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Existence and uniqueness theorem for uncertain differential equations

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Abstract Canonical process is a Lipschitz continuous uncertain process with stationary and independent increments, and uncertain differential equation is a type of differential equations driven by canonical process. This paper presents some methods to solve linear uncertain differential equations, and proves an existence and uniqueness theorem of solution for uncertain differential equation under Lipschitz condition and linear growth condition.

Keywords Uncertain process · Differential equation · Existence and uniqueness theorem

1 Introduction

Uncertainty theory (Liu 2007) is a branch of mathematics based on normality, monotonicity, self-duality, countable subadditivity, and product measure axioms. The uncertainty theory has become a new tool to study uncertainty in human systems. As an application of uncertainty theory, Liu (2009b) proposed a spectrum of uncertain programming which is a type of mathematical programming involving uncertain variables, and applied uncertain programming to system reliability design, facility location problem, vehicle routing problem, project scheduling problem, finance and so on. In

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addition, Li and Liu (2009) presented uncertain logic in which the truth value is defined as the uncertain measure that the proposition is true. Following that, uncertain entailment was developed by Liu (2009d) as a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. Furthermore, uncertain inference was pioneered by Liu (2009a) as a process of deriving consequences from uncertain knowledge or evidence via the tool of conditional uncertainty. Other researchers have also done a lot of theoretical work related to uncertainty theory, such as Chen and Ralescu (2009), Gao (2009), Gao and Ralescu (2009), Qin and Kar (2009), Qin et al. (2009), You (2009), Peng (2009), Peng and Iwamura (2009), and Zhu (2009), etc. For exploring the recent developments of uncertainty theory, the readers may consult Liu (2009c).

Uncertain process is a sequence of uncertain variables indexed by time or space. The study of uncertain process was started by Liu (2008). As a counterpart of Brownian motion, Liu (2009a) designed a canonical process that is a Lipschitz continuous uncertain process with stationary and independent increments. Following that, uncertain calculus was initialized by Liu (2009a) in 2009 to deal with differentiation and integration of functions of uncertain processes.

Differential equations have been widely applied in physics, engineering, biology, economics and other fields. With the development of science and technology, practical problems require more and more accurate description. A wide range of uncertainties are added to the differential equation system, thus producing stochastic differential equations and fuzzy differential equations. Furthermore, uncertain differential equation, a type of differential equations driven by canonical process, was defined by Liu (2008). In this paper, we discuss how to solve linear uncertain differential equations. However, in many cases, it is difficult to find analytic solution of uncertain differential equation. In order to provide a theoretical foundation of numerical solution methods, we will prove an existence and uniqueness theorem of solution for uncertain differential equation under Lipschitz condition and linear growth condition.

The rest of the paper is organized as follows. Some preliminary concepts of uncertainty theory are recalled in Sect. 2. The method to solve linear uncertain differential equation is presented in Sect. 3. An existence and uniqueness theorem is proved in Sect. 4. At last, a brief summary is given in Sect. 5.

2 Preliminary

Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event.

Definition 2.1 (Liu 2007) A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following four axioms:

Axiom 1 (*Normality*) $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (*Monotonicity*) $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ whenever $\Lambda_1 \subset \Lambda_2$;
Axiom 3 (*Self-Duality*) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ;

Axiom 4 (*Countable Subadditivity*) For every countable sequence of events $\{\Lambda_i\}$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}.$$

.

An uncertain variable is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. The uncertainty distribution of uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$ for $x \in \Re$. The expected value of an uncertain variable ξ is defined by

$$E[\xi] = \int_{0}^{+\infty} \mathcal{M}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite. And the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \min_{1\leq i\leq m} \mathcal{M}\{\xi \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_m$ of real numbers. An uncertain variable ξ is called normal if it has a normal uncertain distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

denoted by $\mathcal{N}(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$. Note that e is the expected value of ξ and σ^2 is the variance of ξ. Suppose that ξ_1 and ξ_2 are two independently normal uncertain variables with expected values e_1 and e_2, variances σ_1^2 and σ_2^2, respectively. Then for any real numbers a and b, the uncertain variable $a\xi_1 + b\xi_2$ is also a normal uncertain variable with expected value $ae_1 + be_2$ and variance $(|a|\sigma_1 + |a_2|\sigma_2)^2$.

Definition 2.2 (*Liu 2008*) Let T be an index set and let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain process is a measurable function from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for each $t \in T$ and any Borel set B of real numbers, the set $\{\gamma \in \Gamma | X_t(\gamma) \in B\}$ is an event.

Now, let us introduce a special uncertain process called canonical process, that plays the role of counterpart of Brownian motion.

Definition 2.3 (*Liu 2009a*) An uncertain process C_t is said to be a canonical process if

(1) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(2) C_t has stationary and independent increments,
(3) every increment $C_{t+s} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

The uncertain process $\mathrm{d}C_t = C_{t+\mathrm{d}t} - C_t$ has the properties that $E[\mathrm{d}C_t] = 0$ and $\mathrm{d}t^2/2 \le E[\mathrm{d}C_t^2] \le \mathrm{d}t^2$. Then $\mathrm{d}C_t$ and $\mathrm{d}t$ are infinitesimals of the same order. It has been proved that $\mathrm{d}C_t/\mathrm{d}t$ is a normal uncertain variable with expected value 0 and variance 1.

Definition 2.4 (*Liu 2009a*) Let X_t be an uncertain process and let C_t be a canonical process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh size is written as

$$\triangle = \max_{1\le i\le k} |t_{i+1} - t_i|.$$

Then the uncertain integral of X_t with respect to C_t is

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\triangle\to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is an uncertain variable.

Let $h(t, c)$ be a continuously differentiable function. Then $X_t = h(t, C_t)$ is an uncertain process. Liu (2009a) proved the following chain rule

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t.$$

3 Uncertain differential equations

Based on Brownian motion, Kiyosi Ito founded stochastic integrals and formulated stochastic differential equation in 1942. Liu (2008) presented a type of uncertain differential equations driven by the canonical process.

Definition 3.1 (*Liu 2008*) Suppose that C_t is a canonical process, and f and g are some given functions. Then

$$\mathrm{d}X_t = f(X_t, t)\mathrm{d}t + g(X_t, t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation. A solution is an uncertain process X_t that satisfies (1) identically in t.

Theorem 3.1 *Let C_t be a canonical process. If u_t and v_t are some continuous functions with respect to t, then the homogeneous linear uncertain differential equation*

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t \tag{2}$$

has a solution

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right).$$

Proof It follows from the chain rule that

$$\mathrm{d}\ln X_t = \frac{1}{X_t}\mathrm{d}X_t = u_t \mathrm{d}t + v_t \mathrm{d}C_t.$$

Integration of both sides yields

$$\ln X_t - \ln X_0 = \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s.$$

Therefore, the solution of (2) is

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right).$$

The proof is complete. □

Theorem 3.2 *Suppose that $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are some continuous functions with respect to t. Then the linear uncertain differential equation*

$$\mathrm{d}X_t = (u_{1t} X_t + u_{2t})\mathrm{d}t + (v_{1t} X_t + v_{2t})\mathrm{d}C_t \tag{3}$$

has a solution

$$X_t = U_t \left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right)$$

where

$$U_t = \exp\left(\int_0^t u_{1r}\mathrm{d}r + \int_0^t v_{1r}\mathrm{d}C_r\right).$$

Proof At first, we define two uncertain processes U_t and V_t via

$$\mathrm{d}U_t = u_{1t}U_t\mathrm{d}t + v_{1t}U_t\mathrm{d}C_t,\ \mathrm{d}V_t = \frac{u_{2t}}{U_t}\mathrm{d}t + \frac{v_{2t}}{U_t}\mathrm{d}C_t.$$

Then we have $X_t = U_t V_t$ because

$$\begin{aligned}\mathrm{d}X_t &= V_t\mathrm{d}U_t + U_t\mathrm{d}V_t \\ &= (u_{1t}V_tU_t + u_{2t})\mathrm{d}t + (v_{1t}V_tU_t + v_{2t})\mathrm{d}C_t \\ &= (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t.\end{aligned} \tag{4}$$

Note that

$$U_t = U_0 \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we obtain the solution of (3) as

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right).$$

where

$$U_t = \exp\left(\int_0^t u_{1r}\mathrm{d}r + \int_0^t v_{1r}\mathrm{d}C_r\right).$$

The proof is complete. □

Corollary 3.1 *Assume that m_t, α_t, and σ_t are continuous functions of t. Then the following uncertain differential equation*

$$\mathrm{d}X_t = (m_t - \alpha_t X_t)\mathrm{d}t + \sigma_t\mathrm{d}C_t, \quad \alpha_t \neq 0 \tag{5}$$

has a solution

$$\begin{aligned}X_t = X_0 \exp\left(-\int_0^t \alpha_r\mathrm{d}r\right) + \int_0^t m_s \exp\left(-\int_s^t \alpha_r\mathrm{d}r\right)\mathrm{d}s \\ + \int_0^t \sigma_t \exp\left(-\int_s^t \alpha_r\mathrm{d}r\right)\mathrm{d}C_s.\end{aligned}$$

Proof The uncertain differential Eq. (5) is essentially a linear one. By Theorem 3.2, we have $U_t = \exp\left(-\int_0^t \alpha_s\mathrm{d}s\right)$ and

$$\begin{aligned}X_t &= U_t\left(X_0 + \int_0^t \frac{m_s}{U_s}\mathrm{d}s + \int_0^t \frac{\sigma_s}{U_s}\mathrm{d}C_s\right)\\&= X_0 \exp\left(-\int_0^t \alpha_r \mathrm{d}r\right) + \int_0^t m_s \exp\left(-\int_s^t \alpha_r \mathrm{d}r\right)\mathrm{d}s\\&\quad + \int_0^t \sigma_t \exp\left(-\int_s^t \alpha_r \mathrm{d}r\right)\mathrm{d}C_s.\end{aligned}$$

When α_t, m_t, and σ_t are replaced with some constants α, m and σ, respectively, the solution can be simplified as

$$X_t = \frac{m}{\alpha} + \left(X_0 - \frac{m}{\alpha}\right)\exp(-\alpha t) + \sigma \int_0^t \exp(\alpha(s-t))\mathrm{d}C_s.$$

The proof is complete. □

4 Existence and uniqueness theorem

In this section, we assume that the coefficients of uncertain differential equation satisfy Lipschitz continuous condition and linear growth condition, and prove that the solution exists and is unique. Before proving the existence and uniqueness theorem, we first show a lemma.

Lemma 4.1 *Suppose that C_t is a canonical process, and X_t is an integrable uncertain process on $[a, b]$ with respect to t. Then the inequality*

$$\left|\int_a^b X_t(\gamma)\mathrm{d}C_t(\gamma)\right| \le K(\gamma)\int_a^b |X_t(\gamma)|\mathrm{d}t$$

holds, where $K(\gamma)$ is the Lipschitz constant of the sample path $X_t(\gamma)$.

Proof Let $a = t_0 < t_1 < \cdots < t_n = b$, $\Delta = \max\limits_{1\le i\le n}(t_i - t_{i-1})$. By the definition of uncertain integral, we get

$$\begin{aligned}\left|\int_a^b X_t(\gamma)\mathrm{d}C_t(\gamma)\right| &= \left|\lim_{\Delta\to 0}\sum_{i=1}^n X_{t_{i-1}}(\gamma)(C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma))\right|\\&\le \lim_{\Delta\to 0}\sum_{i=1}^n \left|X_{t_{i-1}}(\gamma)\right| \cdot \left|C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)\right|.\end{aligned}$$

For each sample γ, it follows from the definition of canonical process that $C_t(\gamma)$ is Lipschitz continuous with respect to t. Thus there exists a finite number $K(\gamma)$ such that $|C_{t_1}(\gamma) - C_{t_2}(\gamma)| \leq K(\gamma)|t_1 - t_2|$ and

$$\begin{aligned}\lim_{\Delta\to 0}\sum_{i=1}^{n}\left|X_{t_{i-1}}(\gamma)\right|\cdot\left|C_{t_i}(\gamma) - C_{t_{i-1}}(\gamma)\right| \\ \leq K(\gamma)\lim_{\Delta\to 0}\sum_{i=1}^{n}\left|X_{t_{i-1}}(\gamma)\right|\cdot|t_i - t_{i-1}| \\ \leq K(\gamma)\int_a^b |X_t(\gamma)|\mathrm{d}t\end{aligned}$$

The theorem is proved. □

Theorem 4.1 (Existence and Uniqueness Theorem) *The uncertain differential Eq. (1) has a unique solution if the coefficients $f(x,t)$ and $g(x,t)$ satisfy the Lipschitz condition*

$$|f(x,t) - f(y,t)| + |g(x,t) - g(y,t)| \leq L|x - y|, \quad \forall x, y \in \Re,\ t \geq 0$$

and linear growth condition

$$|f(x,t)| + |g(x,t)| \leq L(1 + |x|), \quad \forall x \in \Re,\ t \geq 0$$

for some constants L. Moreover, the solution is sample-continuous.

Proof To prove the existence of solution, a successive approximation method will be introduced to construct a solution of the uncertain differential Eq. (1). Define $X_t^{(0)} = X_0$, and

$$X_t^{(n+1)} = X_0 + \int_0^t f\left(X_s^{(n)}, s\right)\mathrm{d}s + \int_0^t g\left(X_s^{(n)}, s\right)\mathrm{d}C_s, \quad n = 0, 1, 2, \ldots$$

For any sample γ, we define

$$D_t^{(n)}(\gamma) = \max_{0\leq s\leq t}\left|X_s^{(n+1)}(\gamma) - X_s^{(n)}(\gamma)\right|, \quad n = 0, 1, 2, \ldots$$

We claim that

$$D_t^{(n)}(\gamma) \leq (1 + |X_0|)\frac{L^{n+1}(1 + K(\gamma))^{n+1}}{(n+1)!}t^{n+1}, \quad n = 0, 1, 2, \ldots, \quad 0 \leq t \leq T$$

where T is a constant. Indeed for $n = 0$, it follows from Lemma 4.1 that

$$
\begin{aligned}
D_t^{(0)}(\gamma) &= \max_{0\le s\le t}\left|\int_0^s f(X_0, v)\mathrm{d}v + \int_0^s g(X_0, v)\mathrm{d}C_v(\gamma)\right| \\
&\le \max_{0\le s\le t}\left|\int_0^s f(X_0, v)\mathrm{d}v\right| + \max_{0\le s\le t}\left|\int_0^s g(X_0, v)\mathrm{d}C_v(\gamma)\right| \\
&\le \max_{0\le s\le t}\int_0^s |f(X_0, v)|\,\mathrm{d}v + K(\gamma)\max_{0\le s\le t}\int_0^s |g(X_0, v)|\,\mathrm{d}v \\
&\le \int_0^t |f(X_0, v)|\,\mathrm{d}v + K(\gamma)\int_0^t |g(X_0, v)|\,\mathrm{d}v \\
&\le (1+|X_0|)\,L\,(1+K(\gamma))\,t \text{ (by the linear growth condition).}
\end{aligned}
$$

This confirms the claim for $n = 0$. Next we assume the claim is true for some $n - 1$. Then

$$
\begin{aligned}
D_t^{(n)}(\gamma) &= \max_{0\le s\le t}\left|\int_0^s \left(f\left(X_v^{(n)}(\gamma), v\right) - f\left(X_v^{(n-1)}(\gamma), v\right)\right)\mathrm{d}v\right. \\
&\quad \left. + \int_0^s \left(g\left(X_v^{(n)}(\gamma), v\right) - g\left(X_v^{(n-1)}(\gamma), v\right)\right)\mathrm{d}C_v(\gamma)\right| \\
&\le \max_{0\le s\le t}\left|\int_0^s \left(f\left(X_v^{(n)}(\gamma), v\right) - f\left(X_v^{(n-1)}(\gamma), v\right)\right)\mathrm{d}v\right| \\
&\quad + \max_{0\le s\le t}\left|\int_0^s \left(g\left(X_v^{(n)}(\gamma), v\right) - g\left(X_v^{(n-1)}(\gamma), v\right)\right)\mathrm{d}C_v(\gamma)\right| \\
&\le \int_0^t \left|f\left(X_v^{(n)}(\gamma), v\right) - f\left(X_v^{(n-1)}(\gamma), v\right)\right|\mathrm{d}v \\
&\quad + K(\gamma)\int_0^t \left|g\left(X_v^{(n)}(\gamma), v\right) - g\left(X_v^{(n-1)}(\gamma), v\right)\right|\mathrm{d}v \qquad (6) \\
&\le L\int_0^t \left|X_v^{(n)}(\gamma) - X_v^{(n-1)}(\gamma)\right|\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+K(\gamma)L\int_0^t \left|X_v^{(n)}(\gamma)-X_v^{(n-1)}(\gamma)\right| \mathrm{d}v \text{ (by Lipschitz condition)}\\
&\le L(1+K(\gamma))\int_0^t \left|X_v^{(n)}(\gamma)-X_v^{(n-1)}(\gamma)\right| \mathrm{d}v\\
&\le L(1+K(\gamma))\int_0^t (1+(|X_0|))\frac{L^n(1+K(\gamma))^n v^n}{n!}\mathrm{d}v\\
&\le (1+|X_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}t^{n+1}. \qquad (7)
\end{aligned}
$$

Note that (6) and (7) are induced from Lemma 4.1 and the inductive assumption, respectively. This proves the claim. Therefore,

$$
\begin{aligned}
D_t^{(n)}(\gamma) &= \max_{0\le s\le t}\left|X_s^{(n+1)}(\gamma)-X_s^{(n)}(\gamma)\right|\\
&\le (1+|X_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}t^{n+1}
\end{aligned}
$$

holds for all $n \ge 0$. It follows from Weierstrass' criterion that, for each sample γ,

$$
\begin{aligned}
&\sum_{n=0}^{+\infty}(1+|X_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}t^{n+1}\\
&\quad\le \sum_{n=0}^{+\infty}(1+|X_0|)\frac{L^{n+1}(1+K(\gamma))^{n+1}}{(n+1)!}T^{n+1} < +\infty.
\end{aligned}
$$

Thus $X_t^{(k)}(\gamma)$ converges uniformly in $t \in [0, T]$. We denote the limit by

$$
X_t(\gamma) = \lim_{k\to\infty} X_t^{(k)}(\gamma),\ \gamma \in \Gamma, t \in [0, T].
$$

Then

$$
X_t = X_0 + \int_0^t f(X_s, s)\mathrm{d}s + \int_0^t g(X_s, s)\mathrm{d}s.
$$

Therefore X_t is the solution of (1) for all $t \ge 0$ since T is arbitrary.

Next, we will prove that the solution of uncertain differential Eq. (1) is unique. Assume that both of X_t and X_t^* are solutions of (1) with the same initial value X_0.

Then for each $\gamma \in \Gamma$, we have

$$
\begin{aligned}
&|X_t(\gamma) - X_t^*(\gamma)| \\
&= \left| \int_0^t (f(X_v(\gamma), v) - f(X_v^*(\gamma), v)) \mathrm{d}v \right. \\
&\quad \left. + \int_0^t (g(X_v(\gamma), v) - g(X_v^*(\gamma), v)) \mathrm{d}C_v(\gamma) \right| \\
&\le \left| \int_0^t (f(X_v(\gamma), v) - f(X_v^*(\gamma), v)) \mathrm{d}v \right| \\
&\quad + \left| \int_0^t (g(X_v(\gamma), v) - g(X_v^*(\gamma), v)) \mathrm{d}C_v(\gamma) \right| \\
&\le \int_0^t |f(X_v(\gamma) - f(X_v^*(\gamma))| \mathrm{d}v \\
&\quad + \int_0^t |g(X_v(\gamma), v) - g(X_v^*(\gamma), v)| \mathrm{d}v \text{ (by Lemma 4.1)} \\
&\le L \int_0^t |X_v(\gamma) - X_v^*(\gamma)| \mathrm{d}v \\
&\quad + K(\gamma) L \int_0^t |X_v(\gamma) - X_v^*(\gamma)| \mathrm{d}v \text{ (by Lipschitz condition)} \\
&\le L(1 + K(\gamma)) \int_0^t |X_v(\gamma) - X_v^*(\gamma)| \mathrm{d}v.
\end{aligned}
$$

It follows from Gronwall inequality that

$$
|X_t(\gamma) - X_t^*(\gamma)| \le 0 \cdot \exp(L(+K(\gamma))t) = 0
$$

for all γ. Hence $X_t = X_t^*$.

At last, we will prove the sample-continuity of X_t. By the above proof, we get

$$
X_t(\gamma) \le \sum_{n=0}^{\infty} (1 + |X_0|) \frac{(L(1 + K(\gamma)t))^n}{n!} = (1 + |X_0|) \exp(L(1 + K(\gamma))t).
$$

Suppose $t > s > 0$. We have

$$\begin{aligned}
&|X_t(\gamma) - X_s(\gamma)| \\
&= \left| \int_s^t f(X_v(\gamma), v)\mathrm{d}v + \int_s^t g(X_v(\gamma), v)\mathrm{d}C_v(\gamma) \right| \\
&\le \int_s^t |f(X_v(\gamma), v))|\mathrm{d}v + \left| \int_s^t g(X_v(\gamma), v)|\mathrm{d}C_v(\gamma) \right| \qquad (8) \\
&\le \int_s^t |f(X_v(\gamma), v)|\mathrm{d}v + K(\gamma) \int_s^t |g(X_v(\gamma), v)|\mathrm{d}v \\
&\le (1 + K(\gamma))(1 + |X_0|) \exp\left(L(1 + K(\gamma))t\right)(t - s) \qquad (9)
\end{aligned}$$

Note that (8) and (9) are induced from Lemma 4.1 and linear growth condition, respectively. Thus $|X_t - X_s| \to 0$ as $s \to t$. Hence X_t is sample-continuous. The theorem is proved.

5 Conclusions

Uncertain differential equation is an important tool to deal with dynamic systems in uncertain environments. The contribution of this paper to uncertain differential equation theory was first to provide an existence and uniqueness theorem under Lipschitz condition and linear growth condition.

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