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A method for solving fuzzy multicriteria decision problems with dependent criteria

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Abstract We propose a new multi-criteria decision making (MCDM) method based on fuzzy pair-wise comparisons and a feedback between the criteria. The evaluation of the weights of criteria, the variants as well as the feedback between the criteria is based on the data given in pair-wise comparison matrices. Extended arithmetic operations with fuzzy numbers are used as well as ordering fuzzy relations to compare fuzzy outcomes. An illustrating numerical example is presented to clarify the methodology. A special SW-Microsoft Excel add-in named FVK was developed for applying the proposed method. Comparing to other software products, FVK is free, able to work with fuzzy data and utilizes capabilities of widespread spreadsheet Microsoft Excel.

Keywords Multi-criteria decision making · Analytic hierarchy process · Pair-wise comparisons · Feedback · Fuzzy numbers

1 Introduction

In this paper we propose a new method for solving fuzzy multi-criteria decision making (MCDM) problems. In some sense the method is parallel to, or, extension of the Analytic hierarchy process (AHP) with feedback between criteria. Instead of the classical eigenvector prioritization method employed in the AHP, here, a fuzzy approach is based on the logarithmic least squares method, i.e. on the geometric-mean aggregation. In Lootsma (1993), Lootsma (1996),

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it was demonstrated that some notorious occurrences of rank reversal in AHP can be avoided by the geometric-mean aggregation. We also accept the reasons discussed in Buckley et al. (2001), and other papers cited therein, for using geometric mean instead of Saaty's procedures. Our approach is however different to the method by Van Laarhoven and Pedrycz (1983), and Lootsma (1996).

The interface between hierarchies, multiple objectives and fuzzy sets have been investigated by the author of AHP T.L. Saaty as early as in Saaty (1978). Later on, in Saaty (1991), Saaty (2001), the author extends the AHP to a more general process with feedback called Analytic Network Process (ANP). In Mikhailov and Singh (2003) a new method based on ANP and fuzzy data is proposed, which is, however, essentially different to the approach used in this paper. Recently, Büyüközkan et al. (2004), Enea and Piazza (2004) and Mohanty et al. (2005), proposed another versions of fuzzy ANP and applied their methods in practice. Here, we propose a relatively simple method based on the original approaches by Buckley (1985), Buckley et al. (2001) and Van Laarhoven and Pedrycz (1983).

When applying classical MCDM approach on a real decision problem, e.g. when you want to select the best project from the given group of applications, or, to buy the best product for your personal use, say a car or digital camera, one usually meets among others two difficulties:

- when evaluating pair-wise comparisons (e.g. on the 9-point AHP scale) natural uncertainty is not incorporated,
- decision criteria are not independent each other as they should be.

Here, we solve these difficulties by proposing a new method which incorporates uncertainty adopting pair-wise comparisons by triangular fuzzy numbers, and takes into account interdependences between decision criteria.

Our method has the following drawbacks and new features comparing to classical AHP/ANP, eventually "fuzzy versions" of AHP, e.g. the one from Van Laarhoven and Pedrycz (1983), or Buckley (1985), Buckley et al. (2001).

First, in classical AHP the pairs of elements are evaluated on the scale $\{^1/_9, ^1/_8, \ldots, ^1/_2, 1, 2, \ldots, 9\}$. Here, we extend this scale to the closed interval $\left[^1/_\sigma, \sigma\right]$, where σ is arbitrary positive number greater than 1. Moreover, our approach allows for multiple representations of uncertain human preferences: both crisp evaluations, interval, and also fuzzy judgments in the simple form of triangular fuzzy numbers. The proposed method offers a solution from incomplete sets of pair-wise comparisons. By using the geometric average method we derive the resulting weights-relative importance of the elements in the form of triangular fuzzy numbers. When calculating the weights we minimize the measure of uncertainty, i.e. the spreads of fuzzy weights. This feature, which is new comparing to previous methods, is very important as too large fuzziness of the weights could deteriorate the results.

Second, it was already mentioned above, the decision criteria are frequently dependent to each other. For instance, when buying a best product for our personal use, e.g. a car, the price of the car as a decision criterion is interdependent with some other criteria: the power of engine, maintenance cost, quality of design, prestige of the car etc. Evaluating the criteria individually without taking into account dependency between the criteria the result may become misleading. In AHP this problem has been solved by proposing a new methodology—ANP—a general approach for dealing with feedback among decision elements. Here we solve the problem analogically, however, not in the same generality as we do not have suitable tools for handling matrices with fuzzy elements, particularly inverse matrices, eigenvectors etc. We consider only dependences among decision criteria solving the inverse matrix problem by approximating it with the first few terms of Taylor expansion. This approach will help us to overcome the difficulty with interdependent criteria.

The paper is structured as follows: Sect. 2 presents the classical AHP/ANP theory in the suitable vector notation, in Sect. 3 pair-wise comparison matrices with triangular fuzzy elements are introduced, and in Sect. 4 we describe an algorithm for calculating fuzzy weights from fuzzy pair-wise comparison matrices. In Sect. 5 we deal with the problem of inconsistency, we introduce a new inconsistency index for fuzzy pair-wise comparison matrices and finally, in Sect. 6 we analyze an illustrating example—decision making situation with 3 decision criteria and 3 variants. Also, we supply an illustrating example using a SW tool-Microsoft Excel add-in named FVK developed for applying the proposed method to demonstrate capabilities of the method.

2 Multi-criteria decisions and AHP/ANP

In classical hierarchical analysis we usually consider a three-level hierarchical decision system: On the first level we consider a decision goal G, on the second level, we have n independent evaluation criteria: $C_1, C_2, \ldots, C_n$, such that $\sum_{i=1}^{n} w\,(C_i) = 1$, where $w(C_i)$ is a positive real number—a weight, usually interpreted as a relative importance of criterion C_i subject to the goal G. On the third level, m variants (alternatives) of the decision outcomes $V_1, V_2, \ldots, V_m$ are considered such that again $\sum_{r=1}^{m} w\,(V_r, C_i) = 1$, where $w(V_r, C_i)$ is a non-negative real number - an evaluation (weight) of V_r subject to the criterion C_i, $i = 1, 2, \ldots, n$. It is advantageous to formulate the above mentioned weights into the matrix form, see e.g. Saaty (2001).

Let $\mathbf{W}_1$ be the $n \times 1$ matrix (weighing vector of the criteria), i.e. $\mathbf{W}_1 = \begin{bmatrix} w(C_1) \\ \vdots \\ w(C_n) \end{bmatrix}$,

and $\mathbf{W}_3$ be the $m \times n$ matrix

$$\mathbf{W}_3 = \begin{bmatrix} w(C_1, V_1) & \cdots & w(C_n, V_1) \\ \vdots & \ldots & \vdots \\ w(C_1, V_m) & \cdots & w(C_n, V_m) \end{bmatrix}.$$

The columns of this matrix are evaluations of variants according to the criteria. Moreover, $\mathbf{W}_3$ is a column-stochastic matrix, i.e. the sums of columns are equal to one. Then

$$\mathbf{Z} = \mathbf{W}_3 \mathbf{W}_1$$

is an $m \times 1$ matrix, i.e. the resulting priority vector of weights of the variants. The variants can be ranked according to these priorities.

In real decision systems there exist typical interdependences among criteria or variants. Decision systems with dependences have been extensively investigated by the Analytic Network Process (ANP), see Saaty (2001). Consider now the dependences among the criteria. The interdependences of the criteria are characterized by $n \times n$ matrix $\mathbf{W}_2$

$$\mathbf{W}_2 = \begin{bmatrix} w(C_1, C_1) & \cdots & w(C_n, C_1) \\ \vdots & \cdots & \vdots \\ w(C_1, C_n) & \cdots & w(C_n, C_n) \end{bmatrix}. \tag{1}$$

Here, the k-th column of $\mathbf{W}_2$ is interpreted as the vector of weights characterizing influences of the individual criteria on the k-th criterion and $w(C_k, C_k) = 0$ as the influence of C_k onto itself is zero.

Moreover, n-1 elements of this vector of weights form the vector

$$\mathbf{w}^{(k)} = \begin{bmatrix} w(C_k, C_1) \\ \vdots \\ w(C_k, C_{k-1}) \\ w(C_k, C_{k+1}) \\ \vdots \\ w(C_k, C_n) \end{bmatrix}, \tag{2}$$

where the elements of $\mathbf{w}^{(k)}$ are calculated from the following $(n-1) \times (n-1)$ pair-wise comparison matrix

$$\mathbf{B}_k = \begin{bmatrix} b_{1,1} & \cdots & b_{1,k-1} & b_{1,k+1} & \cdots & b_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k-1,1} & \cdots & b_{k-1,k-1} & b_{k-1,k+1} & \cdots & b_{k-1,n} \\ b_{k+1,1} & \cdots & b_{k+1,k-1} & b_{k+1,k+1} & \cdots & b_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n,1} & \cdots & b_{n,k-1} & b_{n,k+1} & \cdots & b_{n,n} \end{bmatrix}. \tag{3}$$

For all $k = 1, 2, \ldots, n$, $b_{k,k} = 1$, as $\mathbf{B}_k$ is reciprocal (see Sect. 3).

In Saaty (2001), it has been shown that if the sum of every column of matrix $\begin{bmatrix} \mathbf{W_2} \\ \mathbf{W_3} \end{bmatrix}$ is equal to one, then

$$\mathbf{Z} = \mathbf{W}_3 \left(\mathbf{I} - \mathbf{W}_2\right)^{-1} \mathbf{W}_1 \tag{4}$$

is the resulting priority vector of weights of variants applicable for the decision making process, i.e. for ranking the variants.

Usually, the matrix $\mathbf{W}_2$ is close to the matrix with zero elements and the dependences among criteria are weak, it can be approximated by the first few terms of Taylor's expansion

$$(\mathbf{I}-\mathbf{W}_2)^{-1}=\mathbf{I}+\mathbf{W}_2+\mathbf{W}_2^2+\ldots \tag{5}$$

Then by substituting the first four terms from (5) to (4) we get

$$\mathbf{Z}=\mathbf{W}_3\left(\mathbf{I}+\mathbf{W}_2+\mathbf{W}_2^2+\mathbf{W}_2^3\right)\mathbf{W}_1, \tag{6}$$

where $\mathbf{I}$ is the unit matrix. In the next section, formula (6) will be used for computing fuzzy evaluations of the variants.

3 Fuzzy numbers and fuzzy matrices

A *triangular fuzzy number* $\tilde{a}$ can be equivalently expressed by a triple of real numbers, i.e.

$$\tilde{a}=\left(a^L;a^M;a^U\right),$$

where a^L is the *Lower number*, a^M is the *Middle number,* and a^U is the *Upper number*, $a^L\leq a^M\leq a^U$. If $a^L=a^M=a^U$, then $\tilde{a}$ is said to be the *crisp number* (*non-fuzzy number*). Evidently, the set of all crisp numbers is isomorphic to the set of real numbers. In order to distinguish fuzzy and non-fuzzy numbers we shall denote the fuzzy numbers, vectors and matrices by the tilde above the symbol. As usual, the membership function of $\tilde{a}$ is assumed to be piece-wise linear.

It is well known that the arithmetic operations $+$, $-$, $*$ and / can be extended to fuzzy numbers by the Extension principle, see e.g. Buckley et al. (2001). In case of triangular fuzzy numbers $\tilde{a}=(a^L;a^M;a^U)$ and $\tilde{b}=(b^L;b^M;b^U)$, $a^L>0$, $b^L>0$, we use special formulae

$$\tilde{a}\tilde{+}\tilde{b}=\left(a^L+b^L;a^M+b^M;a^U+b^U\right)\quad\text{“addition”},$$
$$\tilde{a}\tilde{-}\tilde{b}=\left(a^L-b^U;a^M-b^M;a^U-b^L\right)\quad\text{“subtraction”},$$
$$\tilde{a}\tilde{*}\tilde{b}=\left(a^L*b^L;a^M*b^M;a^U*b^U\right)\quad\text{“multiplication”},$$
$$\tilde{a}\tilde{/}\tilde{b}=\left(a^L/b^U;a^M/b^M;a^U/b^L\right)\quad\text{“division”}.$$

It should be noted, that the above formulae for multiplication and also for division are not obtained from the Extension principle. The above stated operations are only approximate ones, it means that the result of e.g. multiplication is a triangular fuzzy number with piece-wise linear membership function, whereas the membership function of the exact operation defined by the Extension principle is non-linear.

If all elements of an $m \times n$ matrix $\mathbf{A}$ are triangular fuzzy numbers we call $\mathbf{A}$ the *matrix with triangular fuzzy elements* and this matrix is composed of triples as follows:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \left(a_{11}^L; a_{11}^M; a_{11}^U\right) & \cdots & \left(a_{1n}^L; a_{1n}^M; a_{1n}^U\right) \\ \vdots & \ddots & \vdots \\ \left(a_{m1}^L; a_{m1}^M; a_{m1}^U\right) & \cdots & \left(a_{mn}^L; a_{mn}^M; a_{mn}^U\right) \end{bmatrix}. \tag{7}$$

Particularly, let $\tilde{\mathbf{A}}$ be an $n \times n$ matrix with triangular fuzzy elements. We say that $\tilde{\mathbf{A}}$ is *reciprocal*, if the following condition is satisfied: $\tilde{a}_{ij} = \left(a_{ij}^L; a_{ij}^M; a_{ij}^U\right)$ implies $\tilde{a}_{ji} = \left(\frac{1}{a_{ij}^U}; \frac{1}{a_{ij}^M}; \frac{1}{a_{ij}^L}\right)$ for all $i, j = 1, 2, \ldots, n$, i.e.:

$$\tilde{\mathbf{A}} = \begin{bmatrix} (1\ ;\ 1\ ;\ 1) & \left(a_{12}^L; a_{12}^M; a_{12}^U\right) & \cdots & \left(a_{1n}^L; a_{1n}^M; a_{1n}^U\right) \\ \left(\frac{1}{a_{12}^U}; \frac{1}{a_{12}^M}; \frac{1}{a_{12}^L}\right) & (1\ ;\ 1\ ;\ 1) & \cdots & \left(a_{2n}^L; a_{2n}^M; a_{2n}^U\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{1}{a_{1n}^U}; \frac{1}{a_{1n}^M}; \frac{1}{a_{1n}^L}\right) & \left(\frac{1}{a_{2n}^U}; \frac{1}{a_{2n}^M}; \frac{1}{a_{2n}^L}\right) & \cdots & (1\ ;\ 1\ ;\ 1) \end{bmatrix}, \tag{8}$$

where $0 < a_{ij}^L \leq a_{ij}^M \leq a_{ij}^U, i, j = 1, 2, \ldots, n$.

4 Algorithm

The proposed method of finding the best variant (or ranking all the variants) can be characterized in the following three steps:

1. Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrices.
2. Calculate the aggregating triangular fuzzy evaluations of the variants.
3. Find the "best" variant (eventually, rank the variants).

Below we explain in details the individual steps.

Step 1: *Calculate the triangular fuzzy weights from the fuzzy pair-wise comparison matrices.*

We assume that both the importance of the criteria and values of the criteria and also the feedback between the criteria are given by the fuzzy weights calculated from the corresponding pair-wise comparison matrices with triangular fuzzy elements.

Let $\tilde{\mathbf{A}}$ be an $n \times n$ reciprocal pair-wise comparison matrix with fuzzy elements (8). Following Van Laarhoven and Pedrycz (1983), we shall calculate a fuzzy vectors of triangular fuzzy weights $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n$, such that a special distance between $\tilde{\mathbf{A}}$ and weights $\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n$ is minimized. In contrast to Van Laarhoven and Pedrycz

(1983), who used a functional

$$\sum_{i<j}\left(\log\frac{w_i^L}{w_j^U}-\log a_{ij}^L\right)^2+\left(\log\frac{w_i^M}{w_j^M}-\log a_{ij}^M\right)^2+\left(\log\frac{w_i^U}{w_j^L}-\log a_{ij}^U\right)^2,$$

here, we apply a modification of this well known logarithmic least-squares method for calculating w_k^L, w_k^M, w_k^U.

Particularly, we solve the following optimization problem:

$$\sum_{i<j}\left(\log\frac{w_i^L}{w_j^L}-\log a_{ij}^L\right)^2+\left(\log\frac{w_i^M}{w_j^M}-\log a_{ij}^M\right)^2+\left(\log\frac{w_i^U}{w_j^U}-\log a_{ij}^U\right)^2\rightarrow \min; \tag{9}$$

subject to

$$w_k^U \geq w_k^M \geq w_k^L \geq 0, \quad k = 1, 2, \ldots, n. \tag{10}$$

Setting the derivatives of the function in (9) to zero (a necessary condition of optimality), it can be easily shown, that the optimal solution of problem (9), (10) shall satisfy the following relations:

$$w_k^L = C_L \cdot \left(\prod_{j=1}^{n} a_{kj}^L\right)^{1/n}, \; w_k^M = C_M \cdot \left(\prod_{j=1}^{n} a_{kj}^M\right)^{1/n}, \; w_k^U = C_U \cdot \left(\prod_{j=1}^{n} a_{kj}^U\right)^{1/n}, \tag{11}$$

$k = 1, 2, \ldots, n$, where coefficients C_L, C_M, C_U are suitable positive constants satisfying the following requirements.

First, we ask that the middle values w_k^M of the fuzzy weights $\tilde{w}_k = \left(w_k^L; w_k^M; w_k^U\right)$ satisfy the "normalization condition", i.e. $\sum_{k=1}^{n} w_k^M = 1$. Hence, from normalization condition and (11) we obtain:

$$C_M = \frac{1}{\sum_{i=1}^{n}\left(\prod_j a_{ij}^M\right)^{1/n}}. \tag{12}$$

From (10) and (11) we obtain:

$$C_L \leq C_M \min_{i=1,\ldots,n}\left\{\frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^L\right)^{1/n}}\right\}, C_U \geq C_M \max_{i=1,\ldots,n}\left\{\frac{\left(\prod_{j=1}^{n} a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^{n} a_{ij}^U\right)^{1/n}}\right\}. \tag{13}$$

Second, we want to find weights $\tilde{w}_k = (w_k^L; w_k^M; w_k^U)$, $k = 1, 2, \ldots, n$, with the minimal *spread,* i.e. the minimal measure of fuzziness:

$$s_k = w_k^U - w_k^L. \tag{14}$$

Therefore, by (12) and (13) we choose the resulting weights as follows:

$$w_k^L = C_{\min} \cdot \frac{\left(\prod_{j=1}^n a_{kj}^L\right)^{1/n}}{\sum_{i=1}^n \left(\prod_{j=1}^n a_{ij}^M\right)^{1/n}}, \quad \text{where } C_{\min} = \min_{i=1,\ldots,n} \left\{ \frac{\left(\prod_{j=1}^n a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^n a_{ij}^L\right)^{1/n}} \right\} \tag{15}$$

$$w_k^M = \frac{\left(\prod_{j=1}^n a_{kj}^M\right)^{1/n}}{\sum_{i=1}^n \left(\prod_{j=1}^n a_{ij}^M\right)^{1/n}}, \tag{16}$$

$$w_k^U = C_{\max} \cdot \frac{\left(\prod_{j=1}^n a_{kj}^U\right)^{1/n}}{\sum_{i=1}^n \left(\prod_{j=1}^n a_{ij}^M\right)^{1/n}}, \quad \text{where } C_{\max} = \max_{i=1,\ldots,n} \left\{ \frac{\left(\prod_{j=1}^n a_{ij}^M\right)^{1/n}}{\left(\prod_{j=1}^n a_{ij}^U\right)^{1/n}} \right\}. \tag{17}$$

Particularly, if $\tilde{\mathbf{A}}$ is a crisp (i.e. non-fuzzy) matrix, i.e. $a_{ij}^L = a_{ij}^M = a_{ij}^U$ for all i, j, then $C_{\min} = C_{\max} = 1$, hence $w_k^L = w_k^M = w_k^U$ for all k, and, the solution-weights are crisp, too.

Remark 1

1. The above stated method can be applied both for calculating the triangular fuzzy weights of the criteria and for eliciting relative triangular fuzzy values of the criteria for the individual variants. Moreover, it can be used also for calculating feedback impacts of some criteria on the other criteria.
2. The advantage of our approach is that it usually gives smaller spreads (i.e. fuzziness) of the calculated fuzzy weights. This fact follows from the following inequalities:
   $0 < x_i^L \le x_i^M \le x_i^U$ and $\frac{x_i^L}{x_j^U} \le \frac{x_i^L}{x_j^L} \le \frac{x_i^M}{x_j^M} \le \frac{x_i^U}{x_j^U} \le \frac{x_i^U}{x_j^L}$ for all *i*, *j*.
3. Eventually, the values of the criteria can be given directly as triangular fuzzy numbers. Then, by the normalization they are transformed again to fuzzy weights. Particularly, assume that evaluations of variants according to some criterion are expressed by triangular fuzzy numbers, moreover, assume that the criterion is maximizing (i.e. "bigger is better"). Let $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_r$, $\tilde{v}_i = (v_i^L; v_i^M; v_i^U)$, $i = 1, 2, \ldots, r$, be a set of triangular fuzzy numbers, e.g. fuzzy evaluations of variants according to some criterion. We assume also $0 < v_i^L \le v_i^M \le v_i^U$. Then we "normalize" the values to obtain triangular fuzzy weights as follows:

$$\tilde{w}_k = \left(w_k^L; w_k^M; w_k^U\right), k = 1, 2, \ldots, r, \quad \text{where}$$
$$\tilde{w}_k = \left(\frac{v_k^L}{S}; \frac{v_k^M}{S}; \frac{v_k^U}{S}\right) \tag{18}$$

and $S = \sum_{k=1}^{r} v_k^M$.

Step 2: *Calculate the aggregating triangular fuzzy evaluations of the variants.*

Having calculated triangular fuzzy weights from all reciprocal matrices with triangular fuzzy elements as it was described above, we calculate the aggregated triangular fuzzy evaluation of the individual variants. For this purpose we use formula (6) applied to reciprocal matrices with triangular fuzzy elements. We calculate either

$$\tilde{\mathbf{Z}} = \tilde{\mathbf{W}}_3\tilde{\mathbf{W}}_1,$$

if there is no feedback among the criteria, or

$$\tilde{\mathbf{Z}} = \tilde{\mathbf{W}}_3\left(\mathbf{I}\tilde{+}\tilde{\mathbf{W}}_2\tilde{+}\tilde{\mathbf{W}}_2^2\tilde{+}\tilde{\mathbf{W}}_2^3\right)\tilde{\mathbf{W}}_1,$$

if there is a feedback among the criteria.

Here, $\tilde{\mathbf{W}}_1 = \begin{bmatrix} \tilde{w}(C_1) \\ \vdots \\ \tilde{w}(C_n) \end{bmatrix}$ is the vector of fuzzy weights of the individual criteria and the columns of matrix $\tilde{\mathbf{W}}_3 = \begin{bmatrix} \tilde{w}(C_1, V_1) & \cdots & \tilde{w}(C_n, V_1) \\ \vdots & \cdots & \vdots \\ \tilde{w}(C_1, V_m) & \cdots & \tilde{w}(C_n, V_m) \end{bmatrix}$ are fuzzy evaluations of variants according to the criteria. Similarly, $\tilde{\mathbf{W}}_2$ is calculated according to (1)–(3), however, with triangular fuzzy elements. For addition and multiplication of triangular fuzzy numbers we use the fuzzy operations defined earlier.

Step 3: *Select the "best" variant, or, order the variants.*

In Step 2 we calculated m variants described as triangular fuzzy numbers, i.e. by the above formula we obtain m triangular fuzzy numbers $\left(z_1^L; z_1^M; z_1^U\right), \ldots, \left(z_m^L; z_m^M; z_m^U\right)$. Finally, we have to solve the problem of ranking fuzzy variants. As the set of triangular fuzzy numbers is not linearly ordered we have to use some ranking methods. There exist a number of sophisticated methods for ranking fuzzy numbers, for a comprehensive review of ranking methods see e.g. Chen et al. (1992). Here we present three relatively simple and well applicable ranking methods.

The first method for ranking a set of triangular fuzzy numbers is the well known *center of gravity method (COG)*. This method is based on computing the x-th coordinates x_i^g of the center of gravity of every triangle given by the corresponding membership functions of $\tilde{z}_i$, $i = 1, 2, \ldots, n$. Evidently, it holds

$$x_i^g = \frac{z_i^L + z_i^M + z_i^U}{3}. \tag{19}$$

By (19) the variants can be ordered from the best (with the highest value of (19)) to the worst (with the lowest value of (19)).

The other two ranking methods are based on the α-cut of the triangular fuzzy number, where $\alpha \in[0,1]$ is an aspiration level given by DM. As the α-cut of a triangular fuzzy number $\tilde{z}_i$ is an interval $\left[z_i^L(\alpha), z_i^R(\alpha)\right]$, we can rank the variants either by the left end points of the corresponding intervals, or by their right end points. In the former case we obtain in some sense a pessimistic ranking method (called *L-dominant*) whereas in the letter case we get an optimistic ranking method of alternatives (*R-dominant*), see Ramik (2007). Notice that all three above mentioned ranking methods coincide with the usual ordering of real numbers if the variants are crisp (i.e. non-fuzzy).

5 Inconsistency

Inconsistency is a specific problem of pair-wise comparison. Pair-wise evaluations are inconsistent if they violate either transitivity of mutual relation or proportionality of evaluations. Consider firstly an $n \times n$ non-fuzzy reciprocal pair-wise comparison matrix **A** such that:

$$\mathbf{A} = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix}. \tag{20}$$

We say that **A** is *consistent* if for each $1 \le i, j, k \le n$, it holds:

$$a_{ij} \cdot a_{jk} = a_{ik}. \tag{21}$$

If for some *i, j, k,* (21) does not hold, we say that **A** is *inconsistent*. In AHP, we have $\frac{1}{9} \le a_{ij} \le 9$, $1 \le i, j \le n$, see Saaty (1991), and the inconsistency of **A** is measured by the *index of inconsistency* I_S as:

$$I_S = \frac{\lambda_{\max} - n}{n - 1}$$

where $\lambda_{\max}$ is the maximal eigenvalue of **A**. Generally, $I_S \ge 0$.

A is said to be *consistent* if $I_S = 0$, see Saaty (1991), however, in practical decision problems a pair-wise comparison matrix is inconsistent with $I_S > 0$. To provide a measure independent of the order of the matrix, *n*, T. Saaty proposed the *consistency ratio* (*CR*). The *CR* is obtained by taking the ratio between *CI* to its expected value over a large number of positive reciprocal matrices of order *n*. For this consistency measure, he proposed a 10% threshold for the *CR* to accept the estimation. In practical decision situations inconsistency is "acceptable" if $CR < 0, 1$. For the prioritization procedure based on geometric mean, the *geometric consistency ratio GCR* was proposed, with an interpretation analogous to that considered for Saaty's *CR*. The *GCR* was proven to be proportional to *CR*, see Aguarón and Moreno-Jimenéz (2003).

Construction of an inconsistency index of the reciprocal matrix with triangular fuzzy elements is based on the idea of distance of the matrix to the "ratio" matrix measured by a particular metric function.

Let M be a set of $n \times n$ matrices with triangular fuzzy elements, and let Φ be a real function defined on M×M, i.e. $\Phi : M \times M \rightarrow \mathbf{R}$ satisfying the 3 assumptions:

(i) $\Phi(\tilde{A}, \tilde{B}) \geq 0$ for all $\tilde{A}, \tilde{B} \in M$.
(ii) If $\Phi(\tilde{A}, \tilde{B}) = 0$ then $\tilde{A} = \tilde{B}$.
(iii) $\Phi(\tilde{A}, \tilde{B}) + \Phi(\tilde{B}, \tilde{C}) \geq \Phi(\tilde{A}, \tilde{C})$ for all $\tilde{A}, \tilde{B}, \tilde{C} \in M$.

Then Φ is called the *metric function on M*.

Let $\tilde{X} = \left\{\frac{\tilde{x}_i}{\tilde{x}_j}\right\}, i, j = 1, 2, \ldots, n$, be a *ratio matrix* with positive triangular fuzzy numbers $\tilde{x}_k = \left(x_k^L; x_k^M; x_k^U\right)$.

The vector of positive triangular fuzzy numbers $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is called the *vector of fuzzy weights* if

$$\sum_{k=1}^{n} x_k^M = 1. \tag{22}$$

Given $\tilde{A} = \{\tilde{a}_{ij}\} - n \times n$ reciprocal matrix with triangular fuzzy elements ($n > 2$), where the support *supp* $\left(\tilde{a}_{ij}\right) \subset S = [^1/_\sigma, \sigma], \sigma > 1, \tilde{a}_{ij} = \left(a_{ij}^L; a_{ij}^M; a_{ij}^U\right), i, j = 1, 2, \ldots, n$, and let Φ and Ψ be metric functions on M.

New inconsistency index of $\tilde{A}$ is designed in two steps:

Step 1: Solve the following optimization problem:

$$\Phi\left(\tilde{A}, \tilde{X}\right) \rightarrow \min; \tag{23}$$

subject to

$$\sum_{k=1}^{n} x_k^M = 1, x_k^U \geq x_k^M \geq x_k^L \geq 0, k = 1, 2, \ldots, n, \tag{24}$$

where $\tilde{X} = \left\{\frac{\tilde{x}_i}{\tilde{x}_j}\right\}, \tilde{x}_k = \left(x_k^L; x_k^M; x_k^U\right)$.

Step 2: Set the *inconsistency index* I_n of $\tilde{A}$ as

$$I_n\left(\tilde{A}\right) = \inf\left\{\Psi\left(\tilde{A}, \tilde{W}\right);\ \tilde{W} - \text{ optimal solution of } (23), (24)\right\}. \tag{25}$$

Remark 2

1. In the first step, the vector of fuzzy weights $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ such that the corresponding ratio matrix $\tilde{X}$ closest to the original matrix $\tilde{A}$ is calculated.
2. In the second step, the new inconsistency index is defined as the smallest distance between an eventual optimal solution of the first step and the original matrix $\tilde{A}$.
3. If $\tilde{A}$ is crisp positive reciprocal matrix, then $\tilde{A}$ is consistent iff $I_n\left(\tilde{A}\right) = 0$.
4. By the above described two-step procedure the class of inconsistency indices is defined depending on metric functions Φ and Ψ. Moreover, Φ from the first step need not be same as Ψ used in the second step.

Now we specify the above mentioned two-step procedure by setting particular formulae of metric functions Φ and Ψ:

$$\Phi\left(\tilde{A}, \tilde{X}\right) = \sum_{i<j} \left\{ \left(\log \frac{x_i^L}{x_j^U} - \log a_{ij}^L \right)^2 + \left(\log \frac{x_i^M}{x_j^M} - \log a_{ij}^M \right)^2 + \left(\log \frac{x_i^U}{x_j^L} - \log a_{ij}^U \right)^2 \right\}, \tag{26}$$

$$\Psi\left(\tilde{A}, \tilde{W}\right) = \gamma . \max_{i,j} \left\{ \max \left\{ \left| \frac{w_i^L}{w_j^U} - a_{ij}^L \right|, \left| \frac{w_i^M}{w_j^M} - a_{ij}^M \right|, \left| \frac{w_i^U}{w_j^L} - a_{ij}^U \right| \right\} \right\}, \tag{27}$$

where γ is a "normalizing" constant (see below).

Remark 3

1. If $\tilde{A}$ is crisp (non-fuzzy), i.e. $a_{ij}^L = a_{ij}^M = a_{ij}^U$ for all i, j, then $C_{\min} = C_{\max} = 1$, hence $w_k^L = w_k^M = w_k^U$ for all k. Consequently, in the optimal solution of problem (23), (24), with (26), the weights are crisp, too.
2. Metric function (27) in the second step has been chosen with respect to possibility theory (in the sense of the Chebychev metric function).

Now we introduce a particular inconsistency index (25) which will be suitable for measuring (in)consistency of reciprocal matrices with fuzzy triangular elements.

For a given scale $S = \left[{}^1/_\sigma, \sigma\right]$, $\sigma > 1$, we define an*inconsistency index* $IN_n^\sigma(\tilde{A})$ *of* $n \times n$ *reciprocal matrix* $\tilde{A}$ *with fuzzy triangular elements* as follows:

$$IN_n^\sigma(\tilde{A}) = \gamma_n^\sigma . \max_{i,j} \left\{ \max \left\{ \left| \frac{w_i^L}{w_j^U} - a_{ij}^L \right|, \left| \frac{w_i^M}{w_j^M} - a_{ij}^M \right|, \left| \frac{w_i^U}{w_j^L} - a_{ij}^U \right| \right\} \right\}, \tag{28}$$

where w_k^L, w_k^M, w_k^U are given by (23), (24), (26) for all $k = 1, 2, \ldots, n$, and

$$\gamma_n^\sigma = \frac{1}{\max\left\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^2\left(\left(\frac{2}{n}\right)^{\frac{2}{n-2}} - \left(\frac{2}{n}\right)^{\frac{n}{n-2}}\right)\right\}}, \quad \text{if } \sigma < \left(\frac{n}{2}\right)^{\frac{n}{n-2}}, \tag{29}$$

$$\gamma_n^\sigma = \frac{1}{\max\left\{\sigma - \sigma^{\frac{2-2n}{n}}, \sigma^{\frac{2n-2}{n}} - \sigma\right\}}, \quad \text{if } \sigma \geq \left(\frac{n}{2}\right)^{\frac{n}{n-2}}. \tag{30}$$

We say that $\tilde{A}$ is *F-consistent* if $IN_n^\sigma(\tilde{A}) = 0$, otherwise, $\tilde{A}$ is *F-inconsistent*.

It can be easily shown that γ_n^σ is a "normalizing" constant. Particularly, if $\tilde{\mathbf{A}}$ is an $n \times n$ reciprocal matrix with triangular fuzzy elements evaluated from the scale $\left[\frac{1}{\sigma}, \sigma\right]$, then

$$0 \leq IN_n^\sigma\left(\tilde{\mathbf{A}}\right) \leq 1.$$

Remark 4

1. Let $\tilde{A}$= A be a crisp positive reciprocal matrix. Then A is consistent according to definition (21) iff it is also F-consistent.
2. Inconsistency index is always between 0 and 1, this property can be clearly interpreted. Comparing to Saaty's consistency ratio CR, no benchmark is necessary.

Example Consider crisp 7×7 reciprocal matrix with triangular fuzzy elements from the scale $\left[\frac{1}{9}, 9\right]$:

$$\mathbf{A}^* = \begin{bmatrix} (1;1;1) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) & (9;9;9) & (9;9;9) & (9;9;9) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) & (9;9;9) & (9;9;9) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) & (9;9;9) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) & (9;9;9) \\ \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & \left(\frac{1}{9};\frac{1}{9};\frac{1}{9}\right) & (1;1;1) \end{bmatrix}.$$

Here, $n = 7, \sigma = 9$, then $9 > \left(\frac{7}{2}\right)^{\frac{7}{7-2}} = 5.78$. By a way similar to (25), we obtain

$$\frac{w_1}{w_n} - a_{1n} = \sigma^{\frac{2n-2}{n}} - \sigma = 9^{\frac{12}{7}} - 9 = 34.24 \text{ and}$$
$$a_{n1} - \frac{w_n}{w_1} = \sigma - \sigma^{\frac{2-2n}{n}} = 9 - 9^{-\frac{12}{7}} = 8.98.$$

Hence,

$$\max_{i,j}\left\{\left|\frac{w_i}{w_j} - a_{ij}\right|\right\} = \max\{34.24, 8.98\} = 34.24.$$

By (28), (30) we obtain $IN_7^9(A^*) = 1$.

6 Illustrating case study

In this section we analyze an illustrating example—decision making situation with 3 decision criteria and 3 variants. Let us assume that the goal of a local government in a decision situation is to choose the "best" tender from 3 pre-selected ones according to 3 criteria: cost attractiveness, technical capability, and reference. Cost attractiveness (Criterion 1) concerns about proposed costs of the project with respect to allocated budget. Technical capability (Criterion 2) refers to the tenders' ability to carry out the project in a satisfactory manner. References (Criterion 3) is related to the past performance of the tenders involving similar types of projects. The criteria are evaluated by pair-wise comparisons with triangular fuzzy values, moreover, interdependency among them is considered.

First, we apply our method, i.e. the algorithm described in Sect. 4, for solving the decision problem. The evaluation of the weights of criteria, the variants according to criteria as well as the feedback between the criteria is based on the data from fuzzy pair-wise comparison matrices. Here, based on the same arguments as in the classical Saaty's method, we use the scale-interval $\left[\frac{1}{9},\ 9\right]$ for evaluating preferences between alternatives. We apply, however, triangular fuzzy numbers.

Second, for comparison our fuzzy approach with non-fuzzy one, we use also non-fuzzy evaluations in the pair-wise comparisons without feedback. In this case we apply the same approach, i.e. the method based on the geometric-mean aggregation, a particular case of the previous more general situation of crisp evaluations by using the middle numbers a^M, i.e. $a^M = a^L = a^U$. We demonstrate that the results are different comparing to the previous fuzzy approach.

Third, we solve the same problem by applying classical AHP by Expert Choice v. 11.0, i.e. we use non-fuzzy evaluations in the pair-wise comparisons without the feedback. Again, we compare the results with the previous cases.

A special SW (Microsoft Excel add-in) named FVK was developed for applying the proposed method. Comparing to other software products, FVK is able to work with fuzzy data with capabilities of MS Excel. Here, as a demonstration, we use outputs obtained by this SW. The DM should fill in the grey regions in the corresponding tables, the white regions are calculated automatically.

Step 1: Evaluate the pair-wise comparison matrices and calculate the corresponding weights by the geometric-mean aggregation.

The data for relative importance of the criteria are given in the following pair-wise comparison matrix:

Criteria Comparison:

Criteria	Crit 1			Crit 2			Crit 3		
Crit 1	1	1	1	1	2	3	2	3	6
Crit 2	0.333	0.5	1	1	1	1	2	3	4
Crit 3	0.167	0.333	0.5	0.25	0.333	0.5	1	1	1
	L	M	U	L	M	U	L	M	U

By (15)–(17) we calculate the corresponding triangular fuzzy weights, i.e. the relative fuzzy importance of the individual criteria:

$\tilde{\mathbf{W}}_1$ =

	L	M	U		
	0.366	0.528	0.761	Criterion 1:	W(C1)
	0.253	0.333	0.461	Criterion 2:	W(C2)
	0.101	0.140	0.183	Criterion 3:	W(C3)

The character of all three criteria is qualitative, they are evaluated by pair-wise comparisons with triangular fuzzy values given in the following 3 pair-wise comparison matrices:

Evaluation of Variants According to Individual Criteria:

Crit 1	**Var 1**			**Var 2**			**Var 3**		
Var 1	1	1	1	1	2	3	2	3	8
Var 2	0.33	0.5	1	1	1	1	1	2	3
Var 3	0.13	0.33	0.5	0.33	0.5	1	1	1	1
	L	M	U	L	M	U	L	M	U

Crit 2	**Var 1**			**Var 2**			**Var 3**		
Var 1	1	1	1	1/4	1/3	1	1/4	1/3	1
Var 2	1	3	4	1	1	1	1	2	2
Var 3	1	3	4	0.5	0.5	1	1	1	1
	L	M	U	L	M	U	L	M	U

Crit 3	**Var 1**			**Var 2**			**Var 3**		
Var 1	1	1	1	1/4	1/3	1	2	3	6
Var 2	1	3	4	1	1	1	1	2	3
Var 3	0.17	0.33	0.5	0.33	0.5	1	1	1	1
	L	M	U	L	M	U	L	M	U

The corresponding fuzzy matrix $\tilde{\mathbf{W}}_3$ of fuzzy weights—evaluations of variants according to the individual criteria is calculated by (15)–(17) as

$\tilde{\mathbf{W}}_3$ =

L	M	U	L	M	U	L	M	U
0.374	0.540	0.857	0.115	0.140	0.290	0.236	0.297	0.540
0.206	0.297	0.428	0.290	0.528	0.581	0.297	0.540	0.680
0.103	0.163	0.236	0.231	0.333	0.461	0.113	0.163	0.236

The data for evaluations of fuzzy feedbacks between the criteria are given in 3 pair-wise comparison matrices $\mathbf{B}_k$, see formula (3), however, with fuzzy triangular elements. The interpretation of e.g. $\mathbf{B}_1$ is as follows: Criterion 1 is "approximately 2 times more influenced" by Criterion 2 than it is influenced by Criterion 3. The term “approximately 2 times more influenced” is evaluated by the triangular fuzzy number (1; 2; 3).

Feedback:

B_1 =

Crit 2			Crit 3			
1	1	1	1	2	3	**Crit 2**
0.33	0.5	1	1	1	1	**Crit 3**

B_2 =

Crit 1			Crit 3			
1	1	1	1	2	2	**Crit 1**
0.5	0.5	1	1	1	1	**Crit 3**
L	M	U	L	M	U	

B_3 =

Crit 1			Crit 2			
1	1	1	2	4	4	**Crit 1**
0.25	0.25	0.5	1	1	1	**Crit 2**
L	M	U	L	M	U	

By using (15)–(17) we again obtain the corresponding fuzzy weights and arrange these weights into the fuzzy feedback matrix $\tilde{\mathbf{W}}_2$. There are zeros in the main diagonal as we do not expect an impact of the criterion on itself:

$\tilde{\mathbf{W}}_2$ =

0	0	0	0.471	0.667	0.667	0.567	0.8	0.8
0.471	0.667	0.816	0	0	0	0.2	0.2	0.283
0.272	0.333	0.471	0.333	0.333	0.471	0	0	0
L	M	U	L	M	U	L	M	U

Step 2: Calculate the aggregating triangular fuzzy evaluations of the variants.

By computing triangular fuzzy weights and evaluations as it was mentioned earlier, we calculate the aggregated triangular fuzzy evaluation of the individual variants. For this purpose we use the approximate formula (6), applied for matrices with the elements being triangular fuzzy numbers–triples of positive numbers and with the normalized columns:

$$\tilde{\mathbf{Z}} = \tilde{\mathbf{W}}_3(\mathbf{I}\tilde{+}\tilde{\mathbf{W}}_2\tilde{+}\tilde{\mathbf{W}}_2^2\tilde{+}\tilde{\mathbf{W}}_2^3)\tilde{\mathbf{W}}_1. \tag{31}$$

Here, for addition, subtraction and multiplication of triangular fuzzy numbers in (31) we use the fuzzy arithmetic operations defined earlier.

Step 3: Select the “best” variant - rank the variants.

In Step 2 we have found the variants described as triangular fuzzy numbers, i.e. by (31) we calculated the triangular fuzzy vector

$$\tilde{\mathbf{Z}} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = \left(\left(z_1^L; z_1^M; z_1^U\right), \left(z_2^L; z_2^M; z_2^U\right), \left(z_3^L; z_3^M; z_3^U\right)\right)$$

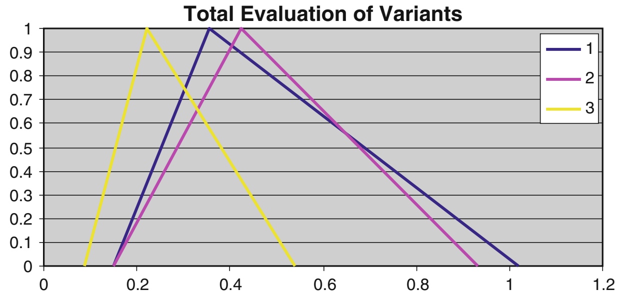

Fig. 1 Total evaluation of fuzzy variants

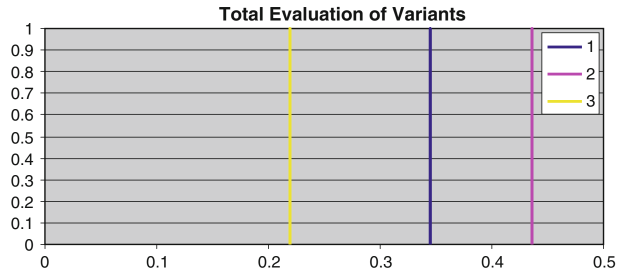

Fig. 2 Crisp case without feedback

given in the following table:

Total Evaluation of Variants:

Z =	L	M	U	Center of gravity	Rank
Var 1	0.150	0.355	1.020	0.508	1
Var 2	0.149	0.424	0.932	0.502	2
Var 3	0.087	0.221	0.540	0.283	3

In the following step we have to rank the evaluations of the above fuzzy variants resulting in the "best" variant by using a proper way of ranking the triangular fuzzy numbers depicted in Fig. 1.

Step 3: Find the "best" variant-rank the variants.

By (19) the variants are ranked from the best (with the biggest value of center of gravity) to the worst (with the lowest value of COG). The situation is graphically depicted in Fig. 1. In this figure we can find that by COG method the best variant is Var 1, whereas by L-dominancy and also R-dominancy with $\alpha > 0.6$, Var 2 is evaluated as the best one.

Now, we solve the same problem applying our method with non-fuzzy evaluations in the pair-wise comparisons and without feedback among the criteria. We apply a particular case of the previous more general situation considering the same data, however, taking into account only middle values of the triangular fuzzy numbers, i.e. $a^M = a^L = a^U$.

Hence, we get

W_1 =	0.528	Criterion 1:	W(C1)
	0.333	Criterion 2:	W(C2)
	0.140	Criterion 3:	W(C3)

By (19) we obtain the crisp evaluations of the variants:

Total Evaluation of Variants:

Z =		**Rank**
Var 1	0.344	2
Var 2	0.436	1
Var 3	0.220	3

The situation is graphically depicted in Fig. 2. In this figure we can see that now the best variant is Var 2. As all three variants are crisp, i.e. real numbers, the above mentioned 3 ranking methods coincide with the natural ordering of real numbers.

Now, we solve the same problem with the data as above applying classical AHP by Expert Choice v. 11.0 (with non-fuzzy evaluations in the pair-wise comparisons and without feedback).

Total Evaluation of Variants:

Z =		**Rank**
Var 1	0.367	2
Var 2	0.418	1
Var 3	0.215	3

We can see that Var 2 is again evaluated as the best one, moreover, numerically the AHP evaluations of the variants are well comparable to the evaluations calculated by our method.

7 Conclusion

Considering feedback dependences between the criteria the total rank of the variants may change as we have demonstrated in the above example. Fuzzy (soft) evaluation of pair-wise comparisons may be considered more comfortable and appropriate for DM. Moreover, occurrence of dependences among criteria is realistic (and also frequent). A presence of fuzziness in evaluations may change the final rank of variants, too. In this paper we proposed a new MCDM method based on geometric-mean aggregation allowing for fuzzy pair-wise comparisons and a feedback dependences between the criteria. The evaluation of the weights of criteria, the variants as well as the feedback between the criteria is based on the data given in pair-wise comparison matrices. Extended arithmetic operations with fuzzy numbers are used as well as ordering fuzzy relations to rank fuzzy variants. An illustrating numerical example is presented to clarify the methodology. A special SW - Microsoft Excel add-in named FVK was developed by the authors for applying the proposed method. Comparing to other software products, FVK is able to work with fuzzy data as well as capabilities of widespread spreadsheet Microsoft Excel.

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