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PROOF-ANALYSIS AND CONTINUITY

ABSTRACT. During the first phase of Greek mathematics a proof consisted in showing or making visible the truth of a statement. This was the epagogic method. This first phase was followed by an apagogic or deductive phase. During this phase visual evidence was rejected and Greek mathematics became a deductive system. Now epagoge and apagoge, apart from being distinguished, roughly according to the modern distinction between inductive and deductive procedures, were also identified on account of the conception of generality as continuity. Epistemology of mathematics today only remembers the distinction, forgetting where they agreed, in this manner not only destroying the unity of the perceptual and conceptual but also forgetting what could be gained from Aristotelian demonstrative science.

1. AN ARISTOTELIAN MODEL OF DEMONSTRATIVE SCIENCE

Aristotle's *Posterior Analytics* is the first elaborated theory in the Western philosophical and scientific traditions of the nature and structure of science and its influence reaches well into our times. It has long been accepted with such a degree of unanimity that nobody even thought of imputing special merit to Aristotle for his establishment of it. Indeed, its impact has been so profound that only Kuhn's *The Structure of Scientific Revolutions* could be seen as its contemporary counterpart. In a contribution to a conference on Kuhn's work Lakatos wrote:

"For centuries knowledge meant proven knowledge [...]. Wisdom and intellectual integrity demanded that one must desist from unproven utterances [...]. The proving power of the intellect or the senses was questioned by the skeptics more than two thousand years ago; but they were brow-beaten into confusion by the glory of Newtonian physics. Einstein's results again turned the tables and now very few philosophers or scientists still think that scientific knowledge is, or can be, proven knowledge. But few realize that with this the whole classical structure of intellectual values falls in ruins and has to be replaced: one cannot simply water down the ideal of proven truth – as some logical empiricists [...] or [...] some sociologists of knowledge do" (Lakatos, 1970, p. 92).

We believe that these claims, although being correct in some details, are fundamentally disoriented. First of all, in the mathematical case at least, the notion of proof has lost nothing of its importance; on the contrary! Second, all major epistemological conceptions of modernity – Kant's epistemology and theory of science in response to the Newtonian achievement as well the philosophical digestion of Einstein's results by the logical empiricists – formulated their contributions within the Aristotelian framework, with the complementarity of analysis and synthesis occupying a central place. Lakatos, who believes that Popperian falsificationism has substituted Aristotelian demonstrative science, misses the point of these debates, which refers to the circular connection between theoretical structure and its intended applications; or between ideal and concrete objects.

Lakatos writes: "Belief may be a regrettably unavoidable biological weakness to be kept under the control of criticism: but commitment is for Popper an outright crime" (Lakatos, 1970, p. 92). We think this to be a rather naive view of science and scientific progress (taking into account Quine's "Two Dogmas of Empiricism") and a view distorted by Lakatos' blinded fighting against Marxism and Freudianism (such that he does not draw the appropriate conclusions even with respect to his own analysis of the problem). One must, however, also admit that it had been Lakatos, not least through his very popular and influential *Proofs and Refutations*, who had helped to free the philosophy of mathematics from the straight jacket of analytical philosophy and logical empiricism. Such that the weaknesses of his philosophical and historical perspectives were at the same time their strength (see also Kvasz, 2002).

Finally, trying to educate the younger generation within to-days technological "knowledge society", it seems worthwhile to remember that knowledge fulfills two major roles in human society: a practical one and a philosophical one, as well as that general education cannot limit itself to the training of practical skills. Education is to be based on proven scientific knowledge not the least because "it seems that science came into being with the requirement of [...] coherence and that one of the functions it performs permanently in human culture consists in unifying [...] practical skills and cosmological beliefs, the *episteme* and the *techne* [...] despite all changes that science might have undergone, this is its permanent and specific function which differentiates it from other products of human intellectual activity" (Amsterdamski, 1975, p. 43/44).

Everybody agrees that mathematics distinguishes itself from the rest of the sciences by the conception of reliable proof and that there is in general unanimity or consensus of the informed with respect to whether some mathematical truth has in fact been established by valid proof or not. Proof is, however, also intended to be a vehicle of introducing the student or newcomer to mathematics and the mathematical way of seeing the world, and it therefore has been since the 19th century made more and more rigorous and has been formalized until finally questions of meaning and development had been lost sight of. Formal proof seems absolutely reliable, but it is not quite certain any more "what it is reliable about", as Lakatos once said, paraphrasing a wellknown *bonmot* of Russell. Proof was constructed as a developmental and didactical device, but in the course of its improvement it lost exactly these creative and explanatory qualities. With respect to the Aristotelian conception of proof the knower's and the listener's status are at least as important as that of the known. And concerning the latter it never was sufficient to give something as a dead thing or strictly individual existent. Aristotle would never separate science or mathematics from philosophy, and it has been claimed indeed that to understand his work one would have to consider "the four discourses": Poetics, Rhetoric, Dialectics and Logic as but variants of only one unified science (De Carvalho, 1996, p. 29).

We are not suggesting, to restore the Aristotelian world view. That would be a futile undertaking – but it is worthwhile to remember that Peirce had called himself "an Aristotelian of the scholastic wing". What we would like to defend, however, is the thesis that all the technical devices which modern logic has invented to deal with the self-reference of meaning and truth – Russell's type hierarchies, for example, or Tarski's distinction between object-language and meta-language – are by no means sufficient, or rather, are technologies and as such, do not prescribe or orient their own application.

There is a paradox of formal proof, for example: Either a proof is just a machine a mere causal compulsion or algorithmic procedure, then one cannot see how new knowledge can be brought about by just reducing the new to the already given. A proof can prove something only inasmuch knowledge possesses a firm tautological structure, proof consisting, in the last instance, in a reduction of the new and unproven to the already known and proven. But then nothing new can arise and no new understanding is stimulated. As Lakatos once said: "No logic can infallibly increase content" (Lakatos, 1970, p. 95).

Or the proof leaves room for interpretative behavior and then it is faced with the request of proving its correctness. And the proof of the correctness of the proof again meets the same requirement and the proof of the correctness of the correctness of the proof also, etc. To interrupt the infinite regress of proving and explaining by sheer force or mere outer compulsion leads back to the other horn of the dilemma.

Traditionally, there have therefore always been two ways of dealing with proof or demonstration, two methods, which Churchman has called the maximum- resp. minimum-loop strategy. "The maximum-loop principle is based on a monistic philosophy. There is one world of interconnected entities, not many. ... The principle is also teleological. For the mind to know itself, it must also know the destiny of all minds as well as all matters. Indeed the principle comes straight down to us from Plato" (Churchman, 1968, p. 114).

The minimum-loop principle for its part is mirrored by the so-called 'immediacy assumption' for formal systems. On such a basis Hilbert has called formal logic self-evident. "The problem of logic is a very direct one: how can a proposition talk about itself?" (Churchman, 1968, p. 112). The starting point for this problem is the assumption that every proposition immediately implies itself. If I say "p" this implies "p". And this in turn means "p is true". And this brings back the maximal loop. The predicate "is true" does not really add something to the status of the original affirmation, although "p" and "p is true" are in general different sentences. Truth is undefinable, at least in its common understanding (see Rucker, 1981, p. 148). Thus one might want to settle with the view that "p is true", really implies "p is true". Therefrom results the 'immediacy assumption' for formal systems.

The minimal loop orientation seems to have guided the intellectual efforts of Descartes and Spinoza in the seventeenth century. "For Descartes the problem was to find a proposition that leads directly to its own validity" (Churchman, 1968, p. 113). The very same spirit stimulated Leibniz to create the idea of a completely formal proof and to base truth on formal proof rather than on meaning (Hacking, 1980). These ideas went hand in hand with Leibniz' representationalism and his belief that everywhere the ideal has to determine the existent, as well as, that coherence is as important to truth as is compliance with the phenomena. Considering the fact that if there were in cognition a direct access to the object, an immediate relationship to it, this relationship would exist in an automatic quasi-mechanized form also, one realizes that these two alternatives – the Cartesian and the Leibnizean, the intuitive and the mechanical–really do not make a difference. Cartesian intuitionism and Leibnizean mechanism amount, on certain accounts, really to the same thing, only in different clothing. In both cases no questions of meaning could arise, because an affirmation would just be a fact, as if reality spoke for itself. But why then search for proofs at all, as any analytical proposition seems to be valid in its own right (A. J. Ayer, 1981).

But nothing seemed stable and real anymore, neither facts nor ideas. Consider the situation of Descartes. "We have usually read him as an ego, trapped in the world of ideas, trying to find out what corresponds to his ideas, and pondering questions of the form, 'How can I ever know?' Underneath his work lies a much deeper worry. Is there any truth al all, even in the domain of ideas? The eternal truth, he tells us, are 'perceptions [. . .] that have no existence outside our thought'. But on our thought they are, in a sense, isolated perceptions. They may be systematized by synthesis but this has nothing to do with their truth. The body of eternal truths which encompassed mathematics, neo-Aristotelian physics and perhaps all reality was a closely knit self-authenticating system of truth, linked by demonstration. For Descartes there are only perceptions which are ontologically unrelated to anything and moreover are not even candidates for having some truth outside my mind. One is led, I think, to a new kind of worry. I cannot doubt an eternal truth when I am contemplating it clearly and distinctly. But when I cease to contemplate, it is a question whether there is truth or falsehood in what I remember having perceived. Bréhier suggested that demonstrated propositions may go false. It seems to me that Cartesian propositions, rendered lone and isolated, are in an even worse state. Perhaps neither they nor their negations have any truth at all. They exist in the mind only as perceptions. Do they have any status al all when not perceived? When demonstration cannot unify and give 'substance' to these truths, the constancy of a veracious God as the arena in which the essences of possible worlds compete for existence, [...] is needed not just to guarantee our beliefs, but also to ensure that there is some truth to believe" (Hacking, 1980, p. 176/177).

The whole edifice of rational knowledge therefore rested on the socalled Ontological Argument for the existence of God. The kernel of this argument of the Rationalists of the 17th century was to claim that the notion of the nonexistence of God is a contradiction; for God is perfect and existence is perfection, so God must exist. Without God there would be no truth, nothing general in fact. Leibniz "held that the realm of essences would have no being at all, if it were not eternally contemplated by the mind of God. 'Every reality must be based upon something existent; if there were no God there would be no objects of mathematics"' (Lovejoy, 1936/1964, p. 147).

Generalization thus became the main interest and strength of "modern" mathematics. And since Descartes and Leibniz generalization seems to mean foremost the postulation or introduction of ideal objects, or rather representations of them, rather than mere abstraction and broadening the extensions of pre-given concepts. Mathematics by this process of hypostatic abstraction became meta-mathematics, it became a theory of mathematical practice. The topologist Salomon Bochner conceives of the iteration of abstraction as of the distinctive feature of the mathematics of the scientific revolution of the 17th century.

"In Greek mathematics, whatever its originality and reputation, symbolization … did not advance beyond a first stage, namely, beyond the process of idealization, which is a process of abstraction from direct actuality, … However … full-scale symbolization is much more than mere idealization. It involves, in particular, untrammeled escalation of abstraction, that is, abstraction from abstraction, abstraction from abstraction from abstraction, and so forth; and, all importantly, the general abstract objects thus arising, if viewed as instances of symbols, must be eligible for the exercise of certain productive manipulations and operations, if they are mathematically meaningful" (Bochner, 1966, p. 18). Generalization thus has to be understood in a quasi Platonic sense, with modern instrumentalism and mentalism added.

Still one might claim that the twofold nature of the general – as predicative general, on the one hand, and as continuity, on the other, had played a role in Antiquity too. Even though the nominalist and instrumentalist twist had not yet developed. Aristotle is most often regarded as the great representative of a logic and mathematics, which rests on the assumption of the possibility of clear divisions and rigorous classification. "But this is only half the story about Aristotle; and it is questionable whether it is the more important half. For it is equally true that he first suggested the limitations and dangers of classification, and the non-conformity of nature to those sharp divisions which are so indispensable for language […]" (Lovejoy, 1964, p. 58). Aristotle thereby became responsible for the introduction of the principle of continuity into natural history. "And the very terms and illustrations used by a hundred later writers down to Locke and Leibniz and beyond, show that they were but repeating Aritotle's expressions of this idea" (Lovejoy loc.cit.).

It also seems that in Greek mathematics occurred two different kinds of proof. "During the first phase of Greek mathematics there a proof consisted in showing or making visible the truth of a statement". This was the epagogic method. "This first phase was followed by an apagogic or deductive phase. During this phase visual evidence was rejected and Greek mathematics became a deductive system" (Koetsier, 1991, p. 180f; and the bibliographic reference given there). Epagogic proof primarily verifies and apagogic proof also generalizes in the sense that a statement's meaning is evaluated with respect to a whole system of statements.

Epagoge is usually translated as "induction". But it is perhaps not quite what we think of as induction, but is rather taking one individual as prototypical for the whole kind. Aristotle writes with respect to epagoge: "The consideration of similarity is useful both for inductive arguments and for hypothetical reasoning [. . .] It is useful for hypothetical reasoning, because it is an accepted opinion that whatever holds good of one or several similars, holds good also for the rest" (Topics 108b 7). Aristotle's attention is more on what it is to be of a certain type *A* (something taken for granted in our standard induction). One might say that investigation of a (new) species is a matter of looking carefully at a number of specimens, checking whether features are in fact common to the species, possibly discounting some variation as accidental, possibly even deciding to reclassify some putative specimens as a different species. Then one illustrates the truth of the (provisionally) final classificatory statement by appeal to undisputed and representative exemplars of the kind in question. Epagogic proof depends on some law of the uniformity of nature or some continuity principle. Fundamental starting-points, like axioms in geometry, have to be grasped through the epagogic process by a faculty Aristotle calls nous. There is not only predicative generality in our thinking, but also generality which cannot be defined (Metaphysics 1048a 25).

Apagogic proof demonstrates the consequences of the axioms. Now, the first proof we know of, it is said, is the apagogic proof of the irrationality of $\sqrt{2}$. It is one which Aristotle will call reduction to the absurd. An apagogic proof then is one that proceeds by disproving the proposition, which contradicts the one to be established, in other words, that proceeds by reduction to the absurd. Such a proof depends on certain existence claims, on the affirmation that there are some completely determinate existents, because the notion of an entity not wholly determinate is "imaginary" or merely possible. The continuity principle in contrast negates that there were definite existents, or that the continuum is composed from points. The opposition between geometric substantialism against relationalism played a very important role throughout history and in the controversy between nominalism and realism (with respect to mathematical philosophy, see: Burgess/Rosen 1997, 97).

The opposition of the epagogic and apagogic comes down ontologically to the famous antinomy of the indivisibility and the infinite divisibility of Time, Space and Matter, which had been known from the days of Xenon and had been taken up and again through history in the service of skepticism. This antinomy consists in the fact that discreteness must be asserted just as much as continuity. "The one-sided assertion of discreteness gives infinite or absolute dividedness, hence an indivisible, for principle: the one-sided assertion of continuity, on the other hand, gives infinite divisibility" (Hegel, *Science of Logic*). The antinomy is expressed in the opposition between Leibniz two fundamental principles, the Principle of the Identity of Indiscernibles, on the one hand, and the Principle of Continuity, on the other. Leibniz tried to resolve the conflict by distinctions between the real and the ideal, or the factual in contrast to the merely possible. Kant rejected Leibniz' conceptualization of the possible in terms of the continuity principle and he used this antinomy to state the principal limitations of human conceptual thinking. Apagogic proof depends on existence claims and these, according to Kant cannot be established by conceptual reasoning, but depend on (pure) intuition. The epagogic proof, Kant calls "ostensiver Beweis".

Ever since, these two sides of cognition, the intuitive or figurative and the logical and operative, have remained opposed to one another and even the process of mathematical knowledge has a dualistic or complementary structure or quality, which evolves over time. The duality seems reproduced nowadays in the duality of the axiomatical versus constructive methods. Axiomatic thinking in the Hilbertian sense is full-blown relational thinking. And relational thinking has since Leibniz at least, been firmly linked to the continuity principle (with respect to Leibniz' geometry see Chasles, 1835; Cassirer, 1962). Relational thinking begins, however, already with Eudoxus' theory of proportions as presented in book V of Euclid. As mathematicians we know that there is no theoretical entity really defined without giving criteria of its identity. Definition 5 of book V of Euclid presents Eudoxus' definition of equality between two relations: $a/b = c/d$. This definition in a

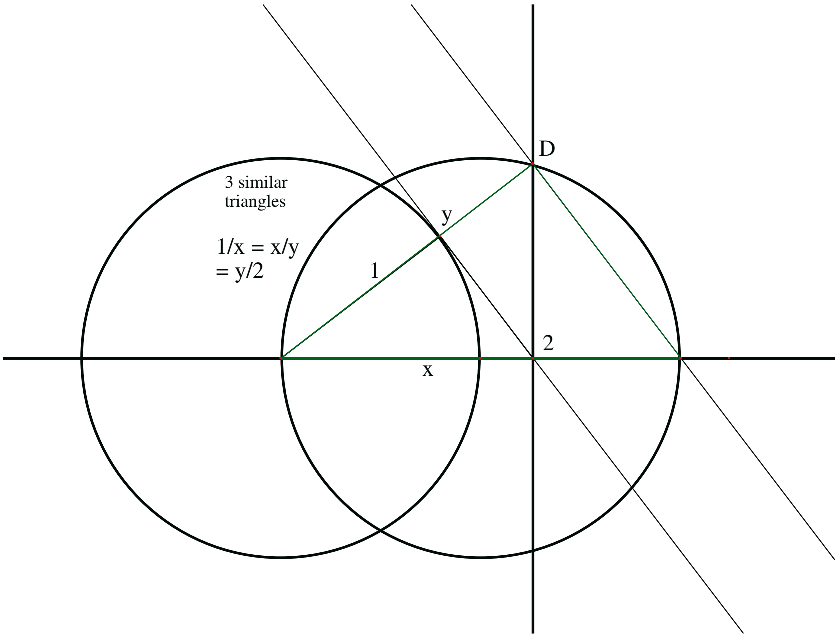

Figure 1.

sense uses the principle of continuity, as Dedekind's definition of real number was to reveal much later. Eudoxus theory is an element alien to Euclid's conception of geometry.

On the basis of relational thinking the continuity principle served also to assure the existence of certain entities (as points of intersection of two or more continuous curves). For instance, the so-called Delian problem, the duplication of the cube (while preserving the cube's shape) demanded by Apollo in the oracle – a central construction problem of Greek geometry – cannot be solved constructively with ruler and compass alone. If we look at the diagram of Figure 1 – provided by Hippocrates of Chios and used in Antiquity to analyze the Delian problem – we come to see also how essential the principle of continuity is to construct this diagram which is intended to find the essential point D.

In Euclidean mathematics the continuity principle was not admitted as a means of proof and the epagogic proof seems forgotten. It was then really eliminated and buried by the arithmetization of pure mathematics, since Bolzano until Hilbert's new axiomatics revitalized it. Bolzano proved the intermediate value theorem by an apagogic proof. Epagoge or abduction is used in hypothetico-deductive reasoning, which in turn is based on hypostatic abstractions and general objects, by means of which subject–predicate expressions are transformed into relations.

Aristotle's, different from Euclid's reluctance, had a much broader view of apagogic proof, namely also admitting the continuity principle. This principle appears in Euclid only once namely in the notion of geometrical similarity as it is established by Eudoxus' theory of proportionality. This appearance of the continuity principle has given birth to a second meaning of the term apagoge (or reduction) in Aristotle. For instance we can by means of similarity interpret length or area as a relation and thus conceive of measurement in terms of invariance of relations, rather than in terms of numerical approximation. A classical example is furnished by the beautiful similarity proof of Pythagoras' theorem. The method reappears in Leibniz' invention of the calculus and in Grassmann's theory of linear extension or vector calculus.

Now this meaning of epagoge or agagoge is prominent also in Aristotle (see Prior Analytics, II, XXV (2)). In his Prior Analytics Aristotle describes the reductive use of the continuity principle in the very same terms that later Proclus was to use, dealing with the Delian problem of doubling the cube, respective its reduction by Hippocrates of Chios as in figure 1. Proclus writes: "Reduction (=apagoge) is a transition from a problem or a theorem to another one which, if known or constructed, will make the original proposition evident. For example to solve the problem of the duplication of the cube geometers shifted their inquiry to another on which this depends, namely, the finding of two mean proportionals" (Proclus (Morrow, 1970, p. 167)). Reduction thus does not necessarily mean reduction to the absurd. Rather reduction or apagoge here requires a creative or metaphorical postulation of some intermediate terms (or mean proportionals) in order "to bring us nearer to knowledge" (Aristotle loc. cit). Aristotle's interpretation of apagogic proof is much wider than is Euclid's, which has come down on us. This means that Aristotle knew that proofs have to be explanations as well as verifications. Contrary to the views of Aristotle the use of the continuity principle was admitted as a mathematical proof strategy only after the construction of the continuum of real numbers enabled its reformulation in Euclidean terms.

But as was said already, one could conceive of the general in two ways either predicatively in terms of (propositional) functions or in terms of free variables or concrete universals. We have explained elsewhere (Otte, 1993, p. 87–89)how the construction of a geometrical measurement theory, combining these two conceptualizations of the general (or the continuum) proceeds. In summary we conclude that the continuity principle and the relation between general and particular that is represented by this principle appears in two very different ways, one justifying epagoge as well as Aristolelian reduction, the other connected to apagoge in the traditional Euclidean understanding.

The development of the axiomatic method from Euclid to Hilbert is illustrative with respect to the changes in the interrelations between epagogic and apagogic reasoning. Modern axiomatics reintroduces epagoge and uses the continuity principle on the meta-mathematical level. Hilbert had to postulate, for instance, to be able to make use of the logical law of excluded middle, an exemplary or paradigmatic instantiation for every property, and this amounts to assuming a, "Law of the Uniformity of Nature", or a "Principle of Continuity", like in the inductive empirical sciences. The difference between the theoretical and empirical thereby becomes a matter of degree (as it was for Leibniz), rather than being one of kind (as it was for Kant). The axiomatic approach expresses a monistic philosophical attitude.

Now a proof of axiomatical theory refers to some (perhaps none, if the system is inconsistent) possible world. Leibniz used the continuity principle to find out whether it states a truth with respect to the actual world. This is not so bad an idea, given Hilbert's efforts, as well as the fact that most people "form an unduly simplistic idea of what consistency (compatibility) of conditions is. One thinks of the compatibility of conditions as something to be directly read off the complex of conditions, such that one need only analyze and sort out the content of the conditions clearly in order to see whether they are in agreement or not. In fact, however, the role of the conditions is that they are effective in functional use and by combination. The result obtained in this way is not contained in what is directly given through the conditions. It is probably the erroneous idea of such inference that gave rise to the view of the tautological character of mathematical propositions" (Bernays, 1976, p. 98).

The Principle of Continuity applied empirically and without further qualification is doomed to failure. This was already clear to Hume and to Kant. Kant therefore emphasized that mathematics (like knowledge in general) is an activity and he drew a sharp distinction between discursive and intuitive knowledge. "Philosophical cognition is the cognition of reason by means of conceptions; mathematical cognition is cognition by means of the construction of conceptions. The construction of a conception is the presentation a priori of the intuition, which corresponds to the conception" (B 741). Thus Kant's notion of "pure intuition" is nothing but a subjective or mentalistic version of the continuity principle.

Mathematical judgments, according to Kant are intuitive, and are thus apodictic and synthetic, whereas for Aristotle or Bolzano they always had also to give a reason why they are true. Here we have the opposition between minimum- and maximum-loop principle back. The constructive approach of modernity, in contrast to the axiomatic method, is dualistic and follows a minimum-loop strategy, assuming that the cognitive subject as well as the means of its activity are completely known and transparent to herself. We have privileged access to ourselves, as well as to the constituents and motives of our activity it is assumed. Thus our explanations and constructions are supposed to be true or necessary. By this property Lakatos characterized what he called Euclidean theories. Riemann complained that Euclid's system was not thoroughly constructive.

Mathematical proof and explanation in general contains ideal objects or concrete universals, as well as particular existents. Without generals there were no universality and no prediction; without particular existents, – remember that to exist in mathematics means to exist with respect to some model – there would be no truth (or: truth in the model; to achieve truth in the common sense, one would than further have to evaluate the adequacy of the model). Cognitively a mathematical proof rests on metaphorical associations of ideas and analogical reasoning, as well as, on immediate verification. Semiotically a proof seems a combination of icons and indices. A metaphor is based on an icon and is therefore something not strictly defined. Metaphor is a means to enlarge our language by creating new meanings, whereas indexical expressions and strict definitions intend to make it more precise. A proof has to be rigorous as well as meaningful.

The classification of proof strategies resp. notions of proof given so far, is very schematic and it is not completely clear as the meaning of terms, like "logic" or "continuity principle" changed over time; but it is still useful. It is useful also to analyze Lakatos' description of mathematical development, as presented in his *Proofs and Refutations*. Lakatos' description of mathematical behavior and proof activity seems defective upon being compared with the reality (Koetsier, 1991; Kvasz, 2002). The reason for this is, we believe, to be attributed to his concentration on object centered proof, proof that wants to prove or verify, or on apagogic proof. This becomes not only clear from the strategies of mathematical development, which Lakatos describes–monster-barring, exception-barring, lemma incorporation–but also from the way he understands and interprets these strategies. One could, for instance, understand monster-barring also as meaning that there are more premises to be taken into account than one had perceived, thus taking the perspective of proof-analysis rather than interpret the situation referentially.

With respect to the first: It has been pointed out that Lakatos omitted important methods, for instance, "lemma exclusion" (Kvasz, 2002) from his catalogue of strategies. These are typically strategies resulting from proof-analysis, rather than from the approach which uses counterexamples and verifications. One example of lemma exclusion, Kvasz says, can be found in Koetsier's book *Lakatos' Philosophy of Mathematics*. "We have in mind the proof of the interchangeability theorem for partial differentiation by H.A. Schwarz. Schwarz first stated the theorem with six conditions, proved it, and then attempted to drop as many of the conditions as possible. He succeeded to drop three of the conditions, and ended with a much stronger theorem than the one which he proved at the beginning" (Koetsier, 1991, p. 268ff; Kvasz, 2002). We shall present a similar strategy with respect to a geometrical theorem by Euler in the last part of this paper.

2. ARISTOTELIAN PROOF

Let us, before we come back to Aristotle's conception of demonstrative science and his description of mathematical proof, summarize the main points of our argument.

Modernity begins, as is sometimes affirmed, with Kant's "sharp discrimination of the intuitive and the discursive processes of the mind" (Peirce CP 1.35). "Our knowledge springs from two main sources in the mind", Kant says, namely, concepts and intuitions. Concepts are, according to Kant, always applied to mental representations of objects, rather than on the objects themselves. By intuition or mental representation an object is given, but remains undetermined, whereas by means of concepts it is thought relatively to the intuitive representation. When Kant claimed that we have to establish the objectivity of our definitions by means of intuition, he had in mind the fact that in general we do not recognize something as something by means of definitions or their verification (in difference to "The Man who mistook his wife for a hat" (O. Sacks)). Therefrom results that the general occurs in two forms, namely predicative generality on the one hand, and perceptive continuity on the other. Both were present in Aristotelian thought already.

Peirce, calling himself "an Aristotelian of the scholastic wing", describes them thus:

"The old definition of a general is *Generale est quod natum aptum est dici de multis*. This recognizes that the general is essentially predicative and therefore of the nature of a representamen. ... In another respect, however, the definition represents a very degenerate sort of generality. None of the scholastic logics fails to explain that sol is a general term; because although there happens to be but one sun yet the term sol *aptum natum est dici de multis*. But that is most inadequately expressed. If sol is apt to be predicated of many, it is apt to be predicated of any multitude however great, and since there is no maximum multitude, those objects, of which it is fit to be predicated, form an aggregate that exceeds all multitude. Take any two possible objects that might be called suns and, however much alike they may be, any multitude whatsoever of intermediate suns are alternatively possible, and therefore as before these intermediate possible suns transcend all multitude. In short, the idea of a general involves the idea of possible variations", or continuity (CP 5.102–103). Continuity or similarity is the basis of the particular representing generality.

This twofold character of the general is expressed in the history of mathematics by two different interpretations of the continuity principle, two interpretations over which Cauchy and Poncelet quarreled (Belhoste, 1991), when the idea of pure mathematics was at stake, although they had been present since Antiquity. It seems, indeed, that these interpretations occurred in two different kinds of proof in Greek mathematics. During the first phase of Greek mathematics a proof consisted in showing or making visible the truth of a statement. This was the epagogic method. This first phase was followed by an apagogic or deductive phase. During this phase visual evidence was rejected and Greek mathematics became a deductive system. Now epagoge and apagoge, apart from being distinguished, roughly according to the modern distinction between inductive and deductive procedures, were also identified on account of the conception of generality as continuity. Epistemology of mathematics today only remembers the distinction, forgetting where they agreed, in this manner not only destroying the unity of the perceptual and conceptual but also forgetting what could be gained from Aristotelian demonstrative science.

"Aristotelianism admitted two modes of being", says Peirce (CP 2.116). This position was attacked by nominalism since William Ockham, on the ground that one kind sufficed to account for all the phenomena. "The hosts of modern philosophers, to the very Hegels, have sided with Ockham in this matter" (Peirce, loc.cit.). "The categories now come in to aid us materially", Peirce continues, "and we clearly make out three modes or factors of being, which we proceed to make clear to ourselves". And the logic of relations, being superior to Ancient subject-predicate logic shows that there are three types of reasoning, abduction, induction and deduction, as well as, that deductive reasoning must build on the two others. This new logic appeared forcefully during the Scientific Revolution of the 17th century, but was already present in Aritotle.

A proof, according to Aristotle, has, for example, not only to show that something is true but has rather also to provide reasons why it is true. "Demonstrations are also explanations. Thus proofs do more than indicate logical relations among propositions" (McKirahan, 1992, p. 4). Now a demonstration "why" in the Aristotelian understanding belongs to logic in the narrower sense of the term as well as to poetics and rhetoric, or logic in a wider sense depending on whether the "why" refers to the objective foundations of the assertion or to its persuasive and didactical justifications. Aristotle would never separate science or mathematics from philosophy, and it has been claimed indeed that to understand his work one would have to consider "the four discourses": Poetics, Rhetoric, Dialectics and Logic as but variants of only one unified science (O. de Carvalho, 1996, p. 29). A proof contains essentially two kinds of moves, logical deductions as well as inductive, or rather abductive arguments. The latter can be called representative generalizations, as some particular is representing a general truth, like in geometrical diagrams.

It seems best to render the problem of proof and explanation in semiotic terms. An argument then is nothing but a representation: A is represented as B; or is seen as B. A representation is a representation of A (of Pegasus, for example) either because it looks like A (iconic sign), or because it is caused by A (indexical sign) or because it can, having been established by means of some conventions, be used to express thoughts about A. Now conventions themselves depend on something perceivable such that one has two basic types of representations constituted by similarity or contiguity, respectively.

Our idea is to connect Aristotle to Peirce by asking how meaning is to be conceived of in mathematics, and in particular how the basic constituents of meaning, intension and extension are to be interpreted. The intensions are simply the axiomatical structures, determined up to isomorphism; but what about the objective or extensional aspects of mathematical meanings. The objective side of a sign seems twofold. One the one side, a sign expresses a possible interpretation of another sign (having the role of the object) and thus expresses the meaning of the original sign qua sign. On the other hand, a sign is, as Cassirer or Peirce have said, merely a possible, and as such a general, rather than an actually existing object, determined in every respect. The term possibility in this description has to be understood as meaning that there are constraints on this process of interpretation-representation or interpreting representation, (possibilities as described in probability theory for example); such that the interpretation has nevertheless an objective character, it is in a sense determined by the sign, rather than by the interpreter. Finally, the sign to be interpreted is referring to some object, which also partakes in the objective meaning of it and which in the case of mathematics is an element of some model (usually framed in set theoretic terms).

Mathematical deduction, for example, could be understood in this way as interpretation or translation of one sign into another. And doing so one might say that deductive reasoning unfolds the intensions of the theoretical terms, as fixed in the premises and axioms. This process is objectively constrained, but is not totally determined or pre-programmed. We would like to say that it is subject to a principle of continuity. The deductive process is split up into so many small steps that the conclusions that lead from one step to the next in the argument become obvious and perceivable. Each step results in a statement of the form A=B.

Peirce himself writes:
"That truth is the correspondence of a representation with its object is, as Kant says, merely the nominal definition of it. Truth belongs exclusively to propositions. A proposition has a subject (or set of subjects) and a predicate. The subject is a sign; the predicate is a sign; and the proposition is a sign that the predicate is a sign of that of which the subject is a sign. [. . .] thought is of the nature of a sign. In that case, then, if we can find out the right method of thinking and can follow it out – the right method of transforming signs – then truth can be nothing more nor less than the last result to which the following out of this method would ultimately carry us. [. . .]Truth is the conformity of a representamen to its object, its object, ITS object, mind you. [. . .] What the sign virtually has to do in order to indicate its object – and make it its – all it has to do is just to seize its interpreter's eyes and forcibly turn them upon the object meant: it is what a knock at the door does, or an alarm or other bell, or a whistle, a cannon-shot, etc. It is pure physiological compulsion; nothing else. [. . .] So, then, a sign, in order to fulfill its office, to actualize its potency, must be compelled by its object" (Peirce, CP 5.553–554).

Relations of similarity or analogical reasoning shall thus never lead to truth. Truth depends on objects or rather on indexical signs. We may thus conclude that a demonstration in the sense of Aristotle contains logical or syllogistic as well as metaphorical and rhetorical elements. The very first page of the *Posterior Analytic* already indicates nearly the whole conception:

"All teaching and learning of an intellectual kind proceeds from pre-existent knowledge. This is evident if we consider the different branches of learning, ... Similarly too with arguments, both deductive and inductive (abductive); both effect instruction through what we already know, the former making assumptions as though granted by an intelligent audience, and the latter proving the universal from the self-evident nature of the particular. [...] It is necessary to have two kinds of previous knowledge. (1) For some things it is necessary to suppose in advance that they are; (2) for other it is necessary to understand what the thing being said is; and for yet others, both. Thus you already knew that every triangle has angles equal to two right angles; but you got to know that this figure in the semicircle is a triangle at the same time as you were being led to the conclusion. In some cases learning occurs in this way, and the last term does not become known through the middle term - this occurs when the items are in fact particulars and are not said of any underlying subject" (the translation is a compilation based on the translations by Barnes, McKirahan and Tredennick: I.71a, 1–29).

The situation is illustrated by Figures 2–5. Figure 2 presents the problem, which is represented by Figure 3. If we already know that the sum of the three angles in a triangle amounts to two right angles we can proceed to Figure 5, otherwise we need the mediation by Figure 4.

Every such diagram represents and thereby interprets the foregoing in a certain way. This is abductive reasoning, seeing A as B and for Aristotle it belongs more to rhetoric than to logic proper. With respect to each particular diagram the reasoning proceeds by formal deductions or inductions, as described in the quotation.

"One of the central themes of Aristotle's *Rhetoric* is that in order to persuade ordinary people to do ordinary things (such as to defend their country) reason should be used but not demonstrative arguments. Ordinary people do not take to .... systems

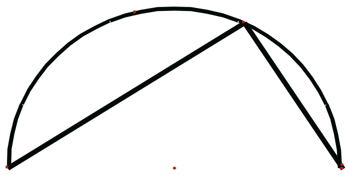

Figure 2.

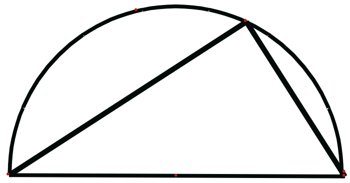

Figure 3.

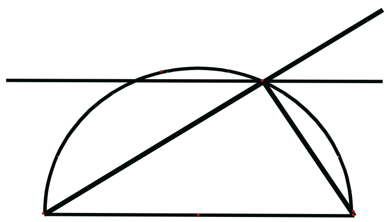

Figure 4.

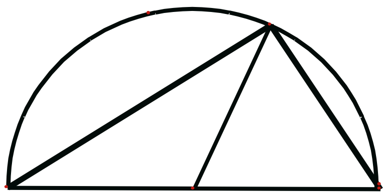

Figure 5.

of syllogisms complete with self-evident first principles. Aristotle recommended the use of arguments which are easily understood, even if weak, such as analogies, arguments from signs, likelihoods, . . . " "(Tursman, 1987, p. 97). Similarly, Peirce assigned to rhetoric the task "to ascertain the laws by which in every scientific intelligence one sign gives birth to another, and especially one thought brings forth another" (CP 2.229). And to bring forth this task of generalization analogy, metaphor and similarity of ideas are important means. On similar grounds Aristotle may have come to the opinion that there must be counterparts to what a logician calls a proof.

Now abductive inferences and their conception generating power generally occur as part of analysis and analysis is applied by Aristotle foremost in the investigation of nature. "Upon surveying the treatises which are especially devoted to positive research (De Caelo, Meteorologica, works on natural science), we soon realise the important place occupied by a form of reasoning whose logical character Aristotle nowhere examines. It consists of ascertaining the nature of one fact, which cannot be directly understood from another fact whose cause is obvious. In his many attempts at explanation the scientist often resorts to observations that bears on cases analogous to the one his research is concerned or which are more open to external view and therefore whose significance is in general merely analogical" (Bourgey, 1975, p. 175).

Peirce has similar views again and in addition to that he indicates the essential role of the continuity principle in bringing forth generalizations and new concepts. He writes:

"I desire to point out that it is by taking advantage of the idea of continuity, or the passage from one form to another by insensible degrees, that the naturalist builds his conceptions. Now, the naturalists are the great builders of conceptions; there is no other branch of science where so much of this work is done as in theirs; and we must, in great measure, take them for our teachers in this important part of logic. And it will be found everywhere that the idea of continuity is a powerful aid to the formation of true and fruitful conceptions. By means of it, the greatest differences are broken down and resolved into differences of degree, and the incessant application of it is of the greatest value in broadening our conceptions" (2.646).

This breaking down of differences thereby making keen metaphors more accessible to the interpreter or learner is thus developing a logic of abductive reasoning. Geometry is based on insight, and this involves two procedures, analysis and synthesis. Analysis was essential to provide insight into the structure of mathematical problems, especially in cases were construction or synthesis failed. For instance, the so-called Delian problem, the duplication of the cube cannot be solved with ruler and compass alone, but can be solved if one admits conics as legitimate instruments of construction (as Descartes did). If we look at the diagram of Figure 1 – provided by Hippocrates of Chios and used in Antiquity to analyze the Delian problem – we immediately realize that the analysis depends on the insertion of an additional similar triangle, that is of two mean-proportionals.

A straightforward identification of these processes is furnished by the Cartesian solution to the Delian problem. Let x be the edge sought, then $x^3 = 2$. This gives $x^4 = 2x$.

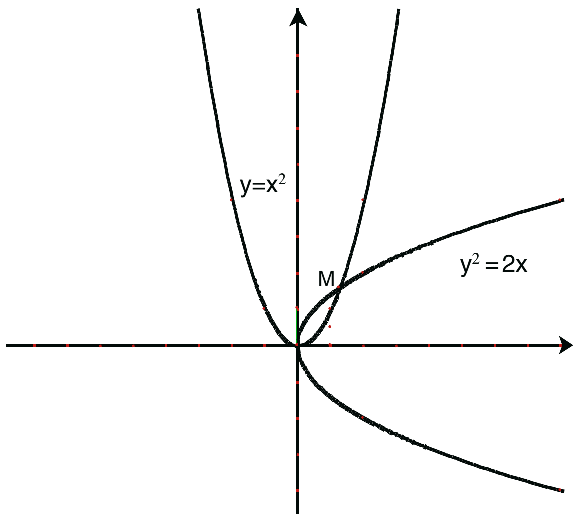

Figure 6.

Now substituting $y=x^2$ (1) yields $y^2=2x$ (2). This means that x is constructed as the intersection point of the parabolas (1) and (2) (see Figure 6).

Descartes entire new analysis and analytical geometry was derived from his examination of proportionality. The discovery of one or more mean-proportionals through the appropriate association of known terms with unknown terms is the paradigm of ingenious discovery and construction in terms of figurative representation that Descartes explains in the *Regulae*, in particular in the second part, beginning with Rule 14.

Now this process of insertion of mean proportionals is derived from analysis in the Aristotelian understanding, that is signifying the breaking up of inferences into smaller and smaller steps by inserting mean concepts and in this manner achieving a "condensation" of the original reasoning, making it appear more natural and more obvious and secure. So if A a C (meaning A applies on all C) we have to look for a mean concept B such that A = B and B = C together imply A a C; and so forth (Detel, 1993, p. 302f).

This analysis is actually indeed based on a compression or introduction of intermediate terms between assumption and conclusion, something which later also determined the Cartesian analysis which is based precisely on introducing mean proportionals. This method is essential if one tries to understand how mathematical proofs, which indeed consist, according to logical understanding, in reducing the new to the old, can serve to obtain or convey new insights at all. With the Aristotelian method of inference, we may assume that compression by inserting intermediate terms is necessary to make similarities obvious, and thus turn the demonstrative method into an intuitive and apodictic process. In such reasoning, equalities are regarded as similarities, and respectively quite different aspects may then underlie the similarities or equalities of this chain. This is what we have illustrated using the example of proving the theorem about the Eulerline (see part 3).

The connection of the individual terms of the chain of inference, in my own terminology $A = B_1,\ B_1 = B_2,\ \ldots,\ B_n = C$, shall remain logically undetermined here. Generally speaking, it is always a matter of regarding an A as a B_1, and a B_1 as a B_2, etc., respectively of representing A by B_1, etc. this relation of representation being supported by an idea. In the last instance, one can say that this is a case of triadic sign relations in Peirce's sense, i.e. of representations mediated via an idea.

Figures 2–5 provide, as was shown already, an example of this kind of reasoning.

3. PROOF-ANALYSIS AND THE DEVELOPMENT OF GEOMETRICAL THOUGHT

Modernity began with Kant, we have said. Piaget was a Kantian, he adhered to Kantianism, as he often affirmed, but to a Kantianism "that is not static, that is, the categories are not there at the outset, it is rather a Kantianism that is dynamic that is, with each category raising new possibilities, which is something completely different. I agree that the previous structure by its very existence opens up possibilities, and what development and construction do in the history of mathematics is to make the most of these possibilities, to convert them into realities, to actualize them" (Piaget, 1980, p. 150).

Piaget lost, however, also something of the Kantian heritage namely the importance of continuity and intuition. Piaget describes the process of mathematical development in terms of hypostatic abstractions, which he calls "reflective abstractions", but he separated them too sharply from empirical abstraction and perception. Kant making the continuity of space and time subjective faculties had thereby at the same time discovered the objectivity of the subject. Piaget appreciated this and his "dynamical" version of Kant's epistemology is nothing but a particular further elaboration of Kant's fundamental insight. Only that Piaget conceives of this objectivity exclusively in terms of formal structure and logical necessity.

Piaget characterizes the historical development of geometry as a succession of three periods of *intrafigural, interfigural*, and finally, *transfigural* or *structural* thought. The second stage marks the appearance of relational thinking during the Scientific Revolution. And with the third stage mathematics seems to be transformed into an analytical science, based on logical consistency alone. This is not Kantianism any more. Describing the three developmental stages Piaget writes:

"Geometry begins with Euclid – with a period during which the object of study is geometrical properties of figures and solids seen as internal relations between elements of figures and solids. No consideration is given to space as such, or consequently, to the transformations of these figures within a space that contains them all. We shall call this period *intrafigural* – an expression already used in developmental psychology to account for the development of geometrical concepts in the child.

The following period is characterized by efforts to find relationships between the figures. This manifests itself specifically in the search for transformations relating the figures according to various forms of correspondence. However, these transformations are not yet subordinated to structured sets. This is the period where projective geometry predominates. We shall call this period *interfigural*.

Following next is a third period, which we call *transfigural*. It is characterized by the predominance of structures. The work most characteristic of this period is the *Erlanger Programm* of Felix Klein" (Piaget and Garcia, 1989, Chapter III, Conclusions, p. 109).

We shall discuss an example to illustrate Piaget's conception of geometrical development and we shall simultaneously provide a particular interpretation of it in the light of what has been exhibited in Part 2 of this paper. Our conclusion will be that Piaget's three stages of cognitive development – from the consideration of individual objects, to the orientation towards actions and transformations and finally to structures – seems *grosso modo* correct. Piaget, however, makes too radical a distinction between acting and perceiving and between empirical and reflective abstraction. The reason lies exactly in his structuralism. Piaget therefore seems unable to grasp the dynamics of the development, providing rather a mere static description of it.

Piaget points out the importance of the concept of transformation for the development of geometrical thought, and he understands Descartes algebraization of mathematics as the essential force behind it. "It was to require a long period of uninterrupted work in algebra and infinitesimal calculus [...] to finally come to a conceptualization of the very idea of geometrical transformation without going through algebra or analysis" (Piaget and Garcia, 1989, p. 106). Piaget seems to pay no attention to the fact, however, that at least infinitesimal analysis and the function concept essentially depend on the very idea of space and the principle of continuity as well. The concept of mathematical function or transformation has a double root, algorithm and objective relation, as exemplified by the regularities of nature. As we have stated as a thesis in the last section, this complementarity might be fundamentally important for the transition to the interfigural and structural stages of development.

To understand mathematical functions means to understand the complementarity of formula and relation, as well as the self-referentiality that governs its evolution, as became apparent in Cauchy's definition of a continuous function. In the mathematics of the 17th/18th centuries, discontinuous functions could not be represented, because a function was an analytical law. A geometrical curve, on the other hand, was called continuous if it could be represented by a(n) (analytical) function (Euler, 1748, Vol. II). But this characterization proved to be incoherent.

Cauchy, after having demonstrated the inconsistency of these efforts (Grattan-Guinness, 1970), revised the whole approach on the basis of the principle of continuity, transforming mathematics into extensional theory. A function in Cauchy's or Dirichlet's sense can be seen as an equivalence class of analytic expressions or formulae, where the equivalence relation is based on the axiom of extensionality. This switch from an intensional to an extensional view has made it possible since Cauchy to single out sets of functions by certain of their properties, and in general to reason about them without representing them explicitly. For instance, instead of giving a linear function directly by $f(x) = ax$, Cauchy proves that a continuous function having the property $f(x + y) = f(x) + f(y)$ can be represented as above (Cauchy, 1821, p. 99/100). Now this kind of reasoning on the mathematical concept itself came to dominate mathematics at the very time when proof-analysis became its basis.

Still, strictly speaking we cannot operate on the concept as such, because it has to be represented anyway to become accessible. A concept is not to be conceived as a completely isolated and distinct entity in Platonic heaven, but must not on the other hand be confused with any set of intended applications. Two predicates or concepts or functions (or functions of functions) are to be considered as different even if they apply to exactly the same class of objects because they influence mental activity differently and may lead to different developments. The extensional view on mathematics ignores these facts completely.

Piaget himself accepts the idea that setting up correspondences on the one hand and operational constructions on the other might be two processes "common to all fields of knowledge" (Piaget and Garcia, 1989, p. 11). It seems indeed to be important to be aware of the fact that the Cartesian innovation already had a twofold nature from the very beginning, represented by the combination of number and variable on the one hand, and of space and quantity on the other; "for the extension in length breadth and depth, which constitute space is plainly the same as that which constitutes body", says Descartes. Cartesian mathematics is not algebraic in our sense, it is not "a science of pure structure", but is based on an interaction of number and geometric visualization. This duality which is, in our view, the basis of a complementarist understanding of algebra proper, (i.e. understanding algebra as a system blending those two lines) becomes crucial as soon as we consider relations between bodies, or in Piagetian terms, if we pass from the intrafigural to the interfigural.

Let us come back to the idea of proof. Leibniz invented formal proof, as we know it today (Hacking, 1980). He got the essential ideas from Descartes' algebraization of geometry, which was at the same time a geometrization of algebra, because of the continuity principle used in the process. The continuity principle presupposes a reasoning in terms of general ideal objects, that are distributive not collective in their character, that means things are considered as species of a kind and evaluated on grounds of the general conditions of their possibility or genesis. We have above already indicated this. Now today's philosophy of mathematics considers the implied conception of proof as a spatio-temporal entity "as quite outlandish" (Friedman, 1992, p. 57). It believes that formal proof begins were every reference to intuition has been eliminated (Schlick, Tarski).

We have seen that the maximal loop strategy of explanation, proving and justification suggests exactly the contrary. We want to profit from new computer programs that allow ample use of the continuity principle and illustrate some important truths of mathematical epistemology by means of specific example. Now Dynamic Geometry Systems (DGS) are apt to make the principle of continuity operative and thus to foster the growth of fertile hypotheses. Representational systems like DGS, having revitalized this principle, play a very important role in cognitive development because they realize an intimate and indissoluble interaction between observation and reasoning.

The example, which is the principal means of our argument, concerns Euler's theorem, stating that the concurrency points of the perpendicular bisectors, the medians and the altitudes of any triangle were collinear.

"Theorem 1 (Euler 1765): The orthocenter O, centroid CG and circumcenter M of any triangle are collinear. The line passing through these points is called the Euler line of the triangle. The centroid divides the distance from the orthocenter to the circumcenter in the ratio 2:1."

By analyzing the proofs of this theorem as presented by textbooks of elementary geometry (see for example: Coxeter and Greitzer, 1967, p. 18ff), one might hit upon the idea that the theorem is not at all about the relations between different properties of a single triangle, but rather is an affirmation about the relation

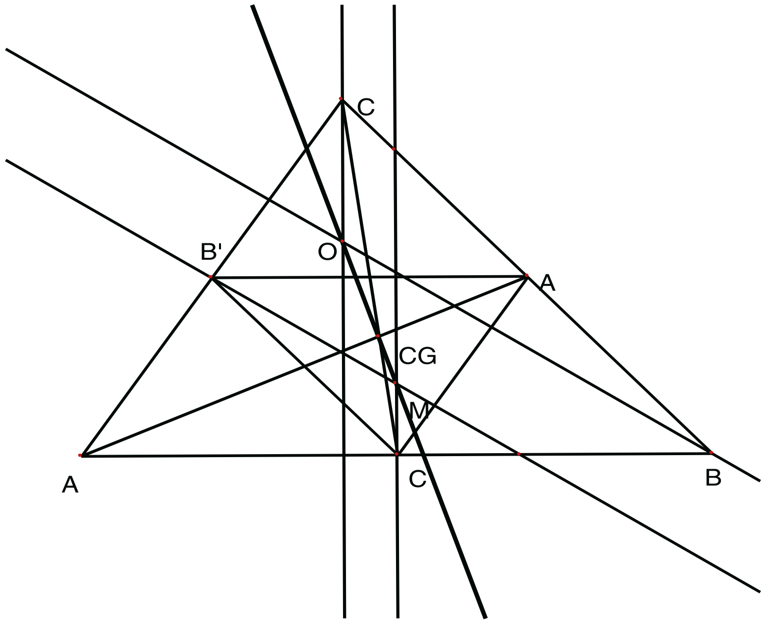

Figure 7.

between one and the same property (namely the location of the orthocenter) of two different triangles (the original one and its medial triangle, the triangle formed by joining the midpoints of the sides of the given triangle). In this manner we proceed from the intrafigural to the interfigural perspective (see Figure 7).

Now these two triangles are related to one another by means of a rotation of 180°about the centroid of the given triangle and an additional shrinking of the rotated triangle towards the centroid to half its size. Thus the image point X' of any point X of the plane under this transformation lies on the line that contains X and the centroid, the center of the transformation, to the other side of the centroid and half the distance from it. We shall call such a transformation a DST, just for the sake of convenience.

Let us pause for a moment and reflect about what has happened. The argument of our new transformational proof rests on analogy or on some principle of continuity according to which similar things in the givens are mapped on to similar things again. Such a view immediately opens the doors to further generalization. Whereas the traditional synthetic or "Euclidean" proofs, for instance, use all the premises of the theorem in the most

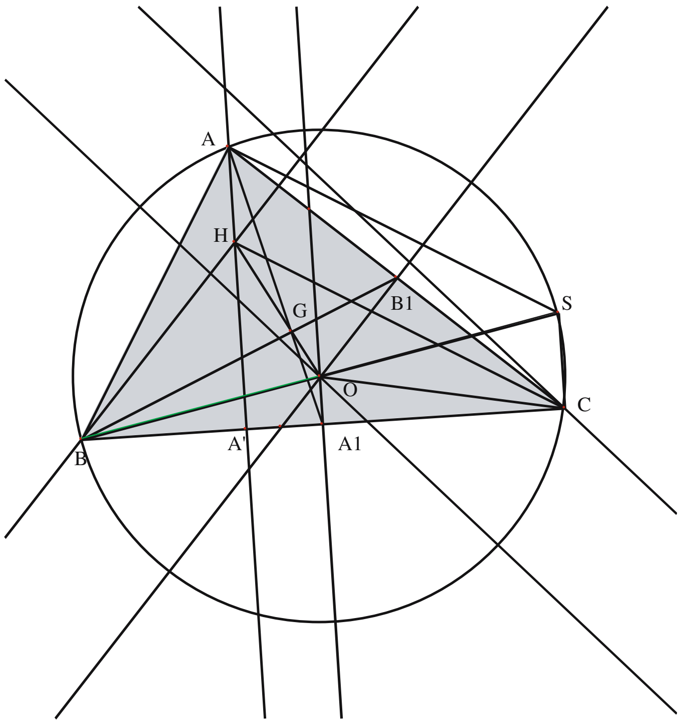

Figure 8.

intricate and ingenious manner (another such proof is indicated in Figure 8) the new proof does not.

It must be considered to be a general proof scheme, rather than a particular proof. Our interfigural perspective, in fact, not only yields a proof of the original theorem, but also proofs of some other ones. The theorem about the nine-point center, i.e. the fact that the center F of the Feuerbach circle, or nine-point circle, also lies on the Euler line, can immediately be established in a like manner (Figure 9) by observing that F is just the circumcenter of the transformed triangle A′ B′ C′.

Finally the proof also provides a first generalization of our original theorem 1, because the consideration of the intersection point of any cevians of the given triangle, and not just the orthocenter, leads to a similar theorem (see Figure 10).

"Euler's theorem 2: Take any two cevians and their point of intersection (for the sake of visual clarity, we shall in the sequel use only two lines of each type, but two already determine the important collinear points) and construct par-

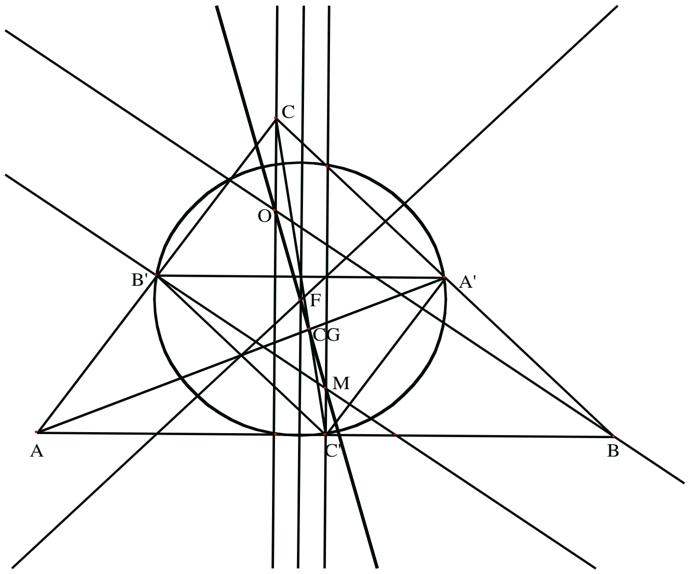

Figure 9.

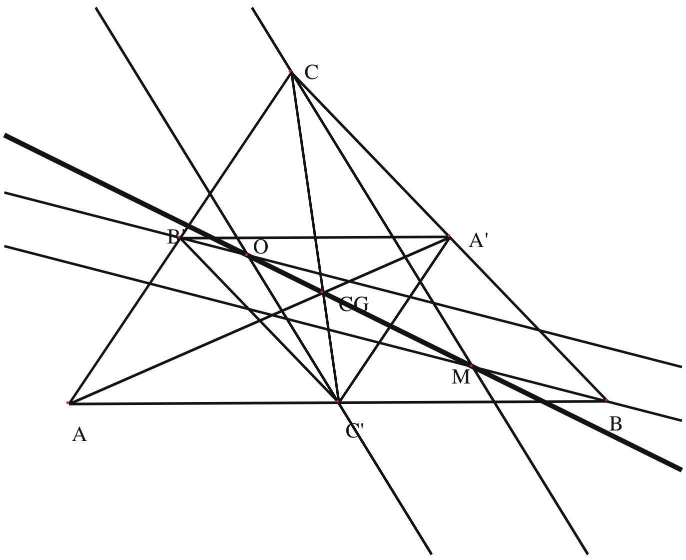

Figure 10.

allels to these lines through the midpoints of the corresponding opposite sides of the given triangle as well as their intersection point. Then the line through these two intersection points contains also the centroid of the triangle."

Looking once more at our proof by means of a geometrical transformation of type DST we realize that the factor 2 can also be substituted by any other number, that is the two similar triangles in question need not necessarily be related in the ratio 2:1. This means that the centroid CG of the given triangle which is also the center of the DST may be substituted by any other "center of gravity", as long as the cevians CC′ and BB′ passing through it intersect the sides of the triangle at points C′ and B′ such that the line B'C' through these points remains parallel to the third side BC of the given triangle (see Figure 11).

Stated differently, the two similar triangles should have parallel sides. Are we not already now taking a glimpse of a Desarguen configuration?

Our configuration essentially consists of three pairs of parallel lines. Mark two points on both lines of the first pair (C and B resp. C′ and B′) and let the other two pairs of parallel lines pass through these points respectively (see Figure 12).

The pairs of parallel lines thus are determined by AC and A′C′; AB and A′B′; AC and A′C′. We thus arrive at a Desarguen configuration, were the respective sides of the two triangles ABC and A′B′C′ in perspective intersect on the line at infinity, that is remain parallel. The existence of the "Eulerline" AA′ is now guaranteed by the inverse of Desargues theorem, stating that if the intersec-

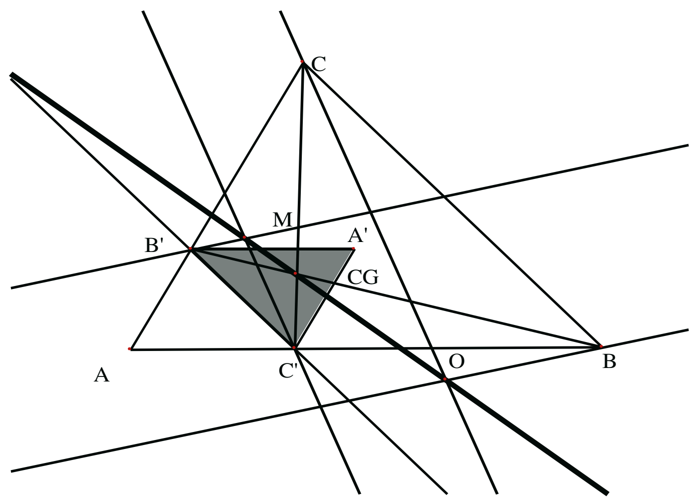

Figure 11.

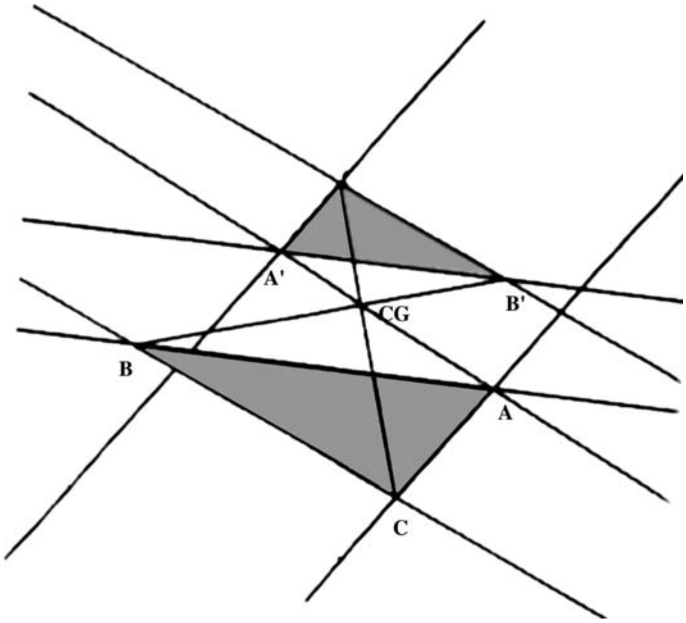

Figure 12.

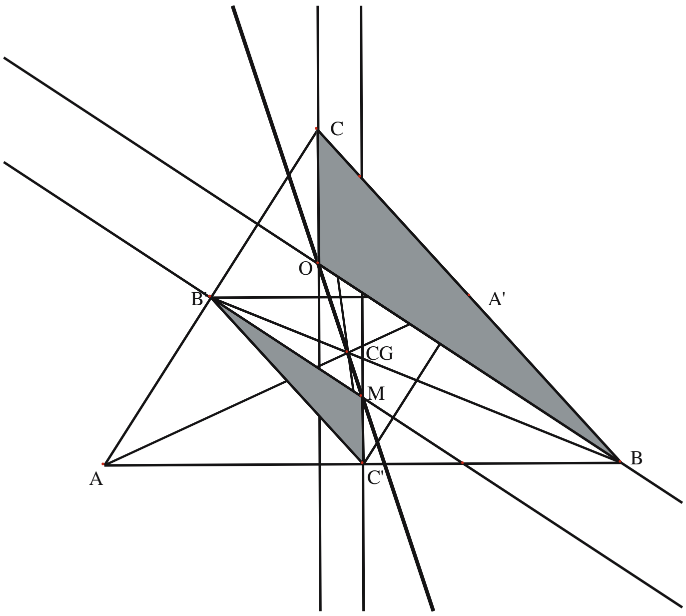

Figure 13.

tions of corresponding sides of two different triangles ABC and A′B′C′ (or the prolongations) lie on the same line, then the lines through corresponding vertices all pass through the same point CG. Now in our case this same line is the line at infinity, but a simple transformation of the coordinate system gives the general statement, thus generalizing once more our original theorem.

We gain also, interpreting Figure 12 in the light of these new insights, another proof of our original theorem. The triangles CBO and C′B′M have parallel sides. Define CG as the intersection of the lines CC′ and BB′ then Desargues' theorem says that the line joining the third vertices of our triangles, namely O and M, also passes through this point of intersection CG. There certainly might exist extremely talented persons, who would have immediately hit upon this new proof idea, thereby shortcutting the whole process of generalization. But such a thing might not happen frequently and our proof and the diagram on which it was based, marking a more natural path rather impedes such a radical idea. That means that there exists a logic of abduction and generalization, which is firmly connected with our cognitive means and representational systems (see Figure 13).

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