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Hamilton, Hamiltonian Mechanics, and Causation

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Abstract

I show how Sir William Rowan Hamilton's philosophical commitments led him to a causal interpretation of classical mechanics. I argue that Hamilton's metaphysics of causation was injected into his dynamics by way of a causal interpretation of force. I then detail how forces are indispensable to both Hamilton's formulation of classical mechanics and what we now call Hamiltonian mechanics (i.e., the modern formulation). On this point, my efforts primarily consist of showing that the contemporary orthodox interpretation of potential energy is the interpretation found in Hamilton's work. Hamilton called the potential energy function the "force-function" because he believed that it represents forces at work in the world. Various non-historical arguments for this orthodox interpretation of potential energy are provided, and matters are concluded by showing that in classical Hamiltonian mechanics, facts about the potential energies of systems are grounded in facts about forces. Thus, if one can tolerate the view that forces are causes of motion, then Hamilton provides one with a road map for transporting causation into one of the most mathematically sophisticated formulations of classical mechanics, viz., Hamiltonian mechanics.

Keywords Hamilton · Hamiltonian mechanics · Causation · Potential energy · Force

> "Now how in general anything can be altered, how it is possible that upon a state in one point of time an opposite one could follow in the next—of these we have *a priori* not the least concept. For this acquaintance with actual forces is required, which can only be given empirically, e.g., acquaintance with moving forces, or, what comes to the same thing, with certain successive appearances (as motions) which indicate such forces…Now every alteration has a cause, which manifests its causality in the entire time during which the alteration proceeds…All alteration is therefore possible only through a continuous action of causality…"[1]
>
> Immanuel Kant (1724–1804)

[1] (Kant 1998, 314–315 B253-254).

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"…Very glad you have the Kant from Coleridge for me: try and send it soon. I have read a large part of the *Critique of Pure Reason*, and find it wonderfully clear, and generally quite convincing. Notwithstanding some previous preparation from Berkeley, and from my own thoughts, I seem to have learned much from Kant's own statement of his views of Space and Time. Yet, on the whole, a large part of my pleasure consists in recognizing, through Kant's works, opinions, or rather views, which have been long familiar to myself, although far more clearly and systematically expressed and combined by him."[2]
Sir William Rowan Hamilton (1805–1865)

"All mechanicians agree that reaction is equal and opposite to action, both when one body presses another, and when one body communicates motion to another. All reasoners join in the assertion, not only that every observed change of motion has had a cause, but that every change of motion must have a cause. Here we have certain portions of substantial and undoubted knowledge…We have, in the Mechanical Sciences, certain universal and necessary truths on the subject of causes…Axioms concerning Cause, or concerning Force, which as we shall see, is a modification of Cause, will flow from an Idea of Cause, just as axioms concerning space and number flow from the ideas of space and number or time. And thus the propositions which constitute the science of Mechanics prove that we possess an idea of cause, in the same sense in which the propositions of geometry and arithmetic prove our possession of the ideas of space and of time or number…the relation of cause and effect is a condition of our apprehending successive events, a part of the mind's constant and universal activity, a source of necessary truths; or, to sum all this in one phrase, a Fundamental Idea."[3]
William Whewell (1794–1866)

"…in Whewell at Cambridge, I thought with delight that I perceived a philosophical spirit more deep and true than I had dared to hope for."[4] "…Whewell has come round almost entirely to my views about the laws of Motion."[5]
Sir William Rowan Hamilton

1 Introduction

Sir William Rowan Hamilton affirmed that every dynamical evolution is a causal evolution. His causal dynamics may seem out of place to the contemporary inquirer into the foundations of physics because most everyone in that subdiscipline now agrees with David Papineau's remark that "there is strong reason to doubt that causation is constituted by basic dynamical processes".[6] These same thinkers also typically agree with Michael Redhead's comment that "physicists long ago gave up the notion of cause as being of any particular

[2] From W.R. Hamilton to Viscount Adare, July 19th, 1834. (Graves 1885, 96).

[3] (Whewell 1858, 182–183).

[4] From W.R. Hamilton to Aubrey De Vere, Observatory, May 7th, 1832. (Graves 1882, 554).

[5] From W.R. Hamilton to Lord Adare dated April, 1834. (Graves 1885, 83).

[6] (Papineau 2013, 127). See (Earman 1976, 6); (Earman 2011, 494); (Field 2003, 435); (Ismael 2016, 134; *cf.* 113, 117, and 136); (Kutach 2013, 266, 272-273, 282 for "culpable causation", the causation of interest to metaphysicians); (Loewer 2007); (Russell 1912-1913); (Schaffer 2008, 92); (Sider 2011, 15-17); (Spohn 2006, for whom all causal laws ultimately reduce to mere regularities ibid., 116); (van Fraassen 1989, 282). See also (Norton 2007a; 2007b, 2021). Farr and Reutlinger (2013, 216) state that "[m]any philosophers of physics today support…[the] claim that causal relations do not belong to the ontology suggested by fundamental physics.".

interest! In physics, the explanatory laws are laws of functional dependence…"[7] Alongside this majority opinion is a somewhat prevalent and friendly (to the majority opinion) story (call it the prevalent story) about causation and the historical development of physics and its supporting mathematics, a story alluded to by Redhead. It says that as physics grew more and more mathematically sophisticated, less and less room remained for causal relations in the ontologies of our best physical theories. As Willard Van Orman Quine (1908–2000) argued,

> Now it is an ironical but familiar fact that though the business of science is describable in unscientific language as the discovery of causes, the notion of cause itself has no firm place in science. The disappearance of causal terminology from the jargon of one branch of science and another has seemed to mark the progress in understanding of the branches concerned.[8]

The highly respected philosopher of physics Jennan Ismael under a subsection entitled "Causation Lost" remarked,

> At first, the notion of cause retained its close connection with mechanical ideas. A cause was something that brought about its effect by a transfer of force. But when Newton published his Principia, causation didn't appear in the presentation of his theory at all. What he put in the place of causal relations were mathematical equations, usually expressed in the form of differential equations that give the rate of change of a quantity over time…[9]

In Ismael's summary of her position, we read:

Footnote 6 (continued)

Some resist the majority opinion. See the powers, capacities, and dispositions group: (Bird 2007, 168; Ellis 2002, 23-24, 159; Mumford 2004, 150, 188). *Cf.* (Ney 2009, 759-761); (Kistler 2013); and (Bartels 1996) *inter alios*.

Some members of the group that promotes the role of mechanism in natural philosophy hesitate to allow causation into fundamental physics: (Glennan 1996, 61, 64; 2010, 367); (Glennan 2017). On the other hand, some seem to encourage its insertion into physics: (Machamer et. al. 2000); (Andersen 2011).

Frisch (2005, 2007, 2009, 2014) gives one some reason to put causation in fundamental physics, but he admits his position is consistent with an instrumentalist attitude about causation there. Frisch's position also seems compatible with a strictly functionalist theory of causation as in (Woodward 2014).

I believe Hamilton provides us with key insights that recommend how to get causation into the ontology of modern Hamiltonian mechanics in a manner that extends beyond mere instrumentalist and/or functional viewpoints about microphysical causation, *if* (as Hamilton believed) *forces can be properly understood as causes of motions*. While I do not spend time arguing for the emphasized antecedent, it is a significant enough task to show how forces are essential to the ontology of Hamiltonian mechanics, for Hamiltonian mechanics is usually regarded as an energy-based theory in which forces play no essential role (*q.v.*, the quotation of North at note 21).

7 (Redhead 1990, 146).

8 (Quine 2004, 205); (Quine 1976, 229); *Cf.* (Chatti 2011, 6).

9 (Ismael 2016, 114).

> A view of causation as an intrinsically directed relation among events, by which one event brings about the other, is not part of the physicist's worldview. We know too much to say such a thing; in the same way that we know too much about how vision works to think that invisibility is an intrinsic property of things that explains why we can't see them…If we are going to take physics seriously and literally, we can't impose our own ideas of cause and law onto it. We have to take physics on its own terms.[10]

Ismael perceives a shift away from a causal interpretation of physics as early as the work of Isaac Newton. The onset of presenting the laws "in the form of differential equations" led to the abandonment of the causal interpretation of those laws.

John D. Norton (2007a, 2007b) argues that inserting causation (not just the concept but the phenomenon) into the ontology of fundamental physics ought to support some type of constraint on its content. If it does not restrict physical possibilities, then causation "is revealed as an empty honorific" and "[c]ausal talk…amounts to little more than an earnest hymn of praise to some imaginary idol; it gives great comfort to the believers, but it calls up no forces or powers."[11] By means of a very brief historical discussion, Norton tries to show that Aristotle's suggested causal constraint (with its inclusion of final causation) on the ontology of physics failed and that none other than Newton began to turn away from a causal physics by denouncing (or so it is claimed) the efficient causal interpretation of the gravitational force (Norton 2007a 15–16).

Marius Stan argues that during the interval that is 1750–1790 natural philosophical thought advanced to such an extent that our modern conception of dynamical laws was born. According to that modern understanding, dynamical laws are differential equations whose content are functional dependency relations and whose primary role is to strictly imply laws of motion and thereby explain and predict natural regularities.[12] These laws are not causal. Stan says,

> …around 1750 the laws of motion…turned into mere assertions of equality between magnitudes, and served *just* [emphasis in the original] to entail (differential) equations of motion for specific mechanical situations. Consequently, *in the Enlightenment* [my emphasis], the laws no longer made claims about natures, and they became epistemically opaque, in that it was no longer clear what counted as strong evidence for the basic laws of mechanics.[13]
>
> …the laws…[viz., "the basic laws of Enlightenment mechanics"] merely assert *equalities between magnitudes*, or that some quantity equals an algebraic combination of other quantities…However, assertions of equality are neither material implications, nor if-then statements, nor any of the logical-syntactic vehicles used to predicate cause-and-effect links or dependencies. And so the basic laws no longer count as causal principles after 1790; nor do their local corollaries, the equations of motion.

[10] (Ismael 2016, 134–135).

[11] (Norton 2007a, 15, 19–20).

[12] Stan remarked:

"Thus an inflection point occurred circa 1750 as the science of motion exited its adolescence. Specifically, we see a deep shift in the form and status of the laws of motion. The shift is where early modern mechanics turns into classical mechanics as we know it." (2022, 405).

[13] (Stan 2022, 388).

> In effect, the laws no longer state facts about the basic causal powers of bodies, material substances, or their ontological analogues. If late-Enlightenment mechanics is still home to causal knowledge, it is not contained in the laws of that discipline.[14]

Stan would go on to remark that the "deep shift in the nature of laws had long-term epistemological consequences", in that "it is not clear what count***ed*** [my emphasis] as evidence for basic laws any more." According to Stan, "[t]his grave problem—that laws lacked even *prima facie* warrant—had two causes."[15] Stan then reveals that the first cause was the fact that "*the laws no longer mention bodies or their causal actions*" [my emphasis] and so "potential rationalist evidence…is irrelevant".[16] It is clear from context that Stan is here discussing the viewpoint of the late-Enlightenment thinkers themselves for in the immediate context Stan notes how at that time certain ways of justifying the laws no longer work because of scale differences (see ibid., n. 35) and a lack of accessible data *at the time* (*i.e.*, *before 1800*, an era referenced at ibid., n. 36). According to Stan then, late-Enlightenment thinkers transformed our conception of dynamical law (Stan calls it a "reconceptualization" (ibid., 388)) moving us away from the causal interpretation of dynamical laws at least after 1790.

The structure of Stan's argumentation is well-illustrated by his comparison of Joseph-Louis Lagrange's (1736–1813) *Analytical Mechanics* with Kant's *Metaphysical Foundations of Natural Science*. According to Stan, Kant did not pay heed to "the most general, cutting-edge science of moving bodies—Lagrange's mechanics" (Stan 2022, 402), and so developed a mistaken mechanics of body (one that incidentally maintained a causal interpretation). Lagrange's mechanics was sufficiently general and sophisticated, such that (according to Stan) Lagrange's *Analytical Mechanics* featured an "absence of any explicit notion of body" (ibid., 402). The point is clearly that Lagrange's approach to mechanics embodied the more modern (re)conceptualization of dynamical law while Kant's did not. Concerning Kant, Stan's language is particularly negative. After quoting a passage from Kant's *Metaphysical Foundations* wherein Kant presents his causal interpretation, Stan remarks.

> …Kant's confident talk above [talk of forces causing motions etc.] runs into the same problem as his two predecessors, Wolff and Émile Du Châtelet. Specifically, all three use the vocabulary of legislation—*leges motus, lois du mouvement*, and *mechanische Gesetze*—but they leave unclear what these laws are for; i.e., what they do within mechanical theory. More regrettably, Kant, the only philosopher to live through the great transformation in the laws' form and function described above, missed the chance to notice and reflect on it—an unfortunate victim of his deficient schooling.[17]

[14] (Stan 2022, 402–403).

[15] See (Stan 2022, 403) for both quotations.

[16] (Stan 2022, 403). Stan is best interpreted as discussing how the laws were regarded during the late Enlightenment era and onward. How else could the past ("count***ed***") epistemic opacity in question *be caused by* the failure of the laws to "mention bodies or their causal actions"? Stan is clearly committed to the hypothesis that the dynamical laws were no longer causally interpreted under the reconceptualization of them during the late-Enlightenment era.

[17] (Stan 2022, 402).

I will argue that **the prevalent** story is multiply flawed.[18] As I've already said, and as I will argue in Sect. 2, Hamilton's (to say nothing of William Whewell's) nineteenth century mechanics was deeply informed by Kant's causal interpretation of mechanics (as was Whewell's approach by the way) *inter alios*. That Hamilton, the great mathematician, shared Kant's causal outlook on the laws despite his intimate familiarity with differential calculus, the calculus of variations, the principle of least action, mechanical variational principles in general, and the more modern Eulerian conception of a function and functional equations cuts directly and especially against Stan's project in obvious ways.

What appears to be all the rage these days notwithstanding, I will also argue that Hamilton's two most important and revolutionary papers on classical mechanics (Hamilton 1834, 1835) provide persuasive ammunition for a causal interpretation of the laws of *modern* classical (non-relativistic) physics, *if it is plausible to believe that forces are causes of motion*. Admittedly, that's a big 'if'. But there are arguments for that antecedent and plausible responses to objections to it.[19] That forces are causes was certainly the majority opinion among the fashioners of what we now call classical mechanics.[20] But my target in this work is the ***prevalent story***, and I take on as an additional (*primary*) task the goal of establishing that forces are indispensable to the ontology of modern Hamiltonian mechanics. These tasks are cumbersome enough because it is usually maintained that Hamiltonian mechanics is an energy-based theory that explains motions and interactions in terms of the energies of systems. If forces enter, they do so in a secondary way that is dispensable or derivative.[21]

[18] One can show why some prominent eighteenth century thinkers were not at all "oblivious", but were instead rationally justified in their continued promulgation of causal interpretations of physical laws even if those laws are restricted to Stan's [1] (Euler's statement of the second law of motion) and [3] (Lagrange's version of the principle of least action) in (Stan 2022). Indeed, in this essay, I provide reasons for causally interpreting the principle of least action (*q.v.*, n. 122).

[19] See, *e.g.*, (Weaver 2019) for the positive case. I'm not alone. James Woodward said,

"…to guard against misunderstanding, let me say unequivocally that it is *not* part of my argument that causal notions play no role in or are entirely absent from fundamental physics. I see no reason to deny, for example, that forces cause accelerations." (Woodward 2007, 68 emphasis in the original).

See also (Loew 2017) who responds to the objections from locality and time-reversal.

[20] It was the view of Galileo, Huygens, Hooke, Newton, Leibniz, Boscovich, Coulomb, Cavendish, Daniel Bernoulli, Varignon, Hermann, Fourier, d'Alembert, Euler, Lagrange, Laplace, Maxwell, Helmholtz, and Boltzmann. See (Pourciau 2020, 185); (Weaver 2019, 67–71); (Weaver 2021); (Westfall 1971).

[21] As North wrote,

> "Newtonian mechanics 'describes the world in terms of forces and accelerations (as related by the second law)' (Taylor 2005, 521), where 'force is something primitive and irreducible' (Lanczos, 1970, 27). Lagrangian and Hamiltonian mechanics describe systems in terms of energy, with force being 'a secondary quantity' derivable from the energy (Lanczos, 1970, 27). According to Newtonian mechanics, the world is fundamentally made up of particles that move around in response to various forces between them. According to Lagrangian and Hamiltonian mechanics, particles move around and interact as a result of their energies. Although energy and force functions are inter-derivable in ways that physics books will show (albeit under certain contestable assumptions…), these are nonetheless prima facie different pictures of the world, built up out of different fundamental quantities, with correspondingly different explanations of the phenomena." (North 2022, 29).

Although North's characterization is a common one, what Lanczos (1970, 27) actually says is that according to Lagrangian and Hamiltonian mechanics, "it is not the force but the *work* done by the force which is of primary importance, while the force is a secondary quantity derived from the work." (emphasis in the original) Work is not equal to energy. It is minus work that is related by mathematical identity to change in potential energy, given certain background assumptions about the system being described.

While, I will provide motivation for a force-laden Hamiltonian mechanics, I should note an admitted limitation. Most of the discussion that ensues will pertain to conservative, holonomic particle systems (defined in Sect. 3) of classical (non-relativistic) mechanics. I'll say nothing about other system-types and almost nothing about other theories endowed with various types of fields and the like (although *q.v.*, Sect. 5 on electrostatics). Again, I believe it is substantial enough to show how forces are indispensable to the ontologies of Hamiltonian models of such systems, for *if* forces are causes of motion, causation will follow suit. My hope is that by looking to Hamilton in the manner that the current project does, those who disagree with the consensus eliminativist attitude about causation in physics will here find an aid in the form of a blueprint and steppingstone for ushering causation back into one of the most sophisticated versions of classical mechanics viz., Hamiltonian mechanics.

Before proceeding as planned, I should address one concern the reader might possess. Despite the fact that I am not here arguing that forces are causes, one might wonder nonetheless which type of causation is important to the larger project. Which type of causation *can* be injected into the best interpretation of Hamiltonian mechanics? My answer: Any type of causation, whether it is reductionist, anti-reductionist, or primitivist. The type of causation need only be consistent with the idea that forces are causes of motion. As noted earlier, my project seeks to object to an *eliminativist* attitude about causation. Causal reductionism is the view that causation is completely determined by, grounded in, or nothing over and above non-causal law-governed, non-causal physical history (Schaffer, 2008). Reductions are asymmetric relations. The (reductive) causal relation cannot therefore be *identified* with constant conjunction, (non-causal) nomological dependence, energy transference, or some other reductive base. It likewise cannot be eliminated because reduction relations are not eliminations. Reduction relations have *relata*. So, what precise sense or theory of causation suits the larger project? Any theory that is consistent with the thesis that forces are causes of motion and that commits to the existence of causes no matter their place in the hierarchy of being.

2 Hamilton's Metaphysical Dynamics

I will now proceed to show that Hamilton possessed a genuine interest in the metaphysical foundations of dynamics and that lessons he drew from four key individuals (viz., Kant, Lagrange, Whewell, and perhaps Leibniz) led him to a causal interpretation of classical dynamics in general, but forces in particular. What follows in this section is not intended to provide novel contributions to Kant, Lagrange, or Whewell scholarship, nor is it intended to argue for the truth of the various metaphysical theories of natural science proffered by these individuals. Rather, with primary and secondary literature in hand, I aim to accurately report that (a) each influencer affirmed their own versions of a causal dynamics, while (b) all agreed that forces were causes of motion. Again, (c) Hamilton's dynamics was informed by facts (a) and (b). It is the way I establish fact (c) that is my novel contribution in this section.[22]

[22] Some Kant scholars may find some of my claims about Kant contentious. I have therefore devoted more time and space to Sect. 2.1.1 than any other historical section.

2.1 Influences and Methodology: A Sample

2.1.1 Kant

Kant influenced Hamilton's approach to both mathematics[23] and dynamical science.[24] To see this, I note first that Hamilton is best viewed as having affirmed something like a distinction between pure and applied algebra (*MPH3*, 5; see the 1832 memorandum quoted at Hankins, 1980, 272–273). The latter is used to model and describe dynamical evolutions of systems in physical as opposed to pure time. For Hamilton, pure time is naked time. Agreeing with Kant,[25] it is time stripped of objective mind-independent chronology and (objective) causation in the external world.[26] In dynamics, parts of accurate models of *applied* algebra represent *objective succession* in ***physical time*** (Hankins, 1980, 273; Winterbourne, 1982, 199; Hamilton spoke in terms of objective order). But recall that according to Kant (and therefore Hamilton who endorsed Kant's conception of succession), the succession involved here is causal. Every substance undergoing change is causally connected to something else. There are two motifs in Kant's work that motivate this picture of his viewpoint. The first is Kant's very idea of succession. The second is Kant's Second Analogy of Experience which drew upon Kant's earlier idea of succession.

Kant presented the "Second Analogy of Experience" in his monumental work, *Critique of Pure Reason*. According to that analogy, any and all transmutation or change must be accompanied by obtaining causal relations.[27] As Kant's "principle of temporal sequence

[23] As Hamilton's biographer and contemporary, Robert Perceval Graves's (1810–1893) summarized:

"…we regard him [Hamilton] as having made a decidedly Kantian movement, when he conceived and published that view of algebraic science, including the various calculi…" (Graves 1842, 107); see also (Graves 1885, 141–143).

Hankins (1980, 460) agrees: Hamilton

"built his metaphysics of mathematics on a direct reading of Kant's works. An appreciation of Hamilton's arguments about the foundations of algebra therefore requires a plunge into Kant's critical philosophy."

When Hamilton spoke of "Algebra" (he liked to capitalize the term) he was referring to both algebra (as we know it) and calculus or analysis (Halberstam and Ingram 1967, xv). For more on Hamilton's algebra including work on quaternions, see (Crowe 1985, 17–46, 117–124); (Fisch 1999); (Hamilton 1837; 1853 or *MPH3*, 117–155); (Hankins 1980, 268–275); (Mathews 1978); (Merzbach and Boyer 2011, 510–512); (Øhstrøm 1985); and (Winterbourne 1982).

[24] Here's some evidence for the claim in the main text. First, Hamilton quotes Kant, sometimes in the original German (Hamilton *MPH3*, 118). Hamilton appeals to Kant again at (Hamilton *MPH3*, 125, the lengthy footnote that extends to p. 126). Second, we learn from Hamilton's personal correspondence and memoranda that Hamilton both read and studied Kant and secondary literature about Kant (see Hankins 1980, 268–275). Third, Hamilton explicitly acknowledged his indebtedness to Kant in the development of the algebra discussed at (Hamilton 1837) in the 1853 preface to the *Lectures on Quaternions* (1853), a work that is one of the most important contributions to the history of mathematical physics being in part a seed for the creation of modern vector calculus (Hendry 1984, 63). Fourth, Hamilton's contemporaries understood him to be developing Kantian motifs in some of his scholarly work (Graves 1885, 141–143). Fifth, ideological dependence can be demonstrated in other ways besides pointing to direct quotations. If we group together reasons (1)-(4) and then see relevant ideological similarities and motifs throughout portions of the work of two individuals, the best explanation is that the temporally later thinker drew upon the former (especially if the temporally later thinker admits to doing so).

[25] (Kant 1998, 180–181, B49 B51). Kant said, "[t]ime is therefore merely a subjective condition of our (human) intuition (which is always sensible, i.e., insofar as we are affected by objects), and in itself, outside the subject, is nothing." (Kant 1998, 181, B51).

[26] This idea appears in Hamilton's 1853 exposition of his earlier views (Hamilton *MPH3*, 117).

[27] (Kant 1998, 304–316; B232-B258). See also (Smith 1918, 363–381); (Watkins 2005, 185–217, 237; 2019, 61–65).

according to the law of causality" (a synthetic a priori judgment) stated, "[a]ll alterations occur in accordance with the law of the connection of cause and effect."[28]

In 1755, some 26 years before the first *Critique*, Kant published the *New Elucidation*. The last section of that work explicated Kant's Principle of Succession:

> No change can happen to substances except in so far as they are connected with other substances; their reciprocal dependency on each other determines their reciprocal change of state.[29]

The connection with which Kant is concerned is the connection between cause and effect. That relation is not reducible to Humean regularity or constant conjunction (see Friedman, Causal Laws 1992, 162; Kant, 1933, 4–6). For Kant, the connection in succession bespeaks a modal tie because causation is a modal tie indicative of the natural necessitation of an effect by its cause. As Kant remarked in the *Prolegomena to Any Future Metaphysics* (1783):

> For this concept [of causation] positively requires that something A be such that something else B follow from it *necessarily* and *in accordance with an absolutely universal rule*…the effect is not merely joined to the cause, but rather is posited *through* it and results *from* it.[30]

Universal and naturally necessary rules or laws back causal relations and are for Kant, therefore, causal laws.[31] To ensure a causal interpretation of the laws, Kant believed that causation enters physics through (*inter alia*) causally efficacious quantities like forces (Kant, 1903, 171, 222).[32]

Hamilton (in Hamilton, 1837) appropriated Kant's view that dynamical science is about non-Humean causal connection. That is why he equated "Dynamical Science", with a science that is conducted using "reasonings and results from the notion of cause and effect".[33]

[28] (Kant 1998, 304; B232). I have removed the emphasis of the first quotation in this sentence.

[29] (Kant 1992, 37). For Kant, this is a principle of "metaphysical cognition" (ibid.). Its flavor reminds one of Isaac Newton's (1643–1727) third law of motion (*i.e*., the action-reaction principle). In the *Prolegomena to Any Future Metaphysics*, Kant presented three *a priori* principles about the cognition of appearances in experience. The last of these invokes an action-reaction principle (reminiscent of Newton's third law of motion and the reciprocity mentioned in the quotation in the main text) understood as a type of epistemic guide or tool (Kant 1933, 66). Of course, all of this resembles ideas found within Kant's third analogy of experience in the *Critique of Pure Reason*.

[30] I quote here the translation of Kant's *Prolegomena to Any Future Metaphysics* (A91-92/B123-124) provided by Michael Friedman in (Friedman, Causal Laws 1992, 161–162) emphasis in the original. For standard English translations of Kant's *Prolegomena*, see (Kant 1933) and (Kant 2004).

Compare: "the concept of cause implies a rule, according to which one state follows another necessarily" (Kant 1933, 76).

[31] There is a question about whether Kant's laws of mechanics are identical to Newton's laws of motion. For the view that they are identical, see (Friedman 2013). For the view that they are distinct, but perhaps similar, see (Stan 2009, 43–44); (Watkins 2005). I should add that Watkins's views have undergone an evolution. Compare (ibid.) to (Watkins 2019).

[32] My reading is within the confines of well-regarded Kant scholarship. See (Guyer 1987); (Friedman Causal Laws 1992); (Friedman 2013, 118, 265); (Watkins 2019, 90). According to Friedman (2014), Kant also maintained that Newton's universal law of gravitation was a kind of *causal* law endowed with a kind of necessity.

[33] (Hamilton, 1837, 7).

Furthermore, Hamilton also affirmed that forces are causally efficacious quantities just as Kant did before him (see Sect. 2.1.3). For Hamilton, the very objectivity of physical time was motivated by appeal to Kant's Second Analogy of Experience (Hankins, 1980, 273–275, although Hankins speaks in terms of subjective and objective order and not pure and physical time) and its underlying idea of causal succession. Thus, both Hamilton and Kant would have affirmed that (quoting one Kant scholar) "the exact role of the causal relation is to constitute the earlier time as necessarily advancing to or determining the later time…"[34] The succession of sensible events and appearances in our experience possesses an objective temporal order, a flow that must include obtaining causal relations.

2.1.2 Lagrange

To discern even more clearly how Hamilton ensured that his mechanics remained committed to a causal ontology, look beyond the influence of Kant to see that of Lagrange.

Hamilton asserted that mechanics is the "the science of force, or of power acting by law in space and time" and that it has.

> …undergone already another revolution and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginable conditions, which Lagrange so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points…[35]

Hamilton here characterizes mechanics as the "science of force" and of acting "power". And while in one respect, Hamilton appeared to be distancing himself from Lagrange, in another respect and within the same work (Hamilton, 1834), he also drew closer to him. The *first* revolution to which Hamilton alluded is the analytical revolution of Lagrange's 1788 *magnum opus*, the *Analytical Mechanics*.[36] Hamilton builds upon that revolution by making direct use of Lagrange's axiom of mechanics, *viz.*, the **Principle of Virtual Velocities** (which was later called the **Principle of Virtual Work** (**PVW**)). For a conservative system and in modern vector notation, given that $\delta\mathbf{r}$ is virtual displacement (*i.e.*, the displacement need not be realized or actual), m is inertial mass, U is potential energy, and dots are time derivatives, the **PVW** states:

$$\sum \mathbf{F} \bullet \delta\mathbf{r} = \sum m\ddot{\mathbf{r}} \bullet \delta\mathbf{r}$$

or:

$$\sum m\ddot{\mathbf{r}} \bullet \delta\mathbf{r} + \delta U = 0$$

Hamilton so builds by deriving his own fundamental axiom of mechanics, *viz.*, **the law of varying action** (**LVA**), introduced and discussed in Sect. 4.1, from the **PVW** *inter alia*.

34 (Melnick 2006, 227).

35 (Hamilton 1834, 247). See (Kargon 1965) on this passage and the possible allusion to Roger Joseph Boscovich (1711–1787) within it.

36 In one place, Hamilton refers to (Lagrange 1788) as "a kind of scientific poem" (as quoted by Truesdell 1968, 86).

That Hamilton builds upon Lagrange's revolution is important because Lagrange affirmed a causal interpretation of (non-fictitious) forces, forces that were fundamental in his mechanics.

At the beginning of Lagrange's *Analytical Mechanics*, Lagrange stated that "[i]n general, force or power is the cause, whatever it may be, which induces or tends to impart motion to the body to which it is applied."[37] My reading of Lagrange may come as a shock, for it is usually emphasized that Lagrangian mechanics is and has always been an energy-based approach to mechanics, and that Lagrange's analytical revolution is best viewed as a turn away from Newton and Leonhard Euler's (1707–1783) force-laden (and thereby causation-laden) mechanics.[38] The opinion is mistaken. In (Lagrange 1788), Lagrange derives his equations of motion (those well-known equations used in Lagrangian mechanics as equations of motion even today) both with and without constraints from his axiom of mechanics, *viz.*, the **PVW**, wherein forces are featured (Caparrini & Fraser, 2013, 375–376; Fraser, 1985, 173). Lagrange did not interpret U as a representative of potential energy (the idea hadn't yet entered physics!) but instead regarded it as a function to be used in "a convenient method for calculating the components of force in various coordinate systems."[39] And as is the case with Euler's laws of mechanics (*e.g*., what are often called the Newtonian equations), the **PVW** is demonstrably more general than Lagrange's equations of motion.[40] The **PVW** is fundamental. The equations of motion that are down-stream from it are not. It's no surprise then that Lagrange scholars have judged that in both his *Theory of Analytical Functions* and his *Analytical Mechanics*, Lagrange turned back to Newton with new analytical equipment in hand. As Marco Panza attests,

> …the foundation of mechanics sketched in the *Théorie des fonctions analytiques* makes apparent his effort to reduce the role of non-Newtonian principles in mechanics and to identify the subject with a quasi-algebraic deductive system based on a general Newtonian analysis of forces. This shows the difference between Lagrange's interpretation of his own results and any modern evaluation of them.[41]

I should add that despite Hamilton's aversion to the notions of "solidity and cohesion" in the work of Lagrange, it is not a coincidence that Lagrange's characterization of *dynamics* in his *Analytical Mechanics* (1788) is remarkably close to Hamilton's (previously quoted):

> Dynamics is the science of accelerating or retarding **forces** and the diverse motions which they **produce**…the discovery of the infinitesimal calculus enabled geometers to reduce the laws of motion for solid bodies to analytical equations and the research

[37] (Lagrange 1997, 11) emphasis in the original.

[38] Save some philosophers, virtually everyone believes that for Newton forces are causes of motion (*q.v*., n. 46 if you require source citations). Both (Ismael 2016) and (Norton 2007a) were wrong to see in Newton an abandonment of causation-laden laws of physics. Ronald S. Calinger, a foremost Euler scholar, has recently said "[t]he concept of force was crucial to Euler's mechanics, and he treated it as an external entity to a body causing change in motion." (Calinger 2016, 126).

[39] (Archibald 2003, 198).

[40] (Truesdell 1968, 133). Truesdell also listed other problems with Lagrange's equations of motion, problems that result from "obscuring the forces" (ibid.).

[41] (Panza 2003, 151–152). For a sample of Lagrange's high view of Newtonian forces within the *Analytical Mechanics*, *q.v.,* n. 37 and n. 40.

> on forces and the motion which they produce has become the **principal object** of their work.[42]

For both Lagrange and Hamilton, dynamics is the science of causally productive forces and the motions that are their effects.

2.1.3 Whewell

Hamilton wasn't just influenced by Kant and Lagrange. He also had numerous and important interactions with the great polymath William Whewell (1794–1866).[43] Whewell was an immensely important intellectual in the nineteenth century. His influence on the development of physics can not only be seen in the central role he played in the process of the invention of terms like 'physicist', 'anode', 'cathode', and 'electrolysis', but it can also be seen in the impact he left on the thought of Michael Faraday (1791–1867), James Clerk Maxwell (1831–1879), and, of course, Hamilton.[44]

Whewell wrote extensively on classical mechanics. He even articulated a unique formulation and interpretation of the three classical laws of motion and sought to justify those laws with their accompanying interpretations by appeal to an elaborate metaphysics of causation.[45] Like Newton[46] and Lagrange,[47] Whewell believed that forces cause motions. He wrote that "the term *Force*" denotes "that property which is the cause of motion produced, changed, or prevented,"[48] and that:

> By Cause we mean some quality, power, or efficacy, by which a state of things produces a succeeding state. Thus the motion of bodies from rest is produced by a cause which we call Force: and in the particular case in which bodies fall to the earth, this force is termed Gravity. In these cases, the Conceptions of Force and Gravity receive their meaning from the Idea of Cause which they involve: for Force is conceived as the Cause of motion.[49]

Hamilton and Whewell maintained an important and mutually beneficial professional relationship. And while I cannot now explore all of the details of their interactions (see Graves, 1882, 1885, 1889; Hankins, 1980, 174–180), I note here that the two seemed to agree on

42 (Lagrange 1997, 169) emphasis mine.

43 On Whewell, see (Butts 1968); (Fisch and Schaffer 1991); (Whewell 1836, 1858, 1967) and for valuable correspondence, see (Todhunter, vol. 1 and vol. 2 1876a, 1876b).

44 For a discussion of Whewell's influence on Maxwell, see (Smith 1998, 305). For a discussion of Whewell's influence on Faraday, see (Darrigol 2000, 83–86, 97). For a discussion of Whewell's influence on Hamilton, see (Hankins 1980, 172–180) on which my discussion here leans in part.

45 See (Whewell 1836, 138–161); (Whewell 1967, 573–594).

46 Newton said, "…the forces…are the causes and effects of true motions." (Newton 1999, 414, cf. 407, 575, 794). See also (Dobbs 1992, 207–209); (M. Jammer 1957); (McGuire 1968); (McGuire 1977); (B. Pourciau 2006, 188–189); (R. Westfall 1971).

47 *Q.v.*, n. 37.

48 (Whewell 1858, 205) emphasis in the original.

49 (Whewell 1858, 173). See also the comments at (Smith and Wise 1989, 362).

how best to interpret the laws of motion. *After* authoring (Hamilton, 1834),[50] Hamilton sent a letter to Whewell in which he stated:

> The Paper of your own On the Nature of the Truth of the Laws of Motion has been as yet so hastily read by me, that I can only say it seems to be an approach, much closer than of old, between your views and mine. Whether this approach is a change on my part or on yours, and if both, in what proportion, and how much or how little it wants of a perfect agreement, I dare not suddenly decide.[51]

The following month, Hamilton wrote to Lord Adare as follows: "Whewell has come round almost entirely to my views about the laws of Motion."[52] Hamilton believed that the interpretation of the laws of motion in Whewell's "On the Nature of the Truth of the Laws of Motion"[53] (published in 1834) matched his own interpretation of those laws. In that essay, Whewell wrote:

> The science of Mechanics is concerned about motions as determined by their causes, namely, forces; the nature and extent of the truth of the first principles of their science must therefore depend upon the way in which we can and do reason concerning *causes*.[54]

Whewell went on to state what he calls "Axiom 1": "*Every change is produced by a cause*."[55] Axiom 2 says that: "*Causes are measured by their effects*."[56] Axiom 3 was about action and reaction. These axioms are used to motivate and interpret the laws of motion. The ensuing discussion also very clearly and explicitly and continually interprets forces causally, characterizing them, again and again, as causes of motion.[57] Because Hamilton agreed with the views of Whewell here articulated, we can safely conclude that Hamilton adopted a causal metaphysics of physics amidst the two years he published his two most important papers on classical mechanics.[58]

2.2 Hamilton's Idealism: A Sample

Hamilton was an idealist about the natural world. As I have already pointed out, Hamilton's views about the role of causation in dynamics were heavily influenced by Kant's first *Critique*. It is therefore unsurprising to see the influence of the first *Critique* on Hamilton's idealism as well. There are also important similarities between Hamilton's idealism and philosophical worldview, and the idealism and worldview of Gottfried Wilhelm Leibniz's (1646–1716) *Monadology* (Leibniz, 1898 originally published in 1714). Leibniz maintained

[50] The paper was received by *Philosophical Transactions of the Royal Society* on April 1st, 1834 and read April 10th, 1834. However, at the end of Hamilton's introductory remarks, Hamilton includes a date of March, 1834, and because his letter to Whewell references (Hamilton 1834 although it was not yet published) we can infer that Hamilton's letter (dated March 31st, 1834) was authored after Hamilton had completed but not yet published (Hamilton 1834).

[51] From W.R. Hamilton to Dr. Whewell, Observatory, Dublin, March 31, 1834 in (Graves 1885, 82).

[52] As quoted in (Graves 1885, 83).

[53] (Whewell 1967, 573–594).

[54] (Whewell 1967, 574) emphasis in the original.

[55] (Whewell 1967, 574) emphasis in the original.

[56] (Whewell 1967, 575) emphasis in the original.

[57] The idea is all over the essay, but see (ibid., 581) for just one (more) example among many.

[58] *Cf.* the conclusions in (Hankins 1980, 178).

that there are scientific efficient causal explanations of motion and change in the kingdom of power (science/physics) and divine final causal explanations of everything in the kingdom of wisdom (philosophy/theology).[59] There is something approaching this bifurcation in the work of Hamilton, for in that work one finds efficient causal scientific explanations divorced from his idealism on the one side, and deeper more metaphysical explanations dependent upon his idealism but ultimately bottoming out in God on the other.[60] There were for Hamilton (quoting Hankins) "two separate sciences joined only by the benevolent act of God."[61]

In the kingdom of wisdom, Leibniz had fundamental simple monads metaphysically explain a great deal. Hamilton had his fundamental and simple energies or powers play a similar role in the Hamiltonian worldview.[62]

As with Hamilton's indebtedness to Kant, a thorough study explicating all of the important similarities and differences should be pursued, but I will not take up that task here. I wish only to disclose that Hamilton's fundamental and simple energies possess *causal powers*. Indeed, sometimes Hamilton speaks as if the fundamental energies are identical to causal powers (Hankins, 1977, p. 179). As Hamilton affirmed, "[p]ower, acting by law in Space and Time, is the ideal base of an ideal world, into which it is the problem of physical science to refine the phenomenal world…".[63]

3 A (Very) Brief Sketch of Modern Hamiltonian Mechanics

Hamilton's methodology, intellectual influences, and philosophical worldview all led him to a causal interpretation of classical mechanics. Forces in that mechanics causally produce changes of motion. The modern philosopher of physics will object. Nowhere in the fundamental equations of *contemporary* Hamiltonian mechanics (at least) do we see an indispensable role for forces. Hamilton's views are his own. How could and why should his ideas influence our modern understanding? Contemporary Hamiltonian mechanics is an *energy*-based theory which seeks to describe and explain mechanical systems by means of the Hamiltonian H. H is equal to the sum of the kinetic T and potential energy U of the modeled system when potential energy is velocity independent and what are called generalized coordinates (introduced below) are natural in the sense that their relationship to the relevant coordinates is time-independent (Marion & Thornton, 1988, pp. 218–219). And

[59] See (Leibniz 1989, 223).

[60] Hamilton stated, "[i]n seeking for absolute objective reality I can find no rest but in God…" From Hamilton to H.F.C. Logan, June 27, 1834 in (Graves 1885, 87).

[61] (Hankins 1980, 179), although Hankins does not relate Hamilton's work to that of Leibniz.

[62] Leibniz's *Monadology* maintained that corporeal substances are phenomenal depending for their existence upon quasi-mental simple entities called monads. Leibniz said, "simple things alone are true things, the rest are only beings through aggregation, and therefore phenomena, and, as Democritus used to say, exist by convention not in reality." (Leibniz's letter to Burchard de Volder (1643–1709) (June 20th of 1703, sent a second time). As quoted by (Garber 2009, 368)). And elsewhere Leibniz wrote,

> "…if there are only monads with their perceptions, primary matter will be nothing other than the passive power of the monads, and an entelechy will be their active power…" (Leibniz and Des Bosses 2007, 274–275 emphasis mine).

The equivalent of monads in Hamilton's system are fundamental simple things called energies or powers (see the main text). Like Leibniz's monads, Hamilton's simple powers/energies give rise to the external world. Hankins calls Hamilton's view an "idealized version of atomism" (Hankins 1977, 182).

[63] From Hamilton to H.F.C. Logan, June 27, 1834 in (Graves 1885, 87).

so, when assuming such naturalness of coordinates and when assuming that the target system (call it SYS) is an energetically isolated holonomic n-particle system, it follows that:

$$H = T + U$$

and that the number of generalized coordinates and the number of degrees of freedom of SYS equal one another.

According to the Hamiltonian formulation, the configuration of SYS (in three-dimensions) is given by n-generalized coordinates: $(q_1, \ldots, q_n)$. Following (Goldstine, Poole and Safko 2002), (Marion & Thornton, 1988, 222–229), (Taylor, 2005, 529–531), and (Thorne & Blandford, 2017, 158–160) one can represent n-generalized coordinates with $\mathbf{q}$, and n-generalized momenta[64] with $\mathbf{p}$ such that,

$$\mathbf{q} = \left(q_1, \ldots, q_n\right), \mathbf{p} = (p_1, \ldots, p_n)$$

We can represent n-generalized velocities, or the time derivatives of n-generalized coordinates with $\dot{\mathbf{q}}$ and the time-derivatives of n-generalized momenta $\dot{\mathbf{p}}$, such that,

$$\dot{\mathbf{q}} = (\dot{q}_1, \ldots, \dot{q}_n), \dot{\mathbf{p}} = (\dot{p}_1, \ldots, \dot{p}_n)$$

Quantities $\mathbf{p}$ and $\mathbf{q}$ and their time derivatives (which can also be represented with a single general subscript (i set equal to $1, \ldots, n$)) are n-dimensional vectors that can be represented in abstract mathematical spaces described by higher-dimensional geometry.[65] I will say almost nothing about these spaces and what "transpires" in them. The common procedure one uses

[64] All discussion of modern physics will use the SI unit system.

Generalized momenta can also be stated in terms of the Lagrangian $\mathscr{L}$ (the difference between kinetic and potential energy) and generalized velocity (Penrose 2005, 476, I'm citing in this case because some suggest otherwise):

$$\mathbf{p} = \frac{\partial \mathscr{L}}{\partial \dot{q}}$$

[65] In technical discussions of Hamiltonian mechanics in the work of physicists, mathematicians, and some philosophers, one will see: (a) higher-dimensional phase spaces the points of which represent possible states of the system modeled (because they encode information about the positions and momenta of constituents of the system), (b) phase space orbits or flows tracing out (c) curves in phase space understood as representations of possible evolutions of the modeled system given by solutions to Hamilton's equations, (d) Liouville's theorem, (e) measures, (f) Poisson brackets, etc. For all of that, see (Dürr and Teufel 2009, 12–26); (Mann 2018, 167–201); (Torres del Castillo 2018, 103–228) and pair it with (Healey 2007, 248–251). I skip that stuff here in the interest of brevity. My central argument will remain unaffected by details about cotangent bundles or phase spaces that are symplectic manifolds, measure preserving flows, and symplectic geometry. What gives you the curves that represent evolutions in the phase space are solutions to Hamilton's equations. In addition, the "dynamical evolution of a system can…be geometrically encapsulated in a single scalar function (namely the Hamiltonian)" (Penrose 2005, 484). So, the important questions are: How should one interpret Hamilton's equations and their solutions? How should one interpret the Hamiltonian?

Later, I will make much of Galilean invariance in classical mechanics. Some might therefore object to precluding a discussion of the geometry of Hamiltonian mechanics because both canonical transformations and canonical invariants (or canonical form-invariants) are important to Hamiltonian and Hamilton–Jacobi mechanics. In order to appreciate canonical invariants and transformations (especially those that have to do with time), one must study symplectomorphisms and that study will require that one give attention to cotangent bundles and symplectic geometry. Canonical invariants and transformations have to do with tracking systems in a higher-dimensional phase space. But we need not worry about any of that. Hamilton's equations of motion are canonical form-invariant, and (again) their solutions provide one with the motions of systems modeled by the Hamiltonian apparatus (points orbiting in the higher-dimensional phase space). Once again, the question is, how should we interpret Hamilton's equations and their solutions?

to model basic physical systems in Hamiltonian mechanics makes no direct or explicit appeal to higher-dimensional geometry. For example, you don't have to explicitly do symplectic geometry on a $6n$-dimensional phase space to use Hamilton's equations (and their solutions) to model a particle's motion under the influence of a central force field (see Taylor, 2005, 532–533 whose discussion does not mention symplectic manifolds). Hamilton's own dynamical theory was not geometric. One "finds all through the work of Hamilton that the directly geometrical point of view is made subservient to the analytical" (Conway & Synge, 1931, xix). As Sect. 4 will make clear, Hamilton "employed variational ideas and techniques" and "his analysis was developed within the established theory of analytical dynamics" (Fraser, 2003, 377). Higher-dimensional geometry did not come to the fore until after 1869.[66] This explains why not even Carl Gustav Jacob Jacobi's (1804–1851) improvements on Hamilton's framework were geometric. His work also rested upon an *analytical* point of view (Lützen, 1995, 21–22; Thiele, 1997, 286). It is my opinion that the higher-dimensional geometry illustrates the dynamics. What governs the dynamics are the fundamental equations and their solutions. Interpreting those equations and solutions is what concerns me here.

Assuming that $(i = 1, \dots, n)$ and, when applicable, that $(j = 1, \dots, m)$ here and throughout, one can write Hamilton's equations (of motion) as:

(Hamilton's Equations of Motion):

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \dot{q}_i = \frac{\partial H}{\partial p_i}$$

Add to the above, a specification of the (full) time derivative of the Hamiltonian as follows:

$$\frac{dH}{dt} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i \right) + \frac{\partial H}{\partial t}$$

And keep in mind that the full or total derivative of the Hamiltonian with respect to time features $2n+1$ terms (as can be read off of the equation just stated) and that:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

which can be inferred from the equations of motion and our specification of the full time derivative of the Hamiltonian.[67] And while the above expressions equal one another, they are *conceptually* distinct. The equality will hold for systems like SYS because the Hamiltonian describing them is a constant of motion.

Solutions to Hamilton's equations give one the evolution of the system or subsystems modeled. The contemporary philosopher of physics will point out that forces do not seem to enter any of the above relations or equations. Likewise, forces do not appear to enter solutions to Hamilton's equations of motion. How then can the contemporary natural

[66] According to (Lützen 1995, 15–16), higher dimensional spaces were referenced by Carl Friedrich Gauss (1777–1855) in 1816, by his student August Ritter (1826–1908) in 1853 (with some reservation), and by Jean-Gaston Darboux (1842–1917) in 1869. However, systematic treatments of mechanics with the equipment of higher dimensional geometry cannot be found until after 1870 (Lützen 1995, 18).

[67] Hence, the logical ordering of the equations in the main text. We can also affirm:

$$-\frac{\partial \mathscr{L}}{\partial t} = \frac{\partial H}{\partial t}$$

philosopher insist on a causal interpretation of Hamiltonian mechanics? Indeed, if Hamilton's understanding was anything like our own, how could Hamilton have done so?

As we shall see, there are powerful reasons for insisting that forces belong to the ontology of modern Hamiltonian mechanics. Those reasons shall issue forth from reflecting upon precisely why those same forces never left the ontology of Hamilton's mechanics despite the privileged place of (what we call) the Hamiltonian in his formulation. The crucial piece to my argumentation will actually turn out to be the Hamiltonian. A proper interpretation of that quantity along with a correct statement of its metaphysical grounds (*according to physics*) provides much of what is needed for the articulation and defense of a robust case for injecting forces into both past and present versions of Hamiltonian mechanics.

4 Hamiltonian Causal Mechanics

In this section, I will lay out Hamilton's dynamics. I will show that his law of varying action (to Hamilton, 1834) was an early primary principle of that dynamics and that it essentially uses the Hamiltonian which is specified in terms of the potential function. I will then show how Hamilton's mature dynamics (1835) led to the transformed law of varying action which would become Hamilton's final principal axiom of mechanics. It too is specified in terms of the Hamiltonian (and so also the force function) and the form of the auxiliary function used to state it takes a form expressed in terms of the initial and (temporally) later potential functions governing the system. *The main purpose of this section is to highlight the indispensable role that the potential function plays in Hamilton's dynamics* and to show how *the potential function was understood as one that simply encodes information about force interactions*. Thus, forces are the true fundamental quantities in Hamilton's dynamics.

4.1 Hamilton's Dynamics: A Brief Exposition

In (Hamilton, 1834), the **law of varying action** for SYS (*i.e.*, a system of n-point masses with an ith member[68]) is stated as follows:

(**Law of Varying Action (LVA)**[69]):

$$\delta V = \sum_{i=1}^{n} m_i\left(\dot{x}_i\delta x_i + \dot{y}_i\delta y_i + \dot{z}_i\delta z_i\right) - \sum_{i=1}^{n} m_i\left(\dot{a}_i\delta a_i + \dot{b}_i\delta b_i + \dot{c}_i\delta c_i\right) + t\delta H$$

given that the characteristic function V can be specified as follows:

$$V = \int_0^t 2T dt$$

and that m is inertial mass, t is time, x_i, y_i, and z_i are Cartesian coordinate variables, a_i, b_i, and c_i give the initial positions of the n-point masses, and $\dot{a}_i$, $\dot{b}_i$, and $\dot{c}_i$ give the initial

68 My gloss on the formalism here follows (Hamilton 1834; 1835) only in part. It is more in line with the notational style of contemporary historians of mathematics, Hamilton scholars, and historians of physics (*e.g.*, Goldstine 1980, 176–189; Lützen 1995; Nakane and Fraser 2002 etc. on which my exposition leans). My discussion will rely upon my own reading of Hamilton, but it owes much to the sources just cited along with (Fraser 2003); (Graves 1842); (Cayley 1890); and (Hankins 1980, 181–198).

69 Hamilton also calls this the "equation of the characteristic function" (Hamilton 1834, 252).

velocities of the same point-masses with regard to select coordinate directions. The 'δ' symbol was originally invented by Lagrange to track some of the maneuvers of Euler. Later on, in both the work of Lagrange and Hamilton, it was used to represent a variation or specific type of differential change of a function or functional, although it can act on operators like the integral as well.[70] By using Lagrange's formalism and notation, Hamilton adopted what is sometimes called the δ-formalism for the calculus of variations.

To see what **LVA** means one need only understand the content and significance of the characteristic function and a little calculus of variations. Function V (quoting Hankins) "completely determines the mechanical system [SYS] and gives us its state at any future time once the initial conditions are specified".[71] The **LVA** is the means whereby Hamilton shows that "the solution of the equations of motion" are "reduced to finding and differentiating a single function V" (Nakane & Fraser, 2002, 165). It is therefore *the* law which governs the evolutions of mechanical systems of the same type as SYS.

The **LVA** was a central tool that Hamilton used to build his dynamics. It is also central to my case for the historical judgment that the dynamical models of (Hamilton 1834; 1835) were correctly causally interpreted by Hamilton. Thus, reflection upon the **LVA** and how it is derived will reveal precisely how causation entered Hamilton's dynamics and by consequence it will also reveal how causation enters modern Hamiltonian mechanics.

Hamilton derives the **LVA** from his specification of the characteristic function integral and manipulations of the formula governing energy for conservative systems, or:

$$H = T + -U$$

assuming with Hamilton that U is "the negative of the potential energy".[72] (More on this function below.) Hamilton picked out the above sum with the letter 'H' in honor of Christiaan Huygens (1629–1695).[73]

It follows that for SYS, T's value is given by what Hamilton called "the celebrated law of living forces" (*q.v.*, n. 74):

(**Law of Living Forces (LLF)**[74]):
$T = H + U$

which Jacobi was able to derive, and which Hamilton was only able to assume. In addition:
$T = \frac{1}{2}\sum_{i=1}^{n} m_i\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)$

The variation on T is expressed in this context as:
$\delta T = \delta U + \delta H = \sum_{i=1}^{n} m_i\left(\dot{x}_i\delta\dot{x}_i + \dot{y}_i\delta\dot{y}_i + \dot{z}_i\delta\dot{z}_i\right)$

Again, V in **LVA** is what Hamilton called the "characteristic function", but he also referred to it as the "action of the system", and the "accumulated living force".[75] It had all of

[70] (Goldstine 1980, 111); (Fraser 2003, 361–363).

[71] (Hankins 1980, 186).

[72] (Hankins 1980, 183). This was a common assumption at the time.

[73] (Dürr and Teufel 2009, 16).

[74] (Hamilton 1834, 250).

[75] (Hamilton 1834, 251–252). Hamilton had already written about the characteristic function in (Hamilton 1828) a work on optics published when Hamilton was only 21 years of age. He said there, "[i]n every optical system, the *action* may be considered as a characteristic function, from the form of which function may be deduced all the other properties of the system" (Hamilton 1828, 79 emphasis in the original). According to Sir Edmund Whittaker (1873–1956), Hamilton discovered the function at the age of 16 (Whittaker 1954, 82).

these titles because it tracks or characterizes dynamical properties of systems. It does this by being a function of H along with x, y, z, a, b, and c. It must satisfy, according to Hamilton, the following two fundamental partial differential equations (derived from the **LVA**):

(**Constraining Fundamental PDEs**):

$$\begin{cases} \frac{1}{2}\sum_{i=1}^{n}\frac{1}{m_i}\left\{\left(\frac{\partial V}{\partial a_i}\right)^2+\left(\frac{\partial V}{\partial b_i}\right)^2+\left(\frac{\partial V}{\partial c_i}\right)^2\right\}=T_{t_0} \\ \frac{1}{2}\sum_{i=1}^{n}\frac{1}{m_i}\left\{\left(\frac{\partial V}{\partial x_i}\right)^2+\left(\frac{\partial V}{\partial y_i}\right)^2+\left(\frac{\partial V}{\partial z_i}\right)^2\right\}=T \end{cases}$$

where the top constraining fundamental PDE gives the living force (kinetic energy) of the system at the start of the evolution, and where the bottom equation gives the living force of the system at the end. Hamilton (1834, 253) says that one must use these equations to retrieve the form of the characteristic function V. That is to say, if one simultaneously solves them, one will retrieve V and thereby solve dynamical problems.

If one looks again to $\delta T = \delta U + \delta H$, one might wonder why Hamilton varies the Hamiltonian. Hamilton shows an interest not in changes of position along a path (as one would acquire by differentiating the **LLF**), but in imagined changes to the path itself, along with (perhaps) changes to initial positions as well. By so proceeding, Hamilton was relying upon his earlier work in optics. Hamilton wrote,

> The quantity H may, however, receive any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data…[76]

This imagined path-change is why variations to the kinetic and potential energy are likewise important to retrieve. But the adroit reader has probably already noticed that I have failed to explicate the meaning and significance of one choice function in $\delta T = \delta U + \delta H$, the **LLF**, and in (indirectly via the Hamiltonian) the **LVA**, viz., the potential energy function U. Concerning that function, Hamilton wrote, "[t]he function which has been here called U, may be named the *force-function* of a system…it is of great utility in theoretical mechanics, into which it was introduced by Lagrange…"[77] As I've already noted, Hamilton's indebtedness to Lagrange becomes evident after reflecting upon the fact that Hamilton specified the force-function's variation by means of the dynamical principle of virtual work:

(**Dynamical Principle of Virtual Work (DPVW)**):

$$\delta U = \sum_{i=1}^{n} m_i\left(\ddot{x}_i\delta x_i + \ddot{y}_i\delta y_i + \ddot{z}_i\delta z_i\right)$$

[76] (Hamilton 1834, 250).

[77] (Hamilton 1834, 249) emphasis in the original. It was very common during Hamilton's time to call the potential function U, the "force-function" (Nakane and Fraser 2002, 163). One primary example cited by Nakane and Fraser is Jacobi who would in some ways improve Hamilton's work. Interestingly, Euler identified what Lagrange thought of as the potential with force *effort* (Euler 1753, 173–175); (Boissonnade and Vagliente 1997, xxxvii).

which is basically Lagrange's **principle of virtual work (PVW)** (sometimes called d'Alembert's principle).[78] The variation of the potential energy function represents changes to work performed over time by the point masses of SYS because (at least in part) potential energy represents work performed by the constituents of SYS.[79] Thus, Hamilton affirms:

(**Force-Function (FF)**):

$$U = \sum_{1\leq i<j\leq n}^{n} m_i m_j f\left(r_{ij}\right)$$

where $f\left(r_{ij}\right)$ here represents a force law specifying details about a repulsive or attractive force.[80] The potential function therefore represents operating forces in SYS because it represents work. That is why Hamilton calls it the "force-function". This is how forces enter Hamilton's approach to classical dynamics even after Hamilton introduces his auxiliary function S around which (Hamilton 1835), is centered. For when Hamilton transforms the characteristic function V into the auxiliary function S via a Legendre transformation, he specifies it thus:

$$S = V - tH$$

where S is a function of x, y, z, a, b, c (space), and t (time), whose variation is:

(**Transformed LVA (T-LVA)**):

$$\delta S = -H\delta t + \sum_{i=1}^{n} m_i \left((\dot{x}_i \delta x_i - \dot{a}_i \delta a_i) + (\dot{y}_i \delta y_i - \dot{b}_i \delta b_i) + (\dot{z}_i \delta z_i - \dot{c}_i \delta c_i) \right)$$

And so, Hamilton transformed the **LVA** into the **T-LVA** using S (see proposition W^7 in (Hamilton, 1834, p. 307); (*qq.v.*, n. 81 and n. 83)). This new auxiliary function S, which in (Hamilton, 1835, 95) is called the "principal function", harbors within both H and V, and thereby includes the force-function U. It too must satisfy two fundamental PDEs that are really specifications of the force-function at the beginning (time t_0) and end of an evolution[81]:

(**New Constraining Fundamental PDEs (or the Hamilton–Jacobi Equations)**):

$$\begin{cases} \frac{\partial S}{\delta t} + \sum_{i=1}^{n} \frac{1}{2m_i} \left\{ \left(\frac{\partial S}{\partial a_i}\right)^2 + \left(\frac{\partial S}{\partial b_i}\right)^2 + \left(\frac{\partial S}{\partial c_i}\right)^2 \right\} = U_{t_0} \\ \frac{\partial S}{\partial t} + \sum_{i=1}^{n} \frac{1}{2m_i} \left\{ \left(\frac{\partial S}{\partial x_i}\right)^2 + \left(\frac{\partial S}{\partial y_i}\right)^2 + \left(\frac{\partial S}{\partial z_i}\right)^2 \right\} = U \end{cases}$$

Just as the **Constraining Fundamental PDEs** give the form of the characteristic function, so too do the Hamilton–Jacobi Equations give the form of the principal function. If one knows the form of S, then by looking to the Hamilton–Jacobi Equations one should be able to

78 See (Hankins 1980, 184–186); (Nakane and Fraser 2002, 163); (Langhaar 1962, 13).

79 (Hankins 1980, 184).

80 Following (Nakane and Fraser 2002, 163). Hamilton's statement is restricted at (Hamilton 1834, 273) to a conservative system of but two point-masses. At (Hamilton 1835a), the formula for the variation of the force-function is suitably generalized. Moreover, at (ibid.) it is stated (as a well-known fact) that *the equations of motion follow from considering the variation of the force-function*.

81 (Hamilton 1834, 307–308); (Goldstine 1980, 178, 183); (Nakane and Fraser 2002, 180–181).

discern "by its partial differential coefficients" every "intermediate and" every final integral "of the known equations of motion"(*MPH2*, 212, *British Association Report* of 1834).

In (Hamilton, 1834) and (Hamilton, 1835), Hamilton defines S via:

$$S = \int_0^t (T + U)dt = \int_0^t \mathscr{L}dt$$

where $\mathscr{L}$ is the Lagrangian within which the force-function (*inter alia*) resides (Hankins, 1980, 192; Lützen, 1995, 10).[82] This statement of S is inferred from the integral statement of the characteristic function V. Hamilton also used the principal function S to state what has become known as **Hamilton's principle** or what some call **the principle of least action**[83]:

(**Hamilton's Principle (HP)**):

$$\delta S = \delta \int_0^t \mathscr{L}dt = 0$$

HP is part of both modern Hamiltonian and Lagrangian mechanics.[84] Hamilton derives **HP** from the **T-LVA** on the assumption that the path is fixed (so that the variation of the Hamiltonian vanishes yielding: $\delta H = 0$), and that the end and initial points are fixed (so that variations on the coordinate variables vanish resulting in $\delta V = 0$). And because, $\delta S = \delta V - t\delta H$, δS vanishes. Obviously, if one works with just the **LVA**, on the same assumptions, it also follows that $\delta V = 0$, a result Hamilton called the law of least (or stationary) action (Hamilton, 1834, 252); (Hankins, 1980, 186). For Hamilton, both the **T-LVA** and **LVA** are respectively upstream from the **HP** and **law of least action**. That is to say, the law*s* of varying action are fundamental, and the **HP** and **law of least action** are derivative.

In (Hamilton, 1834), Hamilton used the force-function to present his equations of motion (the **DPVW** is an equation of motion). Hamilton's canonical equations of motion in (Hamilton, 1835) likewise use the force-function.[85] Moreover, for systems like SYS, modern Hamiltonian mechanics specifies the Hamiltonian in terms of the force-function (see Sect. 3), and for some systems unlike SYS, the Hamiltonian is ordinarily specified in part in terms of the Lagrangian which (again) harbors U.

4.2 The Potential Energy Function Represents Forces

I have shown that the force-function was *used* by Hamilton to present central formulae of his dynamics, to state the Hamiltonian H, and to state the principal function that is S. I have

[82] Just to be clear, Hamilton did not use the Lagrangian. He used its mathematical equivalent.

[83] (Hankins 1980, 194); (Nakane and Fraser 2002, 184).

[84] See (Feynman, Leighton and Sands 2010, 19–8); (Taylor 2005, 239). There are many titles and names of principles thrown about in the literature. For example, Richard Feynman (1918–1988) called the specification of the action integral $S = \int_0 \mathscr{L}dt$ "the principle of least action" at (Feynman, Leighton and Sands 2010, 19–8). There Feynman is concerned with the relativistic limit, but $\mathscr{L}$ becomes the difference between kinetic and potential energy in the classical and non-relativistic limit (compare Feynman, Leighton and Sands 2010, 19–3).

[85] (Nakane and Fraser 2002, 163).

demonstrated that both the force-function for the initial time of an evolution of a physical system, as well as the force-function for the end-time of that same evolution were used to provide constraints on the principal function S thereby helping to determine the intermediate and final integrals for the equations of motion (*q.v.*, the Hamilton–Jacobi Equations). In addition, I showed how the force-function enters the action that is "minimized" in **HP**. I have described in what way the force-function is *used* to present modern Hamiltonian mechanics. Furthermore, I have argued that *Hamilton* interpreted the force-function in such a way that it represents work and so also forces. That forces were for Hamilton, causes of motion, is a fact established in Sect. 2. But why think that forces play an indispensable role in correctly interpreting potential energy in modern Hamiltonian dynamics? In what follows, I shall begin to address this question by first showing (via five reasons) that the potential energy function (the mathematical object) represents forces at work. I will call this understanding of the potential energy function in classical mechanics **the orthodox interpretation of the potential energy function**. As we shall see (*q.v.*, n. 92), it is and always has been the predominant view in classical non-relativistic physics. I should quickly add that I am *not* here attempting to conceptually analyze potential energy. Rather, I'm arguing that the potential energy function (the mathematical object) *represents* forces at work. That the potential energy function represents working forces is one line of evidence in favor of the thesis that (Θ) facts about forces determine/ground potential energy facts. So, I have five considerations in favor of the orthodox interpretation of the potential energy function, and that orthodox interpretation itself supports Θ.

First, in the context of modern classical non-relativistic mechanics (as during Hamilton's time), energy in general is technically defined as a measure of a physical system's power or ability to bring about work.[86] Work, however, is force multiplied by displacement. Thus, if this characterization is correct, forces are essential to the interpretation of the energy function in general. And so, it becomes less surprising to see among the standard technical definitions of kinetic and potential energy in (classical but non-relativistic) physics and physical chemistry the assertion that kinetic and potential energy (or changes of such quantities) represent work performed by physical systems (Atkins et. al. 2019, 38). Admittedly, this evidence is far from conclusive. I therefore turn to the second line of support.

The SI derived unit of measurement for both energy (whether potential, kinetic, or mechanical) and work is the joule **J**. One joule just is a unit of measurement about a force, more specifically, the work performed or executed by one newton (**N**) of force over one meter (**m**) in a single direction, viz., the direction of the force impressed. In the CGS (centimeter-gram-second or the Gaussian) unit system, the unit of both energy and work is the **erg**. The abbreviation 'erg' derives its meaning from the Greek ἔργον, which means work. It is therefore unsurprising that the erg, like the joule, is a unit of measurement about a force, more specifically, the work performed or executed by one **dyne** (**dyn**; the unit of force in the CGS system) of force over one centimeter (**cm**; the unit of length in the CGS system) in a single direction, viz., the direction of the force impressed. These facts (when restricted to the unit of measurement of potential energy) are best explained by the orthodox interpretation of the potential energy function. Notice also that the units for energy and work both reference forces explicitly. This fact fits extremely well with the supposition that the numerous energy functions as well as the work function represent (are about) forces.

[86] The *Oxford Dictionary of Physics* defines energy as "[a] measure of a system's ability to do work" (Rennie 2015, 180).

Third, the following expresses a law of classical mechanics, whether Lagrangian, Newtonian, or Hamiltonian:

(**Potential Energy Identity (PEI)**):

$$U(\mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \bullet d\mathbf{r}'$$

where U is now positive, the **r**s are position vectors, and the triple bar expresses a mathematical identity. **PEI** says that if a point mass under the influence of a conservative force travels along a curve from an initial position $\mathbf{r}_0$ to another position $\mathbf{r}$, $U(\mathbf{r})$ (*which is really a change in potential energy*) will be minus the work done by the force **F** during the evolution. **PEI** is more than a law of classical mechanics. It is the technical definition of potential energy for the relevant system-types.[87] The mathematical training of contemporary physicists attests that (quoting Trapp, 2019, 388) "… potential energy is defined in terms of work, which is a line integral!" (Again) The work involved is virtual if the referenced displacement due to force-influence is nonactual or unrealized. This should not detract from the view that this specific argument intends to promote, for even if the work is virtual, the definition that is **PEI** underwrites the position that the potential function *qua* mathematical object has content about work and so represents forces.

Fourth, the orthodox interpretation of the potential energy function U fits comfortably within the most accurate depiction of the development of the potential function V in the history of the mathematics of potential theory. That history reveals how from its inception, the potential function was a convenient mathematical device used by theoreticians to help them specify or represent working forces. Indeed, very early on, the potential was even more strongly associated with forces. When Daniel Bernoulli (1700–1782) first used

[87] See, *e.g*., (Taylor 2005, 111) who says that "[w]e define $U(\mathbf{r})$" in terms of, $-W(\mathbf{r}_0 \to \mathbf{r})$. But,

$$-W(\mathbf{r}_0 \to \mathbf{r}) \equiv -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

where W is the work function. That we can define $U(\mathbf{r})$ in terms of work is enough for my purposes because, again, work has to do with force. Hecht remarked,

"…only changes in *PE* are defined, and these are defined as the work done on a system by conservative forces. Such work is measurable as it is being done. However, once work is done it no longer exists and is no longer measurable; if ΔPE is *only* defined by work done (e.g., mgh, or $1/2kx^2$, or $1/2CV^2$) it cannot be measured in stasis while it supposedly exists." (Hecht 2019a, 500 emphasis in the original).

Some maintain that potential energy and work are identical. Coopersmith stated, "[p]otential energy and work were eventually seen to be one and the same (an integration of force over distance)" (Coopersmith 2015, 115).

It is common to understand the work quantity in such a way that it is deemed more fundamental than potential energy. In his classic graduate level text on classical mechanics, Cornelius Lanczos (1893–1974) stated,

"[t]he really fundamental quantity of analytical mechanics is not the potential energy but the work function…In all cases where we mention the potential energy, it is tacitly assumed that the work function has the special form $W = W(q_1, q_2, \ldots, q_n)$, together with the connection $U = -W$" (Lanczos 1970, 34)

I have rephrased Lanczos's equations by using my own notation. The first equation in the quotation is inserted here in place of Lanczos's reference to Eq. (17.6). I argue that forces are more fundamental than potential energy in Sect. 4.3 below. Lanczos argues that work is more fundamental than force (Lanczos 1970, 27). I disagree. Work is technically defined in terms of force.

the equivalent of the term 'potential' he used it to denote the magnitude of a force.[88] Euler called the mathematical equivalent of the potential, force effort (*q.v.*, n. 77). One first sees the use of the potential function, as we understand it (although not by that name), in Lagrange's "Sur l'équation séculaire de la lune" ("On the Secular Equation of the Moon") published in 1773. In that work, Lagrange attempted to track the gravitational attractive force of a physical system with the potential function (Gray, 2015, 132). A precursory reading of that work reveals how the potential function "seems to serve principally as a convenient method for calculating the components of force in various coordinate systems."[89] It is an instrumentally useful device.

One of the earliest developers of potential theory was George Green (1793–1841). He emphasized (a) that the potential represents forces, (b) that it is a device of convenience, and (c) that forces are prior to the potential function in that the latter in some sense "arises" from the former. With respect to (a), Green wrote,

> It is well known, that nearly all the attractive and repulsive forces existing in nature are such, that if we consider any material point p, the effect, in a given direction, of all the *forces acting upon that point*, arising from any system of bodies S under consideration, *will be expressed* by a partial differential of a certain function [the potential function] of the co-ordinates which serve to define the point's position in space.[90]

What Green says here is that the potential expresses the force by helping to represent its components in partial differentials. With respect to (b), Green wrote that the potential "gives in so simple a form the values of the forces" (Green, 1871, 22), asserting over and over again (and thereby addressing point (c)) that the values of the potential arise from electric charges of bodies and external forces (ibid., 70, 76, "the value of the potential function arising from the exterior force" (ibid., 78)); The potential function is "due to the exterior forces" (ibid., 107), and it arises "from the magnetic state induced in it by the action of the forces…" (ibid., 92)).

In 1782, Pierre-Simon Laplace (1749–1827) presupposed that the potential function V associated with a spheroid satisfied an elliptical partial differential equation of the second order, an equation that now bears his name (in Cartesian coordinates):

Laplace's Equation (LE):

$$\nabla^2 V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

where ∇^2 is what we now call the Laplacian operator. With this presupposition in hand, Laplace was able to represent the gravitational force resulting from the spheroid system by means of the potential function (Gray, 2015, 132). But the potential was used by Laplace because it (quoting Gray) "simplified his analysis of problems having to do with gravitational attraction…".[91] In 1813, Siméon Denis Poisson (1781–1840) generalized Laplace's equation:

88 See (Archibald 2003, 198) and (Dunnington 1955, 160).

89 (Archibald 2003, 198).

90 (Green 1871, p. 9) emphasis mine.

91 (Gray 2008, 71).

$$\nabla^2 V = -4\pi\rho$$

where ρ is density. And so for gravitation we retrieve **Poisson's equation**:

$$\nabla^2 \phi = -4\pi G\rho$$

where ϕ is a scalar potential often thought to represent the gravitational field. When ρ vanishes, Laplace's equation is recovered. It was soon realized by Poisson and Charles-Augustin de Coulomb (1736–1806) that the state of affairs involving a mass distribution system $\rho(\mathbf{r})$ exerting a force-influence over a system of unit mass at a spatial point $\mathbf{r}'$ is represented by (Gray, 2015, 132; 2008, 71; *q.v.*, n. 135 below):

$$\mathbf{F}(\mathbf{r}') = \int \rho(\mathbf{r}) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r$$

But as Gray (2015, 132; *cf*. 2008, 71–72) notes, it was realized that the potential function $V(\mathbf{r}')$ could *greatly simplify matters* (device of convenience) because:

$$V(\mathbf{r}') = \int \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} d^3 r$$

from which one can derive:

$$\mathbf{F} = -\nabla V$$

This last equation provides a segue to my fifth reason for interpreting the potential energy function U as one that is about (represents) forces at work, for a very similar equation (identical in form) holds true in classical mechanics. Suppose once again that there is but one point mass traveling along a curve in three dimensions of space due to a conservative force (call this target system SYS). Assume that SYS is conservative and holonomic. In modern Hamiltonian mechanics, the force in SYS will be what is sometimes called the corresponding force of the potential energy function. This corresponding force in SYS, is the referent of $\mathbf{F}$ in the integral statement of the potential energy function that is **PEI** above. The relation of correspondence with respect to SYS can be stated quantitatively as:

$$\mathbf{F} = -\nabla U$$

which is obviously related to **PEI**. This equation says that the minus partial derivatives of the potential energy function express the components of the corresponding force in particular directions.[92] With respect to a single coordinate direction, *e.g*., the z-direction, this equation entails that $F_z = -\frac{\partial U}{\partial z}$. Thus, the sharper or deeper the dip in the gradient of the potential energy function, the more robust the involved force is, given that F_z faces toward $-z$ and that the potential increases when there's an increase in z.

It should now be clear. The potential energy function represents forces. It should now also be clear precisely how the potential energy function represents forces. The minus partial derivatives of the potential energy function represent the components of a force in appropriate coordinate directions. This understanding was handed down to us at the start

92 See (Goldstine 1980, 176); (Gray 2008, 71); (Taylor 2005, 111, 116–117).

of potential theory.[93] The appropriate representation fact is surprising if force-facts do not ground potential energy facts. It is unsurprising if force-facts do indeed ground potential energy facts.

4.2.1 Forces Beyond the Newtonian Formulation

Have I reverted to the Newtonian formulation of classical mechanics by explicitly connecting the potential energy function with forces? Not at all. You will recall that Hamilton (1834, 249) forged the same type of connection when he affirmed **FF**. In addition, I remind the reader that Hamilton called the potential energy function the force-function. Even today, we think of potential energy as the "energy of interaction".[94] It should be no surprise then that forces remain part of the ontology of modern Hamiltonian mechanics. There is nothing distinctively anti-Hamiltonian about them. When a point mass is under sway of a central force field, modern Hamiltonian mechanics specifies the involved radial force (in polar coordinates) as follows[95]:

$$\mathbf{F}_{radial} = -\frac{dU}{dr}$$

What one should remember is that in many different contexts, Hamiltonian mechanics is correctly used to arrive at the same precise equations one can discover using force-laden Newtonian mechanics.[96] In other words, modern Hamiltonian mechanics does in fact provide important information about forces.

Some will accept that forces are central to modern Hamiltonian mechanics. At the same time, they will insist that for the general case of an n-particle system, Hamiltonian mathematical models use generalized forces and (quoting North) "[i]t isn't clear that they count as regular forces of the Newtonian kind".[97] A generalized (conservative) force with n-components is a higher dimensional vector that lives in a higher dimensional abstract mathematical space.[98] But in analytical mechanics of either the Lagrangian or Hamiltonian varieties, a generalized (conservative) force Q_i depends upon and is asymmetrically determined by real-world (North's "Newtonian") forces whether due to either external fields or interactions between constituents of systems. Consider now my argument for this conclusion.

If there are real-world forces $\mathbf{F}_i$ acting between point-particles with masses given by m_i, we say that their total work is[99]:

(**Total Work or DPVW***):

[93] If the reader is tempted to understand potential energy in such a way that it is basic, I ask that that reader digest the arguments of Sect. 4.3 as well as the analogical consideration in Sect. 5. I am here only trying to show that the potential *function* represents forces. This is a claim about the mathematical object and not a claim about potential energies in the world.

[94] (Coopersmith 2015, 337).

[95] (Taylor 2005, 531–532).

[96] (ibid., 532).

[97] (North 2022, 29).

[98] I will assume that generalized force Q_i has the dimension of a force and not that of the moment of a force. On this distinction, see (Langhaar 1962, 17). For more on generalized forces in general, see (Fitzpatrick 2011); (Lanczos 1970, 27-31); (Langhaar 1962, 14-23); (Peacock and Hadjiconstantinou 2007); (Sommerfeld vol. 1 1964, 187-189); (Stewart 2016, 16–21). In places, my discussion follows these sources.

[99] Following (Lanczos 1970, 28).

$$dW = \sum_{i=1}^{n} (X_i dx_i + Y_i dy_i + Z_i dz_i)$$

assuming that the particle constituents of the system have Cartesian coordinates that change by the arbitrary infinitesimal amounts represented by dx_i, dy_i, and dz_i, and that X_i, Y_i, and Z_i give the components of the real-world forces $\mathbf{F}_i$ in appropriate orthogonal directions (*qq.v.*, n. 98, n. 99, and n. 101). A quick comparison reveals that this equation is equivalent to Hamilton's **DPVW** above. Modern analytical (*e.g.*, Hamiltonian) mechanics exploits the fact that the Cartesian coordinates are functions of the generalized coordinates $\left(q_1, \ldots, q_n\right)$ such that the generalized coordinate variables have an invariant first-order differential form that is linear[100] and expressed by[101]:

$$dW = (F_1 dq_1 + F_2 dq_2 + F_3 dq_3 + \ldots etc \cdots + F_n dq_n)$$

The coefficients F_1, F_2, F_3, etc. are not the components of the real-world forces $\mathbf{F}_i$, but are instead the components of generalized force Q_i, that higher dimensional abstract vector previously mentioned (more on these components soon). I will now use the symbol: $\mathfrak{F}_i$ to represent the generalized force components.

Q_i can be thought of as an abstract actor in the n-dimensional configuration space used to model the system. Points no longer just travel or orbit without reason, they orbit *because* the generalized force with its n-components "acts", "producing" the orbit (Lanczos, 1970, 28–29). Of course, this is not an action indicative of causation. The 'because' involved here is one "without cause". The explanation provided is an explanation by constraint as in (Lange, 2017). However, we have reason to believe that these explanations by constraint in the abstract configuration or phase spaces are determined by causal explanations in the real world.

Consider that the components of Q_i can be related to potential energy and generalized coordinates as follows:

$$\mathfrak{F}_i = -\frac{\partial U}{\partial q_i}$$

But by **PEI**, we know that one can also state the generalized force components in terms of work and generalized coordinates:

$$\mathfrak{F}_i = \frac{\partial W}{\partial q_i}$$

What these equations reveal is that one can calculate generalized (conservative) forces by looking to the potential energy or work functions. This is because the real-world forces lying beneath work (work is force times displacement) and potential energy (identical to minus work) determine generalized force (*q.v.*, **GF** below). But the minus partial derivatives of potential energy give the components of real-world forces (*q.v.*, Sect. 4.2). Of course, matters are not resolved. When the work function is time-dependent, there is no potential energy to look to for the purposes of calculating generalized force. Of course, this bothers my project none because work remains, and residing in work is real-world force. Furthermore:

[100] (Lanczos 1970, 28).

[101] We could use δ_n instead of d_n to highlight the fact that the involved force impression can be virtual (as in Langhaar 1962, 16).

(**Generalized Force (GF)**):

$$Q_j \equiv \sum_{i=1}^{n} \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_j}$$

where here, $\mathbf{r}_i$ gives the position vector for the i^{th} point where a real-world force is applied (assuming that $\mathbf{r}_i$ is a function of the generalized coordinates) such that its variation is given by[102]:

$$\delta \mathbf{r}_i = \sum_{j=1}^{m} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$

Most importantly, $\mathbf{F}_i$ represents real-world forces such as gravitation. A plain implication of all of this (especially **GF**) is that generalized force depends upon real-world applied forces. Indeed, the components of Q_j represent the numerous forces standing behind $\mathbf{F}_i$, *i.e.*, the real-world working forces in the system. Thus, generalized force Q_j is a more abstract representation of force impressions and actions in the physical world. That is why Lanczos (1970, 29) says, "the dynamical action of all the forces can…be represented by a single vector [*i.e.*, the generalized force vector] acting on" a point in a higher-dimensional mathematical space.

4.2.2 Back to Potential Energy

Some will reject the orthodox interpretation of the potential energy function. They will choose instead to affirm that potential energy functions do not represent forces, rather forces represent an absolute and objective potential energy understood as a "capacity to do work".[103] Call this outlook **the capacity interpretation of the potential energy function**. On this view, potential energy just is an *objective* dispositional property of physical systems picked out by U.

While the capacity interpretation is not identical to the orthodox approach, it is nonetheless friendly to my central thesis because it causally interprets the potential energy function by ascribing to certain systems with potential energies causal powers, viz., capacities or dispositions to act according to force impressed over a distance thereby performing work. Ignoring for now the benefits this account would provide for my cause, I should detail how its fatal flaw follows from a point made by some of the very advocates of the capacity interpretation (*i.e.*, Marion & Thornton, 1988, 73). If the capacity interpretation were correct, whether a system has an *objective* potential energy or not would be a fact insensitive to conventions and arbitrariness introduced by physicists. But in any case, there is no means whereby one can measure such an objective potential energy that is an objective capacity.[104] (At this point, I should acknowledge my indebtedness to the work of Eugene Hecht

[102] See (Fitzpatrick 2011); (Peacock and Hadjiconstantinou 2007, 6); (Stewart 2016, 19).

[103] (Marion and Thornton 1988, 72).

[104] What is said in the main text holds true for both internal and external potential energy. None "of these potential energies is independently measurable" (Hecht 2016, 10); (Ryder 2007, 58). What I say here also holds true for static electric field energy (Hecht 2019b, 3) (*q.v.*, Sect. 5).

Albert Einstein (1879–1955) said,

"[b]ut if every gram of material contains this tremendous energy, why did it go so long unnoticed? The answer is simple enough: so long as none of the energy is given off externally, it cannot be observed. It is as

(2016, 2019a, 2019b and *q.v.*, n. 113). His papers cite several of the sources I cite in this work). The quantity physicists call "potential energy" in the three main formulations of classical mechanics is a quantity whose differences alone ultimately matter because the differences are what can be measured. Of course, the fact that a system's evolution over time can be described by appeal to differences in potential energy presupposes that it at least makes sense to ascribe to said system potential energy full stop (no differences). However, the problem is that the full stop attributions in classical mechanics are arbitrary. Let me explain.

Consider the following well-known and often repeated example used to help teach students about potential energy. Let spatial point x at height $h(x)$ be the summit of Mauna Kea (the tallest mountain on earth measured from base to summit). Put a simple particle system or body b there. One now states the (gravitational) potential energy of the system as follows:

(**Gravitational Potential Energy (Approx.) (GPEA)**):

$$U_g = mgh(x)$$

But here g is gravitational acceleration due to an acting gravitational force,[105] m is the gravitational mass (equivalent to the inertial mass) of a body b at that height, and $mgh(x)$ is interpreted as the *work* required to get b to $h(x)$ given that b started somewhere else, *e.g.*, sea level (you will notice that work is now being set equal to the gravitational potential energy) without imparting kinetic energy to b while laboring against gravity. But sea level here was chosen arbitrarily to facilitate acquisition of the differences one needs to make sense of U_g.[106] Sea level is arbitrarily determined to be the place where U_g vanishes (French, 1971, 378–379). How then can the potential energy referenced here be an objective property of the system when its quality and quantity are determined by the arbitrary decisions of physicists? Of course, this arbitrariness will infect the quantity that is (virtual) work too, but the infection is not overly contagious. The mg portion (weight) of **GPEA** expresses the force pulling the object toward the center of the earth.[107] The only way to get b to $h(x)$ without the impartation of kinetic energy is to exert an external force on b (French, 1971, 378). When any conservative force acts thereby producing an acceleration, what transpires is a purely objective affair. More reasons will be given for this judgment soon (*q.v.*, Sect. 4.3).

That the most important points in the preceding discussion of this subsection (viz., those about (a) arbitrariness and potential energy in general, (b) arbitrariness and gravitational potential energy in particular, and (c) the **GPEA**) hold true in both Lagrangian and Hamiltonian formulations of classical mechanics is well-known.[108] There is therefore no

Footnote 104 (continued)

though a man who is fabulously rich should never spend or give away a cent; no one could tell how rich he was." (Einstein 1954, 340).

105 Sometimes it is interpreted as the gravitational force field.

106 Absolute potential energy does not fail to make sense because it is not measurable. There is no verificationism afoot here.

107 It is therefore no surprise that the SI unit of measurement for both weight and force is one and the same, viz., the newton.

108 (Fecko 2006, 517–518); (Penrose 2005, 475); (Pletser 2018, 55); *cf.* (Taylor 2005, 542 Eq. 13.51). The Hamiltonian would be similarly specified in the Hamilton-Jacobi formulation as well. See (Pletser 2018, 55); (Torres del Castillo 2018, 256, 266–267).

escape from my reasoning by appeal to a shift in formalism. But we can say something more general. In almost every context, the specification of the potential energy function does not change across the three major formulations of classical mechanics.

Let us now entertain an objection to my argument from arbitrariness. The GPEA is but an approximation even in classical mechanics. It holds true for bodies near to the earth, but as one moves away from the earth, one will need to use a different expression for gravitational potential energy, one that follows from Newton's famous universal law of gravitation and the *definition* that is **PEI**:

(**Gravitational Potential Energy (GPE)**):

$$U_g = -\frac{GMm}{r}$$

where M is the gravitational mass of the Earth (the body exerting the gravitational force), m is the gravitational mass of body b, r is the distance between b and the Earth, and big G is the gravitational constant.

Switching to **GPE** does not help matters. That law is usually posited given yet another arbitrary choice. This time it is to set the gravitational potential energy of b that is U_g equal to zero when b is situated at a position infinitely far removed from the earth. Potential energy decreases as b moves closer to the earth and so is negative.[109] Physicists will usually add that this arbitrary decision is reasonable, natural, or intuitive, but I have been unable to find an argument for such a choice that would render that choice principled and non-arbitrary.[110] I do not question that it is convenient for calculations, but that does not mean it informs us about what the world is like even given a robust scientific realism. There are fictional devices aplenty in physics that help us with our calculations, but no one would commit to the existence of these devices. Who likes Gaussian surfaces?[111] Potential energy, yes, even gravitational potential energy, can only be specified up to an *arbitrary* additive constant. (I am not interested here in quantum physics. I leave matters pertaining to potentials and arbitrariness in that theory purposefully unaddressed.)

4.3 Force is More Fundamental than Potential Energy

There are those who wonder whether there is any reality to potential energy at all.[112] I will not go so far as to challenge the existence of potential energy. I will instead argue that potential

[109] All of this remains as I am presenting it in both Newtonian and Hamiltonian mechanics. See (Meyer and Offin 2017, 61–62) for a discussion of gravitation and classical non-relativistic Hamiltonian mechanics.

[110] I should add that I believe that the arbitrariness is to blame for the negativity. The arbitrariness is also the reason why negative gravitational potential energies should not confound the metaphysician of physics (Mann 2018, 17). If energies are but measures or convenient ways of representing what's really happening with forces, then the signs of the convenient devices need not bother one. It's what's fundamental (or, in this case at least, what's more fundamental) that matters.

[111] The reference to Gaussian surfaces wasn't just for the purposes of being humorous. Gaussian surfaces are *arbitrarily* specified closed surfaces introduced by the physicist to help with calculations in electrodynamics.

[112] (Sullivan 1934, 247–248).

energy is less fundamental than force.[113] This conclusion seems to be in line with a well-represented position among physicists.[114] And like the discussion in Sect. 4.2, it also fits nicely with the thesis that (Θ) facts about forces determine/ground potential energy facts. This is because on modern theories of grounding, necessarily, if fact [x] grounds fact [y], it follows that fact [y] is more derivative than fact [x] (Schaffer, 2009). One reason why force-facts might be understood as facts that ground potential energy facts is because of the further quantity-grounding relation, viz., that potential energies are grounded by forces or force interactions.

As I have already noted, potential energy has no absolute or objective status in classical mechanics. One can only specify its mathematical representative up to an arbitrary additive constant. To evaluate facts about potential energy, one must admit arbitrariness, and even after the necessary arbitrariness is in play, differences in potential energies are all that one can have access to. By contrast, when a force is impressed thereby producing an acceleration, the resulting situation is absolute and invariant in the sense that that causal fact will hold relative to every inertial frame of reference in Hamiltonian mechanics. Conservative forces are Galilean invariant quantities, and the equations of motion which relate those forces to resulting motions are Galilean covariant.[115] However, potential energy is not Galilean invariant.[116] There is no guarantee that performing a Galilean transformation on the mathematical statement of potential energy for a system in an inertial frame will yield an equation of the same form preserving the same value for the potential energy of that system. We should deem invariant quantities more fundamental than non-invariant ones. Some have gone farther. For example, Saunders (2003a, b, 299; 2013a, b) has argued that we should regard as real only those quantities that are "invariant under the symmetries of" Newtonian mechanics. Again, I will not go so far as to challenge the existence of non-invariant quantities. Rather, I maintain that in classical mechanics invariant quantities are more fundamental than non-invariant ones. At a non-relativistic classical world, we should deem Galilean covariant laws more fundamental than non-covariant laws. Is this not a lesson taught by the

[113] I believe this is the best philosophically sophisticated characterization of the position defended in (Hecht 2003, 2007, 2016, 2019a, 2019b).

[114] As Robert Mills (of Yang-*Mills* theory) said, "the idea of potential energy is not truly fundamental and that it breaks down in the relativistic world…" (Mills 1994, 152). *Cf.* (Lanczos 1970, 34) already quoted; and (Thomson 1888, 15). Coopersmith (2015, 339) argues that because kinetic energy has the same form for every system and potential energy does not, the former is more fundamental than the latter (citing (Maxwell 1871) in support).

[115] (Raju 1994, 64). See also (Maudlin 2011, 174).

[116] As Jennifer Coopersmith remarked in correspondence:

"Even outside of Einstein's Relativity theories, it is not always true that the potential energy is invariant. A requirement for Galilean invariance is that the potential energy depends only on 'relative' coordinates (e.g. the difference between two positions) and not on 'absolute' coordinates (the position of the old oak tree at the corner of the street)." (11/08/2020).

See (Diaz et. al. 2009, 271–272) who remarked,

"…we illustrated that after a change of reference frame, the work done by each force also changes (even if the transformation is Galilean). Consequently, the corresponding potential energies change when they exist." (ibid., 272).

These authors give an argument for their conclusions at (ibid., 271–272).

Horzela et. al. (1991) stated that,

"The explicit expressions of the potential energy as functions of the position $\vec{x}(t)$ all have noncovariant meaning and therefore may be valid only in one inertial reference frame." (ibid., 12, their argument for this starts on page 11).

With respect to force and acceleration, Tefft and Tefft (2007) state,

"…quantities, such as acceleration and force, are invariant or, to use Newton's term, 'absolute' between inertial reference frames. Such quantities have the same values in any inertial frame." (ibid., 220).

successes of our best physical theories? The laws of classical mechanics that have passed empirical muster are precisely those laws that are appropriately constrained by meta-laws that are symmetry or invariance principles.[117] It is at least highly intuitive that the empirically successful laws are the best candidate laws for putting us in touch with the world (assuming scientific realism). These laws recommend certain invariant quantities. And surely the invariant quantities are the objective quantities, the absolute quantities, the quantities whose important features do not shift and change because of a shift in the observer's inertial frame of reference. Let me further explicate the argument in play.

Robert Nozick (1938–2002) wrote,

> Amalie Emmy Noether showed that for each symmetry/invariance that satisfies a Lie group, there is some quantity that is conserved. Corresponding to invariance under translation in space, momentum is conserved…and to invariance of the law under the addition of an *arbitrary constant* to the phase of the wave function, apparently electrical charge is conserved. So it is not surprising that laws that are invariant under various transformations are held to be more objective. Such laws correspond to a quantity that is conserved, and something whose amount in this universe cannot be altered, diminished, or augmented should count as (at least tied for being) the most objective thing there actually is.[118]

I have cut out that portion of the quotation which reads: "Corresponding…to invariance under translation in time, [mechanical] energy is conserved" (Nozick, 2001, 81). I have done this to emphasize that Nozick's invitation to his reader is for that reader to conclude that mechanical energy is one of "the most objective" things "there actually is." On the contrary, while it is true that for isolated systems featuring only conservative forces total mechanical energy is conserved, that fact does not entail the Galilean invariance of total mechanical energy, potential energy, or kinetic energy. None of these quantities are Galilean invariant. That is why neither the Hamiltonian nor the Lagrangian are either.[119] However, for the types of systems with which I have been concerned, the Hamiltonian becomes total mechanical energy which, again, is conserved. Thus, not all conserved quantities are Galilean invariant quantities. And so, as Schroeren put matters, "Noether's theorem concerns the way particles behave *under temporal evolution*, i.e., whether certain physical

[117] *Cf.* the discussion in (Earman 2004, 1230).

[118] (Nozick 2001, 81). Earman (2004) likes the type of inference invited by Nozick in the context of classical mechanics, but believes it suffers important setbacks when dealing with general relativity.

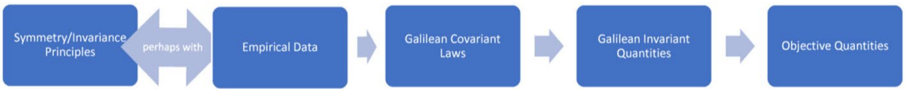
"The implementation of part of Nozick's formula *objectivity* = *invariance* by means of the constrained Hamiltonian formalism goes swimmingly: in case after case it yields intuitively satisfying results. But the application to Einstein's GTR yields some surprising and seemingly unpalatable consequences." (ibid., 1234).

[119] As Coopersmith stated, "energy is not an invariant quantity" (2015, 342). The Hamiltonian is not invariant under boost operations (Butterfield 2007, 6). Butterfield said, "the Hamiltonian of a free particle is just its kinetic energy, which can be made zero by transforming to the particle's rest frame; *i.e.* it is not invariant under boosts." (ibid.) This is true even in quantum mechanics (Lombardi et. al. 2010, 99). In classical mechanics, the Lagrangian is invariant under rotations and translations, but not under boosts (Finkelstein 1973, 106–107). Be careful. Landau and Lifshitz's famous text on mechanics says that "the Lagrangian is [Galilean] invariant", but their argument only demonstrates that Lagrange's equations of motion are covariant under Galilean transformations (Landau and Lifshitz 1976, 7).

Interestingly, Coopersmith (2015, 239) says that the "potential energy function" just "is the Hamiltonian… and it is the function that determines the entire dynamics of the system." If Coopersmith is right, it would not be surprising that potential energy fails to be a Galilean invariant quantity (*cf.*, ibid., 313).

I must add here that Green's function is not guaranteed to be invariant either (Appel 2007, 165).

quantities are conserved"; it does not concern whether those quantities are Galilean invariant.[120] Yet, Nozick was supposed to help us commit to Paul Dirac's (1902–1984) dictum that "[t]he important things in the world appear as the invariants…"[121] *These* important things are the objective things. At a non-relativistic classical world (which is what my discussion is limited to), it is Galilean invariance (using Nozick's wording) that "is [or should be]…connected to something's being an objective fact".[122] But while Hamilton's laws or equations of motion are covariant under Galilean transformations, as is the work-energy theorem, both those laws and that theorem fail to "correspond to a quantity that is conserved" and that is one of "the most objective thing[s] there actually is", *if* the quantity in question is taken to be the Lagrangian, the Hamiltonian, mechanical energy, potential energy, kinetic energy, or even work.[123] Nozick's inference runs from invariance or symmetry principles (perhaps together with or closely followed by empirical data), to invariant (or covariant) laws, to conserved quantities, and then to objective quantities. But that inference is a bad one. Nozick's inference should have had the following structure for the non-relativistic classical case:



What explains the Galilean covariance of the laws of classical mechanics (including Hamilton's equations of motion) is the nature of operating forces. That is why Tim Maudlin was right to stress that "…the necessary and sufficient condition for any Newtonian [classical but non-relativistic] theory to be Galilean invariant is that the force F be the same in all

[120] (Schroeren 2020, 52) emphasis in the original. Schroeren goes on to correctly note how "every property linked to a symmetry in the relevant sense is invariant under that symmetry" (ibid.). So, some type of thesis regarding the invariance of energy may be saved. However, that type of invariance cannot be indicative of that which is one of "the most objective thing[s] there actually is" for, again, total mechanical energy is conserved but not objective at least because of the arbitrariness that sneaks into potential energy (*q.v.*, Sect. 4.2). I do not know if Schroeren would agree with my conclusions.

[121] (Dirac 1995, 456). Nozick himself uses this quotation at (Nozick 2001, 76).

[122] (Nozick 2001, 76).

Someone may ask: But what about the action that is minimized (or taken to equal an extremum) in Lagrangian and Hamiltonian mechanics? That quantity *is* invariant under Galilean transformations, and it is typically understood in terms of the Lagrangian multiplied by a small change in time (usually flanked by the integration symbol (the "action integral")). It is difficult to discern the metaphysical nature of that quantity, but it is far from potential energy alone. My view of the relationship between force and action belonged to both Euler and Lagrange. The action and action integral track the evolution of the system by indirectly representing its dynamical force interactions. That is why when you shift from one system with an operating force F_n to a system with different operating forces, the form of the action changes. Euler recognized that if the system involves accelerations, the action is a minimum (or extremum), given that the system being modeled is acted upon by forces (Euler 1744, 311–312). The correct act of integration yields the system's trajectory only under the assumption that a force has acted and that the form of the integral is appropriately specified in light of that force-action. This same point was made by Lagrange. See (Lagrange 1867, 365–468) and the translated quotation at (Boissonnade and Vagliente 1997, xxxiii). So, the explanation of the invariance of the action integral arises from the invariance of the acting forces, those same forces that the form of the action integral is sensitive to.

[123] The incorporation of work into the list should not be surprising. As I've already argued, energy should be technically defined in terms of work. Energy is not Galilean invariant, and so neither is work.

inertial frames."[124] Forces *are* invariant and objective. So too are their resulting accelerations. Potential energy is neither invariant nor objective. (Quite intuitively) The objective things are more fundamental than the non-objective things. Thus, force is therefore more fundamental than potential energy.

5 Potential Energy and Classical Electrostatics

There exists a weighty analogical consideration in favor of the claim that potential energy in classical mechanics is grounded in working forces. The argument starts out by referencing the known marked similarities between, on the one side, potential energy U_E in electrostatics and the electrostatic potential Φ (in that same science), and on the other side, the potential energy function U in classical mechanics.[125]

Consider a conservative isolated system of classical electrostatics (SYS_{E}). Here the electric potential Φ "is the potential energy of a unit charge"[126] q, such that:

$$U_E = q\Phi$$

I have already shown how potential energy is related to work in mechanics. To show off that same relationship-type in electrostatics, add to SYS_{E} an electric field $\mathbf{E}$.[127] In electrostatics, this field will stand in the same relation to Φ that $\mathbf{F}$ stands in relation to U in classical mechanics.

$$\mathbf{E} = -\nabla\Phi$$

Electric force becomes,

(**Simple Electric Force (SEF)**):

$$\mathbf{F}_E = \mathbf{E}q = -\nabla(q\Phi) = -\nabla U_E$$

Think about what this says. Think about how physicists are trained from multivariable calculus courses onward. They are taught (correctly) that this equation interprets "the electric field E as the *force per unit charge* of the system" (Trapp, 2019, 169 emphasis in the original). The point charge determines the field (see also **SEFi** below). The very elementary equations we've explored thus far have well illustrated consequences given by the standard vector field representation of the gradient of the electric vector field (where our point charge resides at the origin) provided in the figure at (Trapp, 2019, 230). Is it a coincidence that the field exhibits such structure relative to the point charge? Obviously not.

Now if ϵ_0 is the electric or vacuum permittivity constant, and there are two point charges q_1 and q_2, then the force between them (which will yield an equal and opposite force) will be given by Coulomb's law:

(**Coulomb's Law (CL)**)[128]:

[124] (Maudlin 2011, 174).

[125] Please note that I am not here discussing the Lorentz force of classical relativistic electromagnetism, but the simple electric force of classical electrostatics (see **SEF** in the main text).

[126] (Shankar 2016, 97).

[127] I assume that SYS_{E} is one for which it is true that the curl of the electric field equals zero.

[128] We can, of course, simplify many of these expressions (as in Shankar 2016, 19–41), but I like to explicitly state the relations as they appear in the main text thereby following the well-regarded exposition of electrostatics in (Jackson 1999, 24–56).

$$\mathbf{F}_E = \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{\mathbf{r}_1 - \mathbf{r}_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|^3}$$

CL says that the electric force is determined by the locations and properties of point charges. And again, in classical electrostatics, the electric field is likewise determined by point charges. That is why one can motivate the following expression for the electric field by appeal to **CL** (*q.v.*, also n. 129):

(**Simple Electric Field (SEFi)**):

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} q_1 \frac{\mathbf{r} - \mathbf{r}_1}{\left|\mathbf{r} - \mathbf{r}_1\right|^3}$$

which specifies the electric field **E** at **r**. This equation says that **E** is *generated* by point charge q_1.[129]

When a field **E** is in play, the electrostatic potential is reified, and one introduces to $\mathrm{SYS_E}$ an array of $(n-1)$ charges q_j situated in spatial region r_j, that array will *generate* a potential $\Phi\left(\mathrm{r}_i\right)$, such that[130]:

(**Potential from Array (PFA)**):

$$\Phi\left(\mathbf{r}_i\right) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n-1} \frac{q_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

And this helps us support the judgment that the total potential energy of the system featuring the point charge array, electric field, and potential is:

(**Potential Energy and Work (P&W)**[131]):

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i q_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}$$

Even though I've here specified the potential energy of the system (*i.e.*, the potential energy of every charge under the influence of every force in the system), my inclusion of the work variable W on the left is purposeful. In this context when you give the work in electrostatics you give the potential energy. Once again, this is precisely what you'd expect if potential energy were correctly characterized in terms of work. But why should one believe that forces wrought by the point charges are more fundamental than potential energy? Why should one believe that forces ground potential energy?

As with potential energy in classical mechanics, the potential energy of systems like $\mathrm{SYS_E}$ can only be specified up to an arbitrary additive constant.[132] The same is true of the electrostatic potential. Thus, any specification of these quantities will involve some arbitrariness and it is because of this arbitrariness that orthodoxy in electrostatics typically treats the potential as a conventional and derivative device that helps with calculations. For

129 (Jackson 1999, 24–25, 29).

130 (Jackson 1999, 40).

131 (Jackson 1999, 41); (Greiner 1998, 28).

132 See the proof in (Langhaar 1961, 19). As in the case of celestial gravitational potential energy, the arbitrary reference point needed is usually assumed to reside infinitely far away (Trapp 2019, 389).

example, in David Griffiths' widely used textbook, we have: "[p]otential as such carries no real physical significance, for at any given point we can adjust its value at will by a suitable relocation of…[the reference point]".[133] Olness and Scalise remarked, "…the potential itself is not a physical quantity. In particular, we can shift the potential by a constant…and the physical quantities will be unchanged."[134]

Orthodoxy in classical electrostatics has it that U_E is closely connected to work and force. Equation **P&W** and my subsequent commentary make that clear. And because $U_E = q\Phi$, the electrostatic potential is also strongly associated with both work and force. As Feynman said,

> (a) the "electric potential…is related to the work done in carrying a charge from one point to another."[135]

But Feynman likewise expounded orthodoxy when he added that:

> (b) "The existence of a potential, and the fact that the curl of **E** is zero, comes really only from the symmetry and direction of the electrostatic forces."[136]

Orthodox tenets (a) and (b) are both plainly supported by the picture I have painted with the help of the preceding equations. U_E and Φ are strongly related such that: $U_E = q\Phi$ (and **SEF**). The electric field and the electrostatic potential are strongly associated with one another, hence: $\mathbf{E} = -\nabla\Phi$ and **SEF**. But both **E** and Φ are determined by forces between point charges, hence the use of **CL** to motivate **SEFi** (*q.v.*, n. 135), and in addition (considered separately) the fact that the **PFA** holds.[137] **P&W** tells us that the work of a system depends on the positions and natures of point charges. The positions and natures of point charges determine the electric forces at work. This is not surprising because, again, physics teaches that work should be characterized in terms of forces. According to the **P&W**, that work just is the system's potential energy and so, orthodoxy in electrostatics says that "the potential energy…depends on the [acting] forces…"[138] And because there are so many known similarities between potential energy in electrostatics U_E and potential energy U in classical mechanics (some of which I have tried to highlight here), we can infer that it is likely that potential energy in classical mechanics is likewise downstream from and determined by forces.

133 (Griffiths 2017, 81).

134 (Olness and Scalise 2011, 309). In Maxwell's *An Elementary Treatise on Electricity*, he said that "[t]he electric potential…which is the analogue of temperature is a mere scientific concept. We have no reason to regard it as denoting a physical state" (Maxwell 1888, 53).

135 (Feynman, Leighton and Sands 2010, 4–4). *Q.v.*, Appendix 1.

136 (Feynman, Leighton and Sands 2010, 4–7). The fact that the curl of the electric field equals zero, or:

$$\nabla \times \mathbf{E} = 0$$

can be derived from a generalized version of Coulomb's law:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'$$

where d^3r' gives the 3D volume element at $\mathbf{r}'$, and where $\rho(\mathbf{r}')$ gives the volume charge density at $\mathbf{r}'$ (a well-known fact mentioned in numerous places, but see (ibid, 4–3, 4–7) and (Jackson 1999, 24–25, 29)). This derivation-fact supports Feynman's claim.

137 Charges near conductors likewise create electrostatic potentials (Appel 2007, 165).

138 (Shankar 2016, 82).

Any good theory of electrostatics must be relativistic. There is therefore an important objection to the analogical argument just given. The Coulombic or electrostatic force in relativistic electrostatics is not Lorentz invariant, and neither is the electrostatic potential or the electrostatic potential energy (Steane, 2012, 109). Forces in Hamiltonian mechanics are Galilean invariant. I argued that forces in Hamiltonian mechanics are more fundamental than potential energies in that same mechanics because the former are invariant while the latter are not. There is no analogous move in the context of relativistic electrostatics. But recall that nowhere in my discussion of classical electrostatics did I appeal to the property of invariance. Rather, my report on the physics of electrostatics noted how the arbitrariness associated with the electrostatic potential and electrostatic potential energy is analogous to the arbitrariness which infects potential energy in Hamiltonian mechanics. Both electrostatic quantities can only be specified up to an arbitrary constant. This is not true of the electrostatic or Coulombic force. Normally (or defeasibly), given two quantities ξ and κ, if ξ's mathematical representative can only be specified up to an arbitrary constant, and κ's mathematical representative is not similarly associated with arbitrariness, then *ceteris paribus*, ξ is more derivative than κ.

6 Conclusion

I have shown how Hamilton's philosophical commitments led him to a causal interpretation of classical mechanics. I argued that Hamilton's metaphysics of causation was injected into his dynamics by way of a causal interpretation of the force quantity. I then detailed how forces remain indispensable to both Hamilton's formulation of classical mechanics and what we now call Hamiltonian mechanics (*i.e.*, the modern formulation). On this point, my efforts primarily consisted of showing that the orthodox interpretation of potential energy is none other than that interpretation found in Hamilton's work. Hamilton called the potential energy function the force-function because he believed that it represented forces at work in the world. Multifarious non-historical arguments for the orthodox interpretation of potential energy were provided, and matters were concluded by showing that in classical Hamiltonian mechanics, facts about the potential energies of systems are grounded in facts about forces. *If one can tolerate the view* that forces are causes of motion (a thesis I have not argued for here), then one should understand Hamilton's mechanics as having provided the road map for transporting causation into one of the most mathematically sophisticated formulations of classical mechanics, viz., modern Hamiltonian mechanics.

Appendix 1: The Potential Energy—Work—Forces—Causation Link: Nothing New

The orthodox interpretation of potential energy is provided by **PEI**, though it may need some tinkering in order to handle various other system-types. I maintain that because orthodoxy defines potential energy in terms of work, and work in terms of forces, causation enters mechanics through the potential energy function, *if forces really are causes of motion*. Because Hamilton believed forces are causes, he was able to causally interpret classical mechanics by looking to the potential energy function.

Interestingly, my (and Hamilton's) analysis of matters can be found in some of the very earliest work in which potential energy was first used in physics. For example, William Rankine (1820–1872) who coined the term 'potential energy' in 1853, remarked,

> In this investigation the term *energy* is used to comprehend every affection of substances which constitutes or is commensurable with a power of producing change in opposition to resistance, and includes ordinary motion and mechanical power, chemical action, heat, light, electricity, magnetism, and all other powers, known or unknown, which are convertible or commensurable with these. All conceivable forms of energy may be distinguished into two kinds; actual or sensible and potential or latent…*Potential energy*…is measured by the amount of a change in the condition of a substance, and that of the tendency or force whereby that change is produced (or, what is the same thing, of the resistance overcome in producing it), taken jointly.[139]

Rankine is here associating energy (in general) with productive power (causation), but he is also associating potential energy with work, work with forces, and forces with productive power or causation. Seven years later, William Thomson's (or Lord Kelvin's) discussion of the electric potential function (which is used in classical electrostatics to state the potential energy function (*q.v.*, Sect. 5)) strongly associated that potential with work. He wrote,

> *Electric potential*. —The amount of work required to move a unit of electricity from any one position to any other position, is equal to the excess of the electric potential of the second position above the electric potential of the first position.[140]

Thomson here says that differences are what matter, and that the electrostatic potential (and so the electrostatic potential energy) just is work (whether virtual or not) required to complete a task.

Both Rankine and Thomson's views of energy are important because they influenced the work of Rudolf Clausius (1822–1888), James Clerk Maxwell, and Ludwig Boltzmann (1844–1906). These mechanicians thought of entropy as a quantity that tracks how the energy (as understood by Rankine and Thomson) transforms over time.[141] Modern thermodynamics and statistical mechanics has thereby inherited the work-laden and so also force-laden notion of energy. It is therefore unsurprising to see in the work of modern thermodynamicists, such as Klein and Nellis, the following: "…the property entropy is introduced in order to quantify the quality of energy"[142] and "[t]he Second Law states that the quality of energy, i.e., the capability to do work, is reduced in all real processes."[143] All of this is as one would expect given the truth of my interpretation of potential energy.

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139 (Rankine 1853, 106) emphasis in the original.

140 (Thomson 1860, 334) emphasis in the original.

141 See on this point (Weaver 2021, Sect. 5).

142 (Klein and Nellis 2012, 237).

143 (Klein and Nellis 2012, 350).

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References

Andersen, H. K. (2011). Mechanisms, laws, and regularities. *Philosophy of Science., 78*, 325–331.

Appel, W. (2007). *Mathematics for physics and physicists*. Princeton University Press.

Atkins, P., de Paula, J., & Keeler, J. (2019). *Atkins' physical chemistry: Molecular thermodynamics and kinetics*. Oxford University Press.

Bartels, A. (1996). Modern essentialism and the problem of individuation of spacetime points. *Erkenntnis, 45*(1), 25–43.

Bird, A. (2007). *Nature's Metaphysics: Laws and Properties*. Oxford: Oxford University Press.

Boissonnade, A. and Victor N. Vagliente. (1997) "Translator's introduction". In J.L. Lagrange. *Analytical Mechanics: Translated from the Mécanique analytique, novelle édition of 1811,* Auguste Boissonnade, Victor N. Vagliente (Eds.). Translated by Auguste Boissonnade and Victor N. Vagliente. xi-xli. Dordrecht: Kluwer Academic Publishers.

Butterfield, J. (2007). On symplectic reduction in classical mechanics. In J. Butterfield & J. Earman (Eds.), *Philosophy of Physics: Handbook of the Philosophy of Science* (pp. 1–132). North-Holland an imprint of Elsevier.

Butts, R. E. (1968). Introduction. In R. E. Butts (Ed.), *William Whewell's Theory of Scientific Method* (pp. 3–29). University of Pittsburgh Press.

Calinger, R. S. (2016). *Leonhard Euler: Mathematical genius in the enlightenment*. Princeton University Press.

Caparrini, S., & Fraser, C. (2013). Mechanics in the Eighteenth Century. In J. Z. Buchwald & R. Fox (Eds.), *The Oxford Handbook of the History of Physics* (pp. 358–405). Oxford University Press.

del Castillo, T., & Gerardo, F. (2018). *An introduction to hamiltonian mechanics*. Birkhäuser.

Cayley, A. (1890) "Report on the recent progress of theoretical dynamics", Report of the British Association for the Advancement of Science, 1857, pp. 1–42 in Arthur Cayley, in *Mathematical Papers: The Collected Mathematical Papers of Arthur Cayley*. Vol. 3. Cambridge: Cambridge University Press., pp. 156–204.

Chatti, S. (2011). Extensionalism and scientific theory in quine's philosophy. *International Studies in the Philosophy of Science., 25*(1), 1–21.

Conway, A. W., & Synge, J. L. (1931). Introduction. In A. W. Conway & J. L. Synge (Eds.), *William Rowan Hamilton The Mathematical Papers of Sir William Rowan Hamilton Geometrical Optics.* Cambridge University Press.

Coopersmith, J. (2015). *Energy, the subtle concept: the discovery of Feynman's Blocks from Leibniz to Einstein* (Revised). Oxford University Press.

Crowe, M. J. (1985). *A history of vector analysis: The evolution of the idea of a vectorial system*. Dover Publications Inc.

Darrigol, O. (2000). *Electrodynamics from Ampère to Einstein*. Oxford University Press.

Diaz, R., Herrar, W. J., & Manjarrés, D. A. (2009). Work and energy in inertial and noninertial reference frames. *American Journal of Physics., 77*, 270–273.

Dirac, PA.M. (1995). In R. H. Dalitz (Eds.), *The Collected Works of P.A.M Dirac: 1924–1948*. Cambridge: Cambridge University Press.

DobbsTeeter, B. J. (1992). *The janus faces of genius: the role of alchemy in newton's thought*. Cambridge University Press.
Dunnington, G. W. (1955). *Carl Friedrich Gauss: Titan of science: A study of his life and work*. Exposition Press.
Dürr, D., & Teufel, S. (2009). *Bohmian mechanics: The physics and mathematics of quantum theory*. Springer.
Earman, J. (1976). Causation: A matter of life and death. *The Journal of Philosophy, 73*(1), 5–25.
Earman, J. (2004). Laws, symmetry, and symmetry breaking: Invariance, conservation principles, and objectivity. *Philosophy of Science., 71*, 1227–1241.
Earman, J. (2011). Sharpening the Electromagnetic Arrow(s) of Time. In C. Callender (Ed.), *The Oxford Handbook of Philosophy of Time* (pp. 485–527). Oxford University Press.
Einstein, A. (1954). *Ideas and opinions*. In C. Seelig (Ed.), New Translations and Revisions by Sonja Bargmann. Crown Publishers.
Ellis, B. D. (2002). *The Philosophy of Nature: A Guide to the New Essentialism*. Montreal: McGill-Queen's University Press.
Euler, L. (1744) Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti". Euler Archive—All Works. 65. https://scholarlycommons.pacific.edu/euler-works/65
Euler, L. (1753) Harmonie entre les principes généraux de repos et de mouvement de M. de Maupertuis. Euler Archive—All Works. 197. https://scholarlycommons.pacific.edu/euler-works/197
Farr, M., & Reutlinger, A. (2013). A relic of a bygone age? causation, time symmetry and the directionality argument. *Erkenntnis, 78*, 215–235.
Fecko, M. (2006). *Differential geometry and lie groups for physicists*. Cambridge University Press.
Feynman, R., Leighton, R. B., & Sands, M. (2010). *Feynman: The Feynman lectures on physics* (Vol. 2). Basic Books.
Field, H. (2003). Causation in a physical world. In M. J. Loux & D. W. Zimmerman (Eds.), *The Oxford Handbook of Metaphysics* (pp. 435–460). Oxford University Press.
Finkelstein, R. J. (1973). *Nonrelativistic mechanics*. W.A. Benjamin Inc.
Fisch, M. (1999). The Making of Peacock's Treatise on Algebra: A case of creative indecision. *Archive for History of Exact Sciences., 54*, 137–179.
Fisch, M., & Schaffer, S. (Eds.). (1991). *William Whewell: A composite portrait*. Clarendon Press.
Fitzpatrick, R. (2011) Generalized Forces. https://farside.ph.utexas.edu/teaching/336k/lectures/node76.html University of Texas at Austin. Downloaded: 11/10/2020.
Fraser, C. (2003) The calculus of variations: A historical survey. In Hans Niels Jahnke (Eds.) *A History of Analysis*. Providence, Rhode Island: American Mathematical Society.
Fraser, C. (1985). J.L. Langrange's Changing Approach to the Foundations of the Calculus of Variations. *Archive for History of Exact Sciences, 32*, 151–191.
French, A. P. (1971). *Newtonian mechanics*. W.W. Norton & Company.
Friedman, M. (1992). Causal laws and the foundations of natural science. In P. Guyer (Ed.), *The Cambridge Companion to Kant* (pp. 161–199). Cambridge University Press.
Friedman, M. (2013). *Kant's construction of nature: A reading of the metaphysical foundations of natural science*. Cambridge University Press.
Friedman, M. (2014). Laws of nature and causal necessity. *Kant Studien., 105*(4), 531–553.
Frisch, M. (2005). *Inconsistency, asymmetry, and non-locality: A philosophical Investigation of classical electrodynamics*. Oxford University Press.
Frisch, M. (2007). Causation, Counterfactuals, and Entropy. In H. Price & R. Corry (Eds.), *Causation, Physics, and the Constitution of Reality* (pp. 351–395). Oxford University Press.
Frisch, M. (2009). The most sacred tenet? Causal reasoning in physics. *The British Journal for the Philosophy of Science., 60*(3), 459–474.
Frisch, M. (2014). *Causal reasoning in physics*. Cambridge University Press.
Garber, D. (2009). *Leibniz: Body, substance, Monad*. Oxford University Press.
Glennan, S. (1996). Mechanisms and the Nature of Causation. *Erkenntnis, 44*(1), 49–71.
Glennan, S. (2010). Mechanisms, causes, and the layered model of the world. *Philosophy and Phenomenological Research., 81*(2), 362–381.
Glennan, S. (2017). *The new mechanical philosophy*. Oxford University Press.
Goldstine, H. H. (1980). *A History of the Calculus of Variations from the 17th through the 19th Century*. Springer-Verlag.
Goldstine, H., Poole, C. P., Jr., & Safko, J. (2002). *Classical mechanics*. Addison-Wesley.
Graves, R. P. (1842) Our Portrait Gallery—NO. XXVI. Sir William R. Hamilton. *The Dublin University Magazine* 19 (January to June): 94–110.

Graves, R. P. (1882). *Life of Sir William Rowan Hamilton* (Vol. 1). Hodges, Figgis, & CO., Grafton-Street.

Graves, R. P. (1885). *Life of Sir William Rowan Hamilton* (Vol. 2). Hodges, Figgis, & CO., Grafton-Street.

Graves, R. P. (1889). *Life of Sir William Rowan Hamilton* (Vol. 3). Hodges, Figgis, & CO., Grafton-Street.

Gray, J. (2008). *Plato's ghost: The modernist transformation of mathematics*. Princeton University Press.

Gray, J. (2015). *The real and the complex: A history of analysis in the 19th Century*. Springer.

Green, G. (1871). *Mathematical papers of the late George Green*, N.M. Ferrers. London: MacMillan and Co.

Greiner, W. (1998). *Classical electrodynamics*. Springer.

Griffiths, D. J. (2017). *Introduction to electrodynamics* (4th ed.). Oxford University Press.

Guyer, P. and Allen W. Wood. (1998). *Critique of Pure Reason: The Cambridge Edition of the Works of Immanuel Kant*. In: Paul Guyer and Allen W. Wood. Translated by Paul Guyer and Allen W. Wood. New York: Cambridge University Press.

Guyer, P. (1987). *Kant and the claims of knowledge*. Cambridge University Press.

Haar Ter, Dirk. (1961). *Elements of Hamiltonian Mechanics*. North-Holland Publishing Company.

Hamilton, W. R. (1828). Theory of systems of rays. *Transactions of the Royal Irish Academy, 15*, 69–174.

Hamilton, W. R. (1931). The mathematical papers of Sir William Rowan Hamilton. In A. W. Conway & J. L. Synge (Eds.), *Geometrical Optics.* Cambridge University Press. Abbreviated: "MPH1".

Hamilton, W. R. (1834). On a general method in dynamics; by which the study of the motions of all free systems of attracting or repelling points is reduced to the search and differentiation of one central relation, or characteristic function. *Philosophical Transactions of the Royal Society., 124*, 247–308.

Hamilton, W. R. (1835). Second essay on a general method in dynamics. *Philosophical Transactions of the Royal Society, 125*, 95–144.

Hamilton, W. R. (1837). Theory of conjugate functions, or algebraic couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time. *The Transactions of the Royal Irish Academy, 17*, 293–423.

Hamilton, W. R. (1940). The Mathematical Papers of Sir William Rowan Hamilton. In A. W. Conway & A. J. McConnell (Eds.), *Dynamics.* Cambridge University Press. Abbreviated: "MPH2".

Hamilton, W. R. (1853) "Preface to 'Lectures on Quaternions. In *MPH3*, pp. 117–155.

Hamilton, W. R. (1967). The Mathematical Papers of Sir William Rowan Hamilton. In A. H. Halberstam & R. E. Ingram (Eds.), *Algebra.* Cambridge University Press. Abbreviated: "MPH3".

Hankins, T. L. (1977). Triplets and Triads: Sir William Rowan Hamilton on the Metaphysics of Mathematics. *Isis, 68*(2), 174–193.

Hankins, T. L. (1980). *Sir William Rowan Hamilton*. The Johns Hopkins University Press.

Healey, R. (2007). *Gauging What's Real: The Conceptual Foundations of Contemporary Gauge Theories*. Oxford University Press.

Hecht, E. (2003). An historico-critical account of potential energy: Is PE really real? *The Physics Teacher., 41*, 486–493.

Hecht, E. (2007). Energy and change. *The Physics Teacher., 45*, 88–92.

Hecht, E. (2016). Relativity, potential energy, and mass. *European Journal of Physics., 37*, 065804.

Hecht, E. (2019a). Understanding energy as a subtle concept: a model for teaching and learning energy. *American Journal of Physics., 87*, 495.

Hecht, E. (2019b). Reply to comment on 'Relativity, potential energy, and mass. *European Journal of Physics, 40*, 028002.

Hendry, J. (1984). The evolution of William Rowan Hamilton's view of algebra as the science of pure time. *Studies in the History and Philosophy of Science, 15*(1), 63–81.

Hitchcock, C. (2007). What russell got right. In H. Price & R. Corry (Eds.), *Causation, Physics, and the Constitution of Reality: Russell's Republic Revisited* (pp. 66–105). Oxford University Press.

Horzela, A., Edward K. and Jaroslaw K. (1991) "On the Galilean covariance of classical mechanics". Report INP No 1556/PH. Downloaded from the INIS Repository: https://inis.iaea.org/collection/NCLCollectionStore/_Public/23/088/23088077.pdf 10/09/2020.

Ismael, J. T. (2016). *How physics makes us free*. Oxford University Press.

Jackson, J. D. (1999). *Classical electrodynamics* (3rd ed.). John Wiley & Sons Inc.

Jammer, M. (1957). *Concepts of force: A study in the foundations of dynamics*. Harvard University Press.

Kant, I. (1903). *Kant's Prolegomena and Metaphysical Foundations of Natural Science*. Translated from the Original with a Biography and Introduction by Ernest Belfort Bax. London: George Bell and Sons.

Kant, I. (1933). *Kant's Prolegomena to Any Future Metaphysics*. Edited in English by Dr. Paul Carus. Chicago: Open Court Publishing Company.

Kant, I. (1992). *Theoretical Philosophy: 1755–1770*. Translated and edited by David Walford in collaboration with Ralf Meerbote. Cambridge: Cambridge University Press.
Kant, I. (1998). *Critique of pure reason: The cambridge edition of the works of Immanuel Kant*. In P. Guyer & A. W. Wood (Eds.), Translated by Paul Guyer and Allen W Wood. Cambridge University Press.
Kant, I. (2004). *Prolegomena to any future metaphysics that will be able to come forward as science: with Selections from the Critique of Pure Reason*. Cambridge University Press.
Kargon, R. (1965). William Rowan Hamilton and Boscovichian Atomism. *Journal of the History of Ideas., 26*(1), 137–140.
Kistler, M. (2013). The interventionist account of causation and non-causal association laws. *Erkenntnis, 78*, 65–84.
Klein, S., & Nellis, G. (2012). *Thermodynamics*. Cambridge University Press.
Kutach, D. (2013). *Causation and its Basis in Fundamental Physics*. Oxford University Press.
Lagrange, J-L. (1997). *Analytical mechanics*: Translated from the Mécanique analytique, novelle édition of 1811, edited by Auguste Boissonnade and Victor N. Vagliente. Translated by Auguste Boissonnade and Victor N. Vagliente. Dordrecht: Kluwer Academic Publishers.
Lagrange, J.-L. (1867). *Œuvres De Lagrange* (Vol. 1). Gauthier-Villars, Imprimeur-Libraire.
Lanczos, C. (1970). *The variational principles of mechanics* (4th ed.). University of Toronto Press.
Landau, Lev D. and Evgeny M. Lifshitz. (1976). *Mechanics: Volume 1*. Translated by J.B. Sykes and J.S. Bell. New York: Pergamon Press Inc.
Lange, M. (2017). *Because without Cause: Non-Causal Explanations in Science and Mathematics*. Oxford University Press.
Langhaar, H. L. (1962). *Energy methods in applied mechanics*. John Wiley and Sons Inc.
Leibniz, G. W. (1898) *The monadology and other philosophical writings*. Translated with Introduction and Notes by Robert Latta. Oxford: Clarendon Press.
Leibniz, G. W. (1989). *Philosophical Essays*. In R. Ariew & D. Garber (Eds.), Translated by Roger Ariew and Daniel Garber. Hackett Publishers.
Leibniz, G. W., & Bartholomeus, D. B. (2007). *The Leibniz-Des Bosses Correspondence*. In B. Look & D. Rutherford (Eds.), Translated by Brandon Look and Donald Rutherford. Yale University Press.
Loew, C. (2017). Causation, physics, and Fit. *Synthese, 194*(6), 1945–1965.
Loewer, B. (2007). Counterfactuals and the second law. In H. Price & R. Corry (Eds.), *Causation, Physics, and the Constitution of Reality: Russell's Republic Revisited* (pp. 293–326). Oxford University Press.
Lombardi, O., Castagnino, M., & Ardenghi, J. S. (2010). The modal-Hamiltonian interpretation and the galilean covariance of quantum mechanics. *Studies in History and Philosophy of Modern Physics., 41*, 93–103.
Lützen, J. (1995). Interactions between Mechanics and Differential Geometry in the 19th Century. *Archive for History of Exact Sciences, 49*(1), 1–72.
Machamer, P., Darden, L., & Craver, C. F. (2000). Thinking about mechanisms. *Philosophy of Science., 67*(1), 1–25.
Mann, P. (2018). *Lagrangian and hamiltonian dynamics*. Oxford University Press.
Marion, J. B., & Thornton, S. T. (1988). *Classical dynamics of particles and systems* (3rd ed.). Harcourt Brace Javanovich Inc.
Mathews, J. (1978). William Rowan Hamilton's Paper of 1837 on the Arithmetization of analysis. *Archive for History of Exact Sciences., 19*, 177–200.
Maudlin, T. (2011). *Quantum Non-Locality and Relativity* (3rd ed.). Wiley-Blackwell.
Maxwell, J. C. (1888). *An elementary treatise on Electricity. Edited by William Garnett*. Clarendon Press.
Maxwell, J. C. (1871). *Theory of heat*. Longmans.
McGuire, J. E. (1977). Neoplatonism and Active Principles: Newton and the *Corpus Hermeticum*. In: Robert S. Westman and J.E. McGuire (Eds.) *Hermeticism and the Scientific Revolution: Papers Read at the Clark Library Seminar, March 9, 1974*. Los Angeles: Clark Memorial Library.
McGuire, J. E. (1968). Force, active principles, and Newton's invisible realm. *Ambix, 15*, 154–208.
Melnick, A. (2004). *Themes in Kant's Metaphysics and Ethics*. The Catholic University of America Press.
Melnick, A. (2006). Kant's proofs of substance and causation. In P. Guyer (Ed.), *The Cambridge Companion to Kant and Modern Philosophy* (pp. 203–237). Cambridge University Press.
Merzbach, U. C., & Boyer, C. B. (2011). *A History of mathematics* (3rd ed.). John Wiley & Sons Inc.
Meyer, K. R., & Offin, D. C. (2017). *Introduction to Hamiltonian dynamical systems and the n-body problem* (3rd ed.). Springer.
Mills, R. (1994). *Space, time and quanta*. W.H. Freeman.
Mumford, S. (2004). *Laws in Nature*. New York: Routledge Publishers.
Nakane, M., & Fraser, C. G. (2002). The early history of Hamilton-Jacobi dynamics 1834–1837. *Centaurus, 44*, 161–227.

Newton, I. (1999). *The Principia: Mathematical Principles of Natural Philosophy*. Translated by I. Bernard Cohen, Anne Whitman and Julia Budenz. Berkeley and Los Angeles, CA: University of California Press.

Ney, A. (2009). Physical causation and difference-making. *The British Journal for the Philosophy of Science., 60*(4), 737–764.

North, J. (2022). Formulations of classical mechanics. In E. Knox & A. Wilson (Eds.), *A Companion to the Philosophy of Physics* (pp. 21–32). Routledge Publishers.

Norton, JD. (2021) The metaphysics of causation: An Empiricist Critique", https://sites.pitt.edu/~jdnorton/papers/Causation_empiricist.pdf downloaded 12/02/2021

Norton, J. D. (2007a). Causation as folk science. In H. Price & R. Corry (Eds.), *Causation, Physics, and the Constitutionof Reality: Russell's Republic Revisited* (pp. 11–44). Oxford University Press.

Norton, J. D. (2007b). Do the causal principles of modern physics contradict causal anti-fundamentalism? In P. Machamer & G. Wolters (Eds.), *Thinking about Cause: From Greek Philosophy to Modern Physics* (pp. 222–234). University of Pittsburgh Press.

Nozick, R. (2001). *Invariance: The structure of the objective World*. Harvard University Press.

Øhstrøm, P. (1985). W.R. Hamilton's View of Algebra as the Science of Pure Time and His Revision of This View. *Historia Mathematica., 12*, 45–55.

Olness, F., & Scalise, R. (2011). Regularization, renormalization, and dimensional analysis: Dimensional regularization meets freshman E&M. *American Journal of Physics., 79*, 306–312.

Panza, M. (2003) The Origins of Analytic Mechanics in the 18th Century. In *A History of Analysis*, 137–153. American Mathematical Society.

Papineau, D. (2013). Causation is macroscopic but not irreducible. In G. Sophie, E. J. Lowe, & R. D. Ingthorsson (Eds.), *Mental Causation and Ontology.* Oxford University Press.

Peacock, T. and Nicolas H. (2007) "Course Materials for 2.003J/1.053J Dynamics and Control I, Spring 2007". MIT OpenCourseWare (http://ocw.mit.edu), Massachusetts Institute of Technology. Downloaded 11/10/2020.

Penrose, R. (2005). *The road to reality: A complete guide to the laws of the universe*. Vintage Books.

Pletser, V. (2018). *Lagrangian and Hamiltonian analytical mechanics: Forty Exercises Resolved and Explained*. Springer.

Pourciau, B. (2006). Newton's Interpretation of Newton's Second Law. *Archive for History of Exact Sciences, 60*(2), 157–207.

Pourciau, B. (2020). The principia's second Law (as Newton Understood It) from Galileo to Laplace. *Archive for History of Exact Sciences., 74*, 183–242.

Quine. W. V. (2004). Quintessence: Basic Readings from the Philosophy of W.V. Quine, (eds.) Roger F. Gibson Jr. Cambridge, MA: Harvard University Press.

Quine, W. V. (1976). *The ways of Paradox and other essays*. Harvard University Press.

Raju, C. K. (1994). *Time: Towards a consistent theory*. Springer.

Rankine, W. (1853). On the general law of the transformation of energy. *Philosophical Magazine., 5*(30), 106–117.

Redhead, M. (1990). "Explanation." Edited by Dudley Knowles. *Royal Institute of Philosophy Supplement, 27*, 135–154.

Rennie, R. (Ed.). (2015). *Oxford dictionary of physics* (7th ed.). Oxford University Press.

Russell, B. 1912–1913. "On the Notion of Cause." *Proceedings of the Aristotelian Society* 13: 1–26.

Ryder, P. (2007). *Classical mechanics*. Shaker Verlag.

Saunders, S. (2003a). "Indiscernibles, General Covariance, and Other Symmetries". In *Revisiting the Foundations of Relativistic Physics: Festschrift in Honour of John Stachel.* (pp. 151–173). Dordrecht: Springer.

Saunders, S. (2003b). "Physics and Leibniz's Principles". In *Symmetries in Physics*, edited by Katherine Brading and Ellena Castellani. 289–308. Cambridge: Cambridge University Press.

Saunders, S. (2013a). "Rethinking Newton's Principia". *Philosophy of Science*, *80*(1), 22–48.

Saunders, S. (2013b). "Indistinguishability". In R. Batterman (ed.) *The Oxford Handbook of Philosophy of Physics* (pp. 340–380). New York: Oxford University Press.

Schaffer, J. (2008). Causation and laws of nature: Reductionism. In T. Sider, J. Hawthorne, & D. W. Zimmerman (Eds.), *Contemporary Debates in Metaphysics* (pp. 82–107). Blackwell Publishers.

Schaffer, J. (2009). On what grounds what. In D. J. Chalmers, D. Manley, & R. Wasserman (Eds.), *Metametaphysics: New Essays on the Foundations of Ontology* (pp. 347–383). Oxford University Press.

Schroeren, D. (2020). The metaphysics of invariance. *Studies in History and Philosophy of Modern Physics., 70*, 51–64.

Shankar, R. (2016). *Fundamentals of physics II: Electromagnetism, optics, and quantum mechanics*. Yale University Press.

Sider, T. (2011). *Writing the book of the world*. Oxford University Press.
Smith, C. (1998). *The science of energy: A cultural history of energy physics in Victorian Britain*. University of Chicago Press.
Smith, N. K. (1918). *A Commentary to Kant's 'Critique of Pure Reason.'* MacMillan and Co., Limited.
Smith, C., & Wise, N. M. (1989). *Energy & empire: A biography study of lord kelvin: Part 1*. Cambridge University Press.
Sommerfeld, A. (1964). *Lectures on Theoretical Physics*: *Volume 1 Mechanics*., translated by Martin O. Stern. New York: Academic Press.
Spohn, W. (2006). Causation: An alternative. *The British Journal for the Philosophy of Science*, *57*, 93–119.
Stan, M. (2022). From Metaphysical Principles to Dynamical Laws. In M. M. David & J. Dana (Eds.), *The Cambridge History of Philosophy of the Scientific Revolution.* Cambridge University Press.
Stan, M. (2009). Kant's early theory of motion: Metaphysical dynamics and relativity. *The Leibniz Review., 19*, 29–61.
Steane, A. M. (2012). *Relativity made relatively easy*. Oxford University Press.
Stewart, IW. 2016. "Advanced Classical Mechanics". Edited by Francisco Machado, Thanawuth Thanathibodee, and Prashanth S. Venkataram. https://ocw.mit.edu/courses/physics/8-09-classical-mechanics-iii-fall-2014/lecture-notes/MIT8_09F14_full.pdf Downloaded 11/10/2020.
Sullivan, J. W. N. (1934). *The limitations of science*. The Viking Press.
Taylor, J. R. (2005). *Classical mechanics*. University Science Books.
Tefft, B. J., & Tefft, J. A. (2007). Galilean relativity and the work-kinetic energy theorem. *The Physics Teacher., 45*, 218–220.
Thiele, R. (1997). On Some Contributions to Field Theory in the Calculus of Variations from Beltrami to Carathéodory. *Historia Mathematica., 24*, 281–300.
Thomson, J. J. (1888). *Applications of dynamics to physics and chemistry*. Macmillan Co.
Thomson, W. (1860). Measurement of the electromotive force required to produce a spark in air between parallel metal plates at different distances. *Proceedings of the Royal Society of London., 10*, 326–338.
Thorne, K. S., & Blandford, R. D. (2017). *Modern classical physics: Optics, fluids, plasmas, elasticity, relativity, and statistical physics*. Princeton University Press.
Todhunter, I. (1876a). *William Whewell, D.D. Master of Trinity College, Cambridge. An Account of His Writings with Selections from His Literary and Scientific Correspondence*. Volume 1. London: MacMillan and Co.
Todhunter, I. (1876b). *William Whewell, D.D. Master of Trinity College, Cambridge. An Account of His Writings with Selections from His Literary and Scientific Correspondence*. Volume 2. London: MacMillan and Co.
Trapp, R. (2019). *Multivariable calculus*. Oxford University Press.
Truesdell, C. (1968). *Essays in the history of mechanics*. Springer-Verlag.
Uffink, J. (2002). Essay review: Time and chance. *Studies in History and Philosophy of Modern Physics., 33*, 555–563.
van Fraassen, B. C. (1989). *Laws and symmetry*. Oxford University Press.
Watkins, E. (2005). *Kant and the metaphysics of causality*. Cambridge University Press.
Watkins, E. (2019). *Kant on laws*. Cambridge University Press.
Weaver, C. (2021). In Praise of Clausius entropy: Reassessing the foundations of Boltzmannian statistical mechanics. *Foundations of Physics, 51*(3), 1–64.
Weaver, C. G. (2019). *Fundamental causation: Physics, metaphysics, and the deep structure of the world*. Routledge Publishers.
Weinberg, S. (2015). *To explain the world: The discovery of modern science*. HarperCollins.
Wellek, R. (1931). *Immanuel Kant in England 1793–1838*. Princeton, NJ: Princeton University Press.
Westfall, R. S. (1971). *Force in Newton's Physics: The Science of Dynamics in the Seventeenth Century*. American Elsevier.
Whewell, W. (1836). *An elementary treatise on mechanics: Designed for the use of Students in the University* (5th ed.). Cambridge: Printed at the Pitt Press by John Smith for J. & J.J. Deighton, Cambridge: and Whittaker & Arnot, London.
Whewell, W. (1858). *History of Scientific Ideas: Being the First Part of the Philosophy of the Inductive Sciences* (3rd ed.). John W. Parker and Son.
Whewell, W. (1967). *The Philosophy of the inductive sciences: Part Two*. Frank Cass & Co. LTD.
Whittaker, E. T. (1954). William Rowan Hamilton. *Scientific American., 190*, 82–87.
Winterbourne, A. T. (1982). Algebra and Pure Time: Hamilton's Affinity with Kant. *Historia Mathematica, 9*, 195–200.
Woodward, J. (2003). *Making things happen: A theory of causal explanation*. Oxford University Press.

Woodward, J. (2007). Causation with a human face. In H. Price & R. Corry (Eds.), *Causation, Physics, and the Constitution of Reality* (pp. 66–105). Oxford University Press.

Woodward, J. (2014). A functional account of causation: Or, A Defense of the Legitimacy of Causal Thinking by Reference to the Only Standard that Matters—Usefulness (as Opposed to Metaphysics or Agreement with Intuitive Judgment). *Philosophy of Science, 81*(5), 691–713.

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