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Mathematics and Argumentation

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Abstract Some authors have begun to appeal directly to studies of argumentation in their analyses of mathematical practice. These include researchers from an impressively diverse range of disciplines: not only philosophy of mathematics and argumentation theory, but also psychology, education, and computer science. This introduction provides some background to their work.

Keywords Argumentation · Mathematical practice

1 Mathematical Practice as Argumentation

Philosophy of mathematics is widely supposed to be concerned exclusively with the foundations of mathematics and the status of mathematical proof. The methodology of mathematics is likewise supposed to be comprised exclusively of formal logic. However, as an increasing number of mathematicians and philosophers of mathematics have complained, these two suppositions ignore much of mathematical practice, including much that is of philosophical importance. Specifically, not all—indeed hardly any—mathematical proofs are strict formally valid logical derivations. Of course, most of them can be restated in this manner, sometimes with comparatively little effort, but this is not something that mathematicians routinely do. To insist on such paraphrase is to misrepresent the nature of mathematical practice. Moreover, there is much that mathematicians do besides proving results, central as that activity may be. Most of this work may still be understood, however, as a species of *argument*.

This thesis requires some justification. I am using 'argument' in an everyday sense, broadly speaking as an act of communication intended to lend support to a claim.[1] Proofs fit

[1] Ian Dove's paper in this issue provides a more thorough treatment of the definition of argument.

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unproblematically within this definition, as almost all commentators agree.[2] Moreover, one of the problems which treatments of mathematical practice face is the proliferation of what one might call ‘proof*’: species of alleged ‘proof’, where there is either no consensus that the method provides proof, or there is broad consensus that it doesn’t, but a vocal minority or an historical precedent which points the other way. These methods include proofs* predating modern standards of rigour, picture proofs*, probabilistic proofs*, computer-assisted proofs*, textbook proofs* which are didactically useful but would not satisfy an expert practitioner, and proofs* from neighbouring disciplines with different standards—most notoriously mathematical physics (many of these topics are discussed by contributors to Mancosu 2008). Whether or not they qualify to lose their asterisks, all of these varieties of proof* can be understood as arguments.[3]

Furthermore, there is much more to the process of proving than its product, whether that be proof or proof*, and much of that process may also be understood as argument. Perhaps the first step in the process is the choice of problem. While this may be subject to social forces and arbitrary whim, mathematicians often have good reasons for choosing the problems they do, even if they seldom make them explicit. At the very least, as James Franklin points out, supervisors of Ph.D. candidates are obliged to take this issue very seriously: the candidate must chose a problem which has not already been solved and isn’t likely to be solved by somebody else any time soon, but which the candidate has a fair chance of solving (Franklin 1987). A second step where the mathematician is also likely to resort to informal arguments is the choice of methods used to tackle the problem. The next step, application of the method to the problem, may ultimately lead to a proof, but this is likely to take shape first in the form of more or less incomplete proof sketches, which are, nonetheless, already arguments. When the mathematician has the proof in a fit state for publication, it must then survive a more explicitly argumentational phase: the dialectic between author and reviewer. While this is frequently superficial, it can be intense and protracted, as the well-known cases of Andrew Wiles’s proof of Fermat’s Conjecture and Thomas Hales’s proof* of Kepler’s Conjecture attest (for details, see Singh 1997, Chap. 7, and Szpiro 2003, respectively). Once a proof has been refereed and published, it may still be the object of critical argumentation if its methods or results are sufficiently controversial. And even after it has become widely accepted, if it attracts sufficient interest, it may spur further arguments as mathematicians seek to generalize it, extend it, transpose it to a different field, simplify it, or manipulate it in some other way.[4]

Beyond the characteristic mathematical practice of proof, there are yet more examples of mathematical practice as argument, many of which are ill-suited to formalization. The choice of axioms is one such issue, where an account of the ‘non-demonstrative arguments’ (Maddy 1990, p. 148) by which such choices become accepted is required, something which has been attempted in terms of argumentation theory (Alcolea Banegas 1998, pp. 144 ff.; see also Dove’s paper in this issue). Of course, comparatively few mathematicians are directly involved in those arguments, but most will have participated in similar arguments concerning the choice of definitions. Again, while the definitions may be employed in formal proofs, and indeed their usefulness in formal proof may be cited in their favour, the appeals to

[2] Erik Krabbe suggests two possible exceptions: *immediate and intuitive proofs* and *formal proofs* (Krabbe 1997, p. 70). Both exceptions could be challenged, and he agrees that the latter are at least models of arguments. In any event, these exceptions would not significantly undercut my thesis.

[3] And so understanding them may tell us something useful: I have argued elsewhere that many of the debates arising from proofs* may be clarified by understanding them in the context of the types of dialogue to which they are intended to contribute (Aberdein 2007b, p. 148).

[4] The pioneering study of these last two phases in the career of a proof is (Lakatos 1976): see Sect. 3.

'fruitfulness', 'explanatoriness', or 'naturalness' by which they gain acceptance must be informal arguments.[5] A further important source of informal mathematical argument is the application of mathematics to science: considerable slippage between the mathematical theory and the empirical data is often inevitable, and it is informal argumentation that bridges the gap (see, for example, Swinnerton-Dyer 2005, p. 2440 or Urquhart 2008, p. 408).

We have seen just some of the aspects of mathematical practice that are comprised of argumentation. In the following sections I will indicate some of the ways in which argumentation has been studied, and how those studies have been brought to bear on mathematics.

2 What is Argumentation Theory?

Argumentation theory is the study of argument. In particular, it emphasizes those aspects which resist deductive formalization. Informal logic and critical thinking are often understood to be subfields of (or pseudonyms for) argumentation theory. Deductive logic is concerned with validity and proof. It has been a formidably successful research programme within that context. However, deductive validity is only one tool for the appraisal of argument. Other tools exist, including tools which permit finer-grained distinctions amongst the arguments classified as deductively invalid.

Until the successful new mathematical approach of the early twentieth century eclipsed all others, many of the themes which concern modern argumentation theorists were central to logic. Indeed, Aristotle's *Organon*, famous as the ur-text of formal logic, actually devotes more attention to informal reasoning. Notably, Aristotle introduces 'enthymemes', oversimplified by later commentators as syllogisms with tacit premisses, to characterize plausible non-deductive inferences. For Aristotle, the premisses of enthymemes were linked to their conclusions by 'topoi', or topics, which comprise a diverse variety of commonplace patterns of more or less persuasive reasoning with widespread application. The study of topics was developed by later authors, including Cicero and many mediaeval and early modern logicians. In recent decades, this and other aspects of argumentation theory have been revived, and have acquired an increasingly thorough intellectual basis.

The modern revival might be dated to 1958. That year saw the publication of two profoundly influential books: Chaim Perelman and Lucie Olbrechts-Tyteca's *La Nouvelle Rhétorique* and Stephen Toulmin's *The Uses of Argument*. Both works exhibit the influence of a greater range of argumentational practice than had become common in formal logic. In particular, they both emphasize jurisprudential over mathematical approaches to reasoning. Perelman and Olbrechts-Tyteca's work began the rehabilitation of the long dormant topics. Toulmin's chief innovation was the 'layout' which analyzes arguments into six components. The *data* (or grounds) provide *qualified* support for the *claim* in accordance with a *warrant*, which may in turn be supported by *backing* or admit exceptions or *rebuttals*.[6] The last component is particularly significant, since it recognizes that arguments may be *defeasible*. Finally, some artificial intelligence researchers have sought to integrate this work with formal

[5] See Tappenden (2008) for discussion. Indeed, Tappenden indirectly alludes to argumentation theory by taking his '*Port Royal* principle' (p. 273), that definition 'depends much more on our knowledge of the subject matter being discussed than on the rules of logic' (Arnauld and Nicole 1996, p. 128), from a work elsewhere praised as a precursor of modern argumentation theory for its characterization of logic as 'not a discipline in its own right but an instrument for understanding, evaluating, and improving the arguments and reasoning of other disciplines' (Finocchiaro 2005, p. 263).

[6] Several contributors to this issue discuss the Toulmin layout in greater detail. See especially the papers by Bart Van Kerkhove and Jean Paul Van Bendegem, and Alison Pease et al.

accounts of defeasible reasoning developed during the explosion of interest in non-classical logics in the last half of the twentieth century (see, for example, Paglieri and Castelfranchi 2006 or Verheij 2005).

Recent work in argumentation theory exhibits a strong interdisciplinary trend, encompassing not only philosophy but also communication theory, artificial intelligence, and law. This has led to a marked diversity of methodologies (Johnson 1996, pp. 43 ff. provides a helpful overview). We may broadly distinguish three overlapping approaches: the historical, the experimental, and the evaluative. According to Maurice Finocchiaro, one of its most distinguished practitioners, '[t]he historical approach begins with the selection of some important book of the past, containing a suitably wide range and intense degree of argumentation … the investigator has somehow to acquire mastery of the content and historical background of the chosen text' (Finocchiaro 2005, pp. 37f.). Finocchiaro's own contribution to this genre is the magisterial *Galileo and the Art of Reasoning* (1980), an analysis of the argumentational structure of Galileo's *Dialogue Concerning the Two Chief World Systems*. On the other hand, the experimental approach gathers its empirical data by a different species of enquiry. This has obvious roots in psychology, and within argumentation theory it is most closely associated with the critical thinking movement. The assessment and encouragement of critical thinking have significant didactic implications, and mark a point of contact with educational theory.

The evaluative approach has been pursued in a variety of different fashions. For example, the 'pragma-dialectic' school of Amsterdam communication theorists Frans van Eemeren and Rob Grootendorst (Van Eemeren and Grootendorst 1992), so called for its emphasis on the pragmatics of dialogues, proposes an explicit set of normative ideals for critical discussion, jokingly referred to as the 'Ten Commandments'. Adherence to the rules is said to ensure a fair outcome to a critical discussion. Other systems are less dogmatic.

Much recent attention has focused on 'argumentation schemes'. These are stereotypical patterns of plausible reasoning which might be seen as a reinvention of Aristotle's topics. This programme has been developed at length by the prolific Canadian logician Douglas Walton and his collaborators (most recently in Walton et al. 2008). One long-standing problem for which argumentation schemes have proved important is the characterization of informal fallacies. Fallacies may be understood as pathological instances of plausible but not invariably sound schemes. This represents an improvement on the 'standard treatment' of fallacy, as an argument which '*seems to be valid* but *is not* so' (Hamblin 1970, p. 12), identified and discredited in Hamblin's now classic study, but still on offer in most introductory logic textbooks. Schemes have also attracted a growing interest from artificial intelligence researchers, specifically as a means of interaction between 'agents' (see, for instance, Rahwan et al. 2005).

3 Pioneering Approaches

As we have seen, the study of mathematical practice needs an account of argument, and that is what argumentation theory seeks to provide. The intersection of the two has the potential to be highly productive, but, with some important exceptions, it has until recently remained unexplored. Nonetheless, there have long been studies of mathematics which show sensitivity to the structure of argument. Works of this sort have tended not to cite argumentation research, but often address closely related questions. Amongst the most influential are several books by the mathematician George Pólya (1954, 1957). He builds on the ancient tradition of 'heuristics' (Groner et al. 1983), procedures for finding solutions to problems, to articulate an account of what he calls 'plausible reasoning' in mathematical practice. His examples, at least, have been widely cited by philosophers of mathematics, especially his account of Euler's informal derivation of $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$, but his principal philosophical successor was Imre Lakatos.[7] *Proofs and Refutations* (Lakatos 1976) is one of the most thorough treatments of mathematical argumentation to date, and one of the most influential. Lakatos provides what he calls a 'rational reconstruction' of the successive proofs of the Descartes–Euler Conjecture, which relates the numbers of vertices, edges, and faces in convex polyhedra.[8]

Lakatos is also the patron saint of the so-called 'Maverick Tradition' in philosophy of mathematics. This broke decisively with the foundationalist approach that had dominated the subject in the twentieth century, and urged a reconnection with the history of mathematics (important sources include Davis and Hersh 1980 and Aspray and Kitcher 1988). Certainly the admonition to widen the narrow diet of examples that had characterized mainstream philosophy of mathematics is one that researchers into mathematical argumentation should observe carefully. While the suggestion that the 'great majority of deductions in geometry' are instances of the syllogisms *Barbara* or *Celarent* (Fetisov 2006, p. 15) may be an exaggeration, it is clear that elementary mathematics is not enough for an understanding of mathematical argument. Moreover, it is an enduring complaint '[t]hat never any knowledge was delivered in the same order it was invented, no not in the mathematic, though it should seem otherwise in regard that the propositions placed last do use the propositions placed first for their proof and demonstration' (Bacon, c. 1603, *Valerius Terminus*, cited in Merton and Barber 2004, p. 274). Studies of mathematical argumentation must pay attention not only to published arguments, but also to the reasoning from which those arguments emerged, and the choices which were made en route. These are lessons which may be drawn from the Mavericks, without endorsing their anti-foundationalism.

There are several other sources for work on mathematical argument. Fallacies have been a long-standing, if not always especially productive inspiration for argumentation theory, and mathematical fallacies have led to comparable studies, primarily in mathematics education (Fetisov 2006; Bradis et al. 1999; Maxwell 1959; Barbeau 2000). Until recently (Aberdein 2007a, 2009; Krabbe 2008), these traditions have not been explicitly connected, although Wilfred Hodges's entertaining paper (1998) on failed attempts to rebut Cantor's diagonalization argument perhaps comes close. Several philosophers of mathematics have explored the role of particular forms of non-deductive inference in mathematics. For example, there have been studies of ambiguity (Byers 2007; Grosholz 2007) and enumerative induction (Baker 2007), as well as advocacy for the use of Bayesian methods to capture mathematical reasoning (Franklin 1987; Corfield 2003). Some researchers working on the automation of theorem proving and checking have also paid attention to informal mathematical argumentation, in the hope of extending their results into the realm of mathematical discovery (MacKenzie 2001, pp. 94 ff.; Kerber and Pollet 2007, pp. 77 f.). More modestly, the emerging field of mathematical knowledge management seeks to facilitate the development of consistent and searchable databases of existing mathematical knowledge. But this has led to the development of protocols general enough to permit the description of informal as well as formal mathematical argumentation, such as Michael Kohlhase's OMDoc (2006).

Recently, more explicit applications of argumentation theoretic techniques to mathematical practice have begun to appear. There have been several substantially independent appli-

[7] Indeed, the methods of both Pólya and Lakatos have been appropriated as topic-neutral theories of argument, see Rhee (2007) and Chang and Chiu (2008), respectively.

[8] For more detail, see Dove (2007) or the paper in this issue by Pease et al.

cations of the Toulmin layout to mathematical reasoning (Alcolea Banegas 1998; Aberdein 2005; Inglis et al. 2007; Pedemonte 2007) as well as attempts to extend the layout better to fit complex mathematical arguments (Knipping 2001; Aberdein 2006). Other studies deploy a variety of different methodologies including pragma-dialectics (Krabbe 1997, 2008; Aberdein 2007b) and other systems (Douek 1999; Dove 2007).

4 Contributions to This Issue

The papers in this issue are intended to broaden and deepen this recent work, while reinforcing hitherto neglected interconnections, and encouraging further research.

There are some fundamental philosophical issues that any approach to informal mathematics will raise. One of the most crucial is that of the success of informal mathematics—if formality is essential to rigour, and rigour central to mathematical success, how does informal mathematical practice work at all? Jody Azzouni's paper addresses this question by a further defence of his view that informal proofs are 'derivation indicators' (see Azzouni 2004). Another difficult concept concerning informal proof is that of 'surveyability'. Ostensibly, long proofs present an obstacle to the requirement that proofs be surveyable, since they may be too long to be practically surveyed. Edwin Coleman's paper offers a characterization of surveyability which tackles this issue. Moreover, he argues that, since all proofs draw on substantial context, there are in a sense no short proofs. This attendance to context is also a central feature of Bart Van Kerkhove and Jean Paul Van Bendegem's paper. They exemplify this point with a careful reconstruction of an early proof of Pólya. This might be seen as an instance of Finocchiaro's historical approach. Another paper in this tradition is David Sherry's. He deploys a careful reading of the diagrams in Saccheri's inadvertent anticipation of non-Euclidean geometry to argue against the prevailing view that diagrams must represent abstract objects.

Diagrammatic reasoning is one of the more conspicuous and widely discussed examples of informal mathematics, and this issue is no exception. Zenon Kulpa's paper is the first part of a longer work defending an account of diagrams as rigorizable. Specifically, his paper tackles one of the principal difficulties for any such account, the 'generalization problem': how can a proof dependent on a specific diagram be generalized to other cases, and how far can it be generalized? Matthew Inglis and Juan Pablo Mejia-Ramos are also concerned with visual reasoning in mathematics. However, their paper represents a very different methodological approach. They report on the results of a series of experiments into the reception of diagram-based proofs by professional and undergraduate mathematicians. As they point out, although their work originates in educational psychology, the empirical examination of the data of philosophical inquiry is also of a piece with recent work in 'experimental philosophy'. Whereas this paper exhibits the influence of argumentation theory in educational research, that of Alison Pease, Alan Smaill, Simon Colton and John Lee is situated against their backgrounds in artificial intelligence. Their paper explores how resources from that field, in particular computational modelling of mathematical reasoning, may be utilized to further develop argumentation theories and philosophies of mathematical practice. Finally, my co-editor Ian Dove's paper offers a variety of evidence for the utility of argumentation theory in the analysis of mathematical practice, and in the resolution of some of the philosophical debates to which that practice has given rise, while also pointing the way to further open questions that remain to be answered by theorists of mathematical argumentation.

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