Fuzzy Optim Decis Making (2009) 8:245–262
DOI 10.1007/s10700-009-9063-4

Prioritized OWA aggregation

Ronald R. Yager

Published online: 11 June 2009
© Springer Science+Business Media, LLC 2009

Abstract We indicate that the problem of prioritized criteria arises in situations in which there exists a relationship between the criteria so that lack of satisfaction by the higher priority criteria cannot be readily compensated for by satisfaction by lower priority criteria. Typical of this situation is the relationship between safety and cost. We consider the problem of criteria aggregation in this environment. Central to our approach is the use of importance weights to enforce this prioritization imperative. We apply our use of priority based importance weights to the case where the scope of the criteria aggregation is an OWA type aggregation.

Keywords Multi-criteria · Aggregation operators · Decision-making · Lexicographic

1 Introduction

Many applications of modern computational technology involve the task of selecting some object from a set of alternatives based upon the satisfaction of specified criteria. With the wide spread use of database technologies underlying many commercial websites this has become an important problem in E-commerce. Search and information retrieval involve this task. Website personalization and customization to take advantage of knowledge of consumer preferences and buying habits is based upon these types of selections. Multi-criteria decision-making is also an application that involves this task.

Central to this task is the aggregation of individual criteria satisfactions to obtain an overall score for an alternative. These aggregated values can then be used to help select between alternatives.

R. R. Yager (✉)
Machine Intelligence Institute, Iona College, New Rochelle, NY 10801, USA
e-mail: yager@panix.com

If $C = \{C_1, \ldots, C_n\}$ are a collection of criteria and $C_j(x) \in [0, 1]$ indicates the satisfaction of alternative x to criteria C_j then we indicate the overall satisfaction of x to the collection of criteria, $C(x)$, as $C(x) = F(C_1(x), \ldots, C_n(x))$. It is some function of these individual satisfactions. In addition to the individual scores the formulation of F depends upon the relationship between the criteria. Do we require "all" the criteria or will just satisfying some of them be sufficient. We shall refer to this as the ***scope*** of the organization of the criteria. Often the scope of the criteria organization can be expressed using linguistic quantifiers Zadeh (1983); Yager (1996) such as "most", "almost all", "some" or "any". Generally the scope associated with a collection of criteria is cardinality based and makes no distinction between the criteria.

The operators $\mathrm{Min}_i[C_i(x)]$, $\mathrm{Max}_i[C_i(x)]$ and $\frac{1}{n} \sum_i C_i(x)$ are aggregation operators reflecting different scopes associated with the criteria. Linguistically they correspond respectively to the quantifiers, "all", "any" and "some".

In Yager (1988) we introduced the OWA operator that provides a parameterized class of aggregation operators that allows for the modeling of a wide array of different expressions of the organizational scope associated with a collection of criteria. As noted the scope associated with the organization of a collection of criteria is simply cardinality based and makes no distinction between the criteria.

Other considerations that can effect the formulation of F are any distinction among the criteria and interrelationship between the criteria. One such distinction is different importance's associated with the criteria.

Our comprehension of the meaning of importance is dependent upon the type of aggregation being performed. In some cases it can be seen as reflecting the strength of a criterion's need for satisfaction. This is particularly the case for "anding" type aggregation. In other cases the importance associated with a criteria can be seen to reflect the ability of a criteria to contribute to the overall satisfaction. This is particularly the case for the averaging operator.

Another interpretation of the concept of importance is as measuring the ability of an increase of one criteria to compensate for a decrease in satisfaction in another criteria. Here it is reflecting an idea of tradeoffs between criteria. From this perspective if w_k and w_i are the importances of criteria C_k and C_i then the importances are telling us we can compensate for a decrease of Δ in satisfaction to criteria C_k by gain $\frac{w_K}{w_i}\Delta$ in satisfaction to criteria C_i.

In some applications we may not want this kind of compensation between criteria. Consider the situation in which an airline pilot is making a decision based on consideration of gasoline cost and safety. In this situation we should not allow a benefit with respect to cost of fuel to compensate for a loss in safety. Here we have a kind of prioritization of the criteria. The relationship between the criteria is that safety has a higher priority than cost.

In Yager (2008), and Yager et al. (2008) we look at aggregation methods which allow for the modeling of this type of prioritized relationships between the criteria. Central to the approach taken in Yager (2008) was to include priorities by using importance weights in which the importance of lower priority criteria was based upon the satisfaction to the higher priority criteria. In Yager (2008) we applied this to the cases of Max, Min and average operator. Here we generalize this technique to the case where the scope of the aggregation is determined by an OWA operator.

2 Prioritized aggregation

In Yager (2008) we introduced a class of aggregation operations called Prioritized Average (PA) operators. These operators allow for an averaging aggregation of criteria satisfaction for the case in which there exists a prioritization ordering among the criteria. In the following we describe this aggregation operator.

Here we assume we have a collection of criteria $C = \{C_1, \ldots, C_n\}$ and that there is a prioritization between the criteria expressed by the linear ordering

$$C_1 > C_2 > C_3 \ldots > C_n.$$

In this ordering criteria C_i has a higher priority than C_k if $i < k$. In addition for any alternative x and criteria C_j the value $C_i(x) \in [0, 1]$ indicates the satisfaction of criteria C_i by x. The PA operator provides a weighted averaging aggregation of these criteria satisfactions that reflects the prioritization of the criteria. For the above

$$C(x) = F(C_i(x)) = \sum_{i=1}^{n} w_i C_i(x)$$

Here the weights, w_i, are obtained using the prioritization relationship and the criteria satisfactions in a method described in the following[1]. It is important here to emphasize that the resulting form for $C(x)$ is non-linear.

To obtain the weights we proceed as follows

1. We denote $S_k = C_k(x)$
2. With each criterion we associate a value $u_i = T_i$, called its un-normalized weight, defined as follows.

(i) $T_i = 1$

(ii) $T_i = \prod_{k=1}^{i-1} S_K$ for $i = 2$ to n $(T_i = S_{i-1} T_{i-1})$

From these un-normalized weights we obtain the normalized importance weights as

$$w_i = \frac{u_i}{\sum_{j=1}^{n} u_j}$$

We see the importance weights of criteria are determined by the satisfaction of the higher priority criteria. This situation enforces the goal of restricting the effect of lower order criteria to compensate for lack of satisfaction of higher order criteria. In particular, except for the first category, the weight of a criterion is proportional to the product of the satisfaction of the criteria in the higher priority categories.

[1] More formally, we should use $w_i(x)$ to represent the weights, as the weights are dependent on x.

We make some observations about this approach. First in calculating T_i we used $T_i = S_{i-1}T_{i-1}$ an alternative is to use $T_i = \text{Min}[S_{i-1}, T_{i-1}]$. In this case T_i is essentially the value of the least satisfied alternative in class higher than C_i. The use of product will allocate more importance to the higher priority items.

In Yager (2008) we showed that this operator has the necessary properties to be a mean operator, it was monotonic, symmetrical and bounded.

In the following in order to get some feel for the performance of the PA operator we look at it for some cases where the scores are drawn from the space {0, 1}. We first consider the case of three criteria A, B, C where the priority is $A > B > C$

A	B	C	$F(A, B, C)$
1	1	1	1
1	1	0	2/3
1	0	1	1/2
1	0	0	1/2
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

We observe that when the satisfaction to A is zero under no conditions do we get any compensation from B and C. In the case when the satisfaction to A is one but the satisfaction to B is zero we don't get any compensation from C even if it is satisfied.

We now look at the case of four criteria with $A > B > C > D$. Again we see lack of satisfaction to A blocks any compensation. Furthermore lack of satisfaction to B blocks any possible contribution from C or D.

A	B	C	D	$F(A, B, C, D)$
1	1	1	1	1
1	1	1	0	3/4
1	1	0	1	2/3
1	1	0	0	2/3
1	0	1	1	1/2
1	0	1	0	1/2
1	0	0	1	1/2
1	0	0	0	1/2
0	1	1	1	0
0	1	1	0	0
0	1	0	1	0
0	1	0	0	0
0	0	1	1	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	0

In the preceding we indicated the values to be aggregated, the $C_i(x)$, must be numbers in the unit interval. It is possible to relax this constraint. Let us assume $C_i(x) \in R_i$ where R_i is a subset of the real line. We can look at R_i as the range of values that $C_i(x)$

can take. We can now associate with each criterion C_i a function $G_i\colon R_i \rightarrow [0, 1]$. Here, $G_i(r)$ indicates our degree of satisfaction with a value of r for criteria C_i. Using this function we can modify our approach to allow for values of $C_i(x)$ not in the unit interval.

In particular for each criterion C_i we calculate $G_i(C_i(x))$. For simplicity we denote this as d_i. Using this we proceed as follows

1. For each criterion we let $S_k = d_k$
2. Again we calculate

(i) $T_1 = 1$

(ii) $T_i = \prod_{k=1}^{i-1} S_K \quad \text{for } i = 2 \text{ to } n$

Using these we then calculate $w_i = \frac{T_i}{\prod_{i=1}^{n} T_i}$ and then finally we obtain as our evaluation of alternative x

$$C(x) = \sum_{i=1}^{n} w_i C_i(x)$$

In this situation while we aggregate the $C_i(x)$ the weights are determined using the $G_i(C_i(x))$.

Our objective now is to extend this prioritized type aggregation to the more general class of OWA operators. In what follows we shall assume unless otherwise noted that the $C_i(x) \in [0, 1]$.

3 OWA operators

In Yager (1988) we introduced a parameterized class of mean type aggregation operators called the OWA operator, Ordered Weighted Averaging Operator. These operators have been widely used in the task of multi-criteria aggregation Chiclana et al. (2000); Herrera et al. (2003); Kacprzyk and Zadrozny (1997); Mitchell and Schaefer (2000); Xu and Da (2002); Yager and Kacprzyk (1997).

Definition An OWA operator of dimension n is a mapping $F\colon R^n \rightarrow R$ so that

$$F(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j$$

where b_j is the jth largest of the a_i and w_j is a weight so that $w_j \in [0, 1]$ and $\sum_j w_j = 1$.

We can associate with an OWA operator a vector W, called the OWA weighting vector, so that w_j is the jth component of W. Here then the OWA operator is parameterized by W. Furthermore we can associate with the OWA aggregator an index function, *ind*, so that ind(j) is the index of the jth largest of the a_i. Using this we can express $F(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j a_{ind(j)}$

Various different aggregation operators can be obtained by appropriate selection of W. Some notable examples are the following. If $w_j = 1/n$ for all j then we get the simple average. If $w_1 = 1$ and $w_j = 0$ for $j \neq 1$ then we get the Max. If $w_n = 1$ and $w_j = 0$ for $j \neq n$ we get the Min.

A useful parameter that can be associated with an OWA aggregator is its attitudinal character. For an OWA aggregator with weighting vector W we define

$$A - C(W) = \sum_{j=1}^{n} w_j \frac{n-j}{n-1}$$

It can be shown that $A - C(W) \in [0, 1]$. In the framework multi-criteria decision making it has been shown the closer $A - C(W)$ is to one the more "orlike" the relationship between the criteria. On the other hand the closer $A - C(W)$ is to zero the more "andlike" the relationship between the criteria. We also note that for the case where $w_j = 1/n$ then we have $A - C(W) = 0.5$.

Another characterizing feature of the weighting vector W is its degree of dispersion

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j).$$

As noted in Yager (1988) this indicates how much of the argument information is used. For $w_j = 1/n$ for all j it assumes its maximal value of $\ln(n)$. For $w_k = 1$ for some argument it assumes the value zero. This is the minimal value. We can introduce $E(W) = \frac{H(W)}{\ln(n)}$ as the normalized version of the measure of dispersion. Here, $E(W) = [0, 1]$.

An important issue in the use of OWA operators is the determination of the weights. Among the most common methods used for obtaining the weights are the following:

1. Direct choice of weight
2. Learn weights from data
3. Select a notable type of aggregation
4. Using a characterizing feature
5. Linguistic-functional specification

We shall just comment on the last three. There are many notable types of aggregation that are OWA operators. As we already indicated the simple average is one of these. Other examples are the Max, Min and Median. In Yager (1993) describes many notable examples of the operator. In these cases the associated OWA weights are easily identified. For example for the Max we have $w_1 = 1$ and all other $w_j = 0$. For the Min we have $w_n = 1$ and all others are zero. The median has $w_j = 1$ for $j = \frac{n+1}{2}$ if n is odd otherwise $w_j = 1/2$ for $j = \frac{n}{2}$ and $j = \frac{n}{2} + 1$. Thus in these special cases we essentially have the associated weights.

The use of characterizing features was introduced by O'Hagan (1988, 1990). In this approach we provide a value $\alpha \in [0, 1]$ as the desired attitudinal characterization of the weighting vector. Using this we determine the weights that have the maximal dispersion. Thus in this method we solve the following mathematical programming problem for the w_j

$$\begin{aligned} \text{Max} \quad & -\sum_{j=1}^{n} w_j \ln(w_j) \\ \text{S/t} \quad & \sum_{j=1}^{n} w_j \tfrac{n-j}{n-1} = \alpha \\ & \sum_{j=1}^{n} w_j = 1 \\ & 0 \le w_j \le 1 \end{aligned}$$

As a result of solving this problem we end up with a set of OWA weights.

The use of linguistic-functional specification was introduced by Yager (1996). Assume f is a BUM[2] function, a mapping f: $[0, 1] \to [0, 1]$ such that $f(0) = 0$, $f(1) = 1$ and $f(x) \ge f(y)$ if $x > y$. In Yager (1996) we showed that we could obtain the set of n weights needed to define an OWA operator from this type of function by assigning

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \quad \text{for } j = 1 \text{ to } n$$

It can be shown that the w_j obtained satisfy the properties required of OWA weights: $w_j \in [0, 1]$ and $\sum_j w_j = 1$.

Thus using this method we can obtain an OWA weighted vector from a specified function. In Yager (1996) we enhanced the useful of this approach. It was suggested that starting with a linguistic specification describing the type of aggregation we can express this linguistic specification as a fuzzy subset of the unit interval. Then the membership function of this fuzzy subset can provide the function needed to generate the weights as described above. A particularly important type of linguistic specification is provided by linguistic quantifiers, such as most, almost, some etc. Thus here we would have a specification such as "***Most*** of the criteria must be satisfied". Using the technology provided by in Zadeh (1983) we would the express ***Most*** as a fuzzy subset of the unit interval. This membership function is then used to obtain the OWA Weights.

Since it is not our purpose here to dwell on the linguistic formulation of aggregation specifications we shall assume the availability of a functional specification of the aggregation imperative.

4 Importance weighted OWA aggregation

When using OWA operators in the construction of multi-criteria aggregation functions we must be able to include importance associated with each of the criteria. Here, we assume a collection $C_1, \ldots, C_n$ of criteria, and an importance weight $v_j \in [0, 1]$

[2] BUM is an acronym for Basic, Unit Interval, and Monotonic.

associated with each criterion. For a given alternative x we allow $C_i(x) \in [0, 1]$ to be its satisfaction to the ith criteria. For notational simplicity, if no confusion arises, we shall denote $C_i(x) = a_i$. In this environment our task becomes the evaluation of

$$C(x) = \text{OWA}((a_1, v_1), (a_2, v_2), \ldots, (a_n, v_n))$$

That is we need provide the OWA aggregation of these tuples.

In Yager (1997) we suggested an approach to performing this type of aggregation for the case in which the scope of OWA aggregation imperative is specified in terms of a BUM function f of the type described above. We now review this approach.

In the following we shall again assume ind is an index function so that ind(j) is the index of the jth largest of the a_i. Here the $a_{\text{ind}(j)}$ is the jth largest of a_i and $v_{\text{ind}(j)}$ is its associated importance weight. We shall let $R = \sum_{j=1}^{n} v_{\text{ind}(j)}$, it is the sum of the importance weights. We are now in the position to obtain the OWA weights, u_j, that will be used in the aggregation. In this situation we calculate

$$u_j = f\left(\frac{R_j}{R}\right) - f\left(\frac{R_{j-1}}{R}\right)$$

where $R_j = \sum_{k=1}^{j} V_{\text{ind (k)}}$. We note $R_0 = 0$. Thus R_j is the sum of the importance weights associated with the jth most satisfied criteria. Using these weights we obtain

$$\text{OWA}((a_1, v_1), \ldots, (a_n, v_n)) = \sum_{j=1}^{n} u_j\ a_{\text{ind}(j)}.$$

We should point out that if $v_{\text{ind}(j)} = 0$ then $R_j = R_{j-1}$ and hence $f\left(\frac{R_j}{R}\right) = f\left(\frac{R_{j-1}}{R}\right)$ and, therefore, $u_j = 0$. Thus we see if the importance of an argument is zero, its OWA weight is also zero.

In the preceding we suggested an approach to formulate the aggregation $\text{OWA}((a_1, v_1), (a_2, v_2), \ldots (a_n, v_n))$ where the scope of the aggregation is guided by a BUM function f. We shall now consider the situation when we start with an OWA weighting vector W. Thus here we have a set of weights w_j, $j = 1$ to n. We must now modify these weights to include the different importance. In Torra (1997); Torra and Narukawa (2007) Torra suggested an approach to obtaining these modified weights.

The basic idea of the approach is described in the following. We assume there is some unknown underlying BUM function f that has generated the original weights, the w_j. In particular $w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right)$. While we don't completely know the function f we know some of its properties. For example $f(0) = 0$ and $f(1) = 1$. Furthermore, we also observe that $f\left(\frac{j}{n}\right) = w_j + f\left(\frac{j-1}{n}\right)$. Since $f(0) = 0$ we obtain that $f\left(\frac{1}{n}\right) = w_1$ and this allows us to obtain $f\left(\frac{2}{n}\right) = w_2 + f\left(\frac{1}{n}\right) = w_2 + w_2$. More generally we can obtain that for any $j = 1$ to n we have $f\left(\frac{j}{n}\right) = \sum_{k=1}^{j} w_k$ and

$f(0) = 0$. Denoting $w_0 = 0$ we can express this as $f\left(\frac{j}{n}\right) = \sum_{k=0}^{j} w_k$ for $j = 0$ to n. Thus from the given weights can obtain the value of $f(x)$ at the points $x = \frac{j}{n}$ for $j = 0$ to n. We are now faced with the question of selecting a function f that satisfies these conditions.

Many possibilities exist for choosing this function. In Torra (1997); Torra and Narukawa (2007) the author suggests modeling f as a piecewise linear function. In particular it is suggested that with $j = 1$ to n we let

$$f(x) = m_j\, x + b_j \quad \text{for} \quad \frac{j-1}{n} \leq x \leq \frac{j}{n}.$$

The parameters (m_j, b_j) are determined so that the following pair of equations are satisfied for each j

$$f\left(\frac{j-1}{n}\right) = \frac{j-1}{n} m_j + b_j$$
$$f\left(\frac{j}{n}\right) = \frac{j}{n} m_j + b_j$$

In this case $f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) = \frac{m_j}{n}$ and since also $f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) = w_j$; we have $\frac{m_j}{n} = w_j$ and therefore $m_j = n\, w_j$. In addition, we have that

$$f\left(\frac{j}{n}\right) = \sum_{k=1}^{j} w_k = n w_j \frac{j}{n} + b_j$$

From this we get

$$b_j = \sum_{k=1}^{j} (w_k - w_j) = \sum_{k=1}^{j-1} (w_k - w_j) = \sum_{k=1}^{j-1} w_k - (n-1) w_j$$

therefore,

$$f(x) = \sum_{k=1}^{j-1} w_k - \mathrm{nw}_j \left(x - \frac{j-1}{n}\right) \quad \text{for} \quad \frac{j-1}{n} \leq x \leq \frac{j}{n}$$

or equivalently

$$f(x) = f\left(\frac{j-1}{n}\right) + w_j(\mathrm{nx} - (j-1)) \quad \text{for} \quad \frac{j-1}{n} \leq x \leq \frac{j}{n}$$

Using this function we can now obtain the weights u_j that take into account both the w_j and individual importance weights of the criteria. In the following we shall assume $r_i = \frac{V_i}{R}$, it is the normalized weight. Thus $r_i \in [0, 1]$ and $\sum_i r_i = 1$. With

$r_{\text{ind}(j)}$ being the normalized importance weight of the jth largest a_i we can calculate $\sum_{k=1}^{j} r_{\text{ind}(k)} = \frac{R_j}{R}$. Using this and our just determined formula for f we obtain as in the preceding

$$u_j = f\left(\frac{R_j}{R}\right) - f\left(\frac{R_{j-1}}{R}\right)$$

Using this we can obtain the OWA aggregation

$$\text{OWA}((a_1, v_1), (a_2, v_2), \ldots, (a_n, v_n)) = \sum_{j=1}^{n} u_j \, a_{\text{ind}(j)}$$

We note for the special case when $w_j = 1/n$ for all j we then have for any $j - 1 \leq x < j$.

$$f(x) = \left(\frac{j-1}{n}\right) + \frac{1}{n}(nx - (j-1)) = x$$

Here then f is simply a linear function, $f(x) = 1$. In this case we obtain that $u_j = \frac{R_j}{R} - \frac{R_{j-1}}{R} = r_{\text{ind}(j)}$. Here then $\text{OWA}((a_1, v_1), \ldots, (a_n, v_1)) = \frac{1}{R}\sum_i v_i a_i$. It is the usual weighted average.

For the case where the importance weights are all the same, $v_j = 1/n$, then $\frac{R_j}{R} = \frac{j}{n}$ and hence $u_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) = w_j$. This is the same as the original when we did not consider individual criteria importance.

We also observe if $r_{\text{ind}(j)} = 0$ then $u_j = 0$, hence objects with zero importance have zero weight.

Another special case is where $w_k = 1$. In this case,

$$\begin{array}{lll} f(x) = 0 & \text{for} & x \leq \frac{k-1}{n} \\ f(x) = \text{nx} - (k-1) & \text{for} & \frac{k-1}{n} \leq x \leq \frac{k}{n} \\ f(x) = 1 & \text{for} & x \geq \frac{k}{n} \end{array}$$

We plot this function in Fig. 1. In this case, $w_j = 0$ for all j such that $R_j \leq \frac{k-1}{n}$ and $R_j \geq \frac{k}{n}$. For those j such that $\frac{k-1}{n} < R_j < \frac{k}{n}$ the weight get proportionally divided

Another special case is where $w_1 = 1 - \alpha$ and $w_n = \alpha$. Here, we get

$$\begin{array}{ll} f(x) = (1-\alpha)(\text{nx}) & 0 \leq x \leq \frac{1}{n} \\ f(x) = 1 - \alpha & \frac{1}{n} \leq x \leq \frac{n-1}{n} \\ f(x) = (1-\alpha) + \alpha\left(x - \frac{(n-1)}{n}\right) & \frac{n-1}{n} \leq x \leq 1 \end{array}$$

In Fig. 2, we plot this form of f.

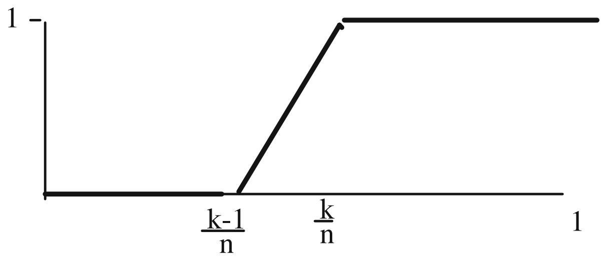

Fig. 1 Form of f when $w_k = 1$

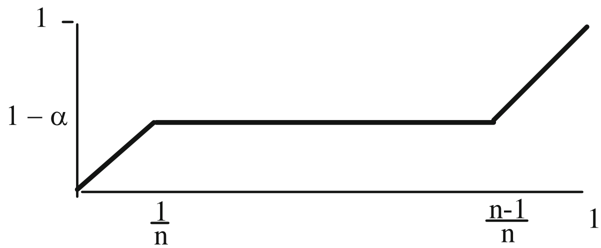

Fig. 2 Form of f when $w_1 = 1 - \alpha$ and $w_n = \alpha$

5 Prioritized OWA aggregation

We now describe the formulation of the prioritized OWA operator, POWA aggregation operator. While we shall embed our discussion within the context of multi-criteria aggregation the formal framework presented here is appropriate for the aggregation of any prioritized collection of arguments.

Here, we assume a collection of criteria $C = \{C_1, C_2, \ldots, C_n\}$ that are prioritized such that $C_i > C_k$ if $i < k$. This is a linear ordering with C_1 having the highest priority. We shall assume for a given alternative x, $C_i(x) = a_i \in [0, 1]$ is the degree of satisfaction to criterion C_i by alternative x. Our interest is to provide $C(x)$, the overall satisfaction of x to the multiple criteria as an OWA aggregation of the individual criteria satisfaction in such a way as to reflect the organization of the criteria with respect to both the scope and priority relationships between the criteria.

Our first step is to obtain the priority induced importance weights of each of the criteria in the case of alternative x. For each criterion we let $S_i = C_i(x)$, the degree of satisfaction of C_i. We then use this to obtain its un-normalized priority based importance $u_i = T_i$ where T_i is such that

$$
\begin{aligned}
T_i &= 1 \\
T_i &= \prod_{k=1}^{i-1} S_k = S_{i-1} T_{i-1} \quad \text{for } i = 2 \text{ to } n.
\end{aligned}
$$

Using this we are able to obtain a normalized priority based importance weight.

$$r_i = \frac{u_i}{\sum_{j=1}^{n} u_j}$$

We note that if $S_k = 0$ then $T_i = 0$ for all $i > k$. From this we see that in the case of $S_k = 0$ we have $r_i = 0$ for $i > k$. In the special case where $S_1 = 0$ then $r_i = 0$ for $i > 1$ and hence $r_1 = 1$.

We further observe that since $T_i \geq T_k$ for $i < k$ then $r_i \geq r_k$ for $i < k$. Thus a criterion can never have a bigger normalized priority importance weight than a criterion that has a higher priority than it.

Once we have the normalized priority based importance we can use our previously described approach for implementing importance weighted OWA aggregations.

Our next step then is to order the criteria by their satisfactions. Here we let ind(k) be the index of the kth most satisfied criteria, it is the criteria with the kth largest of the a_i. We shall let $r_{\text{ind}(k)}$ be its associated priority based importance. We now use this to obtain the OWA weights that reflect the complete organization of the criteria, the scope of the aggregation and priority relationship.

We first consider the case where the scope of the organization of the criteria is expressed in terms of a BUM function f. In this case we calculate $v_k = f\left(\tilde{R}_k\right) - f\left(\tilde{R}_{k-1}\right)$ where $\tilde{R}_k = \sum_{i=1}^{k} r_{\text{ind}(i)}$. Then we calculate $C(x)$ using the OWA operator

$$C(x) = \sum_{k=1}^{n} V_k a_{\text{ind}(k)}$$

In the case where the scope of the criteria organization is described in terms of an n dimension vector W with components w_k we use the method described earlier. We assume f is a piecewise linear function

$$f(z) = \sum_{i=1}^{j-1} w_i + w_j(\text{nx} - (j-1)) \quad \text{for} \quad \tfrac{j-1}{n} \leq z \leq \tfrac{j}{n}$$

Then we proceed as above to obtain $v_k = f\left(\tilde{R}_k\right) - f\left(\tilde{R}_{k-1}\right)$

Example We assume five criteria C_1, C_2, C_3, C_4, C_5 with the following priority ordering $C_1 > C_2 > C_3 > C_4 > C_5$

For alternative x we have

$$C_1(x) = 0.7, C_2(x) = 1.0, C_3(x) = 0.6, C_4(x) = 0.9, C_5(x) = 0.4$$

Using this we have

$$S_1 = 0.7$$
$$S_2 = 1.0$$
$$S_3 = 0.6$$

$$S_4 = 0.9$$
$$S_5 = 0.4$$

From this we get $T_i = T_{i-1} S_{i-1}$

$$T_1 = 1$$
$$T_2 = 0.7$$
$$T_3 = (0.7)(1) = 0.7$$
$$T_4 = (0.6)(0.7) = 0.42$$
$$T_5 = (0.9)(0.42) = (0.38)$$

We see $T = \sum_i T_i = 3.2$. From this we get

$$r_1 = \frac{1}{3.2} = 0.31$$
$$r_2 = \frac{0.7}{3.2} = 0.22$$
$$r_3 = \frac{0.7}{3.2} = 0.22$$
$$r_4 = \frac{0.42}{3.2} = 0.13$$
$$r_5 = \frac{0.38}{3.2} = 0.12$$

We now order the criteria satisfactions and obtain $i_{\text{nd}}(j)$

$$\begin{array}{lll} \text{ind}(1) = 2 & a_{\text{ind}(1)} = 1.0 & r_{\text{ind}(1)} = 0.22 \\ \text{ind}(2) = 4 & a_{\text{ind}(2)} = 0.9 & r_{\text{ind}(2)} = 0.13 \\ \text{ind}(3) = 1 & a_{\text{ind}(3)} = 0.7 & r_{\text{ind}(3)} = 0.31 \\ \text{ind}(4) = 3 & a_{\text{ind}(4)} = 0.6 & r_{\text{ind}(4)} = 0.22 \\ \text{ind}(5) = 5 & a_{\text{ind}(5)} = 0.4 & r_{\text{ind}(5)} = 0.12 \end{array}$$

We shall first all assume our OWA aggregation is guided by the function $f(x) = x^2$. Using this and the normalized priority based weights we obtain the v_j

$$V_1 = (0.22)^2 = 0.048$$
$$V_2 = (0.35)^2 - (0.22)^2 = 0.075$$
$$V_3 = (0.66)^2 - (0.35)^2 = 0.316$$
$$V_4 = (0.88)^2 - (0.66)^2 = 0.338$$
$$V_5 = (1) - (0.88)^2 = 0.223$$

Using these values and the criteria satisfactions we get $C(x) = \sum_{j=1}^{5} v_j a_{\text{ind}(j)} = 0.63$

Now we assume the scope of the aggregation is expressed by a vector W with components

$$w_1 = 0.1,\ w_2 = 0.2,\ w_3 = 0.2,\ w_4 = 0.3,\ w_5 = 0.2$$

Using this we can get our pseudo generating function f such that

$$\begin{array}{ll} f(z) = 0.1(\mathrm{nz}) & 0 \le z < 0.2 \\ f(z) = 0.1 + 0.2(5z - 1) & 0.2 \le z < 0.4 \\ f(z) = 0.3 + 0.2(5z - 2) & 0.4 \le z < 0.6 \\ f(z) = 0.5 + (0.3)(5z - 3) & 0.6 \le z < 0.8 \\ f(z) = 0.8 + (0.2)(5z - 4) & 0.8 \le z \le 1 \end{array}$$

In this example

$$R_0 = 0,\ R_1 = 0.22,\ R_2 = 0.35,\ R_3 = 0.66,\ R_4 = 0.88,\ \text{and } R_5 = 1$$

From this we get

$$f(R_0) = 0,\ f(R_1) = 0.12,\ f(R_2) = 0.25,$$
$$f(R_3) = 0.59,\ f(R_4) = 0.88,\ f(R_5) = 1$$

Using this we get as our modified weight

$$v_1 = 0.12;\ v_2 = 0.13;\ v_3 = 0.34;\ v_4 = 0.29;\ v_5 = 0.12$$

From this we get as our aggregated value

$$\begin{aligned} C(x) = \sum_{k=1}^{5} v_k a_{\mathrm{ind}(k)} &= (0.12)(1) + (0.13)(0.9) + (0.34)(0.7) \\ &+ (0.24)(0.6) + (0.12)(0.4) = 0.697 \end{aligned}$$

Let us consider some special cases of the POWA operator. First consider the situation where $C_1(x) = 0$. In this case $S_i = 0$ for all i and hence $T_1 = 1$ and $T_i = 0$ for all $i > 1$. From this we get $r_1 = 1$ and $r_i = 0$ for all $i > 1$. Let ind(j) be the index of the jth largest payoff. In this case with $C_1(x) = 0$, we can always assign ind$(n) = 1$. From this we see $R_j = \sum_{k=1}^{j} r_{\mathrm{ind}(k)} = 0$ for $j < n$ and $R_n = 1$. Using this we get

$$\begin{array}{ll} v_j = f(R_j) - f(R_j - 1) = 0 & \text{for } j = 1 \text{ to } n - 1. \\ v_n = f(R_n) - f(R_n - 1) = 1 & \end{array}$$

and hence $\mathrm{OWA}(C_1, \ldots, C_n) = \sum_{j=1}^{n} v_j C_{\mathrm{ind(j)}}(x) = v_n C_{\mathrm{ind(j)}}(x) = v_n C_1(x) = 0$. Thus in the case where $C_1(x) = 0$ we always get an aggregated value of zero regardless of the form of f or the satisfaction of the other criteria by x. There is no possibility for any compensation.

An interesting situation is where $C_1(x) = a$ and all other $C_j = 1$. In this case $S_1 = a$ and $S_j = 1$ for all $j > 1$. From this we get $T_1 = 1$ and $T_j = a$ for all $j > 1$.

Using this we have

$$r_1 = \frac{1}{1 + \sum_{j=2}^{n} a} = \frac{1}{1 + (n-1)a}$$

$$r_j = \frac{a}{1 + (n-1)a} \qquad\qquad j = 2 \text{ to } n$$

Furthermore here again we have ind$(n)=1$ and hence we obtain

$$\mathrm{OWA}(C_1, \ldots, C_n) = \sum_{j=1}^{n} v_j C_{\mathrm{ind}(j)}(x) = \sum_{j=1}^{n-1} v_j + v_n a$$

We further observe $\sum_{j=1}^{n-1} v_j = f(R_{n-1}) - f(R_0) = f(R_{n-1})$ We see $R_{n-1} = \frac{(n-1)a}{1+(n-1)a}$. In addition $v_n = f(R_n) - f(R_{n-1}) = 1 - f(R_{n-1})$. Thus, here we get

$$\begin{aligned}\mathrm{OWA}(C_1, \ldots, C_n) = f\left(\frac{(n-1)a}{1+(n-1)a}\right) \\ + \left(1 - f\left(\frac{(n-1)a}{1+(n-1)a}\right)\right) a = a \\ + (1-a) f\left(\frac{(n-1)a}{1+(n-1)a}\right)\end{aligned}$$

We note that this is the largest value the aggregation can take when $C_1(x) = a$.

Let us further assume that a, the satisfaction to the highest priority criteria, is small. In the situation where there are only a few other criteria $a(n-1)$ is also small and hence $(n-1)\ a \approx 0$. In this case then $f\left(\frac{(n-1)a}{1+(n-1)a}\right) \approx f(0) \approx 0$ and hence $\mathrm{OWA}(C_1, \ldots, C_n) = a$. On the other hand, if a is small but $n-1$ is large enough so that $(n-1)\ a \approx 1$ then $\frac{(n-1)a}{1+(n-1)a} \approx 0.5$ and we get that

$$\mathrm{OWA}(C_1, \ldots, C_n) = a + (1-a) f(0.5)$$

In the more extreme case where $(n-1)a \gg 1$ then we get $\frac{(n-1)a}{1+(n-1)a} \approx \frac{(n-1)a}{(n-1)a} \approx 1$ and hence $\mathrm{OWA}(C_1, \ldots, C_n) = a + (1-a) f(1) = a + (1-a) = 1$

But to attain this we need large n, a lot of additional criteria. So we see if the satisfaction of the highest priority criteria is small the ability to compensate has been made difficult.

We continue with this case where $C_1(x) = a$ and all other $C_j(x) = 1$ which has

$$\mathrm{OWA}(C_1, \ldots, C_n) = a + (1-a) f\left(\frac{(n-1)a}{1+(n-1)a}\right).$$

If we assume that f is linear, $f(x) = x$ we get

$$\begin{aligned}\text{OWA}(C_1, \ldots, C_n) &= a + (1-a)\frac{(n-1)a}{1+(n-1)a} \\ &= \frac{a + (n-1)a^2 + (n-1)a - (n-1)a^2}{1+(n-1)a} \\ \text{OWA}(C_1, \ldots, C_n) &= \frac{\text{na}}{1 - \text{na} - a} = \frac{\text{na}}{\text{na} + (1-a)}\end{aligned}$$

More generally if assume $f(x) = x^{\lambda}$ for $\lambda \in [0, \infty]$ then

$$\text{OWA}(C_1, \ldots, C_n) = a + (1-a)\left(\frac{(n-1)}{1+(n-1)^a}\right)^{\lambda}$$

We see that if, $\lambda \rightarrow \quad \infty$, since $(n-1)a < 1 + (n-1)$ a then $\text{OWA}(C_1, \ldots, C_n) = a + (1-a)0 = a$. If $\lambda \rightarrow 0$ then $\left(\frac{(n-1)a}{1+(n-1)a}\right)^{\lambda} \rightarrow 1$ and hence $\text{OWA}(C_1, \ldots, C_n) \approx 1$

For the case where $\lambda = 2$ we get

$$\begin{aligned}\text{OWA}(C_1, \ldots, C_n) &= \frac{a(1+(n-1)(a))^2 + (1-a)(n-1)^2(a^2)}{(1+(n-1)a)^2} \\ \text{OWA}(C_1, \ldots, C_n) &= \frac{(a{+}2(n-1)(a))^2{+}(n-1)^2a^3{+}(n-1)^2a^2 - (n-1)^2(a)^3}{(1{+}(n-1)a)^2} \\ \text{OWA}(C_1, \ldots, C_n) &= \frac{a + (n-1)a^2(2+n-1)}{} = \frac{a(1 + a(n-1)(n+1))}{(1+(n-1)a)^2}\end{aligned}$$

6 Prioritized sugeno integral aggregation

There exists another aggregation operator closely related to the OWA operators which can be seen as a special case of the Sugeno integral Sugeno (1977); Mesiar and Mesiarová (2008). In Yager (1992) we discussed this operator in considerable detail. Again let f be a BUM function $f(0) = 0$, $f(1) = 1$ and $f(x_1) \geq f(x_2)$ if $x_1 \geq x_2$. Let C_i be a collection of criteria where $C_i(x)$ is the satisfaction of C_i by x. Let us denote $C_i(x) = a_i$. Using this we can get an aggregation of the criteria satisfactions as

$$C(x) = \text{Max}_k\left[f\left(\frac{k}{n}\right) \wedge a_{\text{ind}(k)}\right]$$

where $a_{\text{ind}(k)}$ is the kth largest of the $C_i(x)$.

The extension of this aggregation imperative to the case where each of the C_i has importance weight v_i can be easily made. Here, we first let $V = \sum_i v_i$ be the total of the individual criteria importance weights and we can let $r_i = \frac{v_i}{V}$ be the normalized importance weights. We shall further let $r_{\text{ind}(k)}$ be the normalized importance weight associated with the kth largest of the $C_i(x)$. Using this we can let $R_k = \sum_{i=1}^{k} r_{\text{ind}(i)}$ and then we get as our aggregated value

$$C(x) = \text{Max}_k[f(R_k) \wedge a_{\text{ind}(k)}]$$

We can naturally extend this to the situation where the r_i are determined by a priority relationship among the criteria as in the preceding.

7 Conclusion

Fundamental to the construction of multi-criteria decision functions is what we called the organization of the criteria. By this we meant to indicate the body of knowledge that guides how we combine an alternatives satisfactions to the individual criteria to obtain its overall satisfaction. Since there exists a wide variety of ways that a collection of criteria can be organized one of the objectives of computational intelligence is to provide tools to enable the modeling of various possibilities about our knowledge of the organization of a collection of criteria. In this work we focused on the situation in which our knowledge of the criteria organization was expressed in terms of two features. One of these is what we referred to as the scope of the criteria organization. By this we meant information about how many of the criteria we desire to be satisfied. Notable examples of this are "all", "most" or "any." The OWA operators have provided a very robust class of aggregation operators for modeling this type of information. The second feature of the criteria organization considered here was the specification of a prioritization relationship between the criteria reflecting the ability of some criteria to compensate for lack of satisfaction of other criteria. We suggested one approach to building aggregation functions that can capture this type of priority relationship between criteria is to associate with criteria importance weights based on the satisfaction of the higher order criteria. We then described how the importance weighted OWA operator can be used to construct multi-criteria aggregation functions in situations in which our knowledge of the criteria organization is described in terms of scope and a priority relationship over the criteria.

In Yager et al. (2008) we applied the technology described in the paper in the form of a web Personal Evaluation Tool (webPET) to the E-Commerce problem of selecting the most suitable web service that fit a user's needs. The webPet uses the concept of lexicographic preferences and combines user's criteria with customer reviews. The lexicographic preferences allows for mimicking user's attitude that some criteria should be satisfied before other criteria are considered. The criterion satisfaction levels were defined with threshold-based satisfaction level functions built based on two thresholds representing boundaries between acceptable and unacceptable values of attributes of alternatives.

References

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2000). The ordered weighted geometric operator: properties and applications. In *Proceedings of the 8th International Conference on Information Processing and Management of Uncertainty in Knowledge-based systems* (pp. 985–991). Madrid.

Herrera, F., Herrera-Viedma, E., & Chiclana, F. (2003) A study of the origins and uses of the ordered weighted geometric operator in multicriteria decision making. *International Journal of Intelligent Systems*, *18*, 689–707.

Kacprzyk, J., & Zadrozny, S. (1997). Implementation of OWA operators in fuzzy querying for Microsoft Access. In R. R. Yager, & J. Kacprzyk (Eds.), *The ordered averaging operators: Theory and applications* (pp. 293–306). Boston: Kluwer.

Mesiar, R., & Mesiarová, A. (2008). Fuzzy integrals—what are they? *International Journal of Intelligent Systems, 23*, 199–212.

Mitchell, H. B., & Schaefer, P. A. (2000). Multiple priorities in an induced ordered weighted averaging operator. *International Journal of Intelligent Systems, 15*, 317–328.

O'Hagan, M. (1988). Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic. In *Proceedings of the 22nd Annual IEEE Asilomar Conference on Signals, Systems and Computers* (pp. 681–689). Pacific Grove, CA.

O'Hagan, M. (1990). Using maximum entropy-ordered weighted averaging to construct a fuzzy neuron. In *Proceedings of the 24th Annual IEEE Asilomar Conference on Signals, Systems and Computers* (pp. 618–623). Pacific Grove, CA.

Sugeno, M. (1977). Fuzzy measures and fuzzy integrals: A survey. In M. M. Gupta, G. N. Saridis, & B. R. Gaines (Eds.), *Fuzzy automata and decision process* (pp. 89–102). Amsterdam: North-Holland.

Torra, V. (1997). The weighted OWA operator. *International Journal of Intelligent Systems, 12*, 153–166.

Torra, V., & Narukawa, Y. (2007). *Modeling decisions: Information fusion and aggregation operators*. Berlin: Springer.

Xu, Z. S. & Da, Q. L. (2002). The ordered weighted geometric averaging operator. *International Journal of Intelligent Systems, 17*, 709–716.

Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multi-criteria decision making. *IEEE Transactions on Systems, Man and Cybernetics, 18*, 183–190.

Yager, R. R., (1992). Applications and extensions of OWA aggregations. *International Journal of Man-Machine Studies, 37*, 103–132.

Yager, R. R. (1993). Families of OWA operators. *Fuzzy Sets and Systems, 59*, 125–148.

Yager, R. R. (1996). Quantifier guided aggregation using OWA operators. *International Journal of Intelligent Systems, 11*, 49–73.

Yager, R. R. (1997). On the inclusion of importances in OWA aggregations. In R. R. Yager, & J. Kacprzyk (Eds.) *The ordered weighted averaging operators: Theory and applications* (pp. 41–59). Norwell, MA: Kluwer.

Yager, R. R. (2008). Prioritized aggregation operators. *International Journal of Approximate Reasoning, 48*, 263–274.

Yager, R. R., & Kacprzyk, J. (1997). *The ordered weighted averaging operators: Theory and applications*. Norwell, MA: Kluwer.

Yager, R. R., Reformat, M., & Ly, C. (2008). *Using a web PET for lexicographic multi-criteria service selection*. Technical Report# MII-2807. New Rochelle, NY: Machine Intelligence Institute, Iona College.

Yager, R. R., Reformat, M., & Ly, C. (2008). *Web PET: An on-line tool for lexicographically choosing purchases by combining user preferences and customer reviews*. Technical Report# MII-2818. New Rochelle, NY: Machine Intelligence Institute, Iona College.

Zadeh, L. A. (1983). A computational approach to fuzzy quantifiers in natural languages. *Computing and Mathematics with Applications, 9*, 149–184.