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On the resolution and optimization of a system of fuzzy relational equations with sup-T composition

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Abstract This paper provides a thorough investigation on the resolution of a finite system of fuzzy relational equations with sup-T composition, where T is a continuous triangular norm. When such a system is consistent, although we know that the solution set can be characterized by a maximum solution and finitely many minimal solutions, it is still a challenging task to find all minimal solutions in an efficient manner. Using the representation theorem of continuous triangular norms, we show that the systems of sup-T equations can be divided into two categories depending on the involved triangular norm. When the triangular norm is Archimedean, the minimal solutions correspond one-to-one to the irredundant coverings of a *set covering problem*. When it is non-Archimedean, they only correspond to a subset of constrained irredundant coverings of a *set covering problem*. We then show that the problem of minimizing a linear objective function subject to a system of sup-T equations can be reduced into a 0–1 integer programming problem in polynomial time. This work generalizes most, if not all, known results and provides a unified framework to deal with the problem of resolution and optimization of a system of sup-T equations. Further generalizations and related issues are also included for discussion.

Keywords Fuzzy relational equations · Triangular norms · Fuzzy optimization · Integer programming

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1 Introduction

The notion of fuzzy relational equations is associated with the composition of fuzzy binary relations. Fuzzy relational equations have been intensively investigated both from a theoretical standpoint and in view of applications since they were first introduced by Sanchez (1974, 1976). It has been pointed out that fuzzy relational equations play an important role as a uniform platform in many applications of fuzzy sets and fuzzy systems. See, e.g. Pedrycz (1989, 1991), Mordeson and Malik (2002) and Peeva and Kyosev (2004).

Fuzzy relational equations can be presented in many different forms which depend on their interpretation in the specific contexts. Generally, most of these forms can be easily converted to the matrix form of $A \circ x = b$ or $x^T \circ A = b^T$ with composable consistency and a well defined composite operation "$\circ$" when the underlying universes of discourse are finite and discrete. Usually, the coefficient matrix A and the right hand side vector b, as well as the unknown vector x, are defined on the real unit interval [0, 1]. The most fundamental type of fuzzy relational equations is those with sup-T composition, or more accurately max-T composition for finite scenarios, where T is typically a continuous triangular norm among which the *minimum* T_M is the most frequently used one.

The solvability criteria of sup-T equations were first established by Sanchez (1976) for sup-T_M equations and then extended by Pedrycz (1982b, 1985) and Miyakoshi and Shimbo (1985). The structure of the complete solution set of sup-T_M equations was first characterized by Sanchez (1977) and generalized to sup-T equations by Di Nola et al. (1982, 1984). It now becomes well known that the complete solution set of a consistent finite system of sup-T equations can be determined by a maximum solution and a finite number of minimal solutions. The consistency of a system of sup-T equations can be easily verified by checking the potential maximum solution. However, as shown in Chen and Wang (2002, 2007) and Markovskii (2004, 2005), the detection of all the minimal solutions is closely related to the *set covering problem* and hence an NP-hard problem. Various methods have been developed to detect the minimal solutions for sup-T equations with a specific triangular norm. Approaches based on some type of quasi-characteristic matrix were proposed by Peeva (1985, 1992), Han and Sekiguchi (1992), Li (1994), Wang and Hsu (1992) and Peeva and Kyosev (2004, 2007). Some rule-based methods were proposed by Arnould and Tano (1994a, b). More discussions on the minimal solutions may be found in Xu (1978), Prévot (1981), Xu et al. (1982), Czogała et al. (1982), Higashi and Klir (1984), Di Nola (1984), Pappis and Sugeno (1985), Miyakoshi and Shimbo (1986), Bour and Lamotte (1987), Klir and Yuan (1995), Bourke and Fisher (1998), Loetamonphong and Fang (1999), Luoh et al. (2002, 2003) and Wu and Guu (2008).

Some other issues related to the resolution of fuzzy relational equations were also discussed in the literature. The unique solvability of a system of sup-T equations, as well as the existence of the minimum solution, was investigated by Di Nola and Sessa (1983, 1988), Sessa (1984), Lettieri and Liguori (1984, 1985), Cechlárová (1990, 1995), Li (1990), Gavalec (2001) and Gavalec and Plávka (2003). Various estimates of the number of the minimal solutions can be found in Czogała et al. (1982), Wang et al. (1984), Shi (1987), Peeva (1992, 2006) and Peeva and Kyosev (2004, 2007).

The resolution of a system of sup-T equations with infinite number of equations and/or infinite number of unknowns was also investigated by Wagenknecht and Hartmann (1990), Imai et al. (1997, 1998), Perfiliva and Tonis (2000), Wang (2003) and Xiong and Wang (2005). The structure of the complete solution set of a system of sup-T equations in infinite situations may differ from that of finite scenarios. Up to now, it is still a challenge to characterize the solution set of a system of fuzzy relational equations with infinite number of equations and/or infinite number of unknowns.

It has also been noticed that some properties of triangular norms, e.g. associativity and commutativity, are not necessary in defining fuzzy relational equations and hence some general composite operators can be used without losing any particular property of interest. As a consequent, some generalizations have been made by Cheng and Peng (1988), Di Nola et al. (1988), Kawaguchi and Miyakoshi (1998), Han and Li (2005), Li et al. (2005), Wang and Xiong (2005) and Han et al. (2006).

Clearly, fuzzy relational equations can be regarded as a generalization of the classical linear algebraic equations as well as a generalization of Boolean equations, see e.g., Rudeanu (1974, 2001). Moreover, Goguen (1967) indicated that fuzzy sets, as well as fuzzy relations, can be considered in a general mathematical framework of lattices. De Cooman and Kerre (1994) also pointed out that the triangular norms can be defined on bounded lattices. Fuzzy relational equations defined on a general lattice were investigated by Drewniak (1982, 1983), Di Nola (1985, 1990), Zhao (1987), Di Nola and Lettieri (1989), De Baets (1995a, 1998), Wang (2001) and Wang and Xiong (2005).

The theory and applications of fuzzy relational equations developed up to 1989 were well documented by Di Nola et al. (1989) in the first monograph on this issue. De Baets (2000) provided an extensive investigation in a unified framework on the analytical methods for solving different types of fuzzy relational equations on various lattices. The most recent monograph on fuzzy relational equations and their applications is due to Peeva and Kyosev (2004). Good overviews can also be found in Di Nola et al. (1991), Gottwald (1993, 2000), Klir and Yuan (1995), Pedrycz (1989, 1991) and Li and Fang (2008).

The problem of minimizing a linear objective function subject to a consistent system of sup-T_M equations was first investigated by Fang and Li (1999). It was shown that this problem can be decomposed into two subproblems by separating the nonnegative and negative coefficients of the objective function, both of which are subject to the same constraints. The objective function with negative coefficients assumes its optimum at the maximum solution while the objective function with nonnegative coefficients assumes its optimum at one of the minimal solutions which can be determined by solving a 0–1 integer programming problem. Clearly, this observation holds true for any continuous triangular norm utilized in the composition as well as some other composite operators such as arithmetic average. The branch-and-bound method with jump-tracking technique used in Fang and Li (1999) to solve the 0-1 integer programming problem was improved by Wu et al. (2002) and Wu and Guu (2005) by providing the proper upper bounds for the branch and bound procedure.

Following the idea of Fang and Li (1999), the linear optimization problem subject to a system of sup-T_P equations was discussed by Loetamonphong and Fang (2001), Guu and Wu (2002) and Ghodousian and Khorram (2006a), where T_P is the *product* operator. Khorram and Ghodousian (2006) and Wu (2007) considered this problem under the max-average composition. However, it was pointed out by Zimmermann (2007) that one of the algorithms proposed by Khorram and Ghodousian (2006) may not lead to the optimal solution in some cases. Similar deficiencies were detected in Ghodousian and Khorram (2006a, b) and Khorram et al. (2006) and amended by Abbasi Molai and Khorram (2007a, b) in which fuzzy relational equation constrained linear optimization problems were investigated under various composite operations. Some other generalizations on this issue can be found in Wu and Guu (2004a), Pandey (2004), Guo and Xia (2006), Abbasi Molai and Khorram (2007) and Ghodousian and Khorram (2007). Note that the algorithm proposed by Pandey (2004) only works for strict triangular norms but not for general continuous triangular norms as claimed.

Lu and Fang (2001) designed a genetic algorithm to solve nonlinear optimization problems subject to a system of sup-T_M equations. Yang and Cao (2007) considered a special subclass of the problems of this type where the objective functions are posynomials. Wu et al. (2007) explored the linear fractional programming problem subject to a system of sup-T equations where T is a continuous Archimedean triangular norm. Note that the preprocessing procedure proposed in Wu et al. (2007) is not exactly correct. Fuzzy relational equation constrained geometric programming was investigated by Yang and Cao (2005a, b) and Wu (2006), which is a generalization of the so-called latticized linear programming problem considered in Wang and Zhang (1987) and Wang et al. (1991). The multi-objective optimization problem was discussed in Wang (1995), Loetamonphong et al. (2002) and Wu and Guu (2004b).

This paper deals with the problem of solving a finite system of sup-T equations with a general continuous triangular norm T, as well as the linear optimization problem subject to a system of sup-T equations. Some basic results on triangular norms are summarized in Sect. 2. The well known representation theorem of continuous triangular norms is introduced as well, which plays a key role in classifying systems of sup-T equations. In Sect. 3, the resolution of a system of sup-T equations is investigated in detail. The procedure to determine all minimal solutions is introduced respectively based on whether the involved triangular norm is Archimedean or not. In Sect. 4, it is shown that the linear optimization problem can be reduced to a 0–1 integer programming problem in polynomial time. Some generalizations and related issues are discussed as well in Sect. 5.

2 Triangular norms

In this section, we recall some basic concepts and important properties associated with triangular norms, which are indispensable and crucial in the construction and resolution of fuzzy relational equations. However, all proofs are omitted to keep the paper succinct and readable. The reader may refer to the monograph by Klement et al. (2000), as well as the position papers by Klement et al. (2004a, b, c), for a detailed and rather complete overview of triangular norms.

2.1 Basic definitions and properties

Definition 2.1 A triangular norm (t-norm for short) is a binary operator $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following four axioms are satisfied:

(T1) $T(x, y) = T(y, x)$. (commutativity)
(T2) $T(x, T(y, z)) = T(T(x, y), z)$. (associativity)
(T3) $T(x, y) \le T(x, z)$, whenever $y \le z$. (monotonicity)
(T4) $T(x, 1) = x$. (boundary condition)

Definition 2.2 A triangular conorm (t-conorm or s-norm for short) is a binary operator $S : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following four axioms are satisfied:

(S1) $S(x, y) = S(y, x)$. (commutativity)
(S2) $S(x, S(y, z)) = S(S(x, y), z)$. (associativity)
(S3) $S(x, y) \le S(x, z)$, whenever $y \le z$. (monotonicity)
(S4) $S(x, 0) = x$. (boundary condition)

Since t-norms and t-conorms are binary algebraic operators on the real unit interval $[0, 1]$, the infix notations like $x \wedge_t y$ and $x \vee_t y$ are usually used in the literature instead of the prefix notations $T(x, y)$ and $S(x, y)$, respectively, while $x \wedge y$ and $x \vee y$ typically stand for the *minimum* $T_M(x, y)$ and the *maximum* $S_M(x, y)$, respectively.

From an axiomatical point of view, t-norms and t-conorms differ only with respect to their respective boundary conditions. Actually, t-norms and t-conorms are dual to each other in some sense. With the standard negator $N_s(x) = 1 - x, \forall\, x \in [0, 1]$, a t-norm T induces a dual t-conorm via

$$S(x, y) = N_s^{-1}(T(N_s(x), N_s(y))) = 1 - T(1 - x, 1 - y),$$

while a t-conorm S induces a dual t-norm via

$$T(x, y) = N_s^{-1}(S(N_s(x), N_s(y))) = 1 - S(1 - x, 1 - y).$$

It is obvious that a t-norm and its dual t-conorm form a dual pair with respect to the standard negator N_s. Therefore, any concept or property concerned with t-norms can be induced dually for t-conorms.

There exist uncountably many t-norms among which four basic t-norms are remarkable from different points of view and are defined by, respectively,

$T_M(x, y) = \min(x, y)$, (minimum, Gödel t-norm, Zadeh t-norm)
$T_P(x, y) = x \cdot y$, (probabilistic product, Goguen t-norm)
$T_L(x, y) = \max(x + y - 1, 0)$, (bounded difference, Łukasiewicz t-norm)

$$T_D(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1)^2 \\ \min(x, y), & \text{otherwise.} \end{cases} \qquad \text{(drastic product)}$$

Their dual t-conorms with respect to the standard negator N_s are given by, respectively,

$S_M(x, y) = \max(x, y)$, (maximum, Gödel t-conorm, Zadeh t-conorm)

$S_P(x, y) = x + y - x \cdot y,$ (probabilistic sum, Goguen t-conorm)
$S_L(x, y) = \min(x + y, 1),$ (bounded sum, Łukasiewicz t-conorm)

$$S_D(x, y) = \begin{cases} 1, & \text{if } (x, y) \in (0, 1]^2 \\ \max(x, y), & \text{otherwise.} \end{cases} \quad \text{(drastic sum)}$$

Since t-norms are just functions from unit square into the unit interval, they can be compared by the pointwise comparison. A t-norm T_1 is said to be weaker than a t-norm T_2 and is denoted by $T_1 \leq T_2$, if the inequality $T_1(x, y) \leq T_2(x, y)$ holds for all $(x, y) \in [0, 1]^2$. Equivalently, T_2 is said to be stronger than T_1 if $T_1 \leq T_2$. Moreover, the notation $T_1 < T_2$ is used whenever $T_1 \leq T_2$ and $T_1 \neq T_2$. It is well known that the *drastic product* T_D is the weakest and the *minimum* T_M is the strongest t-norm, i.e., for any t-norm T, it holds that

$$T_D \leq T \leq T_M.$$

Furthermore, it is obvious that

$$T_D < T_L < T_P < T_M.$$

The associativity of t-norms allows us to uniquely extend each t-norm T in a recursive way to an n-ary operator, i.e.,

$$T(x_1, \ldots, x_n) = T(T(x_1, \ldots, x_{n-1}), x_n)$$

for each n-tuple $(x_1, \ldots, x_n) \in [0, 1]^n$ with the integer $n \geq 3$. In particular, $T(\overbrace{x, \ldots, x}^{n})$ is denoted as $x_T^{(n)}$ for each $x \in [0, 1]$ and called the n-th power of x with respect to T, with the convention that $x_T^{(0)} = 1$ and $x_T^{(1)} = x$.

A t-norm T, viewed as a real function with two arguments, is said to be continuous if for all convergent sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $x_n, \ y_n \in [0, 1]$, it holds that

$$T\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} T(x_n, y_n).$$

The continuity, as well as the left- and right-continuity, plays an important role in many applications of t-norms. In general, a real function with two arguments, e.g., with domain $[0, 1]^2$, may be continuous in each argument but fail to be continuous on $[0, 1]^2$. However, t-norms are exceptions since a function, which is non-decreasing in its both arguments, is continuous if and only if it is continuous in each argument. Moreover, due to the commutativity of t-norms, the continuity of a t-norm is equivalent to its continuity in one of the arguments, i.e., a t-norm T is continuous if and only if its partial mappings $T(x, \cdot), \forall x \in [0, 1]$ are continuous. Consequently, a t-norm T is said to be left-continuous (right-continuous) if for all non-decreasing (non-increasing) sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [0, 1]$, it holds that

$$T\left(\lim_{n\to\infty} x_n, y\right) = \lim_{n\to\infty} T(x_n, y), \quad \forall\, y \in [0, 1].$$

Clearly, a t-norm T is continuous if and only if it is both left- and right-continuous. The *minimum* T_M, the *product* T_P and the Łukasiewicz t-norm T_L are all continuous while the *drastic product* T_D is right-continuous but not left-continuous.

Another two related concepts are the lower and upper semicontinuity of t-norms. However, they coincide respectively with the left- and right-continuity for t-norms, i.e., a t-norm is lower (upper) semicontinuous if and only if it is left-continuous (right-continuous). This well known result allows us to speak about left-continuous and right-continuous t-norms instead of lower and upper semicontinuous t-norms, respectively. Moreover, note that the left- and right-continuity mean exactly the interchangeability of the supremum operation and the infimum operation, respectively, with the t-norm, which plays a crucial role in the resolution of fuzzy relational equations.

To better understand t-norms, some additional algebraic properties of t-norms are necessary to be presented, most of which are well known from the general theories of semigroups and lattices since t-norms can be viewed as a special class of commutative semigroups.

Definition 2.3 Let T be a t-norm.

(i) An element $a \in [0, 1]$ is called an idempotent element of T if $T(a, a) = a$.
(ii) An element $a \in (0, 1)$ is called a nilpotent element of T if there exists some positive integer n such that $a_T^{(n)} = 0$.
(iii) An element $a \in (0, 1)$ is called a zero divisor of T if there exists some $b \in (0, 1)$ such that $T(a, b) = 0$.

Clearly, no element of $(0, 1)$ can be both idempotent and nilpotent. The numbers 0 and 1 are idempotent elements for each t-norm T and hence called trivial idempotent elements of T while each idempotent element in $(0, 1)$ is called a non-trivial idempotent element of T. Moreover, the idempotent elements of t-norms can be well characterized in the following way:

Theorem 2.1

(i) *An element* $a \in [0, 1]$ *is an idempotent of a* t*-norm* T *if and only if* $T(a, x) = \min(a, x)$ *for all* $x \in [a, 1]$.
(ii) *An element* $a \in [0, 1]$ *is an idempotent of a continuous* t*-norm* T *if and only if* $T(a, x) = \min(a, x)$ *for all* $x \in [0, 1]$.

The *product* T_P, the Łukasiewicz t-norm T_L and the *drastic product* T_D possess only trivial idempotent elements. The set of idempotent elements of the *minimum* T_M is $[0, 1]$. Actually, T_M is the only t-norm with this property. Furthermore, the *minimum* T_M and the *product* T_P have neither nilpotent elements nor zero divisors while each $a \in (0, 1)$ is both a nilpotent element and a zero divisor of the Łukasiewicz t-norm T_L as well as the *drastic product* T_D.

Generally, each nilpotent element of a t-norm T is also a zero divisor of T but not vice versa, which means that the set of nilpotent elements is a subset of the set of zero divisors. However, the existence of zero divisors is equivalent to the existence of nilpotent elements for each t-norm, i.e., a t-norm T has zero divisors if and only if it has nilpotent elements. Moreover, if $a \in (0, 1)$ is a zero divisor of a t-norm T, each number $b \in (0, a)$ is also a zero divisor of T due to the monotonicity of T. This also holds for nilpotent elements.

Definition 2.4

(i) A t-norm T is called strictly monotone if $T(x, y) < T(x, z)$ whenever $x > 0$ and $y < z$.
(ii) A t-norm T is called strict if it is continuous and strictly monotone.
(iii) A t-norm T is called nilpotent if it is continuous and each $x \in (0, 1)$ is a nilpotent element of T.
(iv) A t-norm T is called Archimedean if for each $(x, y) \in (0, 1)^2$ there exists a positive integer n such that $x_T^{(n)} < y$.

An Archimedean t-norm has only trivial idempotent elements. It was shown by Kolesárová (1999) that the left-continuity of an Archimedean t-norm implies its continuity. Furthermore, a continuous t-norm T is Archimedean if and only if $T(x, x) < x$ for all $x \in (0, 1)$, which leads to the fact that each strict and each nilpotent t-norm is Archimedean. Moreover, it turns out that a continuous Archimedean t-norm is either strict or nilpotent. Clearly, the *product* T_P is a strict t-norm and the Łukasiewicz t-norm T_L is a nilpotent t-norm. The *minimum* T_M is a continuous but not Archimedean t-norm. The *drastic product* T_D is a non-continuous Archimedean t-norm for which each $x \in (0, 1)$ is a nilpotent element.

2.2 Construction of triangular norms

From an algebraic point of view, t-norms, as well as t-conorms, are a special class of commutative semigroups and hence can be constructed via various methods.

It is straightforward that, given a t-norm T, any strictly increasing bijection $\psi : [0, 1] \to [0, 1]$ defines a t-norm by

$$T_\psi(x, y) = \psi^{-1}(T(\psi(x), \psi(y))). \tag{2.1}$$

Furthermore, T and T_ψ are called isomorphic in the sense that

$$\psi(T_\psi(x, y)) = T(\psi(x), \psi(y)) \tag{2.2}$$

for all $(x, y) \in [0, 1]^2$. Clearly, T and T_ψ share many common structural features, e.g., the continuity, the Archimedean property and the existence of idempotent and nilpotent elements as well as the existence of zero divisors. The only invariants under arbitrary strictly increasing bijections are the two extremal t-norms, i.e., the *minimum* T_M and the *drastic product* T_D.

A more general method to construct new t-norms involves the pseudo-inverses of monotone functions.

Definition 2.5 Let $f : [a, b] \to [c, d]$ be a monotone function where $[a, b]$ and $[c, d]$ be two closed subintervals of the extended real line $[-\infty, +\infty]$. The pseudo-inverse $f^{(-1)} : [c, d] \to [a, b]$ is defined as

$$f^{(-1)}(y) = \sup \{x \in [a, b] \mid (f(x) - y)(f(b) - f(a)) < 0\}.$$

Obviously, the pseudo-inverse of f coincides with the inverse of f if and only if f is a bijection. For more details on pseudo-inverses of monotone functions, the reader may refer to Klement et al. (1999). With the aid of pseudo-inverses, some t-norms can be constructed via appropriate functions.

Theorem 2.2 *Let $t : [0,1] \to [0,+\infty]$ be a strictly decreasing function with $t(1) = 0$ such that t is right-continuous at 0 and*

$$t(x) + t(y) \in Ran(t) \cup [t(0), +\infty] \tag{2.3}$$

for all $(x, y) \in [0,1]^2$, where $Ran(t) = \{t(x) \mid x \in [0,1]\}$. The function $T : [0,1]^2 \to [0,1]$ defined by

$$T(x, y) = t^{(-1)}(t(x) + t(y)) \tag{2.4}$$

is a t-norm.

A function $t : [0,1] \to [0,+\infty]$ satisfying the conditions in Theorem 2.2 is called an additive generator of the t-norm T. It is obvious that a multiplication of the addictive generator of a t-norm T by a positive constant remains to be an additive generator of T. Moreover, there is a strong connection between the continuity of a t-norm and the continuity of its additive generators.

Theorem 2.3 *Let T be a t-norm with an additive generator $t : [0,1] \to [0,+\infty]$. The following statements are equivalent:*

(*i*) *T is continuous.*
(*ii*) *T is left-continuous in the point $(1,1)$.*
(*iii*) *t is continuous.*
(*iv*) *t is left-continuous in 1.*

It is well known that triangular norms constructed by means of additive generators are always Archimedean. The converse, however, is not true, i.e., an Archimedean t-norm, which is necessarily not continuous, may have no additive generators. However, a t-norm is continuous Archimedean if and only if it has a continuous additive generator which is uniquely determined up to a positive multiplicative constant. Moreover, the value of the continuous additive generator at the point 0 determines whether the induced t-norm is strict or nilpotent.

Theorem 2.4 *Let $t : [0,1] \to [0,+\infty]$ be an additive generator of a continuous Archimedean t-norm T.*

(*i*) *T is strict if and only if $t(0) = +\infty$.*
(*ii*) *T is nilpotent if and only if $t(0) < +\infty$.*

The functions $t(x) = -\ln x$ and $t(x) = 1 - x$ are the additive generators of the *product* T_P and the Łukasiewicz t-norm T_L, respectively, while the *drastic product* T_D can be induced by

$$t(x) = \begin{cases} 2 - x, & x \in [0,1), \\ 0, & x = 1. \end{cases} \tag{2.5}$$

Clearly, no additive generator exists for any t-norm with a non-trivial idempotent element, in particular, the *minimum* T_M. See, e.g., Ling (1965) and Krause (1983).

A similar method to construct t-norms is via so-called multiplicative generators which are completely dual to additive generators in some sense.

Let T be a t-norm with an additive generator $t : [0, 1] \to [0, +\infty]$. The t-norm T can also be induced via

$$T(x, y) = \theta^{(-1)}(\theta(x) \cdot \theta(y)) \tag{2.6}$$

where the function $\theta(x) = e^{-t(x)}$ is called a multiplicative generator of T. Clearly, any additive generator $t : [0, 1] \to [0, +\infty]$ defines a multiplicative generator $\theta : [0, 1] \to [0, 1]$ with $\theta(1) = 1$ and right-continuous at 0 such that

$$\theta(x) \cdot \theta(y) \in Ran(\theta) \cup [0, \theta(0)] \tag{2.7}$$

for all $(x, y) \in [0, 1]^2$. Conversely, such a function $\theta(x)$ also defines an additive generator $t(x) = -\ln \theta(x)$ and hence induces a t-norm.

Another method to construct t-norms also originates from semigroup theory. The basic idea goes back to Clifford (1954) and has been applied for constructing a new t-norm from a family of t-norms. See e.g., Schweizer and Sklar (1963), Ling (1965) and Frank (1979).

Theorem 2.5 *Let $\{T_\alpha\}_{\alpha \in A}$ be a family of t-norms and $\{(a_\alpha, b_\alpha)\}_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $(0, 1)$, where the index set A is finite or countably infinite. The function $T : [0, 1]^2 \to [0, 1]$ defined by*

$$T(x, y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha) \cdot T_\alpha\left(\dfrac{x - a_\alpha}{b_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{b_\alpha - a_\alpha}\right), & \text{if } (x, y) \in [a_\alpha, b_\alpha]^2, \\ \min\{x, y\}, & \text{otherwise.} \end{cases} \tag{2.8}$$

is a t-norm and called the ordinal sum of the summands $\langle a_\alpha, b_\alpha, T_\alpha \rangle, \alpha \in A$. It is usually denoted by $T = (\langle a_\alpha, b_\alpha, T_\alpha \rangle)_{\alpha \in A}$.

Note that the index set A may be empty, in which case the ordinal sum equals to the *minimum* T_M. It is obvious that a summand $T_\alpha = T_M$ in the representation of an ordinal sum can be omitted and each t-norm T can be viewed as a trivial ordinal sum, i.e., $T = (\langle 0, 1, T \rangle)$. Clearly, if $T = (\langle a_\alpha, b_\alpha, T_\alpha \rangle)_{\alpha \in A}$ is an ordinal sum of t-norms, the endpoints $a_\alpha, b_\alpha, \alpha \in A$ are all idempotent elements of the t-norm T. Therefore, any t-norm T defined by a non-trivial ordinal sum $(\langle a_\alpha, b_\alpha, T_\alpha \rangle)_{\alpha \in A}$, i.e., $A \neq \emptyset$ and $(a_\alpha, b_\alpha) \subset (0, 1), \forall\, \alpha \in A$, cannot be Archimedean. Moreover, the continuity of an ordinal sum of t-norms is equivalent to the continuity of all its summands.

Theorem 2.6 *Let $T = (\langle a_\alpha, b_\alpha, T_\alpha \rangle)_{\alpha \in A}$ be an ordinal sum of t-norms with $A \neq \emptyset$. The t-norm T is continuous if and only if T_α is continuous for each $\alpha \in A$.*

Some other methods related to ordinal sums were also developed for constructing t-norms. The reader may refer to Drossos and Navara (1996) and Jenei (2001, 2002) and references therein.

2.3 Representation of continuous triangular norms

There is no universal representation theorem so far for all t-norms, which is actually related to the solution of the still unsolved general associativity equation. See, for instance, Alsina et al. (2006). However, all continuous t-norms can be well characterized and classified. The representation theorems of continuous t-norms by means of additive generators and ordinal sums of Archimedean summands were first developed by Ling (1965) in the framework of triangular norms although they can be derived from the results in Mostert and Shields (1957) in the framework of semigroups.

As an immediate consequence of Theorem 2.4, a continuous Archimedean t-norm T is either strict or nilpotent, which is fully determined by the value $t(0)$ of its continuous additive generator $t : [0, 1] \to [0, +\infty]$. Moreover, each continuous Archimedean t-norm is isomorphic either to the *product* T_P or to the Łukasiewicz t-norm T_L. This fact makes T_P and T_L the most important prototypes of strict and nilpotent t-norms, respectively. Unfortunately, no such characterization exists for continuous non-Archimedean t-norms.

Theorem 2.7
(*i*) *Let T be a strict t-norm with an additive generator t. The t-norm T is isomorphic to the product T_P and*

$$T(x, y) = \psi^{-1}(T_P(\psi(x), \psi(y))) \tag{2.9}$$

for all $(x, y) \in [0, 1]^2$ where $\psi(x) = e^{-t(x)}$.
(*ii*) *Let T be a nilpotent t-norm with an additive generator t. The t-norm T is isomorphic to the Łukasiewicz t-norm T_L and*

$$T(x, y) = \psi^{-1}(T_L(\psi(x), \psi(y))) \tag{2.10}$$

for all $(x, y) \in [0, 1]^2$ where $\psi(x) = 1 - t(x)/t(0)$.

Theorem 2.8 *A t-norm T is continuous if and only if it is an ordinal sum of continuous Archimedean t-norms $(\langle a_\alpha, b_\alpha, T_\alpha\rangle)_{\alpha \in A}$ where the summands $\langle a_\alpha, b_\alpha, T_\alpha\rangle$, $\alpha \in A$ are uniquely determined.*

A remarkable result following Theorems 2.7 and 2.8 is the important classification of continuous t-norms.

Theorem 2.9 *A t-norm T is continuous if and only if exactly one of the following statements is true:*

(*i*) *$T = T_M$.*
(*ii*) *T is strict.*
(*iii*) *T is nilpotent.*
(*iv*) *T is a non-trivial ordinal sum of continuous Archimedean t-norms.*

Consequently, a continuous non-Archimedean t-norm, which definitely has non-trivial idempotent elements, is either the *minimum* T_M or a non-trivial ordinal sum of continuous Archimedean t-norms.

	Without zero divisors	With zero divisors
Archimedean	Strict t-norms T_P	Nilpotent t-norms T_L
Non-Archimedean	T_M T_λ^{DP}, $\lambda \in (0,1)$	T_λ^{MT}, $\lambda \in (0,1)$

Fig. 1 Continuous t-norms with typical representatives of each class

A further classification of continuous t-norms arises for the distinction between t-norms with zero divisors and t-norms without zero divisors. It is clear that a continuous Archimedean t-norm with zero divisors must be a nilpotent t-norm and hence isomorphic to the Łukasiewicz t-norm T_L. For a continuous non-Archimedean t-norm T with zero divisors, it must contain $\langle 0, b_1, T_1\rangle$ in its summands where $b_1 \in (0, 1)$ is the smallest non-trivial idempotent element of T and T_1 is a nilpotent t-norm. Obviously, the converse is also true. An interesting family of such t-norms is the ordinal sums $T_\lambda^{MT} = (\langle 0, \lambda, T_L\rangle), \forall\, \lambda \in (0, 1)$, also known as the Mayor–Torrens t-norms. See Mayor and Torrens (1991). The *minimum* T_M is without doubt a typical but not unique representative of continuous non-Archimedean t-norms without zero divisors. Another example is the family of Dubois-Prade t-norms, $T_\lambda^{DY} = (\langle 0, \lambda, T_P\rangle), \forall\, \lambda \in (0, 1)$, which was introduced by Dubois and Prade (1980). Note that it was sometimes erroneously claimed that the *minimum* T_M is the only continuous t-norm which is not Archimedean. See, e.g., Stamou and Tzafestas (2001) and Shieh (2007). An illustration of the classification of continuous t-norms is shown in Fig. 1. The reader may refer to Klement et al. (2000) for the classification of general t-norms.

2.4 Residual operators of triangular norms

Definition 2.6 The binary residual operators $I_T : [0, 1]^2 \to [0, 1]$ and $J_T : [0, 1]^2 \to [0, 1]$ with respect to a t-norm T are defined, respectively, by

$$I_T(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \le y\}, \tag{2.11}$$

$$J_T(x, y) = \inf\{z \in [0, 1] \mid T(x, z) \ge y\}. \tag{2.12}$$

The residual operator I_T with respect to a left-continuous t-norm is known as a residual implicator or briefly an R-implicator in fuzzy logic while the residual operator J_T has no particular logical interpretation. Recall that when the t-norm T is lower semicontinuous, or equivalently left-continuous, it holds that

$$T\left(a, \sup\nolimits_{x \in X} x\right) = \sup\nolimits_{x \in X} T(a, x) \tag{2.13}$$

for all $a \in [0, 1]$ and $X \subseteq [0, 1]$. Similarly, when the t-norm T is upper semicontinuous, or equivalently right-continuous, it holds that

$$T\,(a, \inf_{x \in X} x) = \inf_{x \in X} T(a, x) \tag{2.14}$$

for all $a \in [0, 1]$ and $\emptyset \neq X \subseteq [0, 1]$. These properties lead to the following important results.

Theorem 2.10 *Let T be a left-continuous t-norm and I_T its associated residual implicator. It holds for all $a, b \in [0, 1]$ that $T(a, x) \leq b$ if and only if $x \leq I_T(a, b)$.*

Theorem 2.11 *Let T be a continuous t-norm and I_T and J_T its associated residual operators. The equation $T(a, x) = b$ has a solution for given $a, b \in [0, 1]$ if and only if $b \leq a$, in which case the solution set of $T(a, x) = b$ is given by the closed interval $[J_T(a, b), I_T(a, b)]$.*

Clearly, it could happen that $T(a, I_T(a, b)) > b$ for some $a, b \in [0, 1]$ with $b \leq a$ when the t-norm T fails to be left-continuous. Similarly, the right-continuity of the t-norm T helps to make the solution set of $T(a, x) = b$ closed from the lower side. Furthermore, given a strictly increasing bijection $\psi : [0, 1] \to [0, 1]$, the equation $T_\psi(a, x) = b$ and the equation $T(\psi(a), x) = \psi(b)$ are isomorphic in the sense that $T_\psi(a, x) = b$ implies $T(\psi(a), \psi(x)) = \psi(b)$ and vice versa.

In the literature of fuzzy relational equations, the residual implicators are also known as φ-operators which were introduced by Pedrycz (1982a, 1985) in a different approach to describe the solutions of sup-T equations. They are essentially the generalization of the α-operation defined by Sanchez (1976, 1977), which now is the specific φ-operator with respect to the *minimum* T_M. The connection between a φ-operator and its corresponding t-norm has been characterized in full generality by Gottwald (1984, 1986). See also Gottwald (1993, 2000), Höhle (1995) and Demirli and De Baetes (1999). The residual operator J_T was also discussed in Di Nola et al. (1989) with a slightly different definition. The infix notations are usually used to denote these two residual operators, i.e., $I_T(x, y) = x\varphi_t y$ and $J_T(x, y) = x\sigma_t y$, respectively. Note that in the context of fuzzy logic, $x \to_t y$ is the more commonly used notation for $I_T(x, y)$. For any $x, y \in [0, 1]$, the element $x\varphi_t y$ is also called the pseudo-complement of x relative to y with respect to the t-norm T. The residual operators I_T and J_T of the three most important continuous t-norms are listed in Table 1. Note that I_T and J_T do not differ too much, which means that the equation $T(a, x) = b$ just has one solution except for some special cases. The fact that each continuous Archimedean t-norm has a continuous additive generator immediately leads to the following important property of continuous Archimedean t-norms.

Theorem 2.12 *Let T be a continuous Archimedean t-norm and $t : [0, 1] \to [0, +\infty]$ its additive generator. The residual operators I_T and J_T can be represented, respectively, by*

Table 1 Residual Operators of the Gödel, Goguen and Łukasiewicz t-norms

T	$I_T(x, y)$	$J_T(x, y)$
T_M	$\begin{cases} 1, & \text{if } x \le y \\ y, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y \\ y, & \text{otherwise.} \end{cases}$
T_P	$\begin{cases} 1, & \text{if } x \le y \\ y/x, & \text{otherwise.} \end{cases}$	$\begin{cases} 1, & \text{if } x < y \\ y/x, & \text{if } 0 < y \le x \\ 0, & \text{otherwise.} \end{cases}$
T_L	$\min(1 - x + y, 1)$	$\begin{cases} 1, & \text{if } x < y \\ 1 - x + y, & \text{if } 0 < y \le x \\ 0, & \text{otherwise.} \end{cases}$

$$I_T(x, y) = t^{(-1)}\left(\max(t(y) - t(x), 0)\right), \tag{2.15}$$

$$J_T(x, y) = \begin{cases} t^{(-1)}\left(\max(t(y) - t(x), 0)\right), & \text{if } y \neq 0, \\ 0, & \text{if } y = 0. \end{cases} \tag{2.16}$$

Theorem 2.13 *Let T be a continuous Archimedean t-norm. The equation $T(a, x) = b$ has a unique and nonzero solution when $b \le a$ and $a, b \in (0, 1]$.*

Theorem 2.13 was stated by Klement et al. (2000) in a slightly different form and restated by Stamou and Tzafestas (2001). It is a direct consequence of Theorems 2.11 and 2.12. It can also be validated simply by checking the *product* T_P and the Łukasiewicz t-norms T_L due to Theorem 2.7. Note that the exception of Theorem 2.13 occurs when $b = 0$ for nilpotent t-norms and $a = b = 0$ for strict t-norms. However, these trivial cases can be easily dealt with when solving a system of sup-T equations. Clearly, Theorem 2.13 suggests a separate consideration of continuous Archimedean t-norms for the resolution and optimization of sup-T equations.

For a continuous non-Archimedean t-norm T, the situation turns out to be a little bit complicated, especially when T is an ordinal sum of Archimedean t-norms and involves a nilpotent summand.

Theorem 2.14 *Let $T = (\langle a_\alpha, b_\alpha, T_\alpha \rangle)_{\alpha \in A}$ be an ordinal sum of continuous Archimedean t-norms. The residual operators I_T and J_T can be obtained, respectively, by*

$$I_T(x, y) = \begin{cases} 1, & \text{if } x \le y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot I_{T_\alpha}\left(\dfrac{x - a_\alpha}{b_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{b_\alpha - a_\alpha}\right), & \text{if } a_\alpha < y < x \le b_\alpha, \\ y, & \text{otherwise.} \end{cases} \tag{2.17}$$

$$J_T(x, y) = \begin{cases} 1, & \text{if } x < y, \\ a_\alpha + (b_\alpha - a_\alpha) \cdot J_{T_\alpha}\left(\dfrac{x - a_\alpha}{b_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{b_\alpha - a_\alpha}\right), & \text{if } a_\alpha < y \le x \le b_\alpha, \\ y, & \text{otherwise.} \end{cases} \tag{2.18}$$

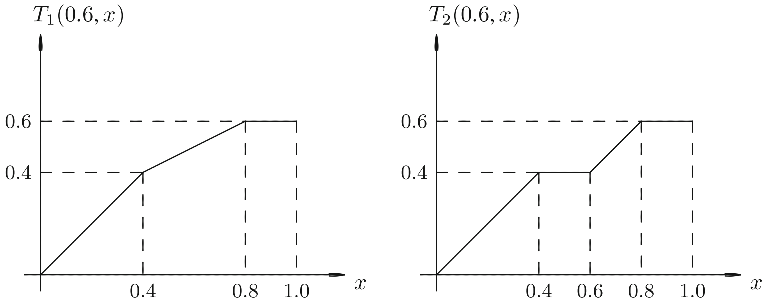


Fig. 2 Illustrations of functions $T_1(0.6, x)$ and $T_2(0.6, x)$

Theorem 2.15 *Let T be a continuous non-Archimedean t-norm. The equation $T(a, x) = b$ with $0 < b \leq a \leq 1$ has multiple solutions only when $b = a < 1$ or $b = a_\alpha \in (0, 1)$ if T involves a nilpotent summand $\langle a_\alpha, b_\alpha, T_\alpha\rangle$ and $a \in (a_\alpha, b_\alpha)$.*

Theorem 2.15 is a direct consequence of Theorems 2.11 and 2.14. Similarly, the behavior of the equation $T(a, x) = 0$ depends on whether T has zero divisors or not. As an example, suppose that $T_1 = (\langle 0.4, 0.8, T_P\rangle)$ and $T_2 = (\langle 0.4, 0.8, T_L\rangle)$, the functions $T_1(0.6, x)$ and $T_2(0.6, x)$ are both piecewise linear and illustrated in Fig. 2. It is obvious that the equations $T_1(0.6, x) = 0.6$, $T_2(0.6, x) = 0.6$ and $T_2(0.6, x) = 0.4$ have multiple solutions, respectively. Note the solution set of the equation $T(a, x) = b$ was characterized by Klement et al. (2000) under the name of the preimage.

It should be noted that the calculation of the residual operators I_T and J_T with respect to a given continuous t-norm T may be not easy although they are well defined theoretically. This is because the evaluation of the value $T(x, y)$ for any pair of $(x, y) \in [0, 1]^2$ may require much computational time when the t-norm T has a complicated structure.

3 Resolution of sup-T equations

In this section, we focus on the resolution of a finite system of fuzzy relational equations $A \circ x = b$ with sup-T composition where T is a continuous t-norm and the coefficient matrix $A = (a_{ij})_{m\times n} \in [0, 1]^{mn}$, the unknown vector $x = (x_j)_{n\times 1} \in [0, 1]^n$ and the right hand side constants $b = (b_i)_{m\times 1} \in [0, 1]^m$, i.e.,

$$\begin{cases} (a_{11} \wedge_t x_1) \vee (a_{12} \wedge_t x_2) \vee \cdots \vee (a_{1n} \wedge_t x_n) = b_1, \\ \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ (a_{m1} \wedge_t x_1) \vee (a_{m2} \wedge_t x_2) \vee \cdots \vee (a_{mn} \wedge_t x_n) = b_m. \end{cases} \tag{3.1}$$

For the convenience of description, two index sets are defined by $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$.

Among all fuzzy relational equations of this type, sup-T_M equations are of the most importance and were first investigated by Sanchez (1974, 1976). As indicated by Zimmermann (2001), the sup-T_M composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value cannot compensate the badness of another value. However, it has been reported by Oden (1977), Thole et al. (1979), Zimmermann and Zysno (1980) and Dubois and Prade (1986) that the *minimum* T_M may not be the best operator for composition and the *product* T_P would be preferred in some situations when the values of a solution vector are allowed to compensate for each other. The Łukasiewicz t-norm T_L turns out to be a special candidate for composition when the general form of the law of noncontradiction is concerned. Moreover, Bellman and Zadeh (1977) stated that the appropriate composite operator strongly depends on the application context and has no universal definition in the situations where the intersection connector acts interactively. Gupta and Qi (1991) studied the performance of the fuzzy logic controllers with various combinations of t-norms and t-conorms implemented and concluded that the performance very much depends on the choice of the composite operators. Van de Walle et al. (1998) and De Baets et al. (1998) discussed the requirements of choosing a suitable t-norm for modeling a fuzzy preference structure. Di Martino et al. (2003) and Loia and Sessa (2005) reported that the Łukasiewicz t-norm T_L would be preferred in image processing. Some outlines for selecting an appropriate t-norm has been provided by Yager (1982). So far, the applications and implementations of fuzzy relational equations are developed mainly for sup-T_M equations. However, they can be extended under some conditions to fuzzy relational equations defined on more general structures or with general compositions. See, for instance, Peeva and Kyosev (2004).

A system of sup-T equations $A \circ x = b$ is said to be in the normal form if its right side elements are arranged in a non-increasing order, i.e.,

$$b_1 \geq b_2 \geq \cdots \geq b_m. \tag{3.2}$$

The notion of normal form of a system of fuzzy relational equations was introduced in Czogała et al. (1982) and Miyakoshi and Shimbo (1986). Any system of fuzzy relational equations can be converted into its normal form in polynomial time, see for instance Cormen et al. (2001). Obviously, systems of sup-T equations with a same normal form have same solutions and therefore are equivalent. Typically for sup-T_M equations, solving sup-T equations in the normal form will possibly offer the convenience of theoretical analysis as well as the reduction of computations.

Furthermore, one can assume that $b_i > 0$ for all $i \in M$ in a system of sup-T equations $A \circ x = b$ as long as the t-norm T has no zero divisors. Otherwise, denote $M_0 = \{i \in M \mid b_i = 0\}$. Any solution of the system $A \circ x = b$ must have $x_j = 0$ for all $j \in N_0$ where $N_0 = \{j \in N \mid a_{ij} > 0,\ i \in M_0\}$. Therefore it is possible to delete the equations with indices from M_0 and the columns of the matrix A with indices from N_0. Each solution of the reduced system can be reconstructed to be a solution of the original system by setting $x_j = 0$ for $j \in N_0$. Moreover, it will be shown later that the existence of zero divisors of the t-norm T actually does not make a lot of trouble. The equations with zero right hand side in a system of sup-T equations can always be eliminated after the characteristic matrix of the system has been obtained. However, the corresponding column reduction cannot be performed any more when the involved t-norm T has zero divisors.

3.1 Solvability and the complete solution set

Given a system of sup-T equations $A \circ x = b$ with a continuous t-norm T, the set of all solutions of $A \circ x = b$ is called its complete solution set and denoted by $S(A, b) = \{x \in [0, 1]^n \mid A \circ x = b\}$. A partial order can be defined on $S(A, b)$ by extending the natural order such that for any $x^1, x^2 \in S(A, b)$, $x^1 \leq x^2$ if and only if $x_j^1 \leq x_j^2$ for all $j \in N$. A system of sup-T equations $A \circ x = b$ is called consistent if $S(A, b) \neq \emptyset$, otherwise, it is inconsistent. Due to the monotonicity of the t-norm involved in the composition, the complete solution set of any consistent system of sup-T equations preserves "convexity of the order", a term used in De Baets (2000), i.e., if $x^1,\ x^2 \in S(A, b)$, any x such that $x^1 \leq x \leq x^2$ is also in $S(A, b)$. Therefore, the attention could be focused on the so called extremal solutions as defined below.

Definition 3.1 A solution $\check{x} \in S(A, b)$ is called a minimal or lower solution if for any $x \in S(A, b)$, the relation $x \leq \check{x}$ implies $x = \check{x}$. A solution $\hat{x} \in S(A, b)$ is called the maximum or greatest solution if $x \leq \hat{x}, \forall\, x \in S(A, b)$.

Definition 3.2 A generalized vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)^T$ with $\tilde{x}_j = [\underline{x}_j, \overline{x}_j]$, $j \in N$ is called an interval solution of the system of sup-T equations $A \circ x = b$ if any vector $x = (x_1, x_2, \ldots, x_n)^T$ with $x_j \in [\underline{x}_j, \overline{x}_j]$, $j \in N$ is a solution of $A \circ x = b$. An interval solution $\tilde{x}$ is called a maximal interval solution if its components are determined by a minimal solution from the left and by the maximum solution from the right.

The maximum solution of a system of sup-T equations $A \circ x = b$, if it exists, is obviously unique. Any solution of $A \circ x = b$ marks a possible way to satisfy all the equations in the system simultaneously, while a maximal interval solution indicates how far each component of the solution vector can be expanded when the input A and the output b are fixed.

Generally, solving fuzzy relational equations is not easy due to the lack of proper inverse operations. However, it is well known how to solve a finite system of sup-T equations and characterize its complete solution set when T is a continuous t-norm.

Theorem 3.1 *Let $A \circ x = b$ be a system of sup-T equations. A vector $x \in [0, 1]^n$ is a solution of $A \circ x = b$ if and only if there exists an index $j_i \in N$ for each $i \in M$ such that*

$$a_{ij_i} \wedge_t x_{j_i} = b_i \quad \text{and} \quad a_{ij} \wedge_t x_j \leq b_i, \quad i \in M,\ j \in N. \tag{3.3}$$

Theorem 3.1 holds in a straightforward way due to the non-interactivity of the *maximum* S_M. As a consequence, it is clear that in any case the inequalities

$$b_i \leq \sup_{j \in N} a_{ij}, \quad i \in M, \tag{3.4}$$

are necessary conditions for the existence of a solution to a system of sup-T equations $A \circ x = b$. They are generally not sufficient conditions unless the t-norm T is continuous and the system $A \circ x = b$ involves only one equation. However, the results in Theorems 2.10 and 2.11 imply a direct way to solve the system $A \circ x = b$, which is well known since the very beginning of the investigation of fuzzy relational equations.

Theorem 3.2 *Let $A \circ x = b$ be a system of sup-T equations where the t-norm T is left-continuous. The system is consistent if and only if the vector $A^T \varphi_t b$ with its components defined by*

$$(A^T \varphi_t b)_j = \inf_{i \in M} I_T(a_{ij}, b_i) = \bigwedge_{i \in M} \left(a_{ij} \varphi_t b_i\right), \quad j \in N, \tag{3.5}$$

is a solution of $A \circ x = b$. Moreover, if the system is consistent, $A^T \varphi_t b$ is the maximum solution in the complete solution set $S(A, b)$.

Therefore, the consistency of the system $A \circ x = b$ can be detected by constructing and checking the potential maximum solution in a time complexity of $O(mn)$. In the context of fuzzy relations, the transposed matrix A^T is also denoted by A^{-1} as the inverse relation of A and hence the potential maximum solution $A^T \varphi_t b$, also known as the principle solution, is denoted by $A^{-1} \varphi_t b$ as well.

Furthermore, if the t-norm T is also right-continuous and hence continuous, the complete solution set $S(A, b)$, when it is not empty, can be well characterized and determined by a unique maximum solution and a finite number of minimal solutions, i.e.,

$$S(A, b) = \bigcup_{\check{x} \in \check{S}(A,b)} \left\{x \in [0, 1] \mid \check{x} \le x \le \hat{x}\right\}, \tag{3.6}$$

where $\check{S}(A, b)$ is the set of all minimal solutions of $A \circ x = b$ and $\hat{x} = A^T \varphi_t b$. This particular structure of the complete solution set $S(A, b)$ is called a finitely generated root system by De Baets (1995b, 2000). It also forms a "join semilattice", see, e.g., Di Nola et al. (1989), in the sense that $x^1 \vee x^2 \in S(A, b)$ for any $x^1,\ x^2 \in S(A, b)$.

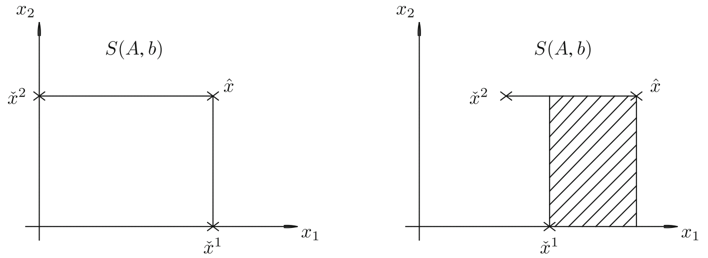
The detection of all minimal solutions is rather complicated and a very interesting issue for investigation although it is sufficient to know the maximum solution of $A \circ x = b$ in most practical considerations. It follows from Theorems 2.11 and 3.1 that the complete solution set of each single equation in a consistent system $A \circ x = b$ is a finitely generated root system with the minimal solutions given by the set

$$\check{S}_i = \{\check{x}^k \mid b_i \le a_{ik},\ k \in N\}, \quad i \in M, \tag{3.7}$$

where the vector $\check{x}^k \in \check{S}_i$, $i \in M$, is defined by

$$\check{x}^k_j = \begin{cases} J_T(a_{ik}, b_i), & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \quad j \in N. \tag{3.8}$$

The complete solution set of $A \circ x = b$ is therefore the intersection of these root systems and remains to be a finitely generated root system. However, a minimal solution of a single equation may not necessarily be a minimal solution of the system. Clearly, as indicated by De Baets (1995b), any algorithm designed in this way for determining the minimal solutions can always be applied independent of the t-norm involved. Of course, this does not means that none of the algorithms could be more efficient for



(a) $S(A, b)$ with an Archimedean t-norm **(b)** $S(A, b)$ with a non-Archimedean t-norm

Fig. 3 The complete solution set of sup-T equations in the 2-dimensional space (**a**) $S(A, b)$ with an Archimedean t-norm (**b**) $S(A, b)$ with a non-Archimedean t-norm

some specific cases. Actually, it will be shown later that sup-T equations with a continuous Archimedean t-norm are somehow easier to solve than those with a continuous non-Archimedean t-norm. However, the problem to obtain all minimal solutions of a consistent system of sup-T equations is inevitably NP-hard in terms of computational complexity. Chen and Wang (2002) provided a proof by transforming polynomially the *minimum covering problem*, which is a well known NP-complete problem, into the problem of solving a system of sup-T_M equations. Markovskii (2004, 2005) showed that determining all minimal solutions of a system of sup-T_P equations is equivalent to the problem of finding all irredundant coverings of a *set covering problem*.

A system of sup-T equations $A \circ x = b$ is called homogeneous if $b = 0$, otherwise it is called nonhomogeneous. Solving a homogeneous system is trivial. It has a unique minimal solution $\check{x} = 0$ and a maximum solution $\hat{x}$ with $\hat{x}_j = \inf_{i \in M} I_T(a_{ij}, 0)$, $j \in N$. Possible shapes of the complete solution set of a nonhomogeneous system of sup-T equations with two variables are illustrated in Fig. 3. Note that for a system of sup-T equations with a continuous Archimedean t-norm, the nonzero elements in each minimal solution can only assume the same values as those in the maximum solution (as $\check{x}^1$ and $\check{x}^2$ shown in Fig. 3a). This phenomenon is a consequence of Theorem 2.13 and has been observed for a long time when solving sup-T_P equations. See, for instance, Loetamonphong and Fang (1999), Markovskii (2004, 2005) and Peeva and Kyosev (2007), as well as Wu and Guu (2004a) for strict t-norms. A version for continuous Archimedean t-norms is due to Stamou and Tzafestas (2001) and restated by Wu et al. (2007) and Wu and Guu (2008). However, when a continuous non-Archimedean t-norm is involved in the composition, the nonzero elements in a minimal solution may assume different values as those in the maximum solution (as $\check{x}^1$ and $\check{x}^2$ shown in Fig. 3b), which without doubt will impose some additional difficulties on the resolution procedure.

Moreover, note that any strictly increasing bijection $\psi : [0, 1] \to [0, 1]$ is order preserving and hence a system of sup-T_ψ equations $A \circ_{T_\psi} x = b$ and a system of sup-T equations $A_\psi \circ_T x = b_\psi$ are isomorphic in the sense that $A \circ_{T_\psi} x = b$ implies $A_\psi \circ_T x_\psi = b_\psi$ and vice versa, where $A_\psi = \left(\psi(a_{ij})\right)_{m \times n}$, $x_\psi = (\psi(x_1), \psi(x_2), \ldots, \psi(x_n))^T$ and $b_\psi = (\psi(b_1), \psi(b_2), \ldots, \psi(b_m))^T$. Therefore, any system of sup-T equations with a continuous Archimedean t-norm T can be converted in principle to either a system of sup-T_P equations or a system of sup-T_L equations. The specific strictly increasing bijection ψ can be determined by Theorem 2.7 as long as the additive generator of T is known.

3.2 Minimal solutions and set covering problems

The close relation between minimal solutions of a system of sup-T equations and some *set covering problems* has been noticed and described from various aspects since the structure of the complete solution set was fully understood. It provides some important information for the analysis of the number of minimal solutions and the development of algorithms to obtain all the minimal solutions. See, for instance, Prévot (1981), Czogała et al. (1982), Higashi and Klir (1984), Wang et al. (1984), Miyakoshi and Shimbo (1986), Peeva (1992, 2006), Wang and Hsu (1992), Cechlárová (1995), Luoh et al. (2002, 2003), Peeva and Kyosev (2004, 2007), Markovskii (2004, 2005) and Chen and Wang (2002, 2007). However, with a small portion for sup-T_P equations, most of these results were developed for sup-T_M equations. In this subsection, a systematic and unified analysis will be performed to illustrate that the minimal solutions of a system of sup-T equations correspond one-to-one to the irredundant coverings of a *set covering problem* when the involved t-norm is Archimedean. However, they just correspond to a subset of constrained irredundant coverings of a *set covering problem* when the t-norm is non-Archimedean. Some concepts and ideas are borrowed from Markovskii (2004, 2005), Peeva (1992, 2006) and Peeva and Kyosev (2004, 2007) and will be extended to the general situations.

For a given system of sup-T equations $A \circ x = b$ with a continuous t-norm T, a variable x_j, $j \in N$ is called essential if $b_j \le a_{ij}$ holds for some $i \in M$, and non-essential otherwise. Clearly, non-essential variables have no influence on the consistency of the system and hence the presence of essential variables is a necessary condition. Moreover, all the non-essential variables can be excluded to simplify a system since they will assume the value 1 in the maximum solution and 0 in any minimal solution.

With the potential maximum solution $\hat{x}$, the characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ of the system $A \circ x = b$ can be defined by

$$\tilde{q}_{ij} = \begin{cases} \left[J_T(a_{ij}, b_i), \hat{x}_j\right], & \text{if } T(a_{ij}, \hat{x}_j) = b_i, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{3.9}$$

and obtained in a time complexity of $O(mn)$. It should be noted that similar types of matrices have been introduced in the literature under various names, e.g., the index sets by Miyakoshi and Shimbo (1986), Fang and Li (1999) and Wu and Guu (2008), the covering matrix by Cheng and Peng (1988), the help matrix by Peeva (1992, 2006) and Peeva and Kyosev (2004, 2007), the solution-base matrix by Chen and Wang (2002, 2007), the matrix pattern by Luoh et al. (2002, 2003), the covering table by Markovskii (2004, 2005) and etc.

It is clear that each element $\tilde{q}_{ij}$ of the characteristic matrix $\tilde{Q}$ indicates all the possible values for the variable x_j to satisfy the ith equation without violating other equations from the upper side. The column corresponding to a non-essential variable contains only empty elements while the column corresponding to an essential variable contains at least one nonempty element. The system $A \circ x = b$ is consistent if and only if each row of $\tilde{Q}$ contains at least one nonempty element. Furthermore, any solution of the system $A \circ x = b$ can be verified via the characteristic matrix $\tilde{Q}$.

Theorem 3.3 *Let $A \circ x = b$ be a system of sup-T equations with a continuous t-norm T. Given its potential maximum solution $\hat{x}$ and its characteristic matrix $\tilde{Q}$, a vector $x \in [0, 1]^n$ is a solution of $A \circ x = b$ if and only if $x \leq \hat{x}$ and the induced matrix $Q_x = (q'_{ij})_{m \times n}$ has no zero rows where*

$$q'_{ij} = \begin{cases} 1, & \text{if } x_j \in \tilde{q}_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.10}$$

Since $J_T(a, 0) = 0$ for all $a \in [0, 1]$, it is obvious that $\tilde{q}_{ij} \subseteq \left[J_T(a_{ij}, 0), \hat{x}_j\right]$ holds for all $i \in M$ and $j \in N$. Hence, all the rows of $\tilde{Q}$ corresponding to the index set $M_0 = \{i \in M \mid b_i = 0\}$ can be removed since the induced matrix Q_x of any vector $x \in [0, 1]^n$ with $x \leq \hat{x}$ has no zero rows if and only if its submatrix, with the rows of the index set M_0 removed, has no zero rows. Consequently, in case that the t-norm T has zero divisors and the index set M_0 is not empty, the corresponding equations can be removed after the characteristic matrix $\tilde{Q}$ has been obtained. However, the corresponding column reduction cannot be performed in this case since the equation $T(a, x) = 0$ with $a \in (0, 1]$ is not trivial due to the existence of zero divisors. Of course, this reduction can be performed before calculating $\tilde{Q}$ if the t-norm T has no zero divisors. A variable is called pseudo-essential if its corresponding column in $\tilde{Q}$ contains only empty elements after the rows of the index set M_0 have been removed. Clearly, the role of pseudo-essential variables is trivial.

Consequently, without loss of generality, one can always assume that $b_i > 0$, $i \in M$ for a system of sup-T equations $A \circ x = b$ if a resolution method based on the characteristic matrix will be applied.

3.2.1 Archimedean property and irredundant coverings

The Archimedean property shows its importance in calculating the characteristic matrix $\tilde{Q}$ of a system of sup-T equations $A \circ x = b$. Theorem 2.13 indicates that for a continuous Archimedean t-norm T, the nonempty elements in $\tilde{Q}$ are always singletons with their values determined by the potential maximum solution. The characteristic matrix $\tilde{Q}$ in this case can be further simplified as $Q = (q_{ij})_{m \times n}$ with

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Definition 3.3 Let $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{mn}$ be a binary matrix. A column j is said to cover a row i if $q_{ij} = 1$. A set of nonzero columns P forms a covering of Q if each row of Q is covered by some column in P. A column j in a covering P is called redundant if the set of columns $P \backslash \{j\}$ remains to be a covering of Q. A covering P

is irredundant if it has no redundant columns. The set of all coverings of Q is denoted by $P(Q)$ while the set of all irredundant coverings of Q is denoted by $\check{P}(Q)$.

It is well known that the set of all coverings $P(Q)$ of a binary matrix Q can be well represented by the feasible solution set of a *set covering problem*, i.e., $\{u \in \{0, 1\}^n \mid Qu \geq e\}$ where $e = (1, 1, \ldots, 1)^T \in \{0, 1\}^m$. The *set covering problem* is known to be one of Karp's 21 NP-complete problems and has been extensively studied. See, for instance, Balas and Padberg (1976), Caprara et al. (2000) and Golumbic and Hartman (2005). The relation between fuzzy relational equations and the *set covering problem* was presented by Markovskii (2004, 2005) for sup-T_P equations, which can be extended without any difficulty to the case of continuous Archimedean t-norms.

Theorem 3.4 *Let $A \circ x = b$ be a system of sup-T equations with a continuous Archimedean t-norm T. Given its potential maximum solution $\hat{x}$ and its simplified characteristic matrix Q, a vector $x \in [0, 1]^n$ with $x \leq \hat{x}$ is a solution of $A \circ x = b$ if $u_x = (\hat{f}(x_1), \hat{f}(x_2), \ldots, \hat{f}(x_n))^T \in \{u \in \{0, 1\}^n \mid Qu \geq e\}$ where*

$$\hat{f}(x_j) = \begin{cases} 1, & \text{if } x_j = \hat{x}_j, \\ 0, & \text{otherwise,} \end{cases} \quad j \in N. \tag{3.12}$$

Conversely, a binary vector u belongs to the set $\{u \in \{0, 1\}^n \mid Qu \geq e\}$ if $x_u = \left(\hat{x}_1 u_1, \hat{x}_2 u_2, \ldots, \hat{x}_n u_n\right)^T$ is a solution of $A \circ x = b$. Moreover, if $S(A, b) \neq \emptyset$, the set of all minimal solutions $\check{S}(A, b)$ corresponds one-to-one to the set of all irredundant coverings $\check{P}(Q)$.

Obviously, determining all irredundant coverings of a binary matrix is very difficult. Actually, it is NP-hard even for finding some specific one, e.g., the minimum weighted covering. However, there does exist a method to represent all irredundant coverings from a point view of propositional calculus. It is well known that a binary vector u can be viewed as a group of atomic propositions and hence each inequality in the system $Qu \geq e$ is equivalent to a disjunctive clause. Therefore, the consistency of the system $Qu \geq e$ can be expressed by the conjunction of these disjunctive clauses which defines a truth function F_Q in conjunctive normal form (CNF), while the irredundant coverings correspond to the irreducible conjunctive clauses in the disjunctive normal form (DNF) of the truth function F_Q. More specifically, F_Q can be represented in CNF as well as DNF, i.e.,

$$F_Q = \bigwedge_{i \in M} \bigvee_{j \in C_i} u_j = \bigvee_{s \in S} \bigwedge_{j \in C'_s} u_j \tag{3.13}$$

where $C_i = \{j \in N \mid q_{ij} = 1\}$ and C'_s is the index set which forms an irredundant covering of Q. Clearly, each irreducible conjunctive clause $\bigwedge_{j \in C'_s} u_j$ defines a minimal solution of $A \circ x = b$. One need to convert the truth function F_Q in CNF to its DNF to obtain all the irreducible conjunctive clauses. This conversion can lead to an exponential explosion of the expression in some cases, i.e., the cardinality of S may increase exponentially as the input size grows. See, e.g., Wengener (1987) and Milterson et al. (2005) for more details.

Example 3.1 Consider a system of sup-T_L equations $A \circ x = b$ with

$$A = \begin{pmatrix} 1 & 1 & 0.5 & 1 & 0.6 & 0.8 \\ 0.9 & 0.2 & 1 & 0.8 & 0.4 & 1 \\ 0.7 & 0.8 & 0.3 & 0.6 & 0.8 & 0.5 \\ 0.2 & 0.7 & 0.9 & 0.3 & 0.5 & 0.5 \\ 0.3 & 0.5 & 0.7 & 0.1 & 0.3 & 0.4 \\ 0 & 0.1 & 0.4 & 0 & 0.1 & 0.3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \\ 0 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{x} = (0.9, 0.7, 0.5, 1, 0.8, 0.7)^T.$$

Hence, the characteristic matrix $\tilde{Q}$ and its simplified form can be obtained, respectively, as

$$\tilde{Q} = \begin{pmatrix} \emptyset & \emptyset & \emptyset & 1 & \emptyset & \emptyset \\ 0.9 & \emptyset & \emptyset & 1 & \emptyset & \emptyset \\ 0.9 & \emptyset & \emptyset & 1 & 0.8 & \emptyset \\ \emptyset & 0.7 & 0.5 & \emptyset & \emptyset & \emptyset \\ 0.9 & 0.7 & 0.5 & \emptyset & \emptyset & \emptyset \\ [0, 0.9] & [0, 0.7] & [0, 0.5] & [0, 1] & [0, 0.8] & [0, 0.7] \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that x_6 is pseudo-essential and can be excluded in the further analysis. The associated truth function F_Q can be expressed in DNF as

$$F_Q = (u_4) \wedge (u_1 \vee u_4) \wedge (u_1 \vee u_4 \vee u_5) \wedge (u_2 \vee u_3) \wedge (u_1 \vee u_2 \vee u_3) \tag{3.14}$$

$$= (u_2 \wedge u_4) \vee (u_3 \wedge u_4). \tag{3.15}$$

Therefore, according to Theorem 3.4, the concerned system $A \circ x = b$ has two minimal solutions, i.e.,

$$\check{x}^1 = (0, 0.7, 0, 1, 0, 0)^T, \tag{3.16}$$

$$\check{x}^2 = (0, 0, 0.5, 1, 0, 0)^T. \tag{3.17}$$

Hence, the two maximal interval solutions are

$$\tilde{x}^1_{max} = \begin{pmatrix} [0, 0.9] \\ 0.7 \\ [0, 0.5] \\ 1 \\ [0, 0.8] \\ [0, 0.7] \end{pmatrix}, \quad \tilde{x}^2_{max} = \begin{pmatrix} [0, 0.9] \\ [0, 0.7] \\ 0.5 \\ 1 \\ [0, 0.8] \\ [0, 0.7] \end{pmatrix}, \tag{3.18}$$

and the union of them forms the complete solution set $S(A, b)$.

Example 3.2 Consider a system of sup-T_P equations $A \circ x = e$ where $A \in \{0, 1\}^{m \times 2m}$ and

$$A = \begin{pmatrix} 1 & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & & & \ddots & \ddots & & \\ & & & & & & 1 & 1 \end{pmatrix}.$$

Clearly, the system is consistent and has the maximum solution $\hat{x} = e$. However, it has 2^m minimal solutions. This example was presented by Markovskii (2005) to illustrate that the total length of all minimal solutions of a system of sup-T equations can considerably exceed the length of the input data.

The resolution technique via the conversion of a truth function was proposed by Markovskii (2004, 2005) and Peeva and Kyosev (2007) for sup-T_P equations which, as has been shown, is valid for continuous Archimedean t-norms. A similar procedure was also proposed by Stamou and Tzafestas (2001). Furthermore, taking the advantage of well developed techniques in integer and combinatorial optimization, some methods can be applied to reduce the complexity of an instance of the *set covering problem*. The most frequently used technique is to remove of the redundant rows.

Definition 3.4 Let $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{mn}$ be a binary matrix.

- (i) A row i is said to be redundant if there is a set of rows such that any covering of these rows also covers row i.
- (ii) A row i_1 is said to dominate a row i_2 if any covering of row i_2 also covers row i_1, or equivalently, if $q_{i_2 j} = 1$ implies $q_{i_1 j} = 1$ for all $j \in N$.

Theorem 3.5 *Let $Q = (q_{ij})_{m \times n} \in \{0, 1\}^{mn}$ be a binary matrix. A row of Q is redundant if and only if it dominates some other row.*

Clearly, redundant rows can be removed from the matrix Q without changing the set of all irredundant coverings $\check{P}(Q)$. Some columns may contain only zero elements after removing the redundant rows. The variables corresponding to such type of columns are called semi-essential since they also assume the value 0 in each minimal solution. Besides, each of the left nonzero columns belongs to at least one of the irredundant coverings of Q.

The elimination of redundant rows has to be applied in a careful way since the identification of all redundant rows may be time consuming for large scale instances. However, it may considerably reduce the size of an instance concerned.

Definition 3.5 Let $Q = (q_{ij})_{m\times n} \in \{0, 1\}^{mn}$ be a binary matrix. The kernel $Ker(Q)$ is the set of columns that belongs to each covering of Q.

It is clear that a column is in $Ker(Q)$ if and only if there exists a row which is covered only by that column. The variables corresponding to $Ker(Q)$ are called super-essential. Consequently, $Ker(Q)$ and the rows covered by $Ker(Q)$ can be removed to reduce the size of Q. Moreover, the following results are straightforward due to Theorem 3.4.

Theorem 3.6 *Let $A \circ x = b$ be a system of sup-T equations with a continuous Archimedean t-norm T and the binary matrix Q its simplified characteristic matrix. The system has a unique solution, i.e., $|S(A, b)| = 1$, if and only if all the variables x_j, $j \in N$ are super-essential while the system has a unique minimal solution, i.e., $|\check{S}(A, b)| = 1$, if and only if $Ker(Q) \in \check{P}(Q)$.*

In Example 3.1, rows 2, 3 and 5 of Q are redundant and column 4 is in $Ker(Q)$. Hence the reduced matrix can be obtained as

$$Q_D = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} u_2 & u_3 \\ 1 & 1 \end{pmatrix}, \quad Ker(Q) = \{u_4\}. \tag{3.19}$$

Clearly, x_4 is super-essential while x_1 and x_5 are semi-essential. Two irredundant coverings $(0, 1, 0, 1, 0, 0)^T$ and $(0, 0, 1, 1, 0, 0)^T$ can be read from Q_D which, by Theorem 3.4, correspond to the minimal solutions $\check{x}^1 = (0, 0.7, 0, 1, 0, 0)^T$ and $\check{x}^2 = (0, 0, 0.5, 1, 0, 0)^T$, respectively.

3.2.2 Non-Archimedean property and constrained irredundant coverings

Consider a system of sup-T equations $A \circ x = b$ with a continuous non-Archimedean t-norm T. Without loss of generality, one can always assume that $b_i > 0, i \in M$ and all the variables x_j, $j \in N$ are essential.

In this case, some nonempty elements of the characteristic matrix $\tilde{Q}$ may not be singletons. Clearly, more efforts are required to figure out all minimal solutions of the system $A \circ x = b$. However, if all the nonempty elements happen to be singletons, the exactly same procedure can be followed to obtain the complete solution set $S(A, b)$ of the system $A \circ x = b$. Such a system is hence called simple. By Theorem 2.15, a system is simple if for all $i \in M$, $b_i \neq a_{ij}, \forall\, j \in N$ and also $b_i \neq a_\alpha$ if the t-norm T involves a nilpotent summand $\langle a_\alpha, b_\alpha, T_\alpha\rangle$ and $a_{ij} \in [a_\alpha, b_\alpha]$ for some $j \in N$. A straightforward check of these sufficient conditions can be implemented in most cases with a time complexity of $O(mn)$. However, particular efforts may be required when the t-norm T involves countably infinite nilpotent summands, for instance, $T = \left(\langle 1/2^{k+1}, 1/2^k, T_L\rangle\right)_{k=0}^{\infty}$.

When a system $A \circ x = b$ is not simple, a variable is called multi-essential if its corresponding column in the characteristic matrix $\tilde{Q}$ contains a nonempty element that is not singleton. A multi-essential variable x_j can assume the value other than $\hat{x}_j$ and 0 in a minimal solution. Actually, by Theorem 2.11, the values x_j can assume in minimal solutions are determined by the set $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$. Moreover, for different values of x_j, the number of equations satisfied by x_j may be different. Therefore, a minimal solution is determined by the combination of the essential variables as well as the values of those multi-essential variables involved. Peeva (1992) and Peeva and Kyosev (2004) proposed a method to obtain all minimal solutions for a system of sup-T_M equations, which is based on the definition of the characteristic matrix and Theorem 3.3 and hence essentially independent of the t-norm involved. Consequently, this method can be extended for general continuous non-Archimedean t-norms. However, it should be noted that this method can be traced back to Cheng and Peng (1988) in which it was developed in a more general framework. The notations used below for the resolution are due to Peeva (1992) and Peeva and Kyosev (2004).

Denote r_j the number of different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$ and $r = \sum_{j \in N} r_j$. Clearly, $r_j = 1$ if x_j is not multi-essential and $r_j \geq 1$ if x_j is multi-essential. Denote $K_j = \{1, 2, \ldots, r_j\}$ and $\check{v}_{jk}, k \in K_j$ the different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$ for all $j \in N$. By definition of the characteristic matrix $\tilde{Q}$, $J_T(a_{ij}, b_i)$ indicates the minimum value that x_j can assume to satisfy the ith equation if $\tilde{q}_{ij} \neq \emptyset$, and hence

$$\bigvee_{j \in C_i} \left\langle \frac{J_T(a_{ij}, b_i)}{x_j} \right\rangle, \quad C_i = \{j \in N \mid \tilde{q}_{ij} \neq \emptyset\} \tag{3.20}$$

indicates the possible ways to satisfy the ith equation with the minimum values of the involved variables. Consequently,

$$F_{\tilde{Q}} = \bigwedge_{i \in M} \bigvee_{j \in C_i} \left\langle \frac{J_T(a_{ij}, b_i)}{x_j} \right\rangle \tag{3.21}$$

provides all lower bound information of the variables to satisfy the system $A \circ x = b$. The function $F_{\tilde{Q}}$ can be regarded as a fuzzy truth function. Clearly, it would reduce to a classical truth function when all the lower bounds are unique for each variable, i.e., $r_j = 1, \forall\ j \in N$. The function $F_{\tilde{Q}}$ is obviously in its fuzzy version of CNF and needs to be converted to its fuzzy version of DNF, i.e.,

$$F_{\tilde{Q}} = \bigwedge_{i \in M} \bigvee_{j \in C_i} \left\langle \frac{J_T(a_{ij}, b_i)}{x_j} \right\rangle = \bigvee_{s \in S} \bigwedge_{j \in C'_s} \left\langle \frac{\check{v}_{jk^s}}{x_j} \right\rangle \tag{3.22}$$

where C'_s is an index set and $\check{v}_{jk^s}$ is the value of x_j such that $\bigwedge_{j \in C'_s} \left\langle \frac{\check{v}_{jk^s}}{x_j} \right\rangle$ forms an irreducible fuzzy conjunctive clause. Some rules are needed to perform this conversion.

Rule 1: commutativity.

$$\left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_2k_2}}{x_{j_2}} \right\rangle = \left\langle \frac{\check{v}_{j_2k_2}}{x_{j_2}} \right\rangle \left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle, \quad j_1 \neq j_2,\ k_1 \in K_{j_1},\ k_2 \in K_{j_2}. \tag{3.23}$$

Rule 2: absorption for conjunction.

$$\left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_1k_2}}{x_{j_1}} \right\rangle = \left\langle \frac{\check{v}_{j_1k_1} \vee \check{v}_{j_1k_2}}{x_{j_1}} \right\rangle, \quad k_1, k_2 \in K_{j_1}. \tag{3.24}$$

Rule 3: distributivity for disjunction.

$$\left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \left(\left\langle \frac{\check{v}_{j_1k_2}}{x_{j_2}} \right\rangle \bigvee \left\langle \frac{\check{v}_{j_3k_3}}{x_{j_3}} \right\rangle \right) = \left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_2k_2}}{x_{j_2}} \right\rangle \bigvee \left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \left\langle \frac{\check{v}_{j_3k_3}}{x_{j_3}} \right\rangle,$$

$$k_1 \in K_{j_1}, k_2 \in K_{j2}, k_3 \in K_{j3}. \tag{3.25}$$

Rule 4: absorption for disjunction.

$$\left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \cdots \left\langle \frac{\check{v}_{j_1k_p}}{x_{j_p}} \right\rangle \bigvee \left\langle \frac{\check{v}_{j_1k'_1}}{x_{j_1}} \right\rangle \cdots \left\langle \frac{\check{v}_{j_1k'_p}}{x_{j_p}} \right\rangle \left\langle \frac{\check{v}_{j_{p+1}k'_{p+1}}}{x_{j_{p+1}}} \right\rangle \cdots \left\langle \frac{\check{v}_{j_{p+q}k'_{p+q}}}{x_{j_{p+q}}} \right\rangle$$
$$= \left\langle \frac{\check{v}_{j_1k_1}}{x_{j_1}} \right\rangle \cdots \left\langle \frac{\check{v}_{j_1k_p}}{x_{j_p}} \right\rangle, \quad \text{if } \check{v}_{j_1k_1} \leq \check{v}_{j_1k'_1}, \ldots, \check{v}_{j_pk_p} \leq \check{v}_{j_pk'_p}. \tag{3.26}$$

Example 3.3 Consider a system of sup-T equations $A \circ x = b$ with $T = (\langle 0.4, 0.8, T_L \rangle)$ and

$$A = \begin{pmatrix} 0.8 & 1 & 0.9 & 1 & 1 \\ 0.9 & 0.7 & 0.8 & 0.8 & 1 \\ 0.4 & 0.4 & 0.6 & 0.5 & 0.2 \\ 0.6 & 0.3 & 0.1 & 0.4 & 0.9 \\ 0.2 & 0 & 0.2 & 0.1 & 0.8 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{x} = (0.6, 1, 1, 1, 0.2)^T.$$

Hence, the characteristic matrix $\tilde{Q}$ can be obtained as

$$\tilde{Q} = \begin{pmatrix} \emptyset & 1 & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & [0.8, 1] & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & \emptyset & \emptyset \\ [0.4, 0.6] & \emptyset & \emptyset & [0.4, 1] & \emptyset \\ [0.2, 0.6] & \emptyset & [0.2, 1] & \emptyset & 0.2 \end{pmatrix}.$$

It is obvious that the system is not simple. The associated fuzzy truth function $F_{\tilde{Q}}$ can be expressed in DNF as

$$F_{\tilde{Q}} = \left(\left\langle\frac{1}{x_2}\right\rangle \bigvee \left\langle\frac{1}{x_4}\right\rangle\right) \bigwedge \left(\left\langle\frac{0.8}{x_3}\right\rangle \bigvee \left\langle\frac{0.8}{x_4}\right\rangle\right) \bigwedge \left(\left\langle\frac{0.8}{x_3}\right\rangle\right)$$

$$\bigwedge \left(\left\langle\frac{0.4}{x_1}\right\rangle \bigvee \left\langle\frac{0.4}{x_4}\right\rangle\right) \bigwedge \left(\left\langle\frac{0.2}{x_1}\right\rangle \bigvee \left\langle\frac{0.2}{x_3}\right\rangle \bigvee \left\langle\frac{0.2}{x_5}\right\rangle\right) \tag{3.27}$$

$$= \left(\left\langle\frac{0.4}{x_1}\right\rangle\left\langle\frac{1}{x_2}\right\rangle\left\langle\frac{0.8}{x_3}\right\rangle\right) \bigvee \left(\left\langle\frac{1}{x_2}\right\rangle\left\langle\frac{0.8}{x_3}\right\rangle\left\langle\frac{0.4}{x_4}\right\rangle\right) \bigvee \left(\left\langle\frac{0.8}{x_3}\right\rangle\left\langle\frac{1}{x_4}\right\rangle\right). \tag{3.28}$$

Therefore, the concerned system $A \circ x = b$ has three minimal solutions, i.e.,

$$\check{x}^1 = (0.4, 1, 0.8, 0, 0)^T, \tag{3.29}$$
$$\check{x}^2 = (0, 1, 0.8, 0.4, 0)^T, \tag{3.30}$$
$$\check{x}^3 = (0, 0, 0.8, 1, 0)^T. \tag{3.31}$$

Hence, the three maximal interval solutions are

$$\tilde{x}^1_{max} = \begin{pmatrix} [0.4, 0.6] \\ 1 \\ [0.8, 1] \\ [0, 1] \\ [0, 0.2] \end{pmatrix}, \quad \tilde{x}^2_{max} = \begin{pmatrix} [0, 0.6] \\ 1 \\ [0.8, 1] \\ [0.4, 1] \\ [0, 0.2] \end{pmatrix}, \quad \tilde{x}^3_{max} = \begin{pmatrix} [0, 0.6] \\ [0, 1] \\ [0.8, 1] \\ 1 \\ [0, 0.2] \end{pmatrix}. \tag{3.32}$$

and the union of them forms the complete solution set $S(A, b)$.

Similarly, the redundant rows of a characteristic matrix $\tilde{Q}$ can be removed to reduce the size of $\tilde{Q}$.

Definition 3.6 Let $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ be the characteristic matrix of a system of sup-T equations $A \circ x = b$.

(i) A row i of $\tilde{Q}$ is said to be redundant if there exists a set of rows of $\tilde{Q}$ such that for all $x \leq \hat{x}$, the corresponding rows of the induced matrix Q_x have no zero rows implies row i of Q_x is also a nonzero row.

(ii) A row i_1 is said to dominate a row i_2 in $\tilde{Q}$ if for all $x \leq \hat{x}$, row i_1 dominates row i_2 in the induced matrix Q_x, or equivalently, $\tilde{q}_{i_2 j} \neq \emptyset$ implies $\tilde{q}_{i_2 j} \subseteq \tilde{q}_{i_1 j}$ for all $j \in N$.

(iii) A column j of $\tilde{Q}$ is said in the kernel $Ker(\tilde{Q})$ if there exists a row i such that $\tilde{q}_{ij}$ is the unique nonempty element of row i.

Clearly, when a continuous Archimedean t-norm is involved for composition, the concepts of redundant rows, dominant rows and the kernel are equivalent to those in Definitions 3.4 and 3.5, respectively. By Theorem 3.5, it is straightforward that a row of $\tilde{Q}$ is redundant if and only if it dominates some other row. Moreover, for a system of sup-T_M equations $A \circ x = b$, if a row i_1 dominates a row i_2, it always holds that $b_{i_1} \leq b_{i_2}$ since $J_{T_M}(a, b) = b$ whenever $0 \leq b \leq a \leq 1$. Hence, the normal form of sup-T_M equations provides some convenience in detecting the redundant rows. However, as shown in Example 3.3, this is not valid in general cases.

Some columns of $\tilde{Q}$ may contain only empty elements after removing the redundant rows. The variables corresponding to such type of columns are called semi-essential and assume the value 0 in each minimal solution. The variables corresponding to the columns in $Ker(\tilde{Q})$ are called sup-essential. However, $Ker(\tilde{Q})$ and the corresponding rows can not be removed in general since the super-essential variables can assume different non-zero values in minimal solutions. Furthermore, the condition that all the variables are super-essential can only guarantee that the uniqueness of the minimal solution. Some discussion on the unique solvability of sup-T_M equations can be found in Sessa (1984), Lettieri and Liguori (1984, 1985), Di Nola and Sessa (1988), Cechlárová (1990, 1995), Li (1990) and Gavalec (2001).

In Example 3.3 rows 2 and 5 of $\tilde{Q}$ are redundant and column 3 is in $Ker(\tilde{Q})$. Hence the reduced matrix can be obtained as

$$\tilde{Q}_D = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ \emptyset & 1 & \emptyset & 1 & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & \emptyset & \emptyset \\ [0.4, 0.6] & \emptyset & \emptyset & [0.4, 1] & \emptyset \end{pmatrix}. \tag{3.33}$$

Clearly, x_3 is super-essential while x_5 is semi-essential. The same result of minimal solutions can be obtained from $\tilde{Q}_D$. Note that in this example the super-essential variable assume a same value in each minimal solution.

Example 3.4 Consider a system of sup-T_M equations $A \circ x = b$ with

$$A = \begin{pmatrix} 1 & 0.4 & 0.6 & 1 \\ 0.6 & 1 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.5 & 0.4 \\ 0.3 & 1 & 0.2 & 0.4 \\ 0.1 & 0.8 & 0.1 & 0.2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0.8 \\ 0.6 \\ 0.4 \\ 0.2 \end{pmatrix}.$$

It can be verified that the system is consistent and the maximum solution is

$$\hat{x} = (1, 0.2, 1, 0.8)^T.$$

Hence, the characteristic matrix $\tilde{Q}$ can be obtained as

$$\tilde{Q} = \begin{pmatrix} 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & 0.8 \\ [0.6, 1] & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & [0.4, 0.8] \\ \emptyset & 0.2 & \emptyset & [0.2, 0.8] \end{pmatrix}.$$

Clearly, row 3 dominates row 1 and row 5 dominates row 4. Columns 1 and 4 are in $Ker(\tilde{Q})$. The reduced matrix can be obtained as

$$\tilde{Q}_D = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 1 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & [0.8, 1] & 0.8 \\ \emptyset & \emptyset & \emptyset & [0.4, 0.8] \end{pmatrix}.$$

The variables x_1 and x_4 are sup-essential while x_2 is semi-essential. The system has two minimal solutions, i.e.,

$$\check{x}^1 = (1, 0, 0.8, 0.4)^T, \tag{3.34}$$

$$\check{x}^2 = (1, 0, 0, 0.8)^T. \tag{3.35}$$

Note that the super-essential variable x_4 assumes different values in the two minimal solutions.

Another method to deal with multi-essential variables is to represent each of them by a set of binary variables.

Recall that $r_j, j \in N$ are the numbers of different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$ and $\check{v}_{jk}, k \in K_j$ are the different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$ for all $j \in N$. Let $\check{v} = (\check{v}_{11}, \ldots, \check{v}_{1r_1}, \ldots, \check{v}_{n1}, \ldots, \check{v}_{nr_n})^T \in [0, 1]^r$ and

$$x_j = \sum_{k \in K_j} \check{v}_{jk} u_{jk}, \quad j \in N \tag{3.36}$$

where $u_{jk} \in \{0, 1\}, \forall\ k \in K_j,\ j \in N$. Obviously, for each $j \in N$, at most one of u_{jk}, $k \in K_j$ can be 1, i.e., $\sum_{k \in K_j} u_{jk} \le 1,\ j \in N$. These restrictions are called the inner-variable incompatibility constraints and can be represented by $Gu \le e^n$, where

$e^n = (1, 1, \ldots, 1)^T \in \{0, 1\}^n$, $u = (u_{11}, \ldots, u_{1r_1}, \ldots, u_{n1}, \ldots, u_{nr_n})^T \in \{0, 1\}^r$ and $G = (g_{jk})_{n\times r}$ with

$$g_{jk} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s, \\ 0, & \text{otherwise.} \end{cases} \tag{3.37}$$

Actually, the incompatibility constraints with $r_j = 1$ are redundant and hence can be removed. Clearly, if $r_j = 1$ for all $j \in N$, all the incompatibility constraints are redundant and no additional difficulties will be imposed. In this case, the values of the nonzero elements in a minimal solution are uniquely determined although they may be different from those in the maximum solution.

As a consequence of this transformation, the characteristic matrix $\tilde{Q}$ can be converted to its augmented characteristic matrix $Q = (q_{ik})_{m\times r} \in \{0, 1\}^{mr}$ where

$$q_{ik} = \begin{cases} 1, & \text{if } \sum_{s=1}^{j-1} r_s < k \le \sum_{s=1}^{j} r_s,\ \check{v}_k \in \tilde{q}_{ij},\ j \in N, \\ 0, & \text{otherwise.} \end{cases} \tag{3.38}$$

Definition 3.7 Let $Q = (q_{ik})_{m\times r} \in \{0, 1\}^{mr}$ and $G = (q_{jk})_{n\times r} \in \{0, 1\}^{nr}$ be two binary matrices. A column k of Q is said to cover a row i of Q if $q_{ik} = 1$. A set of nonzero columns P forms a G-covering of Q if each row of Q is covered by some column in P, i.e., $Qu^P \ge e^m$, and also satisfies $Gu^P \le e^n$ where $u^P = (u_k^P)_{r\times 1}$ and

$$u_k^P = \begin{cases} 1, & \text{if } k \in P, \\ 0, & \text{otherwise.} \end{cases} \tag{3.39}$$

A column k in a G-covering P is called redundant if the set of columns $P\backslash\{k\}$ remains to be a G-covering of Q. A G-covering P is irredundant if P has no redundant columns. The set of all G-coverings of Q is denoted by $P_G(Q)$ while the set of all irredundant G-coverings of Q is denoted by $\check{P}_G(Q)$.

Theorem 3.7 *Let $A \circ x = b$ be a system of sup-T equations with a continuous non-Archimedean t-norm T. Given its potential maximum solution $\hat{x}$ and its characteristic matrix $\tilde{Q}$, denote r_j the number of different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$, $r = \sum_{j\in N} r_j$ and $K_j = \{1, 2, \ldots, r_j\}$. Let $\check{v}_{jk}, k \in K_j$ be the different values in $\{J_T(a_{ij}, b_i) \mid T(a_{ij}, \hat{x}_j) = b_i,\ i \in M\}$ for all $j \in N$, $\check{v} = (\check{v}_{11}, \ldots, \check{v}_{1r_1}, \ldots, \check{v}_{n1}, \ldots, \check{v}_{nr_n})^T \in [0, 1]^r$ and Q and G the corresponding augmented characteristic matrix and the coefficient matrix of the inner-variable incompatibility constraints, respectively. A vector $x \in [0, 1]^n$ with $x \le \hat{x}$ is a solution of $A \circ x = b$ if $u_x = (\hat{f}'(x_1), \hat{f}'(x_2), \ldots, \hat{f}'(x_n))^T \in \{u \in \{0, 1\}^r \mid Qu \ge e^m,\ Gu \le e^n\}$ where $\hat{f}'(x_j) = (u_{j1}, u_{j2}, \ldots, u_{jr_j})$ and*

$$u_{jk} = \begin{cases} 1, & \text{if } k = argmax\{\check{v}_{jk} \mid \check{v}_{jk} \le x_j\}, \\ 0, & \text{otherwise,} \end{cases} \quad j \in N. \tag{3.40}$$

Conversely, a binary vector u *with* $Gu \leq e^n$ *belongs to the set* $\{u \in \{0, 1\}^r \mid Qu \geq e^m\}$ *if*

$$x_u = \left(\sum_{k \in K_1} \check{v}_{1k} u_{1k}, \sum_{k \in K_2} \check{v}_{2k} u_{2k}, \cdots, \sum_{k \in K_n} \check{v}_{nk} u_{nk} \right)^T \tag{3.41}$$

is a solution of $A \circ x = b$. *Moreover, the vector* u *defines an irredundant* G*-covering of* Q *if* x_u *is a minimal solution of* $A \circ x = b$.

Proof If $x \leq \hat{x}$ and $u_x \in \{u \in \{0, 1\}^r \mid Qu \geq e^m, \ Gu \leq e^n\}$, it is clear the induced matrix Q_x has no zero rows and hence x is a solution of $A \circ x = b$ by Theorem 3.3. Conversely, if the binary vector u satisfies $Gu \leq e^n$ and x_u is a solution of $A \circ x = b$, there exists an index j_i for each $i \in M$ such that $T(a_{ij_i}, \sum_{k \in K_{j_i}} \check{v}_{j_i k} u_{j_i k}) = b_i$ and hence $Qu \geq e^n$ according to the definition of the matrix Q. Moreover, if the vector u is not irredundant when x_u is a minimal solution of $A \circ x = b$, there must exist an index $k' \in K_{j'}$ with $u_{j'k'} = 1$ for some $j' \in N$ such that $Qu' \geq e^m$ and $Gu' \leq e^n$ where $u' \in \{0, 1\}^r$ and

$$u'_{jk} = \begin{cases} 0, & \text{if } j = j', \ k = k', \\ u_{jk}, & \text{otherwise.} \end{cases} \tag{3.42}$$

Consequently, $x_{u'}$ with

$$(x_{u'})_j = \begin{cases} (x_u)_j, & \text{if } j \neq j', \\ 0, & \text{if } j = j', \end{cases} \tag{3.43}$$

is also a solution of $A \circ x = b$ and $x_{u'} \leq x_u$. However, $(x_u)_{j'} \neq 0$ since $u_{j'k'} = 1$, which leads to a contradiction. Hence, the vector u defines an irredundant G-covering of Q if $x_u \in \check{S}(A, b)$.

By Theorem 3.7 and the properties of the transformations between $\check{S}(A, b)$ and $\check{P}_G(Q)$, the binary vector u corresponding to a minimal solution defines an irredundant G-covering. Unfortunately, the vector x corresponding to an irredundant G-covering may not necessarily be a minimal solution. Due to the existence of multi-essential variables, the coexistence of some columns in an irredundant G-covering, say $u_{j_1 k_1} = u_{j_2 k_2} = 1$ with $j_1 \neq j_2, k_1 \in K_{j_1}, k_2 \in K_{j_2}$, may fail to induce a minimal solution.

Example 3.5 Consider a system of sup-T_M equations $A \circ x = b$ with

$$A = \begin{pmatrix} 0.8 & 0 & 0.8 \\ 0.6 & 0.6 & 0 \\ 0 & 0.4 & 0.2 \end{pmatrix}, \quad b = \begin{pmatrix} 0.8 \\ 0.6 \\ 0.4 \end{pmatrix}.$$

This example is adapted from the one in Markovskii (2005). Clearly, the system is consistent and has the maximum solution $\hat{x} = (1, 1, 1)^T$. Hence, the characteristic matrix $\tilde{Q}$ can be obtained as

$$\tilde{Q} = \begin{pmatrix} [0.8, 1] & \emptyset & [0.8, 1] \\ [0.6, 1] & [0.6, 1] & \emptyset \\ \emptyset & [0.4, 1] & \emptyset \end{pmatrix}.$$

Therefore, the concerned system has three minimal solutions, i.e.,

$$\check{x}^1 = (0.8, 0.4, 0)^T, \tag{3.44}$$
$$\check{x}^2 = (0.6, 0.4, 0.8)^T, \tag{3.45}$$
$$\check{x}^3 = (0, 0.6, 0.8)^T. \tag{3.46}$$

Let $u = (u_{11}, u_{12}, u_{21}, u_{22}, u_{31})^T \in \{0, 1\}^5$ and

$$\check{v} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31})^T = (0.8, 0.6, 0.6, 0.4, 0.8)^T. \tag{3.47}$$

The augmented characteristic matrix Q can be obtained as

$$Q = \begin{pmatrix} u_{11} & u_{12} & \vdots & u_{21} & u_{22} & \vdots & u_{31} \\ 1 & 0 & \vdots & 0 & 0 & \vdots & 1 \\ 1 & 1 & \vdots & 1 & 0 & \vdots & 0 \\ 0 & 0 & \vdots & 1 & 1 & \vdots & 0 \end{pmatrix}. \tag{3.48}$$

The coefficient matrix G of the inner-variable incompatibility constraints is

$$G = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{3.49}$$

Note that the constraint for u_{31} is redundant and has been omitted. In this case, five irredundant G-coverings can be obtained, i.e.,

$$\check{P}_G^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \check{P}_G^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \check{P}_G^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \check{P}_G^4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \check{P}_G^5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{3.50}$$

However, only the first three irredundant G-coverings induce a minimal solution, respectively. In $\check{P}_G^4$, the variables u_{11} and u_{21} are actually incompatible to induce a minimal solution since the columns corresponding to u_{11} and u_{22} cover the same rows as those corresponding to u_{11} and u_{21} but with a smaller value $\check{v}_{22}$ for x_2 in the induced solution of $A \circ x = b$. The similar situation occurs in $\check{P}_G^5$ where u_{12} and u_{21} can be replaced by u_{21} and u_{22} in order to induce a minimal solution. Furthermore, it is also possible that a group of variables are incompatible but pairwise compatible for inducing a minimal solution. This phenomenon is called the inter-variable incompatibility and essentially equivalent to the rule of absorption for disjunction introduced for converting a fuzzy truth function in CNF to its DNF.

It is possible but not worthwhile to detect all inter-variable incompatibility constraints for a system of sup-T equations since they do not offer any computational convenience in the determination of all minimal solutions. Moreover, the inner-variable incompatibility constraints, which are easy to define, are usually sufficient when a specific minimal solution of the system is concerned.

4 Linear optimization subject to sup-T equations

In this section, we discuss the linear optimization problem subject to a system of sup-T equations with a continuous t-norm T, i.e.,

$$\begin{aligned} \min\ & z = \textstyle\sum_{j \in N} c_j x_j \\ \text{s.t.}\ & \\ & A \circ x = b, \\ & x \in [0, 1]^n, \end{aligned} \tag{4.1}$$

where $c = (c_1, c_2, \ldots, c_n)^T \in R^n$ is the weight (or cost) vector and c_j represents the weight associated with the variable x_j for all $j \in N$. Clearly, it is in general a nonconvex optimization problem. However, it can be polynomially reduced to a classical 0-1 integer programming problem based on the relation between the feasible domain and a *set covering problem.*

Theorem 4.1 *Let $A \circ x = b$ be a consistent system of sup-T equations with a continuous t-norm T. The maximum solution $\hat{x}$ is an optimal solution with respect to the objective function $z = c^T x$ if $c_j \leq 0$ for all $j \in N$. One of the minimal solutions is an optimal solution with respect to the objective function $z = c^T x$ if $c_j \geq 0$ for all $j \in N$.*

Theorem 4.1 was first stated by Fang and Li (1999) for sup-T_M equations, which is valid for general continuous t-norms since it only depends on the structure of the complete solution set $S(A, b)$.

With the aid of Theorem 4.1, any given wight vector $c = (c_1, c_2, \ldots, c_n)^T \in R^n$ can be separated into two parts, i.e., $c^+ = (c_1^+, c_2^+, \ldots, c_n^+)^T$ and $c^- = (c_1^-, c_2^-, \ldots, c_n^-)^T$ such that for all $j \in N$,

$$c_j^+ = \begin{cases} c_j, & \text{if } c_j \leq 0, \\ 0, & \text{if } c_j < 0, \end{cases} \quad \text{and} \quad c_j^- = \begin{cases} 0, & \text{if } c_j \leq 0, \\ c_j, & \text{if } c_j < 0. \end{cases} \tag{4.2}$$

Obviously, $c = c^+ + c^-$ with $c^+ \geq 0$ and $c^- \leq 0$. Two subproblems can be defined as

$$\begin{aligned} &\min\ z^+ = \textstyle\sum_{j \in N} c_j^+ x_j \\ &\text{s.t.} \\ &\qquad A \circ x = b, \\ &\qquad x \in [0, 1]^n, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\min\ z^- = \textstyle\sum_{j \in N} c_j^- x_j \\ &\text{s.t.} \\ &\qquad A \circ x = b, \\ &\qquad x \in [0, 1]^n, \end{aligned} \tag{4.4}$$

By Theorem 4.1, when $S(A, b) \neq \emptyset$, one of the minimal solutions, say $\check{x}^*$, is an optimal solution with respect to $\sum_{j \in N} c_j^+ x_j$ and the maximum solution $\hat{x}$ is an optimal solution with respect to $\sum_{j \in N} c_j^- x_j$. Consequently, an optimal solution with respect to $\sum_{j \in N} c_j x_j$ can be constructed via $\check{x}^*$ and $\hat{x}$.

Theorem 4.2 *Let $A \circ x = b$ be a consistent system of sup-T equations with a continuous t-norm T. For any weight vector $c = (c_1, c_2, \ldots, c_n)^T \in R^n$, the vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ with*

$$x_j^* = \begin{cases} \check{x}_j^*, & \text{if } c_j \geq 0, \\ \hat{x}_j, & \text{if } c_j < 0, \end{cases} \quad j \in N. \tag{4.5}$$

is an optimal solution with respect to $\sum_{j \in N} c_j x_j$ where $\hat{x}$ is the maximum solution and $\check{x}^*$ is the optimal solution with respect to $\sum_{j \in N} c_j^+ x_j$. The optimal value is $z^* = c^T x^* = \sum_{j \in N} (c_j^- \hat{x}_j + c_j^+ \check{x}_j^*)$.

Proof For any $x \in S(A, b) \neq \emptyset$, one has

$$\begin{aligned} c^T x &= (c^+ + c^-)^T x = (c^+)^T x + (c^-)^T x \\ &\geq (c^+)^T \check{x}^* + (c^-)^T \hat{x} = (c^+)^T x^* + (c^-)^T x^* \\ &= c^T x^*. \end{aligned} \tag{4.6}$$

Hence, x^* is an optimal solution with respect to $z = c^T x$.

By Theorem 4.2, for an arbitrary weight vector, solving the linear optimization problem subject to a system of sup-T equations can be decomposed into two subproblems, one of which can be solved analytically while another is not easy to handle. The subproblem with nonnegative weight vector is inevitably an NP-hard problem since the classical *set covering problem* can be regarded as a special scenario of this problem. On the other hand, this problem can be polynomially reduced to a *set covering problem* or a *constrained set covering problem*, where the existence of additional constraints depends on whether the involved continuous triangular norm is Archimedean or not.

Theorem 4.3 *Let $A \circ x = b$ be a consistent system of sup-T equations with a continuous Archimedean t-norm T and Q its associated simplified characteristic matrix. For any given weight vector $c = (c_1, c_2, \ldots, c_n)^T$, the following problem*

$$(LO\text{-}Ar) \qquad \begin{aligned} \min\ & z_x = \textstyle\sum_{j \in N} c_j x_j \\ \text{s.t.}\ & \\ & A \circ x = b, \\ & x \in [0, 1]^n, \end{aligned} \tag{4.7}$$

is equivalent to the set covering problem

$$(SCP) \qquad \begin{aligned} \min\ & z_u = \textstyle\sum_{j \in N} (c_j \hat{x}_j) u_j \\ \text{s.t.}\ & \\ & Qu \geq e, \\ & u \in \{0, 1\}^n, \end{aligned} \tag{4.8}$$

in the sense that any optimal solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$ to (SCP) defines an optimal solution $x^* = (\hat{x}_1 u_1^*, \hat{x}_2 u_2^*, \ldots, \hat{x}_n u_n^*)^T$ to (LO-Ar) and any optimal solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ to (LO-Ar) defines an optimal solution $u^* = (\hat{f}(x_1^*), \hat{f}(x_2^*), \ldots, \hat{f}(x_n^*))^T$ to (SCP) where

$$\hat{f}(x_j^*) = \begin{cases} 1, \text{ if } x_j^* = \hat{x}_j, \\ 0, \text{ otherwise}, \end{cases} \quad j \in N. \tag{4.9}$$

Besides, the optimal values of both problems are equal, i.e., $z_x^* = z_u^*$.

Proof It is straightforward by Theorems 3.4 and 4.2.

Theorem 4.4 *Let $A \circ x = b$ be a consistent system of sup-T equations with a continuous non-Archimedean t-norm T and Q and G its associated augmented characteristic matrix and the coefficient matrix of the inner-variable incompatibility constraints,* *respectively. For any given weight vector* $c = (c_1, c_2, \ldots, c_n)^T$, *the following problem*

$$(LO\text{-}nAr) \qquad \begin{aligned} \min\ & z_x^+ = \textstyle\sum_{j \in N} c_j^+ x_j \\ \text{s.t.}\ & \\ & A \circ x = b, \\ & x \in [0, 1]^n, \end{aligned} \tag{4.10}$$

is equivalent to the constrained set covering problem

$$(CSCP) \qquad \begin{aligned} \min\ & z_u^+ = \textstyle\sum_{j \in N} \sum_{k \in K_j} (c_j^+ \check{v}_{jk}) u_{jk} \\ \text{s.t.}\ & \\ & Qu \geq e^m, \\ & Gu \leq e^n, \\ & u \in \{0, 1\}^r, \end{aligned} \tag{4.11}$$

in the sense that any optimal solution $\check{u}^* = (\check{u}_{11}^*, \ldots, \check{u}_{1r_1}^*, \ldots, \check{u}_{n1}^*, \ldots \check{u}_{nr_n}^*)^T$ *to (CSCP) defines an optimal solution*

$$\check{x}^* = \left(\sum_{k \in K_1} \check{v}_{1k} \check{u}_{1k}^*,\ \sum_{k \in K_2} \check{v}_{2k} \check{u}_{2k}^*,\ \ldots,\ \sum_{k \in K_n} \check{v}_{nk} \check{u}_{nk}^* \right)^T \tag{4.12}$$

to (LO-nAr) and any optimal solution $\check{x}^* = (\check{x}_1^*, \check{x}_2^*, \ldots, \check{x}_n^*)^T$ *to (LO-nAr) defines an optimal solution* $\check{u}^* = (\hat{f}'(x_1^*), \hat{f}'(x_2^*), \ldots, \hat{f}'(x_n^*))^T$ *to (CSCP) where* $\hat{f}'(x_j^*) = (\check{u}_{j1}^*, \check{u}_{j2}^*, \ldots, \check{u}_{jr_j}^*)^T$ *and*

$$\check{u}_{jk}^* = \begin{cases} 1, \text{ if } k = argmax\{\check{v}_{jk} \mid \check{v}_{jk} \leq \check{x}_j^*\}, \\ 0, \text{ otherwise}, \end{cases} \qquad j \in N. \tag{4.13}$$

Besides, the optimal values of both problems are equal, i.e., $(z_x^+)^* = (z_u^+)^*$.

Proof It is straightforward by Theorems 3.7 and 4.2.

Actually, if there exists an index $j \in N$ such that $c_j < 0$, the corresponding variables $u_{jk}, k \in K_j$ can be aggregated as a single variable with the weight $c_j \hat{x}_j$ in the *constrained set covering problem*. With this modification, one can obtain the optimal solution immediately after solving the *constrained set covering problem*.

Therefore, it is clear that the linear optimization problem subject to a system of sup-T equations with a continuous t-norms T can be polynomially reduced to a 0–1 integer programming problem which is known to be NP-hard. This reduction can be completed in a time complexity of $O(mn)$ if the t-norm T is Archimedean or in a time complexity of $O(m^2 n)$ if it is non-Archimedean. Furthermore, taking the advantage of well developed techniques and clarity of exposition in the theory of integer programming, some methods may be applied to efficiently solve the linear optimization problem subject to a system of sup-T equations.

Example 4.1 Reconsider the system of sup-T_M equations in Example 3.5 with an objective function $z = -2x_1 + 2x_2 + x_3$.

By Theorem 4.2, it is clear that the optimal solution of this problem is $x^* = (1, 0.4, 0)^T$ by evaluating the maximum solution and the three minimal solutions. On the other hand, one has $c^+ = (0, 2, 1)^T$ and

$$\check{v} = (\check{v}_{11}, \check{v}_{12}, \check{v}_{21}, \check{v}_{22}, \check{v}_{31})^T = (0.8, 0.6, 0.6, 0.4, 0.8)^T. \tag{4.14}$$

Hence, the corresponding *constrained set covering problem* can be constructed as

$$\begin{aligned} \min \quad & z_u^+ = 0u_{11} + 0u_{12} + 1.2u_{21} + 0.8u_{22} + 0.8u_{31} \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \\ u_{31} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \\ & u \in \{0, 1\}^5. \end{aligned} \tag{4.15}$$

and the optimal solution can be obtained as $\check{u}^* = (1, 0, 0, 1, 0)^T$ which corresponds to the minimal solution $\check{x}^* = (0.8, 0.4, 0)^T$. Hence, the optimal solution can be constructed as $x^* = (1, 0.4, 0)^T$ with the optimal value $z^* = -1.2$ by Theorem 4.2.

Moreover, since $c_1 < 0$, one can obtain the optimal solution directly by solving the following problem

$$\begin{aligned} \min \quad & z_u = -2u_{11} + 1.2u_{21} + 0.8u_{22} + 0.8u_{31} \\ \text{s.t.} \quad & \\ & \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{22} \\ u_{31} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \\ & u \in \{0, 1\}^4. \end{aligned} \tag{4.16}$$

It is clear that the procedure for solving linear optimization problem subject to a system of sup-T equations can be directly extended to the case that the objective function is separable and monotone in each variable, i.e., $z_x = \sum_{j \in N} f_j(x)$ where $f_j(x)$, $j \in N$

are monotone functions defined on [0, 1]. Without loss of generality, we may assume that $f_j(0) = 0$, $j \in N$. In case that the continuous t-norm T is Archimedean, a system of sup-T equations $A \circ x = b$ with an objective function $z_x = \sum_{j \in N} f_j(x)$ is equivalent to the *set covering problem*

$$\begin{aligned} \min\ & z_u = \textstyle\sum_{j \in N} f_j(\hat{x}_j)u_j \\ \text{s.t.}\ & \\ & Qu \geq e, \\ & u \in \{0, 1\}^n, \end{aligned} \tag{4.17}$$

where Q is the associated simplified characteristic matrix. When T is non-Archimedean, this problem is equivalent to the *constrained set covering problem*

$$\begin{aligned} \min\ & z_u = \textstyle\sum_{j \in N^-} f_j(\hat{x}_j)u_{j1} + \sum_{j \in N^+} \sum_{k \in K_j} f_j(\check{v}_{jk})u_{jk} \\ \text{s.t.}\ & \\ & Q'u \geq e^m, \\ & G'u \leq e^n, \\ & u \in \{0, 1\}^{r'}, \end{aligned} \tag{4.18}$$

where Q' and G' are the associated augmented characteristic matrix and the coefficient matrix of inner-variable incompatibility constraints, respectively, with necessary modification, and $N^- = \{j \in N \mid f_j(x) \text{ is a descresing function.}\}$ and $N^+ = N \backslash N^-$.

It is clear that similar properties as in Theorem 4.2 may still hold when the objective function is monotone in each variable separately. See, e.g., Wang et al. (1991), Yang and Cao (2005a, 2007). However, for general nonlinear objective functions, the optimization problem can be very complicated and deserves further investigation since the reduction used for linear optimization problems may be no longer valid.

5 Concluding remarks

It has been shown in this paper that systems of sup-T equations can be generally divided into two categories based on whether the involved continuous t-norm is Archimedean or not, while the linear optimization problem subject to a system of sup-T equations can be reduced to a 0–1 integer programming in polynomial time. It is clear that the structures of the complete solution set and the characteristic matrix of a system of sup-T equations play the crucial roles in the resolution and optimization procedures. Similar analysis can be performed on a system of fuzzy relational equations or inequalities with sup-O composition where $O : [0, 1]^2 \to [0, 1]$ is some general binary operator, as long as the complete solution set can be characterized by a maximum solution and a finite number of minimal solutions, i.e., preserves the structure of a finitely generated root system. Consequently, the t-norm can be replaced by a weak t-norm, t-seminorm or strong pseudo-t-norm. See Fodor (1991), De Cooman and Kerre (1994), Han and Li (2005), Wang and Xiong (2005) and Han et al. (2006)

for details. More general composite operators were also investigated in Cuninghame-Green (1979, 1995), Kawaguchi and Miyakoshi (1998), Cheng and Peng (1988), Noskoá (2005), Khorram and Ghodousian (2006) and Wu (2007).

Besides, given a solution of a system of sup-T equations, algorithms can be designed based on the associated characteristic matrix to obtain all minimal solutions which are less than or equal to the given solution. See, for instance, Yeh (2008).

On the other hand, it is well known that any system of sup-T equations defines a dual system of inf-S equations where S is the dual t-conorm of T. Therefore, the results derived in this paper can be applied dually on a system of inf-S equations, of which the complete solution set, when it is nonempty, can be determined by a minimum solution and a finite number of maximal solutions.

In the literature, fuzzy relational equations with inf-I composition are also considered, where $I : [0, 1]^2 \to [0, 1]$ is an implicator. See, e.g., Miyakoshi and Shimbo (1985), De Baets (2000), Luo and Li (2004) and Noskoá (2005). Similar to inf-S equations, the complete solution set of a consistent system of inf-I equations is also determined by a minimum solution and a finite number of maximal solutions. Actually, when concerning a model implicator I, any system of inf-I equations can be defined dually by a system of sup-T equations where T is the dual t-norm of the implicator I with respect to a strong negator, particularly, the standard negator N_s. Consequently, the results presented in this paper would apply for inf-I equations as well. The readers may refer to De Baets (1995c, 1997) for more details on model implicators and model coimplicators and to Li and Fang (2008) for a detailed discussion on the relationship between various types of fuzzy relational equations.

Since fuzzy relational equations can be considered in a general framework of lattice, it would be an interesting topic of finding all minimal/maximal solutions, as well as some specific ones, in an efficient manner for a system of fuzzy relational equations defined on a bounded lattice. It certainly depends on the properties of the composite operator and the specific structure of the underlying lattice.

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References

Abbasi Molai, A., & Khorram, E. (2007a). A modified algorithm for solving the proposed models by Ghodousian and Khorram and Khorram and Ghodousian. *Applied Mathematics and Computation, 190*, 1161–1167.

Abbasi Molai, A., & Khorram, E. (2007b). Another modification from two papers of Ghodousian and Khorram and Ghorram et al. *Applied Mathematics and Computation*. doi:10.1016/j.amc.2007.07.061.

Abbasi Molai, A., & Khorram, E. (2007). An algorithm for solving fuzzy relation equations with max-T composition operator. *Information Sciences*. doi:10.1016/j.ins.2007.10.010.

Alsina, C., Frank, M. J., & Schweizer, B. (2006). *Associative functions: Triangular norms and copulas*. Singarpore: World Scientific.

Arnould, T., & Tano, S. (1994a). A rule-based method to calculate the widest solution sets of a max–min fuzzy relational equation. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 2*, 247–256.

Arnould, T., & Tano, S. (1994b). A rule-based method to calculate exactly the widest solution sets of a max-min fuzzy relational inequality. *Fuzzy Sets and Systems, 64*, 39–58.

Balas, E., & Padberg, M. W. (1976). Set partitioning: A survey. *SIAM Review, 18*, 710–760.

Bellman, R. E., & Zadeh, L. A. (1977). Local and fuzzy logics. In J. M. Dunn & G. Epstein (Eds.), *Modern uses of multiple valued logic* (pp. 103–165). Dordrecht: Reidel

Bour, L., & Lamotte, M. (1987). Solutions minimales d'équations de relations floues avec la composition max t-norme. *BUSEFAL, 31*, 24–31.

Bourke, M. M., & Fisher, D. G. (1998). Solution algorithms for fuzzy relational equations with max-product composition. *Fuzzy Sets and Systems, 94*, 61–69.

Caprara, A., Toth, P., & Fischetti, M. (2000). Algorithms for the set covering problem. *Annals of Operations Research, 98*, 353–371.

Cechlárová, K. (1990). Strong regularity of matrices in a discrete bottleneck algebra. *Linear Algebra and Its Applications, 128*, 35–50.

Cechlárová, K. (1995). Unique solvability of max-min fuzzy equations and strong regularity of matrices over fuzzy algebra. *Fuzzy Sets and Systems, 75*, 165–177.

Chen, L., & Wang, P. P. (2002). Fuzzy relation equations (I): The general and specialized solving algorithms. *Soft Computing, 6*, 428–435.

Chen, L., & Wang, P. P. (2007). Fuzzy relation equations (II): The branch-poit-solutions and the categorized minimal solutions. *Soft Computing, 11*, 33–40.

Cheng, L., & Peng, B. (1988). The fuzzy relation equation with union or intersection preserving operator. *Fuzzy Sets and Systems, 25*, 191–204.

Clifford, A. H. (1954). Naturally totally ordered commutative semigroups. *American Journal of Mathematics, 76*, 631–646.

Cormen, T. H., Leiserson, C. E., Rivest, R. L., & Stein, C. (2001). *Introduction to algorithms* (2nd ed.). Cambridge, MA: MIT Press.

Cuninghame-Green, R. A. (1979). *Minimax algebra*, Lecture Notes in Economics and Mathematical Systems, Vol. 166. Berlin: Springer.

Cuninghame-Green, R. A. (1995). Minimax algebra and applications. *Advances in Imaging and Electron Physics, 90*, 1–121.

Czogała, E., Drewiak, J., & Pedrycz, W. (1982). Fuzzy relation equations on a finite set. *Fuzzy Sets and Systems, 7*, 89–101.

De Baets, B. (1995a). An order-theoretic approach to solving sup-$\mathcal{T}$ equations, In D. Ruan (Ed.), *Fuzzy set theory and advanced mathematical applications* (pp. 67–87). Dordrecht: Kluwer.

De Baets, B. (1995b). *Oplossen van vaagrelationele vergelijkingen: een ordetheoretische benadering*, Ph.D. Dissertation, University of Gent.

De Baets, B. (1995c). Model implicators and their characterization. In N. Steele (Ed.), *Proceedings of the First ICSC International Symposium on Fuzzy Logic* (pp. A42–A49). ICSC Academic Press.

De Baets, B. (1997). Coimplicators, the forgotten connectives. *Tatra Mountains Mathematical Publications, 12*, 229–240.

De Baets, B. (1998). Sup-$\mathcal{T}$ Equations: State of the art. In O. Kaynak, et al. (Eds.), *Computational Intelligence: Soft Computing and Fuzzy-Neural Integration with Applications*, NATO ASI Series F: Computer and Systems Sciences (Vol. 162, pp. 80–93). Berlin: Springer-Verlag.

De Baets, B. (2000). Analytical solution methods for fuzzy relational equations. In D. Dubois & H. Prade, (Eds.), *Fundamentals of fuzzy sets, the handbooks of fuzzy sets series* (Vol. 1, pp. 291–340). Dordrecht: Kluwer.

De Baets, B., Van de Walle, B., & Kerre, E. (1998). A plea for the use of Łukasiewicz triplets in the definition of fuzzy preference structures. Part II: The identity case. *Fuzzy Sets and Systems, 99*, 303–310.

De Cooman, G., & Kerre, E. (1994). Order norms on bounded partially ordered sets. *Journal of Fuzzy Mathematics, 2*, 281–310.

Demirli, K., & De Baetes, B. (1999). Basic properties of implicators in a residual framework . *Tatra Mountains Mathematical Publications, 16*, 31–46.

Di Martino, F., Loia, V., & Sessa, S. (2003). A method in the compression/decompression of images using fuzzy equations and fuzzy similarities. In T. Bilgiç, B. De Baets, & O. Kaynak (Eds.), *Proceedings of the 10th International Fuzzy Systems Association World Congress*, Istanbul, Turkey, pp. 524–527.

Di Nola, A. (1984). An algorithm of calculation of lower solutions of fuzzy relation equation. *Stochastica, 3*, 33–40.

Di Nola, A. (1985). Relational equations in totally ordered lattices and their complete resolution. *Journal of Mathematical Analysis and Applications, 107*, 148–155.

Di Nola, A. (1990). On solving relational equations in Brouwerian lattices. *Fuzzy Sets and Systems, 34*, 365–376.

Di Nola, A., & Lettieri, A. (1989). Relation equations in residuated lattices. *Rendiconti del Circolo Matematico di Palermo, 38*, 246–256.

Di Nola, A., Pedrycz, W., & Sessa, S. (1982). On solution of fuzzy relational equations and their characterization. *BUSEFAL, 12*, 60–71.

Di Nola, A., Pedrycz, W., & Sessa, S. (1988). Fuzzy relation equations with equality and difference composition operators. *Fuzzy Sets and Systems, 25*, 205–215.

Di Nola, A., Pedrycz, W., Sessa, S., & Sanchez, E. (1991). Fuzzy relation equations theory as a basis of fuzzy modelling: An overview. *Fuzzy Sets and Systems, 40*, 415–429.

Di Nola, A., Pedrycz, W., Sessa, S., & Wang, P. Z. (1984). Fuzzy relation equations under triangular norms: a survey and new results. *Stochastica, 8*, 99–145.

Di Nola, A., & Sessa, S. (1983). On the set of solutions of composite fuzzy relation equations. *Fuzzy Sets and Systems, 9*, 275–286.

Di Nola, A., & Sessa, S. (1988). Finite fuzzy relational equations with a unique solution in linear lattices. *Journal of Mathematical Analysis and Applications, 132*, 39–49.

Di Nola, A., Sessa, S., Pedrycz, W., & Sanchez, E. (1989). *Fuzzy relation equations and their applications to knowledge engineering*. Dordrecht: Kluwer.

Drewniak, J. (1982). Note on fuzzy relation equations. *BUSEFAL, 12*, 50–51.

Drewniak, J. (1983). System of equations in a linear lattice. *BUSEFAL, 15*, 88–96.

Drossos, C., & Navara, M. (1996). Generalized t-conorms and closure operators. In *Proceedings of the Fourth European Congress on Intelligent Techniques and Sof Computing, EUFIT'96, Aachen, Germany*, pp. 22–26.

Dubois, D., & Prade, H. (1980). New results about properties and semantics of fuzzy set-theoretic operators. In P. P. Wang & S. K. Chang (Eds.), *Fuzzy sets: Theory and applications to policy analysis and information systems* (pp. 59–75). New York: Plenum Press.

Dubois, D., & Prade, H. (1986). New results about properties and semantics of fuzzy set-theoretic operators. In P. P. Wang & S. K. Chang (Eds.), *Fuzzy sets: Theory and applications to policy analysis and information systems* (pp. 59–75). New York: Plenum Press.

Fang, S.-C., & Li, G. (1999). Solving fuzzy relation equations with a linear objective function. *Fuzzy Sets and Systems, 103*, 107–113.

Fodor, J. C. (1991). Strict preference relations based on weak t-norms. *Fuzzy Sets and Systems, 43*, 327–336.

Frank, M. J. (1979). On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. *Aequationes Mathematicae, 19*, 194–226.

Gavalec, M. (2001). Solvability and unique solvability of max-min fuzzy equations. *Fuzzy Sets and Systems, 124*, 385–393.

Gavalec, M., & Plávka, J. (2003). Strong regularity of matrices in general max-min algebra. *Linear Algebra and Its Applications, 371*, 241–254.

Ghodousian, A., & Khorram, E. (2006a). An algorithm for optimizing the linear function with fuzzy relation equation constraints regarding max-prod composition. *Applied Mathematics and Computation, 178*, 502–509.

Ghodousian, A., & Khorram, E. (2006b). Solving a linear programming problem with the convex combination of the max-min and the max-average fuzzy relation equations. *Applied Mathematics and Computation, 180*, 411-418.

Ghodousian, A., & Khorram, E. (2007). Fuzzy linear optimization in the presence of the fuzzy relation inequality constraints with max-min composition. *Information Sciences*. doi:10.1016/j.ins.2007.07.022.

Goguen, J. A. (1967). L-Fuzzy sets. *Journal of Mathematical Analysis and Applications, 18*, 145–174.

Golumbic, M. C., & Hartman, I. B.-A. (2005). *Graph theory, combinatorics and algorithms: Interdisciplinary applications*. New York: Springer-Verlag.

Gottwald, S. (1984). T-normen und ϕ-operatoren als Wahrheitswertfunktionen mehrtiger junktoren. In G. Wechsung (Ed.), *Frege Conference 1984, Proceedings of the International Conference held at Schwerin (GDR)*, Mathematical Research (Vol. 20, pp. 121–128). Berlin: Akademie-Verlag.

Gottwald, S. (1986). Characterizations of the solvability of fuzzy equations. *Elektron. Informationsverarb. Kybernet, 22*, 67–91.

Gottwald, S. (1993). *Fuzzy sets and fuzzy logic: The foundations of application—from a mathematical point of view*. Wiesbaden: Vieweg.

Gottwald, S. (2000). Generalized solvability behaviour for systems of fuzzy equations. In V. Novák & I. Perfilieva (Eds.), *Discovering the world with fuzzy logic* (pp. 401–430). Heidelberg: Physica-Verlag.

Guo, F.-F., & Xia, Z.-Q. (2006). An algorithm for solving optimization problems with one linear objective function and finitely many constraints of fuzzy relation inequalities. *Fuzzy Optimization and Decision Making, 5*, 33–47.

Gupta, M. M., & Qi, J. (1991). Design of fuzzy logic controllers based on generalized T-operators. *Fuzzy Sets and Systems, 40*, 473–489.

Guu, S.-M., & Wu. Y.-K. (2002). Minimizing a linear objective function with fuzzy relation equation constraints. *Fuzzy Optimization and Decision Making, 1*, 347–360.

Han, S. C., & Li, H. X. (2005). Note on "pseudo-t-norms and implication operators on a complete Brouwerian lattice" and "pseudo-t-norms and implication operators: Direct products and direct product decompositions". *Fuzzy Sets and Systems, 153*, 289–294.

Han, S. C., Li, H. X., & Wang, J. Y. (2006). Resolution of finite fuzzy relation equations based on strong pseudo-t-norms. *Applied Mathematics Letters, 19*, 752–757.

Han, S. R., & Sekiguchi, T. (1992). Solution of a fuzzy relation equation using a sign matrix. *Japanese Journal of Fuzzy Theory and Systems, 4*, 160–171.

Higashi, M., & Klir, G. J. (1984). Resolution of finite fuzzy relation equations. *Fuzzy Sets and Systems, 13*, 65–82.

Höhle, U. (1995). Commutative residuated l-monoids. In U. Höhle & E. P. Klement (Eds.), *Non-classical logics and their applications to fuzzy subsets. A handbook of the mathematical foundations of fuzzy set theory* (pp. 53–106). Boston: Kluwer Academic Publishers.

Imai, H., Kikuchi, K., & Miyakoshi, M. (1998). Unattainable solutions of a fuzzy relation equation. *Fuzzy Sets and Systems, 99*, 195–196.

Imai, H., Miyakoshi, M., & Da-Te, T. (1997). Some properties of minimal solutions for a fuzzy relation equation. *Fuzzy Sets and Systems, 90*, 335–340.

Jenei, S. (2001). Continuity of left-continuous triangular norms with strong induced negations and their boundary condition. *Fuzzy Sets and Systems, 124*, 35–41.

Jenei, S. (2002). Structure of left-continuous t-norms with strong induced negations, (III) Construction and decomposition. *Fuzzy Sets and Systems, 128*, 197–208.

Kawaguchi, M. F., & Miyakoshi, M. (1998). Composite fuzzy relational equations with non-commutative conjunctions. *Information Sciences, 110*, 113–125.

Khorram, E., & Ghodousian, A. (2006). Linear objective function optimization with fuzzy relation equation constraints regarding max-av composition. *Applied Mathematics and Computation, 173*, 872–886.

Khorram, E., Ghodousian, A., & Abbasi Molai, A. (2006). Solving linear optimization problems with max-star composition equation constraints. *Applied Mathematics and Computation, 179*, 654–661.

Klement, E. P., Mesiar, R., & Pap, E. (1999). Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. *Fuzzy Sets and Systems, 104*, 3–13.

Klement, E. P., Mesiar, R., & Pap, E. (2000). *Triangular norms*. Dordrecht: Kluwer.

Klement, E. P., Mesiar, R., & Pap, E. (2004a). Triangular norms. Position paper I: Basic analytical and algebraic properties. *Fuzzy Sets and Systems, 143*, 5–26.

Klement, E. P., Mesiar, R., & Pap, E. (2004b). Triangular norms. Position paper II: General constructions and parameterized families. *Fuzzy Sets and Systems, 145*, 411–438.

Klement, E. P., Mesiar, R., & Pap, E. (2004c). Triangular norms. Position paper III: Continuous t-norms. *Fuzzy Sets and Systems, 145*, 439–454.

Klir, G., & Yuan, B. (1995). *Fuzzy sets and fuzzy logic: Theory and applications*. Upper Saddle River, NJ: Prentice Hall.

Kolesárová, A. (1999). A note on Archimedean triangular norms. *BUSEFAL, 80*, 57–60.

Krause, G. M. (1983). Interior idempotents and non-representability of groupoids. *Stochastica, 7*, 5–10.

Lettieri, A., & Liguori, F. (1984). Characterization of some fuzzy relation equations provided with one solution on a finite set. *Fuzzy Sets and Systems, 13*, 83–94.

Lettieri, A., & Liguori, F. (1985). Some results relative to fuzzy relation equations provided with one solution. *Fuzzy Sets and Systems, 17*, 199–209.

Li, J.-X. (1990). The smallest solution of max-min fuzzy equations. *Fuzzy Sets and Systems, 41*, 317–327.

Li, J.-X. (1994). On an algorithm for solving fuzzy linear systems. *Fuzzy Sets and Systems, 61*, 369–371.

Li, P., & Fang, S.-C. (2008). A survey on fuzzy relational equations, Part I: Classification and solvability. *Fuzzy Optimization and Decision Making* (submitted).

Li, H. X., Miao, Z. H., Han, S. C., & Wang, J. Y. (2005). A new kind of fuzzy relation equations based on inner transformation. *Computers and Mathematics with Applications, 50*, 623–636.

Ling, C. M. (1965). Representation of associative functions. *Publicationes Mathematicae Debrecen, 12*, 189–212.

Loetamonphong, J., & Fang, S.-C. (1999). An efficient solution procedure for fuzzy relation equations with max-product composition. *IEEE Transactions on Fuzzy Systems, 7*, 441–445.

Loetamonphong, J., & Fang, S.-C. (2001). Optimization of fuzzy relation equations with max-product composition. *Fuzzy Sets and Systems, 118*, 509–517.

Loetamonphong, J., Fang, S.-C., & Young, R. (2002). Multi-objective optimization problems with fuzzy relation equation constraints. *Fuzzy Sets and Systems, 127*, 141–164.

Loia, V., & Sessa, S. (2005). Fuzzy relation equations for coding/decoding processes of images and videos. *Information Sciences, 171*, 145–172.

Lu, J., & Fang, S.-C. (2001). Solving nonlinear optimization problems with fuzzy relation equation constraints. *Fuzzy Sets and Systems, 119*, 1–20.

Luo, Y., & Li, Y. (2004). Decomposition and resolution of min-implication fuzzy relation equations based on S-implication. *Fuzzy Sets and Systems, 148*, 305–317.

Luoh, L., Wang, W. J., & Liaw, Y. K. (2002). New algorithms for solving fuzzy relation equations. *Mathematics and Computers in Simulation, 59*, 329–333.

Luoh, L., Wang, W. J., & Liaw, Y. K. (2003). Matrix-pattern-based computer algorithm for solving fuzzy relation equations. *IEEE Transactions on Fuzzy Systems, 11*, 100–108.

Markovskii, A. (2004). Solution of fuzzy equations with max-product composition in inverse control and decision making problems. *Automation and Remote Control, 65*, 1486–1495.

Markovskii, A. (2005). On the relation between equations with max-product composition and the covering problem. *Fuzzy Sets and Systems, 153*, 261–273.

Mayor, G., & Torrens, J. (1991). On a family of t-norms. *Fuzzy Sets and Systems, 41*, 161–166.

Milterson, P. B., Radhakrishnan, J., & Wegener, I. (2005). On converting CNF to DNF. *Theoretical Computer Science, 347*, 325–335.

Miyakoshi, M., & Shimbo, M. (1985). Solutions of composite fuzzy relational equations with triangular norms. *Fuzzy Sets and Systems, 16*, 53–63.

Miyakoshi, M., & Shimbo, M. (1986). Lower solutions of systems of fuzzy equations. *Fuzzy Sets and Systems, 19*, 37–46.

Mordeson, J. N., & Malik, D. S. (2002). *Fuzzy automata and languages: Theory and applications*. Boca Raton: Chapman & Hall/CRC.

Mostert, P. S., & Shields, A. L. (1957). On the structure of semi-groups on a compact manifold with boundary. *The Annals of Mathematics, 65*, 117–143.

Noskoá, L. (2005). Systems of fuzzy relation equation with inf-$\longrightarrow$ composition: solvability and solutions. *Journal of Electrical Engineering, 12(s)*, 69–72.

Oden, G. C. (1977). Integration of fuzzy logical information. *Journal of Experimental Psychology, Human Perception and Performance, 106*, 565–575.

Pandey, D. (2004). On the optimization of fuzzy relation equations with continuous t-norm and with linear objective function. In *Proceedings of the Second Asian Applied Computing Conference, AACC 2004*, Kathmandu, Nepal pp. 41–51.

Pappis, C. P., & Sugeno, M. (1985). Fuzzy relational equations and the inverse problem. *Fuzzy Sets and Systems, 15*, 79–90.

Pedrycz, W. (1982a). Fuzzy control and fuzzy systems. *Technical Report 82 14*, Delft University of Technology, Department of Mathematics.

Pedrycz, W. (1982b). Fuzzy relational equations with triangular norms and their resolutions. *BUSEFAL, 11*, 24–32.

Pedrycz, W. (1985). On generalized fuzzy relational equations and their applications. *Journal of Mathematical Analysis and Applications, 107*, 520–536.

Pedrycz, W. (1989). *Fuzzy control and fuzzy systems*. New York, NY: Research Studies Press/Wiely.

Pedrycz, W. (1991). Processing in relational structures: Fuzzy relational equations. *Fuzzy Sets and Systems, 40*, 77–106.

Pedrycz, W. (2000). Fuzzy relational equations: bridging theory, methodolody and practice. *International Journal of General Systems, 29*, 529–554.

Peeva, K. (1985). Systems of linear equations over a bounded chain. *Acta Cybernetica, 7*, 195–202.

Peeva, K. (1992). Fuzzy linear systems. *Fuzzy Sets and Systems, 49*, 339–355.

Peeva, K. (2006). Universal algorithm for solving fuzzy relational equations. *Italian Journal of Pure and Applied Mathematics, 9*, 9–20.

Peeva, K., & Kyosev, Y. (2004). *Fuzzy relational calculus: Theory, applications and software*. New Jersey: World Scientific.

Peeva, K., & Kyosev, Y. (2007). Algorithm for solving max-product fuzzy relational equations. *Soft Computing, 11*, 593–605.

Perfiliva, I., & Tonis, A. (2000). Compatibility of systems of fuzzy relation equations. *International Journal of General Systems, 29*, 511–528.

Prévot, M. (1981). Algorithm for the solution of fuzzy relations. *Fuzzy Sets and Systems, 5*, 319–322.

Rudeanu, S. (1974). *Boolean functions and equations*. Amsterdam: North Holland.

Rudeanu, S. (2001). *Lattice functions and equations*. London: Springer.

Sanchez, E. (1974). *Equations de relation floues*, Thèse de Doctorat, Faculté de Médecine de Marseille.

Sanchez, E. (1976). Resolution of composite fuzzy relation equation. *Information and Control, 30*, 38–48.

Sanchez, E. (1977). Solutions in composite fuzzy relation equations: application to medical diagnosis in Brouwerian logic. In M. M. Gupta, G. N. Saridis, & B. R. Gaines (Eds.), *Fuzzy automata and decision processes* (pp. 221–234). Amsterdam: North-Holland.

Schweizer, B., & Sklar, A. (1963). Associative functions and abstract semigroups. *Plublicationes Mathematicae Debrecen, 10*, 69–81.

Sessa, S. (1984). Some results in the setting of fuzzy relation equations theory. *Fuzzy Sets and Systems, 14*, 281–297.

Shi, E. W. (1987). The hypothesis on the number of lower solutions of a fuzzy relation equation. *BUSEFAL, 31*, 32–41.

Shieh, B.-S. (2007). Solutions of fuzzy relation equations based on continuous t-norms. *Information Sciences, 177*, 4208–4215.

Stamou, G. B., & Tzafestas, S. G. (2001). Resolution of composite fuzzy relation equations based on Archimedean triangular norms. *Fuzzy Sets and Systems, 120*, 395–407.

Thole, U., Zimmermann, H.-J., & Zysno, P. (1979). On the suitability of minimum and product operators for the intersection of fuzzy sets. *Fuzzy Sets and Systems, 2*, 167–180.

Van de Walle, De Baets, B., & Kerre, E. (1998). A plea for the use of Łukasiewicz triples in the definition of fuzzy preference structures. Part I: General argumentation. *Fuzzy Sets and Systems, 97*, 349–359.

Wagenknecht, M., & Hartmann, K. (1990). On the existence of minimal solutions for fuzzy equations with tolerances. *Fuzzy Sets and Systems, 34*, 237–244.

Wang, H. F. (1995). A multi-objective mathematical programming problem with fuzzy relation constraints. *Journal of Multi-Criteria Decision Analysis, 4*, 23–35.

Wang, X. P. (2001). Method of solution to fuzzy relational equations in a complete Brouwerian lattice. *Fuzzy Sets and Systems, 120*, 409–414.

Wang, X. P. (2003). Infinite fuzzy relational equations on a complete Brouwerian lattice. *Fuzzy Sets and Systems, 138*, 657–666.

Wang, X. P., & Xiong, Q. Q. (2005). The solution set of a fuzzy relational equation with sup-conjunctor composition in a complete Brouwerian lattice. *Fuzzy Sets and Systems, 153*, 249–260.

Wang, H. F., & Hsu, H. M. (1992). An alternative approach to the resolution of fuzzy relation equations. *Fuzzy Sets and Systems, 45*, 203–213.

Wang, P. Z., Sessa, S., Di Nola, A., & Pedrycz, W. (1984). How many lower solutions does a fuzzy relation equation have? *BUSEFAL, 18*, 67–74.

Wang, P. Z., & Zhang, D.Z. (1987). *Fuzzy decision making*, Beijing Normal University Lectures, 1987.

Wang, P. Z., Zhang, D. Z., Sanchez, E., & Lee, E. S. (1991). Latticized linear programming and fuzzy relation inequalities. *Journal of Mathematical Analysis and Applications, 159*, 72–87.

Wengener, I. (1987). *The complexity of Boolean functions*. New York: Wieley.

Wu, Y.-K. (2006). Optimizing the geometric programming problem with max-min fuzzy relational equation constraints, *Technical Report*, Vanung University, Department of Industrial Management.

Wu, Y.-K. (2007). Optimization of fuzzy relational equations with max-av composition. *Information Sciences, 177*, 4216–4229.

Wu, Y.-K., & Guu, S.-M. (2004a). A note on fuzzy relation programming problems with max-strict-t-norm composition. *Fuzzy Optimization and Decision Making, 3*, 271–278.

Wu, Y.-K., & Guu, S.-M. (2004b). On multi-objective fuzzy relation programming problem with max-strict-t-norm composition, *Technical Report*, Yuan Ze University, Department of Business Administration.

Wu, Y.-K., & Guu, S.-M. (2005). Minimizing a linear function under a fuzzy max-min relational equation constraint. *Fuzzy Sets and Systems, 150*, 147–162.

Wu, Y.-K., & Guu, S.-M. (2008). An efficient procedure for solving a fuzzy relational equation with max-Archimedean t-norm composition. *IEEE Transactions on Fuzzy Systems, 16*, 73–84.

Wu, Y.-K., Guu, S.-M., & Liu, J. Y.-C. (2002). An accelerated approach for solving fuzzy relation equations with a linear objective function. *IEEE Transactions on Fuzzy Systems, 10*, 552–558.

Wu, Y.-K., Guu, S.-M., & Liu, J. Y.-C. (2007). Optimizing the linear fractional programming problem with max-Archimedean t-norm fuzzy relational equation constraints. In *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 1–6.

Xiong, Q. Q., & Wang, X. P. (2005). Some properties of sup-min fuzzy relational equations on infinite domains. *Fuzzy Sets and Systems, 151*, 393–402.

Xu, W.-L. (1978). Fuzzy relation equation. In *Reports on Beijing Fuzzy Mathematics Meeting*, 1978.

Xu, W.-L., Wu, C.-F., & Cheng, W.-M. (1982). An algorithm to solve the max-min fuzzy relational equations. In M. Gupta & E. Sanchez (Eds.), *Approximate reasoning in decision analysis* (pp. 47–49). Amsterdam: North-Holland.

Yager, R. R. (1982). Some procedures for selecting fuzzy set-theoretic operators. *International Journal General Systems, 8*, 115–124.

Yang, J. H., & Cao, B. Y. (2005a). Geometric programming with fuzzy relation equation constraints. In *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 557–560.

Yang, J. H., & Cao, B. Y. (2005b). Geometric programming with max-product fuzzy relation equation constraints. In *Proceedings of Annual Meeting of the North American Fuzzy Information Processing Society*, 2005, 650–653.

Yang, J. H., & Cao, B.Y. (2007). Posynomial fuzzy relation geometric programming. In P. Melin, O. Castillo, L. T. Aguilar, J. Kacprzyk, & W. Pedrycz (Eds.), *Proceedings of the 12th International Fuzzy Systems Association World Congress* (pp. 563–572). Cancun, Mexico.

Yeh, C.-T. (2008). On the minimal solutions of max-min fuzzy relational equations. *Fuzzy Sets and Systems, 159*, 23–39.

Zhao, C. K. (1987). On matrix equations in a class of complete and completely distributive lattice. *Fuzzy Sets and Systems, 22*, 303–320.

Zimmermann, H.-J. (2001). *Fuzzy set theory and its applications* (4th ed.). Boston: Kluwer.

Zimmermann, K. (2007). A note on a paper by E. Khorram and A. Ghodousian. *Applied Mathematics and Computation, 188*, 244–245.

Zimmermann, H.-J, & Zysno, P. (1980). Latent connectives in human decision-making. *Fuzzy Sets and Systems, 4*, 37–51.