots, 0\right),$$

where $\sum_{j=1}^{j^*-1} \mathbf{w}^T q_{wj} \leq \mathbf{w}^T \mathbf{e}^m \leq \sum_{j=1}^{j^*} \mathbf{w}^T q_{wj}$ and wj^* is the index of the fractional variable.

Let $v(\cdot)$ denotes the optimal value of problem $(\cdot)$. The problem to find a surrogate multiplier vector $\mathbf{w} \in \mathbf{R}_+^m$ that maximizes $v(S_\mathbf{w})$ is called the surrogate dual problem. We assume the reader is familiar with the duality theory in the combinatorial optimization (see, e.g., Parker (1988) for details). The surrogate heuristic is a procedure that approximates the solution of the surrogate dual and provides a lower bound for the set covering problem (1). A subgradient method is employed in the surrogate heuristic to determine the optimal (near optimal) surrogate multipliers for the surrogate dual. The use of subgradient procedure in the context of Lagrangian duality to solve structured combinatorial optimization problems has been shown to be very effective and to work better than classical gradient procedures or column generation techniques Held et al. (1974), Nemhauser and Wolsey (1988). For a given $\mathbf{w} \in \mathbf{R}_+^m$, the subgradient procedure in the surrogate heuristic uses the direction

$$G(\mathbf{w}) = (G_1(\mathbf{w}), G_2(\mathbf{w}), \ldots, G_m(\mathbf{w}))^T \triangleq \mathbf{e}^m - Q\mathbf{u_w},$$

where $\mathbf{u_w}$ is the optimal solution of $(S_\mathbf{w})$ and u_{wj^*} is set to be 0. It generates a sequence of nonnegative surrogate multiplier vectors $\mathbf{w}^{(0)}, \mathbf{w}^{(1)}, \ldots$, where $\mathbf{w}^{(0)}$ is a given initial vector and $\mathbf{w}^{(l+1)}$ is updated from $\mathbf{w}^{(l)}$, $l = 0, 1, 2, \ldots$, by the following formula:

$$w_i^{(l+1)} \leftarrow \max\{0, w_i^{(l)} + \rho \frac{f_{ub} - f_{lb}}{\| G(\mathbf{w}^{(l)}) \|^2} G_i(\mathbf{w}^{(l)})\}, \quad i = 1, 2, \ldots, m,$$

with f_{ub} and f_{lb} being the upper and lower bounds of the set covering problem (1), respectively, and ρ being a parameter associated with the step size. It has been assured that the direction $G(\mathbf{w})$, for $\mathbf{w} \in \mathbf{R}_+^m$, is a subgradient for the Lagrangian function

$$(L_\lambda) \begin{array}{l} \min\ z = \sum_{j \in J} c'_j u_j + \lambda^T (\mathbf{e}^m - Q\mathbf{u}), \\ \text{s.t. } u_j \in \{0, 1\}, \quad j \in J, \end{array}$$

by setting $\lambda = (c'_{wj^*}/\mathbf{w}^T q_{wj^*}) \cdot \mathbf{w}$, and it is also immediate that $v(L_\lambda) = v(S_\mathbf{w})$ Lorena and Plateau (1988). As a consequence it can be conjectured that the surrogate heuristic is also a Lagrangian heuristic. Computational tests in Lopes and Lorena (1994) for large scale set covering problems (up to 1,000 rows and 12,000 columns) indicate the surrogate heuristic approach produces better-quality results than algorithms based on Lagrangian relaxations in terms of final solutions and mainly in computer times.

A surrogate heuristic algorithm based upon continuous surrogate relaxations ($S_{\mathbf{w}}$), and subgradient optimization for solving the set covering problem (1) can be stated as follows.

Surrogate heuristic algorithm

Step 1. Initialize. Let $f_{ub} = +\infty$, $f_{lb} = -\infty$ and $\mathbf{w} = \mathbf{e}^m$ ($\mathbf{w} \geq 0$ and $\mathbf{w} \neq 0$);

Step 2. Solve ($S_{\mathbf{w}}$) and let the solution be $\mathbf{u_w} = (u_{w1}, u_{w2}, \ldots, u_{wn'})^T$ with optimal value $v(S_{\mathbf{w}})$ and wj^* the index of the fractional variable.

Step 3. Construct a feasible solution for the set covering problem (1) using $\mathbf{u_w}$. Set $u_{wj^*} = 1$ and construct a feasible solution $\mathbf{u}_f = (u_{f_1}, u_{f_2}, \ldots, u_{f_{n'}})^T$ for (1) with value $z(\mathbf{w}) = \sum_{j\in J} c'_j u_{f_j}$. This is done by extending $\mathbf{u_w}$ by means of a greedy heuristic.

Step 4. Update f_{ub} and f_{lb}. Let $f_{ub} = \min(f_{ub}, z(\mathbf{w}))$ and $f_{lb} = \max(f_{lb}, v(S_{\mathbf{w}}))$.

Step 5. Check the stopping rules in Sect. 4.1.3. If none holds, go to step 6. Otherwise, stop and output $\mathbf{u}_f$ as the optimal (near optimal) solution for the set covering problem (1).

Step 6. Find the subgradient direction $G(\mathbf{w})$ and the step size $t_{\mathbf{w}}$. Set $u_{wj^*} = 0$ and define $\rho \triangleq \alpha/d_{wj^*}$ for the new step size

$$t_{\mathbf{w}} \triangleq \rho(f_{ub} - f_{lb}),$$

with α being a parameter, and update the vector $\mathbf{w}$ with

$$w_i \leftarrow \max\{0, w_i + t_{\mathbf{w}} G_i(\mathbf{w}) / \parallel G(\mathbf{w}) \parallel^2\}, \quad i = 1, 2, \ldots, m,$$

where

$$G(\mathbf{w}) = \mathbf{e}^m - \sum_{j\in J} q_{wj} u_{wj}.$$

Return to step 2.

4.1 Implementation issues on the surrogate heuristic algorithm

In this section, some implementation issues on the surrogate heuristic algorithm discussed in Lopes and Lorena (1994) are considered.

4.1.1 The construction of a feasible solution

Assume that the columns are ordered in increasing cost order, and the columns with equal cost are ordered in decreasing order of number of rows that they cover. The procedure used for construction of a feasible solution $\mathbf{u}_f$ using $\mathbf{u_w}$ is similar to that one in Beasley (1990). Set $u_{wj^*} = 1$ and for each row i which is uncovered ($\sum_{j\in J} q_{ij} u_{wj} = 0$) set $u_{wk} = 1$, where k is the column corresponding to $\min\{j \mid q_{ij} = 1,\ j \in J\}$. After that, try to reduce the redundancies. To get a better feasible solution of the set covering problem (1), a replacement heuristic Lopes and Lorena (1994) is considered to be applied after constructing a feasible solution $\mathbf{u}_f$ in step 3 of the surrogate heuristic algorithm.

Replacement Heuristic

Step RH1. For each variable equal to one in $\mathbf{u}_f$, set the variable equal to zero and repeat the procedure above to obtain a feasible solution of the set covering problem (1).

Step RH2. A number of different feasible solutions are defined in step RH1. Take the best of these solutions and denoted it by $\mathbf{u}_b$ with cost f_b. If $f_b < z(\mathbf{w})$ then set $\mathbf{u}_f \leftarrow \mathbf{u}_b$, $z(\mathbf{w}) \leftarrow f_b$, and return to step RH1. If $f_b \geq z(\mathbf{w})$ then stop.

4.1.2 The subgradient direction $G(\mathbf{w})$ and the step size $t_{\mathbf{w}}$

Subgradient optimization can be viewed as a generalization of the steepest descent method. It is well known that slow convergence and non-monotonic behavior are two main undesirable features of subgradient methods (Lopes and Lorena (1994)). To prevent these problems, an easily computational expression, suggested in Lorena and Plateau (1988), can be used for the step size

$$t_{\mathbf{w}} = \rho \cdot (f_{ub} - f_{lb}), \quad \text{setting } \rho = \alpha / d_{wj^*}.$$

According to the discussion in Lopes and Lorena (1994), the parameter α is empirically initialized to obtain $t_{\mathbf{w}} / \parallel G(\mathbf{w}) \parallel^2 = 20$, in the first iteration. If f_{lb} has not been increased in the last 6 iterations then set $\alpha = 0.90\alpha$. In some instances the value of the step size can be too large at the initial iterations, and a superior limit of 20 is imposed to $t_{\mathbf{w}} / \parallel G(\mathbf{w}) \parallel^2$.

4.1.3 Stopping rules

The algorithm stops when one of the following conditions is satisfied. These conditions determine when the algorithm stops either because the optimum (near optimum) has been found or because the rate of convergence is too slow.

(a) The number of iterations is greater than 1000; or
(b) $f_{ub} - f_{lb} < \varepsilon$ with a sufficiently small $\varepsilon > 0$, f_{ub} is an optimal value for the set covering problem; or
(c) the value f_{lb} has not increased more than 1% in the last 10 computed values.

4.1.4 An initial reduction

For the implementation of the surrogate heuristic algorithm, an initial reduction of the set covering problem (1) is considered. A number of reduction tests are well known in the literature (Beasley (1987)). Some of those found to be more effective are employed in our implementation. The initial reduction tests are conducted before the surrogate heuristic algorithm.

(i) Column domination. Any column j whose rows $\{i \mid q_{ij} = 1,\ i = 1, 2, \ldots, m\}$ can be covered by other columns for a cost less than c'_j can be deleted from the problem. Assume that the columns are ordered in increasing cost order, and the columns with equal cost are ordered in decreasing order of number of rows that they cover. Let

$$\beta_i = \min\{j \mid q_{ij} = 1,\ j \in J\},\ i = 1, 2, \cdots, m,$$

and

$$H_j = \bigcup_{i=1}^{m} \{\beta_i \mid q_{ij} = 1\},\ j \in J.$$

Then if $\sum_{k \in H_j} c'_k < c'_j$, the column j can be deleted from the problem. This procedure differs from the one suggested by Beasley (1987) since it avoids to add the cost of a column more than once time. It is therefore about 20% more efficient in terms of column reduction, with a very low overhead in computational time (Lopes and Lorena (1994)).

(ii) Column inclusion. If a row $i, i = 1, 2, \ldots, m$, is covered by only one column, this column must be at the optimal solution. Note that when a column is set to be at the optimal solution, all the rows (constraints) that are covered by this column are automatically satisfied and can be deleted from the problem.

(iii) Null column reduction. Applying the tests above may result in columns which cover no rows, i.e. they are null columns. Those columns must be deleted from the problem.

5 Numerical examples

In this section, numerical examples are provided to illustrate the set covering-based surrogate approach for solving the sup-$\mathcal{T}$ equation constrained optimization problems.

Example 5.1 Consider a system of sup-$\mathcal{T}_P$ equations $A \circ \mathbf{x} = \mathbf{b}$ with a linear objective function studied in Loetamonphong and Fang (2001).

$$\begin{aligned}
\min\ z_{\mathbf{x}} &= -4x_1 + 3x_2 + 2x_3 + 3x_4 + 5x_5 + 2x_6 + x_7 + 2x_8 + 5x_9 + 6x_{10} \\
\text{s.t.} & \begin{pmatrix} 0.6 & 0.5 & 0.1 & 0.1 & 0.3 & 0.8 & 0.4 & 0.6 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.9 & 0.6 & 0.8 & 0.4 & 0.5 & 0.3 & 0.5 & 0.3 \\ 0.5 & 0.9 & 0.4 & 0.2 & 0.8 & 0.1 & 0.4 & 0.4 & 0.7 & 0.6 \\ 0.3 & 0.5 & 0.7 & 0.5 & 0.8 & 0.1 & 0.8 & 0.3 & 0.4 & 0.6 \\ 0.7 & 0.8 & 0.5 & 0.4 & 0.8 & 0.2 & 0.4 & 0.1 & 0.9 & 0.6 \\ 0.5 & 0.9 & 0.7 & 0.1 & 0.5 & 0.8 & 0.7 & 0.2 & 0.9 & 0.4 \\ 0.2 & 0.3 & 0.4 & 0.7 & 0.5 & 0.8 & 0.3 & 0.5 & 0.7 & 0.4 \\ 0.8 & 0.8 & 0.7 & 0.5 & 0.8 & 0.3 & 0.4 & 0.7 & 0.2 & 0.8 \end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.48 \\ 0.56 \\ 0.72 \\ 0.56 \\ 0.64 \\ 0.72 \\ 0.42 \\ 0.64 \end{pmatrix} \\
& x_j \in [0, 1], \quad j = 1, 2, \ldots, 10.
\end{aligned} \tag{2}$$

The system of sup-$\mathcal{T}_P$ equations has the maximum solution $\hat{\mathbf{x}}$=(0.8, 0.8, 0.622, 0.6, 0.7, 0.525, 0.7, 0.8, 0.6, 0.8)^T.

The associated characteristic matrix is:

$$\tilde{Q} = \begin{pmatrix} 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.622 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0.525 & 0 & 0 & 0.6 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \end{pmatrix}.$$

The associated set covering problem is:

$$\begin{aligned} \min z_{\mathbf{u}}^{+} &= 0u_1 + 2.4u_2 + 1.244u_3 + 1.8u_4 + 3.5u_5 + 1.05u_6 + 0.7u_7 + 1.6u_8 \\ &\quad +3u_9 + 0.48u_{10} \\ \text{s.t.} &\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \in \{0, 1\}. \end{aligned} \tag{3}$$

Initial reduction tests are conducted for the set covering problem (3) before the heuristic surrogate algorithm with $\beta_1 = 1, \beta_2 = 3, \beta_3 = 2, \beta_4 = 7, \beta_5 = 2, \beta_6 = 2, \beta_7 = 6, \beta_8 = 1$.

Since $\sum_{k \in H_j} c'_k < c'_j,\ \ j = 4, 5, 8, 9, 10,$ the columns $j = 4, 5, 8, 9, 10$, can be deleted from the problem and the reduced problem can be stated as follows:

$$\begin{aligned} \min z_{\mathbf{u}}^{+} &= 0u_1 + 2.4u_2 + 1.244u_3 + 1.8u_4 + 3.5u_5 + 1.05u_6 + 0.7u_7 + 1.6u_8 \\ &\quad +3u_9 + 0.48u_{10} \\ \text{s.t.} &\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_6 \\ u_7 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ &u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \in \{0, 1\}. \end{aligned} \tag{4}$$

Applying the heuristic surrogate algorithm for the set covering problem (4), we first find the solution for the linear surrogate relaxation $(S_{\mathbf{w}})$ of (4).

At iteration #1:

For a given $\mathbf{w} = (1, 1, 1, 1, 1, 1, 1, 1)^T$, the continuous surrogate relaxation $(S_{\mathbf{w}})$ of (4) is

$$\begin{aligned}
\min z_{\mathbf{u}}^+ &= 0u_1 + 2.4u_2 + 1.244u_3 + 1.8u_4 + 3.5u_5 + 1.05u_6 + 0.7u_7 \\
&\quad +1.6u_8 + 3u_9 + 0.48u_{10} \\
\text{s.t. } & 2u_1 + 4u_2 + u_3 + u_6 + u_7 \geq 8 \\
& u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \in \{0, 1\}.
\end{aligned}$$

Compute $d_1 = \frac{0}{2}, d_2 = \frac{2.4}{4}, d_3 = \frac{1.244}{1}, d_6 = \frac{1.05}{1}, d_7 = \frac{0.7}{1}$, we have

$$d_1 \leq d_2 \leq d_7 \leq d_6 \leq d_3,$$

and $d_{w1} = d_1, d_{w2} = d_2, d_{w3} = d_7, d_{w4} = d_6, d_{w5} = d_3$.

Since $\mathbf{w}^T q_1 + \mathbf{w}^T q_2 + \mathbf{w}^T q_7 \leq \mathbf{w}^T \mathbf{e}^8 \leq \mathbf{w}^T q_1 + \mathbf{w}^T q_2 + \mathbf{w}^T q_7 + \mathbf{w}^T q_6$, we have $wj* = 6$. The optimal solution

$$\begin{aligned}
\mathbf{u}_{\mathbf{w}} &= (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})^T \\
&= (1, 1, 0, 0, 0, 1, 1, 0, 0, 0)^T
\end{aligned}$$

with the optimal value $v(S_{\mathbf{w}}) = 4.15$.

To construct a feasible for the set covering problem (4) using $\mathbf{u}_{\mathbf{w}}$ and setting $u_{wj^*} = u_6 = 1$. For row 2 which is uncovered ($\sum_{j \in J} q_{2j} u_{wj} = 0$) set $u_3 = 1$. A feasible solution

$$\mathbf{u}_f = (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})^T = (1, 1, 1, 0, 0, 1, 1, 0, 0, 0)^T$$

is then obtained with $z(\mathbf{w}) = 5.394$.

Update $f_{\mathbf{ub}} \leftarrow 5.394$ and $f_{\mathbf{lb}} \leftarrow 4.15$.

To compute the subgradient direction and step size, set $u_{wj*} = u_6 = 0$ and the direction

$$G(\mathbf{w}) = e^m - \sum_{j \in J} q_j u_{wj} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

The step size

$$t_{\mathbf{w}} = \rho \cdot (5.394 - 4.15), \quad \text{setting } \rho = \alpha / d_6.$$

Since the parameter α is initialized to obtain $t_{\mathbf{w}}/ \parallel G(\mathbf{w}) \parallel^2 = 20$, we have

$$\begin{aligned}
w_1 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1,\\
w_2 &\leftarrow \max\{0, 1 + 1 \cdot 20\} = 21,\\
w_3 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1,\\
w_4 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1,\\
w_5 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1,\\
w_6 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1,\\
w_7 &\leftarrow \max\{0, 1 + 1 \cdot 20\} = 21,\\
w_8 &\leftarrow \max\{0, 1 - 1 \cdot 20\} = 0.
\end{aligned}$$

At iteration #2

For $\mathbf{w} = (1, 21, 1, 1, 1, 1, 21, 0)^T$, the continuous surrogate relaxation $S_{\mathbf{w}}$ is

$$\begin{aligned}
\min z_{\mathbf{u}}^+ &= 0u_1 + 2.4u_2 + 1.244u_3 + 1.8u_4 + 3.5u_5 + 1.05u_6 + 0.7u_7\\
&\quad +1.6u_8 + 3u_9 + 0.48u_{10}\\
\text{s.t. } & u_1 + 3u_2 + 21u_3 + 21u_6 + u_7 \geq 47\\
& u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10} \in \{0, 1\}.
\end{aligned}$$

Compute $d_1 = \frac{0}{1}, d_2 = \frac{2.4}{3}, d_3 = \frac{1.244}{21}, d_6 = \frac{1.05}{21}, d_7 = \frac{0.7}{1}$, we have

$$d_1 \leq d_6 \leq d_3 \leq d_7 \leq d_2,$$

and $d_{w1} = d_1, d_{w2} = d_6, d_{w3} = d_3, d_{w4} = d_7, d_{w5} = d_2$.

Since $\mathbf{w}^T q_1 + \mathbf{w}^T q_6 + \mathbf{w}^T q_3 + \mathbf{w}^T q_7 \leq \mathbf{w}^T \mathbf{e}^8 \leq \mathbf{w}^T q_1 + \mathbf{w}^T q_6 + \mathbf{w}^T q_3 + \mathbf{w}^T q_7 + \mathbf{w}^T q_2$, we have $wj* = 2$. The optimal solution

$$\begin{aligned}
\mathbf{u_w} &= (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})^T\\
&= (1, 1, 1, 0, 0, 1, 1, 0, 0, 0)^T
\end{aligned}$$

with the optimal value $v(S_{\mathbf{w}}) = 5.394$.

To construct a feasible for the set covering problem (4) using $\mathbf{u_w}$ and setting $u_{wj^*} = u_2 = 1$. A feasible solution

$$\begin{aligned}
\mathbf{u}_f &= (u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9, u_{10})^T\\
&= (1, 1, 1, 0, 0, 1, 1, 0, 0, 0)^T
\end{aligned}$$

is then obtained with $z(\mathbf{w}) = 5.394$.

Update $f_{\mathbf{ub}} \leftarrow 5.394$ and $f_{\mathbf{lb}} \leftarrow 5.394$.

Since $f_{ub} = f_{lb}$, $\mathbf{x}^* = (0.8, 0.8, 0.622, 0, 0, 0.525, 0.7, 0, 0, 0)$ is an optimal solution with the objective value $z_{\mathbf{x}} = 2.194$ for the set covering problem (4).

Example 5.2 Consider a system of sup-$\mathcal{T}_M$ equations $A \circ \mathbf{x} = \mathbf{b}$ with a linear objective function studied in Wu et al. (2002).

$$
\begin{aligned}
&\min z_{\mathbf{x}} = 0.7x_1 + x_2 + 1.1x_3 + 1.4x_4 + 1.5x_5 + 2x_6 \\
&\text{s.t.} \begin{pmatrix} 0.6 & 0.5 & 0.6 & 0.6 & 0.6 & 0.2 \\ 0.1 & 0.6 & 0.8 & 0.5 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.5 & 0.8 & 0.2 & 0.8 \\ 0.8 & 0.95 & 0.1 & 0.3 & 0.9 & 0.9 \\ 0.9 & 0.8 & 0.4 & 0.95 & 0.4 & 1 \\ 1 & 0.8 & 0.4 & 1 & 1 & 0.5 \end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 0.95 \\ 1 \end{pmatrix} \\
&x_j \in [0, 1], \quad j = 1, 2, \ldots, 6.
\end{aligned} \tag{5}
$$

The system of sup-$\mathcal{T}_M$ equations has the maximum solution $\hat{\mathbf{x}} = (1, 0.9, 0.7, 1, 1, 0.95)^T$.

The associated characteristic matrix is:

$$
\tilde{Q} = \begin{pmatrix} [0.6, 1] & \emptyset & [0.6, 0.7] & [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & \emptyset & 0.7 & \emptyset & \emptyset & [0.7, 0.95] \\ [0.8, 1] & [0.8, 0.9] & \emptyset & [0.8, 1] & \emptyset & [0.8, 0.95] \\ \emptyset & 0.9 & \emptyset & \emptyset & [0.9, 1] & [0.9, 0.95] \\ \emptyset & \emptyset & \emptyset & [0.95, 1] & \emptyset & 0.95 \\ 1 & \emptyset & \emptyset & 1 & 1 & \emptyset \end{pmatrix},
$$

and $\check{v} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{13}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31}, \check{v}_{32}, \check{v}_{41}, \check{v}_{42}, \check{v}_{43}, \check{v}_{44}, \check{v}_{51}, \check{v}_{52}, \check{v}_{53}, \check{v}_{61}, \check{v}_{62}, \check{v}_{63}, \check{v}_{64})^T = (0.6, 0.8, 1{,}0.8, 0.9, 0.6.0.7, 0.6, 0.8, 0.95, 1{,}0.6, 0.9, 1{,}0.7, 0.8, 0.9{,}0.95)^T.$

The associated set covering problem is

$$
\begin{aligned}
\min z^+_{\mathbf{u}} &= 0.42u_{11} + 0.56u_{12} + 0.7u_{13} + 0.8u_{21} + 0.9u_{22} + 0.66u_{31} + 0.77u_{32} + 0.84u_{41} \\
&\quad + 1.12u_{42} + 1.33u_{43} + 1.4u_{44} + 0.9u_{51} + 1.35u_{52} + 1.5u_{53} + 1.4u_{61} + 1.6u_{62} \\
&\quad + 1.8u_{63} + 1.9u_{64} \\
\text{s.t.} & \begin{pmatrix} 1&1&1&0&0&1&1&1&1&1&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&1&1&1 \\ 0&1&1&1&1&0&0&0&1&1&1&0&0&0&0&1&1&1 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0 \end{pmatrix} \mathbf{u} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
\mathbf{u} &= (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\
&\quad u_{61}, u_{62}, u_{63}, u_{64})^T \in \{0, 1\}^{18}.
\end{aligned} \tag{6}
$$

Initial reduction tests are conducted for the set covering problem (6) before the surrogate heuristic algorithm with $\beta_1 = 11, \beta_2 = 32, \beta_3 = 12, \beta_4 = 22, \beta_5 = 43, \beta_6 = 13$. Since $\sum_{k \in H_j} c'_k < c'_j,\ j = 21, 31, 41, 42, 51, 52, 61, 62$, the associated columns can be deleted from the problem and the reduced problem is stated as follows:

$$
\begin{aligned}
\min z_{\mathbf{u}}^{+} \;=\;& 0.42u_{11} + 0.56u_{12} + 0.7u_{13} + 0.8u_{21} + 0.9u_{22} + 0.66u_{31} + 0.77u_{32} \\
& + 0.84u_{41} \;+\; 1.12u_{42} \;+\; 1.33u_{43} + 1.4u_{44} + 0.9u_{51} + 1.35u_{52} \\
& +1.5u_{53} + 1.4u_{61} + 1.6u_{62} + 1.8u_{63} + 1.9u_{64}
\end{aligned}
$$

$$
\text{s.t.} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{22} \\ u_{32} \\ u_{43} \\ u_{44} \\ u_{53} \\ u_{63} \\ u_{64} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}
$$

$$
\begin{aligned}
& u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\
& u_{61}, u_{62}, u_{63}, u_{64} \in \{0, 1\}.
\end{aligned} \tag{7}
$$

Applying the surrogate heuristic algorithm for the set covering problem (7), we first find the solution for the continuous surrogate relaxation ($S_{\mathbf{w}}$) of (7).

At iteration #1:

For a given $\mathbf{w} = (1, 1, 1, 1, 1, 1)^T$, the continuous surrogate relaxation ($S_{\mathbf{w}}$) of (7) is

$$
\begin{aligned}
\min z_{\mathbf{u}}^{+} =\;& 0.42u_{11} + 0.56u_{12} + 0.7u_{13} + 0.8u_{21} + 0.9u_{22} + 0.66u_{31} + 0.77u_{32} \\
& +0.84u_{41} + 1.12u_{42} + 1.33u_{43} + 1.4u_{44} + 0.9u_{51} + 1.35u_{52} + 1.5u_{53} \\
& +1.4u_{61} + 1.6u_{62} + 1.8u_{63} + 1.9u_{64} \\
& \text{s.t. } u_{11} + 2u_{12} + 3u_{13} + 2u_{22} + 2u_{32} + 3u_{43} + 4u_{44} + 3u_{53} + 3u_{63} \\
& + 4u_{64} \geq 6, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, \\
& \quad u_{52}, u_{53}, u_{61}, u_{62}, u_{63}, u_{64} \in \{0, 1\}.
\end{aligned}
$$

Compute $d_{11}=\frac{0.42}{1}$, $d_{12}=\frac{0.56}{2}$, $d_{13}=\frac{0.7}{3}$, $d_{22}=\frac{0.9}{2}$, $d_{32} = \frac{0.77}{2}$, $d_{43}=\frac{1.33}{3}$, $d_{44} = \frac{1.4}{4}$, $d_{53} = \frac{1.5}{3}$, $d_{63} = \frac{1.8}{3}$, $d_{64} = \frac{1.9}{4}$ we have

$$
d_{13} \leq d_{12} \leq d_{44} \leq d_{32} \leq d_{11} \leq d_{43} \leq d_{22} \leq d_{64} \leq d_{53} \leq d_{23},
$$

and $d_{w1} = d_{13}, d_{w2} = d_{12}, d_{w3} = d_{44}, d_{w4} = d_{32}, d_{w5} = d_{11}, d_{w6} = d_{43}$, $d_{w7} = d_{22}, d_{w8} = d_{64}, d_{w9} = d_{53}, d_{w10} = d_{23}$.

Since $\mathbf{w}^T q_{13} + \mathbf{w}^T q_{12} \leq \mathbf{w}^T \mathbf{e}^6 \leq \mathbf{w}^T q_{13} + \mathbf{w}^T q_{12} + \mathbf{w}^T q_{44}$, we have $wj* = 44$. The optimal solution can be obtained as

$$
\begin{aligned}
\mathbf{u_w} =\;& (u_{12}, u_{13}, u_{44}, u_{11}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{51}, u_{52}, u_{53}, \\
& u_{61}, u_{62}, u_{63}, u_{64})^T = (1, 1, \frac{1}{4}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T
\end{aligned}
$$

with the optimal value $v(S_{\mathbf{w}}) = 1.61$.

To construct a feasible for the set covering problem (7) using $\mathbf{u_w}$ and setting $u_{wj^*} = u_{44} = 1$. By using replacement heuristic, a feasible solution

$$\mathbf{u}_f = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, u_{61}, u_{62}, u_{63}, u_{64})^T = (0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0)^T$$

is then obtained with $z(\mathbf{w}) = 3.07$.

Update $f_{\mathbf{ub}} \leftarrow 3.07$ and $f_{\mathbf{lb}} \leftarrow 1.61$.

To compute the subgradient direction and step size, set $u_{wj*} = u_{44} = 0$ and the direction

$$G(\mathbf{w}) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The step size $t_{\mathbf{w}} = \rho \cdot (3.07 - 1.61)$, setting $\rho = \alpha / d_{44}$.

Since the parameter α is initialized to obtain $t_{\mathbf{w}} / \parallel G(\mathbf{w}) \parallel^2 = 20$, we have

$$\begin{aligned} w_1 &\leftarrow \max\{0, 1 + (-1) \cdot 20\} = 0, \\ w_2 &\leftarrow \max\{0, 1 + 1 \cdot 20\} = 21, \\ w_3 &\leftarrow \max\{0, 1 + (-1) \cdot 20\} = 0, \\ w_4 &\leftarrow \max\{0, 1 + 1 \cdot 20\} = 21, \\ w_5 &\leftarrow \max\{0, 1 + 1 \cdot 20\} = 21, \\ w_6 &\leftarrow \max\{0, 1 + 0 \cdot 20\} = 1. \end{aligned}$$

At iteration #2

For $\mathbf{w} = (0, 21, 0, 21, 21, 1)^T$, the continuous surrogate relaxation $(S_{\mathbf{w}})$ of (7) is

$$\begin{aligned} \min z_{\mathbf{u}}^+ = {} & 0.42u_{11} + 0.56u_{12} + 0.7u_{13} + 0.8u_{21} + 0.9u_{22} + 0.66u_{31} + 0.77u_{32} \\ & + 0.84u_{41} + 1.12u_{42} + 1.33u_{43} + 1.4u_{44} + 0.9u_{51} + 1.35u_{52} + 1.5u_{53} \\ & + 1.4u_{61} + 1.6u_{62} + 1.8u_{63} + 1.9u_{64} \\ & \text{s.t. } u_{13} + 21u_{22} + 21u_{32} + 21u_{43} + 22u_{44} + 22u_{53} + 42u_{63} + 63u_{64} \geq 64, \\ & \quad u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ & \quad u_{61}, u_{62}, u_{63}, u_{64} \in \{0, 1\}. \end{aligned}$$

Compute $d_{13} = \frac{0.7}{1}, d_{22} = \frac{0.9}{21}, d_{32} = \frac{0.77}{21}, d_{43} = \frac{1.33}{21}, d_{44} = \frac{1.4}{22}, d_{53} = \frac{1.5}{22}, d_{63} = \frac{1.8}{42}, d_{64} = \frac{1.9}{63}$, we have

$$d_{64} \leq d_{32} \leq d_{63} \leq d_{22} \leq d_{43} \leq d_{44} \leq d_{53} \leq d_{13}$$

and $d_{w1} = d_{64}, d_{w2} = d_{32}, d_{w3} = d_{63}, d_{w4} = d_{22}, d_{w5} = d_{43}, d_{w6} = d_{44}, d_{w7} = d_{53}, d_{w8} = d_{13}$.

Since $\mathbf{w}^T q_{64} \le \mathbf{w}^T \mathbf{e}^6 \le \mathbf{w}^T q_{64} + \mathbf{w}^T q_{32}$, we have $wj* = 32$. The optimal solution

$$\mathbf{u_w} = (u_{64}, u_{32}, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, u_{61}, u_{62}, u_{63})^T = (1, \frac{1}{21}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$$

with the optimal value $v(S_{\mathbf{w}}) = 1.9367$.

To construct a feasible solution for the set covering problem (7) using replacement heuristic, a feasible solution

$$\mathbf{u}_f = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, u_{61}, u_{62}, u_{63}, u_{64})^T = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)^T$$

is then obtained with $z(\mathbf{w}) = 2.6$.

Update $f_{\mathbf{ub}} \leftarrow 2.6$ and $f_{\mathbf{lb}} \leftarrow 1.9367$.

Taking $\varepsilon = 1$, since $f_{ub} - f_{lb} < \varepsilon$, $\mathbf{x}^{+*} = (1, 0, 0, 0, 0, 0.95)^T$. By Theorem 2.2, an optimal solution to problem (5) can be constructed as $\mathbf{x}^* = (1, 0, 0, 0, 0, 0.95)^T$ with the optimal value $z^* = 2.6$.

Example 5.3 Consider the system of sup-$\mathcal{T}_M$ equations in Example 5.1 with a separable and monotone objective function in each of the variables described as follows:

$$\begin{aligned} &\min z_x = x_1(x_1 - 4) + (x_2)^2 + (x_3)^2 + x_4 + (x_5)^2 + x_6 \\ &\text{s.t.} \begin{pmatrix} 0.6 & 0.5 & 0.6 & 0.6 & 0.6 & 0.2 \\ 0.1 & 0.6 & 0.8 & 0.5 & 0.6 & 0.7 \\ 0.8 & 0.8 & 0.5 & 0.8 & 0.2 & 0.8 \\ 0.8 & 0.95 & 0.1 & 0.3 & 0.9 & 0.9 \\ 0.9 & 0.8 & 0.4 & 0.95 & 0.4 & 1 \\ 1 & 0.8 & 0.4 & 1 & 1 & 0.5 \end{pmatrix} \circ \mathbf{x} = \begin{pmatrix} 0.6 \\ 0.7 \\ 0.8 \\ 0.9 \\ 0.95 \\ 1 \end{pmatrix} \\ &x_j \in [0, 1], \quad j = 1, 2, \cdots, 6. \end{aligned} \tag{8}$$

As shown in Example 5.1, the system of sup-$\mathcal{T}_M$ equations has the maximum solution $\hat{\mathbf{x}} = (1, 0.9, 0.7, 1, 1, 0.95)^T$. Its associated characteristic matrix is

$$\tilde{Q} = \begin{pmatrix} [0.6, 1] & \emptyset & [0.6, 0.7] & [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & \emptyset & 0.7 & \emptyset & \emptyset & [0.7, 0.95] \\ [0.8, 1] & [0.8, 0.9] & \emptyset & [0.8, 1] & \emptyset & [0.8, 0.95] \\ \emptyset & 0.9 & \emptyset & \emptyset & [0.9, 1] & [0.9, 0.95] \\ \emptyset & \emptyset & \emptyset & [0.95, 1] & \emptyset & 0.95 \\ 1 & \emptyset & \emptyset & 1 & 1 & \emptyset \end{pmatrix},$$

In this case, $N^- = \{1\}$, $N^+ = \{2, 3, 4, 5, 6\}$ and

$$\begin{aligned}\check{v} &= (\check{v}_{11}, \check{v}_{12}, \check{v}_{13}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31}, \check{v}_{32}, \check{v}_{41}, \check{v}_{42}, \check{v}_{43}, \check{v}_{44}, \check{v}_{51}, \check{v}_{52}, \check{v}_{53}, \check{v}_{61}, \check{v}_{62}, \check{v}_{63}, \check{v}_{64})^T \\ &= (0.6, 0.8, 1, 0.8, 0.9, 0.6.0.7, 0.6, 0.8, 0.95, 1, 0.6, 0.9, 1, 0.7, 0.8, 0.9, 0.95)^T.\end{aligned}$$

The associated set covering problem is

$$\begin{aligned}\min z_{\mathbf{u}}^+ &= 0 \cdot u_{11} + 0 \cdot u_{12} + 0 \cdot u_{13} + 0.64u_{21} + 0.81u_{22} + 0.36u_{31} + 0.49u_{32} + 0.6u_{41} \\ &\quad + 0.8u_{42} + 0.95u_{43} + u_{44} + 0.36u_{51} + 0.81u_{52} + u_{53} + 0.7u_{61} + 0.8u_{62} \\ &\quad + 0.9u_{63} + 0.95u_{64} \\ \text{s.t.} &\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{u} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\ \mathbf{u} &= (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ &\quad u_{61}, u_{62}, u_{63}, u_{64})^T \in \{0, 1\}^{18}.\end{aligned} \tag{9}$$

Initial reduction tests are conducted for the set covering problem (1) with $\beta_1 = 13, \beta_2 = 32, \beta_3 = 13, \beta_4 = 22, \beta_5 = 64, \beta_6 = 13$. Since $\sum_{k \in H_j} c'_k < c'_j,\ j = 21, 31, 41, 42, 44, 51, 53, 61, 62$, the associated columns can be deleted from the problem. Moreover, after conducting column domination row 6 is covered by only column 13, hence rows 1, 3, 6 and the associated null columns can be deleted from the problem with $u_{13} = 1$. The reduced problem is stated as follows:

$$\begin{aligned}\min z_{\mathbf{u}}^+ &= 0 \cdot u_{11} + 0 \cdot u_{12} + 0 \cdot u_{13} + 0.64u_{21} + 0.81u_{22} + 0.36u_{31} + 0.49u_{32} \\ &\quad + 0.6u_{41} + 0.8u_{42} + 0.95u_{43} + u_{44} + 0.36u_{51} + 0.81u_{52} + u_{53} \\ &\quad + 0.7u_{61} + 0.8u_{62} + 0.9u_{63} + 0.95u_{64} \\ &\text{s.t.} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{22} \\ u_{32} \\ u_{43} \\ u_{52} \\ u_{63} \\ u_{64} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ &\quad u_{61}, u_{62}, u_{63}, u_{64} \in \{0, 1\}.\end{aligned} \tag{10}$$

Applying the surrogate heuristic algorithm for the set covering problem (10), we first find the solution for the continuous surrogate relaxation ($S_{\mathbf{w}}$) of (10).

At iteration #1:

For a given $\mathbf{w} = (1, 1, 1)^T$, the continuous surrogate relaxation ($S_\mathbf{w}$) of (10) is

$$\begin{aligned}\min z_\mathbf{u}^+ &= 0 \cdot u_{11} + 0 \cdot u_{12} + 0 \cdot u_{13} + 0.64u_{21} + 0.81u_{22} + 0.36u_{31} + 0.49u_{32} \\ &\quad +0.6u_{41} + 0.8u_{42} + 0.95u_{43} + u_{44} + 0.36u_{51} + 0.81u_{52} + u_{53} \\ &\quad +0.7u_{61} + 0.8u_{62} + 0.9u_{63} + 0.95u_{64} \\ &\quad \text{s.t.} u_{22} + u_{32} + u_{43} + u_{52} + 2u_{63} + 3u_{64} \geq 3, \\ &\quad u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ &\quad u_{61}, u_{62}, u_{63}, u_{64} \in \{0, 1\}.\end{aligned}$$

Compute $d_{22} = \frac{0.81}{1}, d_{32} = \frac{0.49}{1}, d_{43} = \frac{0.95}{1}, d_{52} = \frac{0.81}{1}, d_{63} = \frac{0.9}{2}, d_{64} = \frac{0.95}{3}$, we have

$$d_{64} \leq d_{63} \leq d_{32} \leq d_{22} \leq d_{52} \leq d_{43}$$

and $d_{w1} = d_{64}, d_{w2} = d_{63}, d_{w3} = d_{32}, d_{w4} = d_{22}, d_{w5} = d_{52}, d_{w6} = d_{43}$.

Since $0 \leq \mathbf{w}^T\mathbf{e}^3 \leq \mathbf{w}^T q_{64}$, we have $wj* = 64$. The optimal solution

$$\begin{aligned}\mathbf{u_w} &= (u_{64}, u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ &\quad u_{61}, u_{62}, u_{63})^T = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T\end{aligned}$$

with the optimal value $v(S_\mathbf{w}) = 0.95$.

To construct a feasible for the set covering problem (10) using $\mathbf{u_w}$, a feasible solution

$$\begin{aligned}\mathbf{u}_f &= (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{31}, u_{32}, u_{41}, u_{42}, u_{43}, u_{44}, u_{51}, u_{52}, u_{53}, \\ &\quad u_{61}, u_{62}, u_{64})^T = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)^T\end{aligned}$$

is then obtained with $z(\mathbf{w}) = 0.95$.

Update $f_{\mathbf{ub}} \leftarrow 0.95$ and $f_{\mathbf{lb}} \leftarrow 0.95$.

Since $f_{ub} = f_{lb}$, $\mathbf{x}^{+*} = (1, 0, 0, 0, 0, 0.95)^T$. An optimal solution to problem (8) can be constructed as $\mathbf{x}^* = (1, 0, 0, 0, 0, 0.95)^T$ with the optimal value $z^* = -2.05$.

Notice that our results in Examples 5.1 and 5.2 are consistent with the results in the literature Loetamonphong and Fang (2001) and Wu et al. (2002), respectively, and indicate that the set covering-based surrogate approach finds the solutions very quickly (at very early iterations).

6 Conclusions

This paper studies the optimal solutions to the sup-$\mathcal{T}$ equation constrained optimization problems, with $\mathcal{T}$ being a continuous triangular norm. Taking advantage of the well developed techniques and clarity of exposition in the theory of integer programming, a set covering-based surrogate approach is proposed to solve the sup-$\mathcal{T}$ equation constrained optimization problem with a separable and monotone objective function in each of the variables. Computational results show that the proposed method can efficiently solve the sup-$\mathcal{T}$ equation constrained optimization problems. In fact, the proposed method has been tested on most examples reported in the literature Fang and Li (1999), Li and Fang (2008), Peeva and Kyosev (2004), and it finds the solutions at very early iterations. This validates the properties of faster convergence and less oscillation of the surrogate heuristic. With the aid of the initial reduction technique and the replacement heuristic in the implementation, the performance of the proposed method can be further improved. This study provides, for the first time, an opportunity to solve the large scale sup-$\mathcal{T}$ equation constrained optimization problems.

Acknowledgments The author would like to thank Dr. Pingke Li for his constructive and valuable suggestions.

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