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A fuzzy closeness approach to fuzzy multi-attribute decision making

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Abstract The aim of this paper is to develop a new fuzzy closeness (FC) methodology for multi-attribute decision making (MADM) in fuzzy environments, which is an important research field in decision science and operations research. The TOPSIS method based on an aggregating function representing "closeness to the ideal solution" is one of the well-known MADM methods. However, while the highest ranked alternative by the TOPSIS method is the best in terms of its ranking index, this does not mean that it is always the closest to the ideal solution. Furthermore, the TOPSIS method presumes crisp data while fuzziness is inherent in decision data and decision making processes, so that fuzzy ratings using linguistic variables are better suited for assessing decision alternatives. In this paper, a new FC method for MADM under fuzzy environments is developed by introducing a multi-attribute ranking index based on the particular measure of closeness to the ideal solution, which is developed from the fuzzy weighted Minkowski distance used as an aggregating function in a compromise programming method. The FC method of compromise ranking determines a compromise solution, providing a maximum "group utility" for the "majority" and a minimum individual regret for the "opponent". A real example of a personnel selection problem is examined to demonstrate the implementation process of the method proposed in this paper.

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1 Introduction

Multi-attribute decision making (MADM) problems are wide spread in real life decision situations (Deng et al. 2000; Lai et al. 1994; Li and Yang 2004; Li 2005a, Li 2005b; Opricovic and Tzeng 2004). A MADM problem is to find a best compromise solution among all feasible alternatives assessed on the basis of multiple attributes, both quantitative and qualitative. Such problems can be dealt with using several existing methods such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) developed by Hwang and Yoon (1981), which is one of the well-known MADM methods. In the TOPSIS method, the decision data are given as crisp values a priori. The basic principle of the TOPSIS method is that the chosen alternative should have the shortest distance from the ideal solution and the farthest distance from the negative, i.e., least, ideal solution. However, while the highest ranked alternative by the TOPSIS method is the best in terms of the ranking index, this does not mean that it is always the closest to the ideal solution (Sect. 4.1).

In addition, in many situations, crisp data are inadequate or insufficient to model real-life decision problems (Chen 2000; Chen and Lu 2001; Chen and Hwang 1992). Indeed, human judgments are vague or fuzzy in nature and thus it may not be appropriate to represent them by precise numerical values. A more realistic approach could be to use linguistic variables to model human judgments (Carlsson and Fuller 2000). In this paper, a new fuzzy closeness methodology for solving MADM problems under fuzzy environments is proposed. In this methodology, linguistic variables are used to capture fuzziness in decision information and decision making processes by means of a fuzzy decision matrix. A multi-attribute ranking index is introduced based on a particular measure of closeness to the ideal solution which is developed from the fuzzy weighted Minkowski distance between triangular fuzzy numbers used as an aggregating function in a compromise programming method. The fuzzy closeness method of compromise ranking determines a compromise solution, providing a maximum "group utility" for the "majority" and a minimum of an individual regret for the "opponent". The implementation process of the fuzzy closeness method proposed in this paper is illustrated with a real example of a personnel selection problem.

2 The TOPSIS method

Suppose there is one decision maker who has to choose one of (or to rank) n feasible alternatives x_j $(j = 1, 2, \dots, n)$ on the basis of the attribute set $Z = \{z_1, z_2, \dots, z_m\}$. Denote the alternative set by $X = \{x_1, x_2, \dots, x_n\}$. In general, Z can be divided into Z^1 and Z^2, where Z^1 is the subset of benefit attributes and Z^2 is the subset of cost attributes. z_{ij} is the value of alternative x_j with respect to attribute z_i, i.e., $z_{ij} = z_i(x_j)$. Denote the decision matrix by $\mathbf{Z} = (z_{ij})_{m\times n}$.

The procedure of the TOPSIS method is as follows.

Firstly, calculate the weighted normalized value v_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ of r_{ij} as follows

$$v_{ij} = \omega_i r_{ij} \tag{1}$$

where the normalized value r_{ij} is calculated as follows

$$r_{ij} = \frac{z_{ij}}{\sqrt{\sum_{j=1}^{n} z_{ij}^2}} \tag{2}$$

and ω_i is the relative weight of attribute $z_i \in Z$, where $\omega_i \geq 0\,(i = 1, 2, \ldots, m)$ and $\sum_{i=1}^{m} \omega_i = 1$.

Then, determine the ideal solution x^{*+} and the negative ideal solution x^{*-}, whose weighted normalized value vectors are denoted by $\boldsymbol{v}^{*+} = (v_1^{*+}, v_2^{*+}, \ldots, v_m^{*+})^{\mathrm{T}}$ and $\boldsymbol{v}^{*-} = (v_1^{*-}, v_2^{*-}, \ldots, v_m^{*-})^{\mathrm{T}}$, respectively, where

$$v_i^{*+} = \begin{cases} \max\limits_{1 \leq j \leq n} \{v_{ij}\} & (z_i \in Z^1) \\ \min\limits_{1 \leq j \leq n} \{v_{ij}\} & (z_i \in Z^2) \end{cases} \tag{3}$$

and

$$v_i^{*-} = \begin{cases} \min\limits_{1 \leq j \leq n} \{v_{ij}\} & (z_i \in Z^1) \\ \max\limits_{1 \leq j \leq n} \{v_{ij}\} & (z_i \in Z^2) \end{cases} \tag{4}$$

Finally, calculate the closeness coefficient of each alternative x_j with respect to the ideal solution x^{*+} as follows

$$C^*(x_j) = \frac{D^{*-}(x_j)}{D^{*+}(x_j) + D^{*-}(x_j)} \tag{5}$$

where the separation of alternative x_j from x^{*+} is given as

$$D^{*+}(x_j) = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_i^{*+})^2} \tag{6}$$

and the separation of x_j from x^{*-} is given as

$$D^{*-}(x_j) = \sqrt{\sum_{i=1}^{m} (v_{ij} - v_i^{*-})^2} \tag{7}$$

The alternatives in set X are ranked in decreasing order of $C^*(x_j)\,(j = 1, 2, \ldots, n)$ with the alternative corresponding to the largest $C^*(x_j)$ being regarded as the best.

3 Fuzzy closeness method

Before describing the method itself, we first introduce some important concepts which are used in the rest of the paper.

3.1 Triangular fuzzy numbers

For the sake of simplicity and without loss of generality, throughout the paper we assume that any fuzzy number is a triangular fuzzy number (TFN).

Let $\tilde{m} = (l, m, r)$ be a TFN, where the membership function $\mu_{\tilde{m}}$ of $\tilde{m}$ is given by

$$\mu_{\tilde{m}}(x) = \begin{cases} \dfrac{x-l}{m-l} & (l \leq x \leq m) \\ \dfrac{r-x}{r-m} & (m \leq x \leq r) \end{cases}$$

Obviously, a TFN $\tilde{m} = (l, m, r)$ is reduced to a real number m if $l = m = r$. Conversely, m can be written as a TFN $\tilde{m} = (m, m, m)$.

3.2 Linguistic variable

A linguistic variable (LV) is a variable whose values are linguistic terms (Chen 2000; Chen and Lu 2001; Cheng 1998; Choobineh and Huishen 1993).

The concept of LV is very useful in situations where decision problems are too complex or too ill-defined to be described properly using conventional quantitative expressions. For example, the performance ratings of alternatives on qualitative attributes could be expressed using a LV which has linguistic values such as "low", "high" and "very high". Such linguistic values can be represented using positive TFNs. For example, "high" can be represented by a positive TFN (0.3, 0.4, 0.5).

3.3 Distance between two TFNs

Let $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ be two TFNs. Then the vertex method is defined to calculate the distance between them as follows (Chen 2000; Li and Yang 2004)

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{\sum_{k=1}^{3} (m_k - n_k)^2}{3}} \tag{8}$$

It is easily seen that the distance measurement $d(\tilde{m}, \tilde{n})$ is identical to the Euclidean distance if both $\tilde{m}$ and $\tilde{n}$ are real numbers and $d(\tilde{m}, \tilde{n}) = 0$ if and only if $\tilde{m}$ and $\tilde{n}$ are identical.

3.4 The normalization method

Suppose the rating of the alternative x_j on attribute z_i given by the decision maker is a TFN $\tilde{z}_{ij} = (a_{ij}, b_{ij}, c_{ij})$, and suppose $\tilde{\omega}_i$ is the relative weight of attribute z_i, where $\tilde{\omega}_i = (\omega_i^l, \omega_i, \omega_i^r)$ is a TFN. A fuzzy multi-attribute decision making (FMADM) problem can be concisely expressed in matrix format as follows

$$\tilde{\boldsymbol{Y}} = (\tilde{z}_{ij})_{m\times n}$$

which is referred to as a fuzzy decision matrix.

To ensure compatibility between the fuzzy (or non-fuzzy) evaluation values (or ratings) of all attributes, the fuzzy (or non-fuzzy) evaluation values of attributes must be converted into a compatible scale (or into dimensionless indices). The normalization formula is chosen as follows

$$\tilde{r}_{ij} = \begin{cases} \left(\dfrac{a_{ij}-a_i^{\min}}{c_i^{\max}-a_i^{\min}}, \dfrac{b_{ij}-a_i^{\min}}{c_i^{\max}-a_i^{\min}}, \dfrac{c_{ij}-a_i^{\min}}{c_i^{\max}-a_i^{\min}}\right)^{p_i} & (z_i \in Z^1) \\ \left(\dfrac{c_i^{\max}-c_{ij}}{c_i^{\max}-a_i^{\min}}, \dfrac{c_i^{\max}-b_{ij}}{c_i^{\max}-a_i^{\min}}, \dfrac{c_i^{\max}-a_{ij}}{c_i^{\max}-a_i^{\min}}\right)^{p_i} & (z_i \in Z^2) \end{cases} \tag{9}$$

where $c_i^{\max} = \max\limits_{1\le j\le n}\{c_{ij}\,|\tilde{z}_{ij} = (a_{ij}\ , b_{ij}, c_{ij})\}$, $a_i^{\min} = \min\limits_{1\le j\le n}\{a_{ij}\,|\tilde{z}_{ij} = (a_{ij}\ , b_{ij}, c_{ij})\}$, and $p_i > 0$ is a distance parameter. Denote $\tilde{r}_{ij}$ by $\tilde{r}_{ij} = (r_{ij}^l, r_{ij}, r_{ij}^r)$. The transform function in Eq. 9 ensures that the range of a normalized TFN $\tilde{r}_{ij}$ belongs to the interval [0, 1]. Hence, $\tilde{\boldsymbol{Y}}$ is transformed into the weighted normalized fuzzy decision matrix $\tilde{\boldsymbol{V}} = (\tilde{v}_{ij})_{m\times n}$, where

$$\tilde{v}_{ij} = (\omega_i^l r_{ij}^l, \omega_i r_{ij}, \omega_i^r r_{ij}^r) \tag{10}$$

is a TFN, denoted by $\tilde{v}_{ij} = (v_{ij}^l, v_{ij}, v_{ij}^r)$.

3.5 Principle and procedure

The fuzzy closeness method determines a compromise ranking-list and the compromise solution for the decision maker. It introduces the multi-attribute ranking index based on the particular measure of closeness to an ideal solution x^{c+}. The compromise ranking is performed by comparing alternatives based on the measure of closeness to x^{c+}. The compromise ranking algorithm of the fuzzy closeness method can be summarized as follows:

Step 1: Determine the ideal solution x^{c+}, whose attribute vector $\tilde{\boldsymbol{v}}^{c+} = (\tilde{v}_1^{c+}, \tilde{v}_2^{c+}, \ldots, \tilde{v}_m^{c+})^{\mathrm{T}}$ is defined as follows

$$\tilde{v}_i^{c+} = (v_i^+, v_i^+, v_i^+)(i = 1, 2, \ldots, m)$$

where

$$v_i^+ = \max_{1\le j\le n}\{v_{ij}^r | \tilde{v}_{ij} = (v_{ij}^l, v_{ij}, v_{ij}^r)\}$$

Step 2: Compute $S(x_j)$ and $R(x_j)$ for each alternative x_j using the relations

$$S(x_j) = \sum_{i=1}^{m} d(\tilde{v}_{ij}, \tilde{v}_i^{c+}), \quad R(x_j) = \max_{1\le i\le m}\{d(\tilde{v}_{ij}, \tilde{v}_i^{c+})\} \tag{11}$$

(where $d(\tilde{v}_{ij}, \tilde{v}_i^{c+})$ is the distance between two TFNs $\tilde{v}_{ij}$ and $\tilde{v}_i^{c+}$) using Eq. 8.

Step 3: Compute $Q(x_j)$ for each alternative x_j by the relation

$$Q(x_j) = \lambda\frac{S(x_j) - S^*}{S^- - S^*} + (1-\lambda)\frac{R(x_j) - R^*}{R^- - R^*} \tag{12}$$

where

$$S^* = \min_{1\le j\le n}\{S(x_j)\}, \quad S^- = \max_{1\le j\le n}\{S(x_j)\} \tag{13}$$

and

$$R^* = \min_{1\le j\le n}\{R(x_j)\}, \quad R^- = \max_{1\le j\le n}\{R(x_j)\} \tag{14}$$

and $\lambda \in [0, 1]$ is introduced to reflect the importance of the decision making strategy of "the majority of attributes," or "the maximum group utility".

Step 4: Rank the alternatives $x_j \in X$, sorting by $S(x_j)$, $R(x_j)$ and $Q(x_j)$ in increasing order to yield three ranked lists.

Step 5: Propose as the compromise solution the alternative $x' \in X$ which is ranked the best by $Q(x_j)$ (where $Q(x') = \min_{1\le j\le n}\{Q(x_j)\}$) if the following two conditions, (a) and (b), are satisfied

(a) "Acceptable advantage":

$$Q(x'') - Q(x') \ge \frac{1}{n-1}$$

where $x'' \in X$ is the alternative in the second position in the list ranked by $Q(x_j)$.

(b) "Acceptable stability in decision making": x' must also be the best ranked by $S(x_j)$ or/and $R(x_j)$. This compromise solution is stable within a decision making process, which could be: "voting by majority rule" (when $\lambda > 0.5$), or "by consensus" (when $\lambda \approx 0.5$), or "with veto" (when $\lambda < 0.5$).
If one of the conditions (c) and (d) is not satisfied, then a set of compromise solutions is proposed, which consists of:

(c) Alternatives x' and x'' if only the condition (b) is not satisfied, or

(d) Alternatives $x', x'', \ldots, x^N$ if the condition (a) is not satisfied where N is the largest j such that $Q(x^j) - Q(x') < 1/(n-1)$, which means that the positions of these alternatives are "in closeness".

The best alternative, ranked by $Q(x_j)$, is the one with the minimum value of $Q(x_j)$. The main ranking result is the compromise ranked list of alternatives, and the compromise solution with the "advantage rate".

The preference stability of the obtained compromise solution may be analyzed using the fuzzy closeness method. Ranking by the fuzzy closeness method may be performed with different ratings of attribute weights, analyzing the impact of attribute weights on the proposed compromise solution. The fuzzy closeness method may be used to determine weight stability intervals. The compromise solution obtained with initial weights $\tilde{\omega}_i$ $(i = 1, 2, \ldots, m)$ will be replaced if the rating of a weight is not within the stability interval. The analysis of the weight stability interval for a single attribute is performed for all attribute functions, with the same (given) initial values of weights.

The fuzzy closeness method is a helpful tool in MADM, particularly in situations where the decision maker is not able, or does not know how to express his/her preference at the beginning of the system design process The obtained compromise solution could be accepted by the decision maker because it provides a maximum "group utility" (represented by $\min\limits_{1\le j\le n} \{S(x_j)\}$ in Eq. 11) of the "majority", and a minimum of the individual regret (represented by $\min\limits_{1\le j\le n} \{R(x_j)\}$ in Eq. 11) of the "opponent". The compromise solutions could be the basis for negotiations, involving the decision maker's preference by attribute weights.

4 Comparative analysis between the fuzzy closeness method and the TOPSIS

The fuzzy closeness method and the TOPSIS method (Hwang and Yoon 1981) as well as the fuzzy extension of the TOPSIS method (Chen 2000) are based on an aggregating function representing closeness to the reference point(s) such as the ideal solution and/or the negative ideal solution. Our comparative analysis points out that both methods introduce different forms of aggregating function (L_p-metric) for ranking of alternatives, where $p \geq 1$ is an arbitrary distance parameter. The fuzzy closeness method introduces functions $Q(x_j)$ $(j = 1, 2, \ldots, n)$ of L_1 and L_∞. The TOPSIS method introduces functions $C^*(x_j)$ $(j = 1, 2, \ldots, n)$ of L_2. Both methods use different kinds of normalization methods to eliminate the units of attribute functions. The fuzzy closeness method uses the p_i-power normalization in Eq. 9, whereas the TOPSIS method uses the vector normalization in Eq. 2.

4.1 Aggregating function

The fuzzy closeness method is based on the following aggregating function derived from the L_p-metric:

$$L_p(x_j) = \sqrt[p]{\sum_{i:z_i \in Z^1} \left(\omega_i \frac{z_i^{b*} - z_{ij}}{z_i^{b*} - z_i^{b-}} \right)^p + \sum_{i:z_i \in Z^2} \left(\omega_i \frac{z_i^{c*} - z_{ij}}{z_i^{c*} - z_i^{c-}} \right)^p}$$

where z_i^{b*} (or z_i^{c*}) and z_i^{b-} (or z_i^{c-}) are the best value and the worst value for each attribute $z_i (i = 1, 2, \ldots, m)$, respectively. The measure $L_p(x_j)$ represents the distance of the alternative x_j from the ideal solution x^{c+}. In the fuzzy closeness method, $S(x_j)$ and $R(x_j)$ in Eq. 11 are introduced as "boundary measures" for each alternative $x_j (j = 1, 2, \ldots, n)$. The solution obtained by $S^* = \min_{1 \le j \le n} \{S(x_j)\}$ is associated with a maximum "group utility" or "majority" rule. The solution obtained by $R^* = \min_{1 \le j \le n} \{R(x_j)\}$ is associated with a minimum individual regret of an "opponent". According to Eq. 12, the fuzzy closeness method's decision result applies only for the given set of alternatives. Inclusion (or exclusion) of another alternative could affect the fuzzy closeness method's ranking of current alternatives. By fixing the best values z_i^{b*} (or z_i^{c*}) and the worst values z_i^{b-} (or z_i^{c-}) for all attributes $z_i (i = 1, 2, \ldots, m)$, this effect could be avoided, but this would require that the decision maker be able to define a fixed ideal solution.

The TOPSIS method uses the aggregating function for ranking in Eq. 5. According to the formulation of the ranking index $C^*(x_j)$ $(j = 1, 2, \ldots, n)$, an alternative x_j is better than x_k if $C^*(x_j) > C^*(x_k)$, i. e.,

$$\frac{D^{*-}(x_j)}{D^{*+}(x_j) + D^{*-}(x_j)} > \frac{D^{*-}(x_k)}{D^{*+}(x_k) + D^{*-}(x_k)}$$

which will hold if one of the following conditions (A) and (B) is satisfied.

(A) $D^{*+}(x_j) < D^{*+}(x_k)$ and $D^{*-}(x_j) > D^{*-}(x_k)$
(B) $D^{*+}(x_j) > D^{*+}(x_k)$ and $D^{*-}(x_j) > D^{*-}(x_k)$, but $D^{*+}(x_j) < D^{*+}(x_k) D^{*-}(x_j) / D^{*-}(x_k)$.

Condition (A) corresponds to the "regular" situation, when an alternative x_j is better than x_k because it is closer to the ideal solution x^{*+} and farther from the negative ideal solution x^{*-}. In contrast, condition (B) allows an alternative x_j to be better than x_k even though x_j is farther from the ideal solution x^{*+} than x_k. For example, let x_k be the alternative with $D^{*+}(x_k) = D^{*-}(x_k)$ and $C^*(x_k) = 0.5$. In this case, all alternatives x_j with $D^{*+}(x_j) > D^{*+}(x_k)$ and $D^{*+}(x_j) < D^{*-}(x_j)$ are better ranked than x_k, although x_k is closer to the ideal solution x^{*+}.

4.2 Normalization effects

Normalization is used to eliminate the units of attribute functions so that values of all the attribute functions are dimensionless. The same attribute function could be evaluated in different units. These "evaluation units" may be related as follows

$$\tilde{y}_{ij} = \alpha \tilde{z}_{ij} + \beta \tag{15}$$

where $\alpha > 0$ and β are constants. Denote the TFN $\tilde{y}_{ij}$ by $\tilde{y}_{ij} = (e_{ij}, f_{ij}, g_{ij})$, i.e.,

$$(e_{ij}, f_{ij}, g_{ij}) = (\alpha a_{ij} + \beta, \alpha b_{ij} + \beta, \alpha c_{ij} + \beta) \tag{16}$$

Does evaluation of the ith attribute function as $\tilde{y}_{ij} = \tilde{y}_i(x_j)$ or $\tilde{z}_{ij} = \tilde{z}_i(x_j)$ affect the decision result of the MADM method? The answer should be NO. However, as we will illustrate below, there are some normalization procedures with which the normalized value and thus the final MADM result can depend on the evaluation unit of the attribution function.

The fuzzy closeness method uses the normalized formula in Eq. 9, i.e.,

$$\tilde{r}_{ij}(\tilde{z}_{ij}) = \begin{cases} \left(\dfrac{a_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}}, \dfrac{b_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}}, \dfrac{c_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}} \right)^{p_i} & (z_i \in Z^1) \\ \left(\dfrac{c_i^{\max} - c_{ij}}{c_i^{\max} - a_i^{\min}}, \dfrac{c_i^{\max} - b_{ij}}{c_i^{\max} - a_i^{\min}}, \dfrac{c_i^{\max} - a_{ij}}{c_i^{\max} - a_i^{\min}} \right)^{p_i} & (z_i \in Z^2) \end{cases}$$

where $c_i^{\max} = \max\limits_{1 \le j \le n} \{c_{ij} \left| \tilde{z}_{ij} = (a_{ij}, b_{ij}, c_{ij}) \right.\}$ and $a_i^{\min} = \min\limits_{1 \le j \le n} \{a_{ij} \left| \tilde{z}_{ij} = (a_{ij}, b_{ij}, c_{ij}) \right.\}$.

The corresponding normalized value of $\tilde{y}_{ij}$ is calculated as follows

$$\tilde{r}_{ij}(\tilde{y}_{ij}) = \begin{cases} \left(\dfrac{e_{ij} - e_i^{\min}}{g_i^{\max} - e_i^{\min}}, \dfrac{f_{ij} - e_i^{\min}}{g_i^{\max} - e_i^{\min}}, \dfrac{g_{ij} - e_i^{\min}}{g_i^{\max} - e_i^{\min}} \right)^{p_i} & (z_i \in Z^1) \\ \left(\dfrac{g_i^{\max} - g_{ij}}{g_i^{\max} - e_i^{\min}}, \dfrac{g_i^{\max} - f_{ij}}{g_i^{\max} - e_i^{\min}}, \dfrac{g_i^{\max} - e_{ij}}{g_i^{\max} - e_i^{\min}} \right)^{p_i} & (z_i \in Z^2) \end{cases} \tag{17}$$

According to Eq. 16, since $\alpha > 0$ and β are constants, it follows that

$$\begin{aligned} g_i^{\max} &= \max_{1 \le j \le n} \{g_{ij} | \tilde{y}_{ij} = (e_{ij}, f_{ij}, g_{ij})\} \\ &= \max_{1 \le j \le n} \{\alpha c_{ij} + \beta | (\alpha a_{ij} + \beta, \alpha b_{ij} + \beta, \alpha c_{ij} + \beta)\} = \alpha c_i^{\max} + \beta \end{aligned}$$

and

$$\begin{aligned} e_i^{\min} &= \min_{1 \le j \le n} \{e_{ij} | \tilde{y}_{ij} = (e_{ij}, f_{ij}, g_{ij})\} \\ &= \min_{1 \le j \le n} \{\alpha a_{ij} + \beta | (\alpha a_{ij} + \beta, \alpha b_{ij} + \beta, \alpha c_{ij} + \beta)\} = \alpha a_i^{\min} + \beta \end{aligned}$$

Combining Eqs. 9, 16 and 17, it follows that

$$\tilde{r}_{ij}(\tilde{y}_{ij}) = \begin{cases} \left(\frac{\alpha a_{ij}+\beta-(\alpha a_i^{\min}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)}, \frac{\alpha b_{ij}+\beta-(\alpha a_i^{\min}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)}, \frac{\alpha c_{ij}+\beta-(\alpha a_i^{\min}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)} \right)^{p_i} & (z_i \in Z^1) \\ \left(\frac{\alpha c_i^{\max}+\beta-(\alpha c_{ij}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)}, \frac{\alpha c_i^{\max}+\beta-(\alpha b_{ij}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)}, \frac{\alpha c_i^{\max}+\beta-(\alpha a_{ij}+\beta)}{\alpha c_i^{\max}+\beta-(\alpha a_i^{\min}+\beta)} \right)^{p_i} & (z_i \in Z^2) \end{cases}$$

It is easy to see that

$$\tilde{r}_{ij}(\tilde{y}_{ij}) = \begin{cases} \left(\frac{a_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}}, \frac{b_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}}, \frac{c_{ij} - a_i^{\min}}{c_i^{\max} - a_i^{\min}} \right)^{p_i} & (z_i \in Z^1) \\ \left(\frac{c_i^{\max} - c_{ij}}{c_i^{\max} - a_i^{\min}}, \frac{c_i^{\max} - b_{ij}}{c_i^{\max} - a_i^{\min}}, \frac{c_i^{\max} - a_{ij}}{c_i^{\max} - a_i^{\min}} \right)^{p_i} & (z_i \in Z^2) \end{cases}$$

Therefore,

$$\tilde{r}_{ij}(\tilde{y}_{ij}) = \tilde{r}_{ij}(\tilde{z}_{ij})$$

This means that the normalized value in the fuzzy closeness method does not depend on the evaluation unit of an attribute function.

The normalized value of $\tilde{z}_{ij}$ in the TOPSIS method (Hwang and Yoon, 1981) is calculated as follows (Eq. 2) when it is a real number (i.e., $\tilde{z}_{ij} = z_{ij}$)

$$r_{ij}(z_{ij}) = \frac{z_{ij}}{\sqrt{\sum_{j=1}^{n} z_{ij}^2}}$$

whereas

$$r_{ij}(y_{ij}) = \frac{y_{ij}}{\sqrt{\sum_{j=1}^{n} y_{ij}^2}} = \frac{\alpha z_{ij} + \beta}{\sqrt{\sum_{i=1}^{m} (\alpha z_{ij} + \beta)^2}}$$

Obviously, the normalized value in the TOPSIS method could depend on the evaluation unit of the attribute function if $y_{ij} = \alpha z_{ij} + \beta$. The equality $r_{ij}(y_{ij}) = r_{ij}(z_{ij})$ holds only if $y_{ij} = \alpha z_{ij}$ or $\beta = 0$.

Linear normalization such as that in Eq. 9 with $p_i = 1$, was subsequently introduced into the TOPSIS method by Lai et al. (1994) as follows

$$r_{ij} = \begin{cases} \frac{z_{ij} - z_i^{\min}}{z_i^{\max} - z_i^{\min}} & (z_i \in Z^1) \\ \frac{z_i^{\max} - z_{ij}}{z_i^{\max} - z_i^{\min}} & (z_i \in Z^2) \end{cases} \tag{18}$$

where $z_i^{\max} = \max_{1 \le j \le n} \{z_{ij}\}$ and $z_i^{\min} = \min_{1 \le j \le n} \{z_{ij}\}$.

The normalized value defined by Eq. 18 in the TOPSIS method does not depend on the evaluation unit of the attribute function.

Chen (2000) proposed the following normalization formula:

$$\tilde{r}_{ijC}(\tilde{z}_{ij}) = \begin{cases} \left(\dfrac{a_{ij}}{c_i^{\max}}, \dfrac{b_{ij}}{c_i^{\max}}, \dfrac{c_{ij}}{c_i^{\max}}\right) & (z_i \in Z^1) \\ \left(\dfrac{a_i^{\min}}{c_{ij}}, \dfrac{a_i^{\min}}{b_{ij}}, \dfrac{a_i^{\min}}{a_{ij}}\right) & (z_i \in Z^2) \end{cases} \tag{19}$$

From Eqs. 16 and 19, it follows that

$$\tilde{r}_{ijC}(\tilde{y}_{ij}) = \begin{cases} \left(\dfrac{\alpha a_{ij} + \beta}{\alpha c_i^{\max} + \beta}, \dfrac{\alpha b_{ij} + \beta}{\alpha c_i^{\max} + \beta}, \dfrac{\alpha c_{ij} + \beta}{\alpha c_i^{\max} + \beta}\right) & (z_i \in Z^1) \\ \left(\dfrac{\alpha a_i^{\min} + \beta}{\alpha c_{ij} + \beta}, \dfrac{\alpha a_i^{\min} + \beta}{\alpha b_{ij} + \beta}, \dfrac{\alpha a_i^{\min} + \beta}{\alpha a_{ij} + \beta}\right) & (z_i \in Z^2) \end{cases}$$

Obviously, the normalized value in the fuzzy extension of the TOPSIS method (Chen 2000) could depend on the evaluation unit of an attribute function if $\tilde{y}_{ij} = \alpha \tilde{z}_{ij} + \beta$. The equality $\tilde{r}_{ijC}(\tilde{y}_{ij}) = \tilde{r}_{ijC}(\tilde{z}_{ij})$ holds only if $\tilde{y}_{ij} = \alpha \tilde{z}_{ij}$ or $\beta = 0$.

4.3 Essentials of the fuzzy closeness method and the TOPSIS

In order to clarify the differences between these two MADM methods, the main features of the fuzzy closeness method and the TOPSIS method are summarized in the following.

(1) *Procedural basis*. These two MADM methods assume that there exists a decision matrix obtained by the evaluation of each of the alternatives in terms of each attribute. Normalization is used to eliminate the units of attribute function values. An aggregating function is formulated and used as a ranking index.

(2) *Normalization*. The difference appears in the normalization procedure used within these two MADM methods. The fuzzy closeness method uses the p_i-power normalization in Eq. 9, and the normalized value does not depend on the evaluation unit of the attribute function. The TOPSIS method uses the vector normalization in Eq. 2, and the normalized value could be different for different evaluation unit of a particular attribute function. A later version of the TOPSIS method uses the linear normalization in Eq. 18. The fuzzy extension of the TOPSIS method (Chen 2000) uses the fuzzy normalization in Eq. 19, and the normalized value could differ depending on the evaluation units used for a particular attribute function.

(3) *Aggregation*. The main difference appears in the aggregation approaches. The fuzzy closeness method introduces an aggregating function representing the

Table 1 Index values $S(x_j)$ and $R(x_j)$

Index values	Alternatives			
	x_1	x_2	x_3	x_4
$S(x_j)$	0.3	0.4	0.1	0.5
$R(x_j)$	0.6	0.2	0.7	0.4

Table 2 Ranking index values $Q(x_j)$ and ranking order of alternatives

Parameter λ	Alternatives				Ranking order
	x_1	x_2	x_3	x_4	
0	0.8	0	1	0.4	$x_2 \succ x_4 \succ x_1 \succ x_3$
1/8	0.7625	0.09375	0.875	0.475	$x_2 \succ x_4 \succ x_1 \succ x_3$
1/4	0.725	0.1875	0.75	0.55	$x_2 \succ x_4 \succ x_1 \succ x_3$
1/2	0.65	0.375	0.5	0.7	$x_2 \succ x_3 \succ x_1 \succ x_4$
3/4	0.575	0.5625	0.25	0.85	$x_3 \succ x_2 \succ x_1 \succ x_4$
7/8	0.5375	0.6563	0.125	0.925	$x_3 \succ x_1 \succ x_2 \succ x_4$
1	0.5	0.75	0	1	$x_3 \succ x_1 \succ x_2 \succ x_4$

distance from the ideal solution. This ranking index is an aggregation of all attributes and a balance between total and individual satisfaction. However, the parameter $\lambda \in [0, 1]$ which is introduced to reflect the importance of the decision making strategy of "the maximum group utility" could affect the final decision result. For example, assume that there are four alternatives x_j ($j = 1, 2, 3, 4$). The corresponding values $S(x_j)$ and $R(x_j)$ ($j = 1, 2, 3, 4$) are listed in Table 1. For different parameter values $\lambda \in [0, 1]$, ranking index values $Q(x_j)(j = 1, 2, 3, 4)$ are computed in Table 2 using Eq. 12.

It is easily seen that the ranking order of the alternatives x_j ($j = 1, 2, 3, 4$) is heavy affected by the parameter λ. The TOPSIS method (Hwang and Yoon 1981) and the fuzzy extension of the TOPSIS method (Chen 2000) introduces the ranking index in Eq. 5, including the distances from the ideal solution and from the negative ideal solution. These distances in the TOPSIS method are simply summed. However, the reference point could be a major concern in decision making, and to be as close as possible to the ideal solution is the rationale of human choice. Being far away from the negative ideal solution could be a goal only in a particular situation. The relative importance of distances $D^{*+}(x_j)$ in Eq. 6 and $D^{*-}(x_j)$ in Eq. 7 was not considered in the TOPSIS method, although this could be a major concern in real life decision making. Lai et al. (1994) considered this issue by introducing the satisfactory level for both criteria of the shortest distance from the ideal solution and the farthest distance from the negative ideal solution, and concluding "The compromise solution will exist at the point where the satisfactory levels of both criteria are the same. In future studies, applying compensatory operators should be emphasized". Thus, the relative importance of these two distances $D^{*+}(x_j)$ and $D^{*-}(x_j)$ remained an open question. The TOPSIS method uses the m-dimensional Euclidean distance that by itself could represent some balance between total and individual satisfaction, but uses it in

Table 3 Linguistic variables for importance of attributes

Linguistic variables	Triangular fuzzy numbers
Very low (LV)	(0,0,0.1)
Low (L)	(0,0.1,0.3)
Medium low (ML)	(0.1,0.3,0.5)
Medium (M)	(0.3,0.5,0.7)
Medium high (MH)	(0.5,0.7,0.9)
High (H)	(0.7,0.9,1)
Very high (VH)	(0.9,1,1)

Table 4 Importance of the attributes and ratings of candidates given by the decision maker

Attributes	Importance	Candidates		
		x_1	x_2	x_3
z_1	ML	MG	G	VG
z_2	VH	G	P	MG
z_3	H	F	VG	P
z_4	VH	VG	LP	F
z_5	M	F	G	G

a method very different that of the fuzzy closeness method, which employs the "advantage weight" $\lambda \in [0, 1]$, introduced in Eq. 12. However, λ could heavily affect the final decision result mentioned in the above.

(4) *Solution*. These two MADM methods each provide a ranked list. The highest ranked alternative by the fuzzy closeness method is the closest to the ideal solution. The main ranking result is the compromise ranked list of alternatives, and the compromise solution with the "advantage rate". The highest ranked alternative by the TOPSIS method is the best in terms of the ranking index, which does not mean that it is always the closest to the ideal solution, as explained in Sect. 4.1.

5 Analysis of a real example

A modification of the real example of the personnel selection problem from Chen and Hwang (1992) is adopted in this section. Suppose that a software company desires to hire a system analysis engineer. After preliminary screening, three candidates (i.e., alternatives) x_1, x_2 and x_3 remain for further evaluation. Denote the candidate set by $X = \{x_1, x_2, x_3\}$. A decision maker (the manager) has to conduct the interview and to select the most suitable candidate. Five benefit attributes are considered, including emotional stability (z_1), oral communication skill (z_2), personality (z_3), past experience (z_4) and self-confidence (z_5). The decision maker uses the linguistic variables shown in Table 3 to assess the importance of the five attributes and present it in Table 4.

The decision maker uses the linguistic variables shown in Table 5 to evaluate the candidates with respect to each attribute. The results are presented in Table 4.

Table 5 Linguistic variables for the ratings of candidates with respect to attributes

Linguistic variables	Triangular fuzzy numbers
Very poor (LP)	(0,0,1)
Poor (P)	(0,1,3)
Medium poor (MP)	(1,3,5)
Fair (F)	(3,5,7)
Medium good (MG)	(5,7,9)
Good (G)	(7,9,10)
Very good (VG)	(9,10,10)

Table 6 The fuzzy decision values of candidates and fuzzy weights of attributes

Attributes	Candidates			Weight
	x_1	x_2	x_3	
z_1	(5,7,9)	(7,9,10)	(9,10,10)	(0.1,0.3,0.5)
z_2	(7,9,10)	(0,1,3)	(5,7,9)	(0.9,1,1)
z_3	(3,5,7)	(9,10,10)	(0,1,3)	(0.7,0.9,1)
z_4	(9,10,10)	(0,0,1)	(3,5,7)	(0.9,1,1)
z_5	(3,5,7)	(7,9,10)	(7,9,10)	(0.3,0.5,0.7)

Table 7 The normalized fuzzy decision values of candidates

Attributes	Candidates		
	x_1	x_2	x_3
z_1	(0,0.4,0.8)	(0.4,0.8,1)	(0.8,1,1)
z_2	(0.7,0.9,1)	(0,0.1,0.3)	(0.5,0.7,0.9)
z_3	(0.3,0.5,0.7)	(0.9,1,1)	(0,0.1,0.3)
z_4	(0.9,1,1)	(0,0,0.1)	(0.3,0.5,0.7)
z_5	(0,0.29,0.57)	(0.57,0.86,1)	(0.57,0.86,1)

Converting the linguistic evaluation shown in Table 4 into triangular fuzzy numbers to construct the fuzzy decision values of candidates with respect to each attribute and determine the fuzzy weight of attributes we obtain the fuzzy values in Table 6.

Using Eq. 9, where $p_i = 1$ for $i = 1, 2, \ldots, 5$, the fuzzy decision values in Table 6 are transformed into the normalized fuzzy decision values as Table 7.

Using Eq. 10, the normalized fuzzy decision values in Table 7 are transformed into the weighted normalized fuzzy decision values as Table 8.

The attribute vector of the ideal solution x^{c+} is $\tilde{\boldsymbol{v}}^{c+} = (0.5, 1, 1, 1, 0.7)^{\mathrm{T}}$. Using Eqs. 11–14, the decision results are obtained as shown in the bottom portion of Table 9.

Chen (2000) extended the TOPSIS method to the fuzzy group multi-attribute decision making under fuzzy environments. The distances and closeness coefficient used

Table 8 The weighted normalized fuzzy decision values of candidates

Attributes	Candidates		
	x_1	x_2	x_3
z_1	(0,0.12,0.4)	(0.04,0.24,0.5)	(0.08,0.3,0.5)
z_2	(0.63,0.9,1)	(0,0.1,0.3)	(0.45,0.7,0.9)
z_3	(0.21,0.45,0.7)	(0.63,0.9,1)	(0,0.09,0.3)
z_4	(0.81,1,1)	(0,0,0.1)	(0.27,0.5,0.7)
z_5	(0,0.15,0.4)	(0.17,0.43,0.7)	(0.17,0.43,0.7)

Table 9 Decision results obtained by the fuzzy closeness method and the fuzzy TOPSIS

Candidates		x_1	x_2	x_3	Ranking order
The fuzzy extension of the TOPSIS (Chen 2000)	$d(x_j, x^-)$	2.702	1.904	2.234	$x_1 \succ x_3 \succ x_2$
	$d(x_j, x^+)$	2.589	3.436	2.902	$x_1 \succ x_3 \succ x_2$
	$C(x_j)$	0.511	0.357	0.435	$x_1 \succ x_3 \succ x_2$
The fuzzy closeness method	$S(x_j)$	2.163	2.713	2.194	$x_1 \succ x_3 \succ x_2$
	$R(x_j)$	0.883	0.968	0.879	$x_3 \succ x_1 \succ x_2$
	$Q(x_j)$	$0.045(1-\lambda)$	1	0.056λ	$x_3 \succ x_1 \succ x_2$ $(0 \le \lambda < 45/101)$
					$x_3 \sim x_1 \succ x_2$ $(\lambda = 45/101)$
					$x_1 \succ x_3 \succ x_2$ $(45/101 < \lambda \le 1)$

are defined as follows

$$d(x_j, x^+) = \sum_{i=1}^{m} d(\tilde{v}_{ij}, \tilde{a}_i^+) = \sum_{i=1}^{m} \sqrt{\frac{(1 - v_{ij}^l)^2 + (1 - v_{ij})^2 + (1 - v_{ij}^r)^2}{3}} \tag{20}$$

$$d(x_j, x^-) = \sum_{i=1}^{m} d(\tilde{v}_{ij}, \tilde{a}_i^-) = \sum_{i=1}^{m} \sqrt{\frac{(v_{ij}^l)^2 + (v_{ij})^2 + (v_{ij}^r)^2}{3}} \tag{21}$$

and

$$C(x_j) = \frac{d(x_j, x^-)}{d(x_j, x^+) + d(x_j, x^-)} \tag{22}$$

where x^+ and x^- are the fuzzy positive ideal solution and the fuzzy negative ideal solution, whose weighted normalized fuzzy vectors are $\tilde{\boldsymbol{a}}^+ = (1, 1, 1, 1, 1)^{\mathrm{T}}$ and $\tilde{\boldsymbol{a}}^- = (0, 0, 0, 0, 0)^{\mathrm{T}}$, respectively.

Using Eqs. 20–22 and Table 8, the decision results shown in the top portion of Table 9 are obtained.

It is easily seen from Table 9 that the decision results by the fuzzy closeness method and the fuzzy extension of the TOPSIS method (Chen 2000) are different. Using the fuzzy extension of the TOPSIS method, the best candidate (alternative) is x_1 and the ranked order of all candidates is $x_1 \succ x_3 \succ x_2$. Using the fuzzy closeness method, the best candidate is x_3 and the ranked order of all candidates is $x_3 \succ x_1 \succ x_2$ if $0 \le \lambda < 45/101$; the best candidate is x_1 and the ranked order of all candidates is $x_1 \succ x_3 \succ x_2$ if $45/101 < \lambda \le 1$; the best candidate is x_1 and x_3, and the ranked

Table 10 The normalized fuzzy decision values of candidates

Attributes	Candidates		
	x_1	x_2	x_3
z_1	(0.5,0.7,0.9)	(0.7,0.9,1)	(0.9,1,1)
z_2	(0.7,0.9,1)	(0,0.1,0.3)	(0.5,0.7,0.9)
z_3	(0.3,0.5,0.7)	(0.9,1,1)	(0,0.1,0.3)
z_4	(0.9,1,1)	(0,0,0.1)	(0.3,0.5,0.7)
z_5	(0.3,0.5,0.7)	(0.7,0.9,1)	(0.7,0.9,1)

Table 11 The weighted normalized fuzzy decision values of candidates

Attributes	Candidates		
	x_1	x_2	x_3
z_1	(0.05,0.21,0.45)	(0.07,0.27,0.5)	(0.09,0.3,0.5)
z_2	(0.63,0.9,1)	(0,0.1,0.3)	(0.45,0.7,0.9)
z_3	(0.21,0.45,0.7)	(0.63,0.9,1)	(0,0.09,0.3)
z_4	(0.81,1,1)	(0,0,0.1)	(0.27,0.5,0.7)
z_5	(0.09,0.25,0.49)	(0.21,0.45,0.7)	(0.21,0.45,0.7)

Table 12 Decision results obtained by the fuzzy closeness method and the fuzzy TOPSIS

Candidates	x_1	x_2	x_3	Ranking order	
The fuzzy extension of the TOPSIS (Chen 2000)	$d(x_j, x^-)$	2.905	1.926	2.278	$x_1 \succ x_3 \succ x_2$
	$d(x_j, x^+)$	2.436	3.314	3.089	$x_1 \succ x_3 \succ x_2$
	$C(x_j)$	0.544	0.368	0.424	$x_1 \succ x_3 \succ x_2$
The fuzzy closeness method	$S(x_j)$	1.965	2.665	2.365	$x_1 \succ x_3 \succ x_2$
	$R(x_j)$	0.742	0.968	0.879	$x_1 \succ x_3 \succ x_2$
	$Q(x_j)$	0	1	$0.606 - 0.035\lambda$	$x_1 \succ x_3 \succ x_2$

order of all candidates is $x_1 \sim x_3 \succ x_2$ if $\lambda = 45/101$. Therefore, the decision results obtained by the fuzzy closeness method depend on the weight λ.

The decision results in Table 9 are obtained by using the normalization in Eq. 9. To compare the effect on the final decision results using different normalization methods, decision data in Table 6 are normalized using Eq. 19 and the decision results are computed by both the fuzzy closeness method and the fuzzy extension of the TOPSIS method.

Using Eq. 19, the fuzzy decision values in Table 6 can be transformed into the normalized fuzzy decision matrix as Table 10.

Using Eq. 10, the normalized fuzzy decision values in Table 10 can be transformed into the weighted normalized fuzzy decision values as Table 11.

The attribute vector of the ideal solution x^{c+} is $\tilde{\boldsymbol{v}}^{c+} = (0.5, 1, 1, 1, 0.7)^{\mathrm{T}}$. Using Eqs. 11–14 (i.e., the fuzzy closeness method) and Table 11, the final decision results are obtained as shown in the bottom portion of Table 12.

According to the fuzzy extension of the TOPSIS method (Chen 2000) and Table 11, the final decision results are obtained as shown in the top portion of Table 12.

As can bee seen, using the normalization in Eq. 19, the decision results by the fuzzy closeness method and the fuzzy extension of the TOPSIS method are the same, i. e., the best candidate is x_1 and the ranking order of all alternatives is $x_1 \succ x_3 \succ x_2$. However, as shown above, using the normalization in Eq. 9, the final decision results in Table 9 obtained by the fuzzy closeness method and the fuzzy extension of the TOPSIS method are different. Therefore, the final decision results are affected by different normalization methods.

6 Concluding remarks

Most MADM problems include both quantitative and qualitative attributes which are often assessed using imprecise data and human judgment. Fuzzy set theory is well suited to deal with such decision problems. In this paper, we find that while the highest ranked alternative obtained by the TOPSIS method (Hwang and Yoon 1981) is the best in terms of the ranking index, this does not mean that it is always the closest to the ideal solution. The fuzzy closeness method is developed to solve FMADM problems. Linguistic variables as well as crisp numerical values are used to assess qualitative and quantitative attributes. The fuzzy closeness method proposed in this paper can be used to generate a consistent and reliable ranked ordering of alternatives as is illustrated with a real example of a personnel selection problem. A comparative analysis of the fuzzy closeness method and the TOPSIS method as well as the fuzzy extension of the TOPSIS method (Chen 2000) are made. Furthermore, different normalization methods may result in different final decision results for a given instance of the personnel selection problem. Our work should be applicable to decision problems in many areas, especially in situations where multiple decision makers are involved and the weights of attributes are not crisp.

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