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Three-Dimensional Isothermal Lava Flows over a Non-Axisymmetric Conical Surface

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Abstract — Within the thin-layer approximation for a highly-viscous heavy incompressible fluid, a hydrodynamic model of a 3D isothermal lava flow over a non-axisymmetric conical surface is constructed. Using analytical methods, a self-similar solution for the law of leading-edge propagation is obtained in the case of a flow from a non-axisymmetric source located at the apex of a conical surface with smoothly varying properties. In the case of a flow over a substantially non-axisymmetric surface, it is shown that there exists a self-similar solution for the free-surface shape and the law of leading-edge motion. This solution is studied numerically for particular examples of the substrate surface and the source. In the general case of a non-self-similar flow over a substantially non-axisymmetric conical surface, a local analytical solution is obtained for the free-surface shape and the velocity field near the leading flow front.

Keywords: similarity, viscous fluid, conical surface, thin-film flow, extrusive eruption.

A lava flow produced by an extrusive volcanic eruption usually develops over a substantially curvilinear underlying surface [1]. However, in hydrodynamic models of lava flows the substrate surface is, as a rule, assumed to be planar for simplicity [2]. The literature contains self-similar solutions obtained using asymptotic lubrication theory which describe the isothermal lava flow front propagation over a horizontal [3] or inclined [4] plane. The effect of curvature of the underlying surface was investigated in a recent study [5], where within the lubrication approximation the steady-state solutions for the free-boundary shape were obtained for axisymmetric isothermal flow over a curved rigid surface. In [6], a flow from a point source located at the apex of a cone with finite angles of inclination of the generatrix to the horizontal was studied within the framework of the asymptotic model [5], under the condition that the entire fluid volume grows with time in accordance with a power (or exponential) law. The analytical self-similar solution obtained for the law of leading-edge propagation differs significantly from the corresponding law for flow over a horizontal plane [3]. In [6], the solution obtained was also generalized to include the case of a flow produced by an axisymmetric source located at the apex of a slightly non-axisymmetric conical surface. In [7], asymptotic models of non-isothermal lava flows with an exponential temperature dependence of the viscosity were constructed.

In this study, we will consider a 3D flow from a non-axisymmetric source located at the apex of an arbitrary non-axisymmetric conical surface. This problem formulation is a generalization of the model of a lava flow over a circular cone proposed in [6]. Below, we will construct asymptotic models of lava flows for a conical surface with smoothly varying properties in the azimuthal direction and for a substantially non-axisymmetric conical surface. The structure of the flow in the close vicinity of a lava flow front will also be studied. In contrast to [6], the present model takes into account the fluid flow in the azimuthal direction due to the non-symmetry of the underlying surface and the source.

1. FORMULATION OF THE PROBLEM

We will consider a 3D unsteady isothermal flow of a thin layer of highly viscous heavy incompressible fluid with a free surface. The origin of the flow is a non-axisymmetric point source located at the apex of an arbitrary non-axisymmetric conical surface. The ambient medium is assumed to be at rest. We introduce an orthogonal curvilinear coordinate system with the origin at the apex O. The axes x and y are directed along the surface generatrix and normal to the surface, respectively. The angle φ is measured counterclockwise in the horizontal plane, starting from a certain fixed direction (thus, the φ-axis is directed along the curvilinear spatial contour $x = \text{const}$, $y = \text{const}$). The velocity components in projection on the x, y, and φ axes are denoted by u, v, and w, respectively. The particular form of the underlying surface is specified by the function $\theta = \theta(\varphi)$, where θ is the angle of inclination of the surface generatrix to the horizontal. The magnitude of θ is assumed to be of the order of unity.

The Lamé coefficients of the x and y axes are equal to unity. Since the thin-layer approximation will be used for deriving the equations of motion, it is sufficient to calculate the Lamé coefficient of the φ-axis for points located on the conical surface ($y = 0$). Using auxiliary spherical coordinates, we obtain the formula for the Lamé coefficient of the φ-axis

$$H = x\sqrt{\theta'^2 + \cos^2\theta}$$

Here and in what follows, the ordinary derivative with respect to φ is denoted by a prime. In projection on the axes of the curvilinear coordinate system, the Navier-Stokes equations in the nondimensional variables with standard boundary conditions on the rigid ($y = 0$) and free ($y = h$) surfaces take the form [8]:

$$\frac{\partial uH}{\partial x} + \frac{\partial w}{\partial \varphi} + \frac{\partial vH}{\partial y} = 0, \qquad \varepsilon \mathrm{Re}\, a_x + \varepsilon \frac{\partial p}{\partial x} = \frac{\varepsilon}{H} L_x + \sin\theta \tag{1.1}$$

$$\varepsilon \mathrm{Re}\, a_\varphi + \varepsilon \frac{\partial p}{\partial \varphi} = \frac{\varepsilon}{H} L_\varphi + \frac{\theta' \cos\theta}{\sqrt{\theta'^2 + \cos^2\theta}}$$

$$\varepsilon \mathrm{Re}\, a_y + \varepsilon \frac{\partial p}{\partial y} = \frac{\varepsilon}{H} L_y - \frac{\cos^2\theta}{\sqrt{\theta'^2 + \cos^2\theta}}$$

$$L_x = \frac{\partial}{\partial y}\left[H\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right] - \frac{\partial}{\partial \varphi}\left[\frac{1}{H}\left(\frac{\partial wH}{\partial x} - \frac{\partial u}{\partial \varphi}\right)\right]$$

$$L_\varphi = \frac{\partial}{\partial y}\left[\frac{1}{H}\left(\frac{\partial wH}{\partial y} - \frac{\partial v}{\partial \varphi}\right)\right] - \frac{\partial}{\partial x}\left[\frac{1}{H}\left(\frac{\partial u}{\partial \varphi} - \frac{\partial wH}{\partial x}\right)\right]$$

$$L_y = \frac{\partial}{\partial x}\left[H\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] - \frac{\partial}{\partial \varphi}\left[\frac{1}{H}\left(\frac{\partial wH}{\partial y} - \frac{\partial v}{\partial \varphi}\right)\right]$$

$$y = h, \quad x > 0: \ v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + \frac{w}{H}\frac{\partial h}{\partial \varphi}, \quad \mathbf{P}_n = 0$$

$$y = 0, \quad x > 0: \ u = w = v = 0$$

$$u^* = Uu, \quad v^* = Uv, \quad w^* = Uw, \quad x^* = Lx, \quad y^* = Ly, \quad h^* = Lh,$$

$$t^* = \frac{L}{U}t, \quad p^* = \frac{\rho U^2}{\mathrm{Re}}p, \quad \mathrm{Re} = \frac{\rho U L}{\mu}, \quad \varepsilon = \frac{\mathrm{Fr}^2}{\mathrm{Re}} = \frac{\mu U}{\rho g L^2}, \quad \mathrm{Fr} = \frac{U}{\sqrt{gL}}$$

Here, a_x, a_φ, a_y, and L_x, L_φ, L_y are the projections of the vectors $d\mathbf{V}/dt$ and $\Delta\mathbf{V}$ on the coordinate axes, $\mathbf{P}_n$ is the stress vector on the free surface, p is the pressure, ρ and μ are the fluid density and viscosity, L is the characteristic length scale, and U is the characteristic scale of the longitudinal velocity component. The dimensional variables are denoted by an asterisk. Introducing the nondimensional variables, we assume that all the characteristic scales are given, except for the velocity scale U which will be determined below.

We assume that the dimensional volume of the moving fluid grows with time in accordance with the power law $2\pi Q t^{*\gamma}$, where Q is a dimensional constant. This condition restricts only the time dependence of the mass supply. In the general case, the source at the apex of the conical surface can be non-axisymmetric. The particular dimensional form of the mass supply dependence on φ is specified by the expression $2\pi Q Q_s(\varphi)(t^*)^{\gamma-1}$ for the mass flux across a small area with base $Hd\varphi$ and height h, perpendicular to the conical-surface generatrix and located at a distance $x \ll 1$ from the apex. In this expression, the azimuthal distribution of the mass supply rate $Q_s(\varphi)$ satisfies the relation

$$\int_0^{2\pi} Q_s(\varphi)\,d\varphi = 1 \tag{1.2}$$

In the case of flow over a circular cone, the evolution of the free-surface shape is described by a hyperbolic equation [6]. The solution of this equation contains a shock at the leading flow front. In the flow over a conical surface, there should also be a discontinuity of the layer thickness at the leading edge. The mass conservation law at the leading edge can be written in integral form as follows:

$$\int_0^h \left(u - \frac{\partial x_r}{\partial t} - \frac{w}{x\sqrt{\theta'^2 + \cos^2\theta}} \frac{\partial x_r}{\partial \varphi} \right) dy = 0 \tag{1.3}$$

Here, $x_r(t,\ \varphi)$ is the law of leading-edge motion.

The formulation of the problem in nondimensional variables contains the small parameter ε, which for real lava flows usually belongs to the range $[10^{-2},\ 10^{-6}]$ [7]. We will seek the solution of the Navier-Stokes and continuity equations with boundary conditions (1.1) in the form of asymptotic series in ε, retaining only the leading terms.

2. SELF-SIMILAR SOLUTIONS FOR SMALL $\theta'(\varphi)$

We will consider a flow over a conical surface, for which the generatrix inclination angle $\theta(\varphi)$ is a weak function of φ, i.e., we will assume that $\theta'(\varphi) \ll 1$. This is valid for a wide range of conical surfaces with smoothly varying properties, for which the variation of θ does not exceed $\pi/3$. In this case, the magnitudes of $\theta'(\varphi)$ are of the order of $1/6$, which can be regarded as a quantity significantly smaller than unity. For conical surfaces satisfying the condition $\theta'(\varphi) \ll 1$, we will seek the solution of Eqs. (1.1) in the form of asymptotic expansions in ε with the leading terms

$$\begin{aligned} &y = \varepsilon^{1/2}\eta, \qquad h = \varepsilon^{1/2}h_0, \qquad v = \varepsilon^{1/2}v_0 \\ &w = \varepsilon^{1/2}w_0, \qquad \theta' = \varepsilon^{1/2}\vartheta', \qquad p = \varepsilon^{-1/2}p_0 \end{aligned} \tag{2.1}$$

The other functions are of the order of unity. In what follows the subscript "0" is omitted. We set $\varepsilon^{1/2}L^3 = Q(L/U)^\gamma$, then U and ε can be expressed in terms of the known dimensional variables in accordance with the formulas [6]

$$U = \left(\frac{\rho g}{\mu}\right)^{1/(2\gamma+1)} Q^{2/(2\gamma+1)} L^{(2\gamma-4)/(2\gamma+1)}$$

$$\varepsilon = \left(\frac{\mu}{\rho g}\right)^{2\gamma/(2\gamma+1)} Q^{2/(2\gamma+1)} L^{-(2\gamma+6)/(2\gamma+1)}$$

We will assume that $\mathrm{Re} = o(\varepsilon^{-1/2})$. Substituting expansions (2.1) in Eq. (1.1) and retaining only the leading terms, we obtain

$$\begin{aligned} &\frac{\partial ux}{\partial x} + x\frac{\partial v}{\partial \eta} = 0, \quad 0 = \frac{\partial^2 u}{\partial \eta^2} + \sin\theta, \quad \frac{\partial p}{\partial \eta} = -\cos\theta, \quad \frac{1}{H}\frac{\partial p}{\partial \varphi} = \frac{\partial^2 w}{\partial \eta^2} + \vartheta' \\ &\eta = h, \quad x > 0: \ v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}, \quad \frac{\partial u}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0, \quad p = 0 \\ &\eta = 0, \quad x > 0: \ u = w = v = 0; \quad x = x_r: \ h = 0 \end{aligned} \tag{2.2}$$

The w velocity component does not enter into the continuity equation or the kinematic boundary condition. Thus, the problem of finding w can be separated, and φ enters into the other equations only as a parameter. This makes it possible to seek the solution of the 3D problem as a one-parameter family of solutions of the 2D problem of flow over a circular cone. Integrating the dynamical equation in projection on the longitudinal coordinate x (2.2) with account for the boundary conditions, we obtain an expression for the u velocity component. Substituting this expression in the integrated continuity equation (2.2), we derive an evolution equation similar to that for a circular cone [6], in which the dependence of θ on φ is parametric

$$\frac{\partial h}{\partial t} + \frac{1}{3x}\frac{\partial (xh^3)}{\partial x}\sin\theta(\varphi) = 0 \tag{2.3}$$

Taking into account the expression for the u velocity component, we write the azimuthal mass supply distribution in the form

$$\lim_{x\to 0}\left(\frac{h^3 x\sin 2\theta}{6}\right) = Q_s(\varphi)t^{\gamma-1} \tag{2.4}$$

The flow from an axisymmetric source is described by the self-similar solution for the leading-front motion [6] in which $\theta = \theta(\varphi)$.

We will now generalize the self-similar solution [6] to include the case of a flow from a non-axisymmetric source with the rate (2.4). The law of flow front motion is given in explicit dimensional form:

$$x_r^*(t^*,\ \varphi) = z_r(\gamma)\left[\frac{Q_s^2(\varphi)\sin\theta(\varphi)}{\cos^2\theta(\varphi)}\right]^{1/5}\left(\frac{\rho g Q^2}{\mu}\right)^{1/5}(t^*)^{(2\gamma+1)/5}$$
$$z_r(\gamma) = \left(\int_0^1 g(\xi)\xi\,d\xi\right)^{-2/5}, \quad (5g^2 - 3\xi)^{1-\gamma}g^{2\gamma+1}\xi = (6\gamma)^{1-\gamma}\left(\frac{3(2\gamma+1)}{5}\right)^{\gamma+1/2} \tag{2.5}$$

The second formula gives an implicit expression for the free-surface shape in the self-similar variables $g(\xi)$, which is related to the layer thickness in the initial variables by the formula

$$h(t,\ x,\ \varphi) = t^{(\gamma-2)/2}\sqrt{z_r(\gamma)}\left(\frac{Q_s}{\sin^2\theta\cos\theta}\right)^{1/5} g(\xi), \quad \xi = x/x_r$$

The expression for the law of flow front motion (2.5) depends explicitly on the angle φ, which makes it possible to predict the most dangerous directions of lava front propagation in the case when the shape of the underlying surface and the azimuthal distribution of the mass supply rate are known. For the sake of illustration, in Figs. 1, 2 we present the flow front positions for a circular cone with a non-axisymmetric source and also in the most general case of a non-axisymmetric surface with a non-axisymmetric source at the apex.

In Fig. 1, for the flow over a cone, the nondimensional dependence of the leading-edge coordinate on φ (2.5) is presented for successive instants of time a constant interval apart. The azimuthal distribution of the mass supply rate is given by $Q_s(\varphi) = 2 + \sin 4\varphi$ (*a*) and $Q_s(\varphi) = 1.1 + \sin\varphi$ (*b*).

In Fig. 2*a*, the shapes of the flow front at successive instants of time are shown for the flow over the surface $\theta(\varphi) = \pi/3 + (\pi/12)\cos(4\varphi)$ from a non-axisymmetric source with the time-independent rate $Q_s(\varphi) = 2 + \sin 4\varphi$. This conical surface has four canyons ($\varphi = n\pi/2$, $n = 0,\dots,\ 3$), which in the case of an axisymmetric source are the directions of maximum rate of flow front propagation. The dependence of the mass supply rate on φ has four maxima ($\varphi = \pi/8 + n\pi/2$, $n = 0,\dots,\ 3$) and four minima. The nonlinear interplay between these effects results in the appearance of directions of maximum rate of flow front propagation, which differ from the directions of the maxima of the generatrix inclination angle $\theta(\varphi)$ and the mass supply rate $Q_s(\varphi)$. In the case considered, the directions of the maximum rate of leading-edge

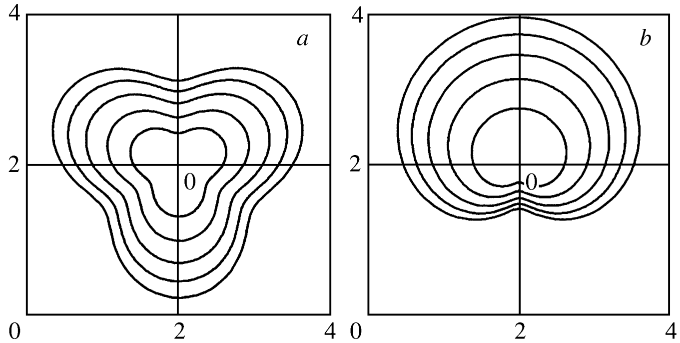

Fig. 1. Dependence of the flow front coordinate on the azimuthal angle φ for the flow over a circular cone from a non-axisymmetric source of constant intensity. The time interval is constant

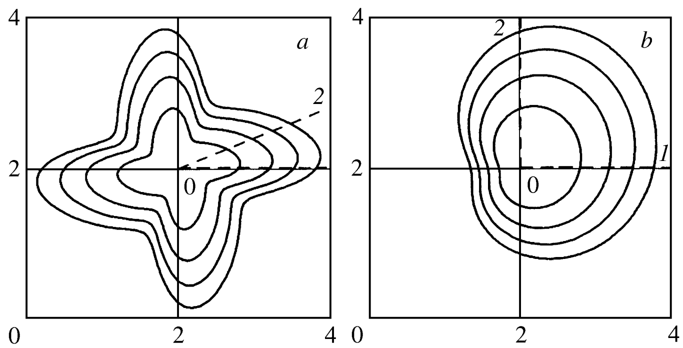

Fig. 2. Dependence of the flow front coordinate on the azimuthal angle φ for the flow over non-axisymmetric conical surfaces from a non-axisymmetric source of constant intensity. The broken curves denote the directions of the maxima of the generatrix inclination angle *1* and the source rate *2*. The time interval is constant

propagation are closer to the directions of the canyons than to the directions of the maxima of the mass supply rate Q_s.

Consider the flow over the surface $\theta(\varphi) = \pi/4 + (\pi/6)\cos(\varphi)$ from a source with the time-independent rate $Q_s(\varphi) = 1.1 + \sin\varphi$. Both the surface inclination angle and the mass supply rate have a single maximum, and $\theta(\varphi)$ is a weak function of φ. In this case, the direction of maximum rate of flow front propagation is closer to the direction of maximum rate of mass supply Q_s (Fig. 2*b*).

We will now find the condition for $\theta(\varphi)$ and $Q_s(\varphi)$, which determines the flow with an azimuthally constant rate of leading-edge propagation. Setting the expression for $\partial x_r/\partial\varphi$ (2.5) equal to zero, we arrive at the equation

$$\frac{Q_s'}{Q_s} = -\frac{1+\sin\theta}{2\sin\theta\cos\theta}\theta'$$

Integrating this equation, we find

$$Q_s(\varphi) = \frac{\cos\theta(\varphi)}{\sqrt{\sin\theta(\varphi)}}$$

3. SELF-SIMILAR SOLUTIONS FOR FINITE $\theta'(\varphi)$

In the case of a substantially non-axisymmetric conical surface with finite angles of inclination of the generatrix to the horizontal, the leading terms of the asymptotic expansions of the unknown functions are of the orders

$$y = \varepsilon^{1/2}\eta, \quad h = \varepsilon^{1/2}h_0, \quad v = \varepsilon^{1/2}v_0, \quad p = \varepsilon^{-1/2}p_0 \tag{3.1}$$

The other functions are of the order of unity. Below, the subscript "0" is omitted. We assume that $\mathrm{Re} = o(\varepsilon^{-1})$. Substituting expansions (3.1) in the Navier-Stokes equations and boundary conditions (1.1) and retaining only the leading terms, we obtain

$$\frac{\partial uH}{\partial x} + \frac{\partial w}{\partial \varphi} + \frac{\partial vH}{\partial \eta} = 0, \quad H = \sqrt{\theta'^2 + \cos^2\theta} \tag{3.2}$$

$$0 = \frac{\partial^2 u}{\partial \eta^2} + \sin\theta, \quad 0 = \frac{\partial^2 w}{\partial \eta^2} + \frac{\theta'\cos\theta}{\sqrt{\theta'^2 + \cos^2\theta}}, \quad \frac{\partial p}{\partial \eta} = -\frac{\cos^2\theta}{\sqrt{\theta'^2 + \cos^2\theta}}$$

$$\eta = h, \quad x > 0: \ v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + \frac{w}{H}\frac{\partial h}{\partial \varphi}, \quad \frac{\partial u}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0, \quad p = 0$$

$$\eta = 0, \quad x > 0: \ u = w = v = 0; \quad x = x_r: \ h = 0$$

Integration of the dynamical equations and the continuity equation over the transverse coordinate with account for the boundary conditions gives the equation

$$\frac{\partial h}{\partial t} + \frac{1}{3x}\frac{\partial (xh^3)}{\partial x}\sin\theta + \frac{1}{3x\sqrt{\theta'^2 + \cos^2\theta}}\frac{\partial}{\partial \varphi}\left(\frac{h^3\theta'\cos\theta}{\sqrt{\theta'^2 + \cos^2\theta}}\right) = 0 \tag{3.3}$$

Here, in the last term the expression in parentheses is the azimuthal mass flux caused by the non-symmetry of the underlying surface. In contrast to Eq. (2.3) describing the flow over a circular cone, this equation is essentially three-dimensional.

The integral mass conservation law in the stretched nondimensional variables takes the form:

$$\int_0^{2\pi}\int_0^{x_r} hx\sqrt{\theta'^2 + \cos^2\theta}\, dx\, d\varphi = 2\pi t^{\gamma} \tag{3.4}$$

With account for the expression for u obtained by integrating (3.2), the azimuthal distribution of the mass supply rate can be written as

$$\lim_{x\to 0}\left(\frac{xh^3\sin\theta}{3}\sqrt{\theta'^2 + \cos^2\theta}\right) = Q_s(\varphi)t^{\gamma-1} \tag{3.5}$$

Equation (3.3) has a self-similar solution of the form:

$$h(t,\, x,\, \varphi) = t^{\alpha} f(\xi,\, \varphi), \qquad x = t^{\beta}\xi, \qquad 0 < \xi < \xi_r(\varphi), \qquad 2\alpha + 1 = \beta \tag{3.6}$$

In the self-similar variables, the law of flow front motion becomes $\xi = \xi_r(\varphi)$. Substitution of expressions (3.6) in (3.3) leads to the first-order quasilinear partial differential equation

$$\begin{gathered} \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \varphi}\left[\frac{f^2\theta'\cos\theta}{\xi D(f^2\sin\theta - \beta\xi)}\right] = -\frac{f}{3\xi}\left(\frac{f^2 A + 3\alpha\xi}{f^2\sin\theta - \beta\xi}\right) \\ A(\varphi) = \sin\theta + \frac{\theta''\cos^3\theta - \theta'^4\sin\theta}{D^2}, \quad D(\varphi) = \theta'^2 + \cos^2\theta \end{gathered} \tag{3.7}$$

The introduction of self-similar variables (3.6) makes it possible to reduce the number of independent variables to two. From the condition that mass conservation law (3.4) admits the introduction of self-similar variables (3.6), we find $\alpha = (\gamma - 2)/5$ and $\beta = (2\gamma + 1)/5$. These formulas coincide with the corresponding expressions for the flow over a circular cone. Thus, for a power-law mass supply, the time dependence of the leading-front coordinate $x_r = \xi_r(\varphi)t^{(2\gamma+1)/5}$ is the same for all conical surfaces. Here, the coefficient $\xi_r(\varphi)$ is determined by the particular form of the underlying surface $\theta(\varphi)$ and the azimuthal distribution of the mass supply rate $Q_S(\varphi)$.

In the self-similar variables, conditions (3.5) and (1.3) take the form:

$$\xi \to 0: \quad f = \left(\frac{3Q_s}{\xi B \sin\theta}\right)^{1/3}, \quad B(\varphi) = \sqrt{\theta'^2 + \cos^2\theta} \tag{3.8}$$

$$\xi = \xi_r(\varphi): \quad \frac{d\xi_r}{d\varphi} = \frac{\xi_r D(f^2 \sin\theta - 3\beta\xi_r)}{f^2\theta'\cos\theta} \tag{3.9}$$

By means of the change of variables $g(\xi,\ \varphi) = f^2/\xi$, Eq. (3.7) with conditions (3.8) and (3.9) can be simplified and reduced to the form:

$$\frac{\partial g}{\partial \xi} + \frac{\partial g}{\partial \varphi}\left[\frac{g\theta'\cos\theta}{\xi D(g\sin\theta - \beta)}\right] = -\frac{g}{3\xi}\left(\frac{gC - 3}{g\sin\theta - \beta}\right) \tag{3.10}$$

$$C(\varphi) = 5\sin\theta + 2\frac{\theta''\cos^3\theta - \theta'^4\sin\theta}{D^2} \tag{3.11}$$

$$\xi \to 0: \quad g = \left(\frac{9Q_s^2}{\xi^5 D\sin^2\theta}\right)^{1/3} \tag{3.12}$$

$$\xi = \xi_r(\varphi): \quad \frac{d\xi_r}{d\varphi} = \frac{\xi_r D(g\sin\theta - 3\beta)}{g\theta'\cos\theta}$$

For solving Eq. (3.10) with conditions (3.11) and (3.12) numerically, it is convenient to map the flow domain with the unknown boundary $\xi_r(\varphi)$ onto the interior of the unit circle using the following change of variables:

$$g(\xi,\ \varphi) = \psi(\zeta,\ \varphi), \qquad \xi = \xi_r(\varphi)\zeta, \qquad 0 < \zeta < 1$$

$$\frac{\partial}{\partial \xi} = \frac{1}{\xi_r}\frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi} - \zeta\frac{\xi_r'}{\xi_r}\frac{\partial}{\partial \zeta}$$

A similar mapping of the flow domain with the unknown boundary onto a fixed region was used, for example, in [9] and [10]. In the new variables, Eq. (3.10) with conditions (3.11) and (3.12) can written as

$$\frac{\partial \psi}{\partial \varphi} = -\frac{\zeta}{\psi}\left[\frac{(\psi\sin\theta - \beta)\xi_r D - \psi\xi_r'\theta'\cos\theta}{\xi_r\theta'\cos\theta}\right]\frac{\partial \psi}{\partial \zeta} - \frac{D(\psi C - 3)}{3\theta'\cos\theta} \tag{3.13}$$

$$\zeta = \zeta_0 \ll 1: \quad \psi = \frac{1}{\zeta^{5/3}}\left(\frac{9Q_s^2}{\xi_r^5 D\sin^2\theta}\right)^{1/3} \tag{3.14}$$

$$\zeta = 1: \quad \frac{d\xi_r}{d\varphi} = \frac{\xi_r D(\psi\sin\theta - 3\beta)}{\psi\theta'\cos\theta} \tag{3.15}$$

The system of equations obtained was investigated with the aim of estimating the effect of the azimuthal fluid flow on the law of flow front propagation. System (3.13)–(3.15) was solved numerically in the domain $[\varphi_0,\ \varphi_1] \times [\zeta_0,\ 1]$. It was assumed that on the interval $[\varphi_0,\ \varphi_1]$ the magnitude of the derivative $\theta'(\varphi)$ varies from a small value to a finite one, and then returns to a small value. Thus, we assume that, on the interval $[0,\ \varphi_0]$, $\theta'(\varphi) \ll 1$ and the self-similar solution is valid, while, on the interval $[\varphi_0,\ \varphi_1]$, $\theta'(\varphi) \sim 1$ and system (3.13)–(3.15) holds. In the numerical calculations, the point φ_0 (as well as the point φ_1) is conventionally assumed to be the boundary separating the domains of applicability of the different approximations.

We used a rectangular grid with steps $\Delta\varphi$ and $\Delta\zeta$. The derivative $\partial\psi/\partial\varphi$ was represented using the first-order right-hand difference, and the derivative $\partial\psi/\partial\zeta$ using the second-order central difference. As a result of discretization, for Eq. (3.13) an implicit two-layer scheme on a four-point template was constructed. Relation (3.14) was employed as the left boundary condition, while at the right end a soft condition was

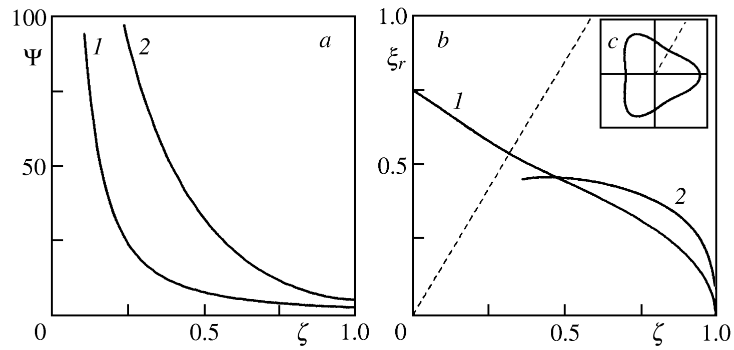

Fig. 3. Numerical solution for the 3D flow from an axisymmetric source over a substantially non-axisymmetric surface. The free-surface shape (*a*) near the crest (*1*) and the canyon (*2*). The shape of the flow front (*b*). The analytical self-similar solution (*1*) and the numerical calculation (*2*). The entire shape of the flow front as predicted by the analytical self-similar solution (*c*)

specified, which means that the derivative $\partial\psi/\partial\zeta$ is equal to zero. As the initial condition for $\psi(\varphi_0,\ \zeta)$ and $\xi_r(\varphi_0)$, we used self-similar solution (2.5) obtained within the approximation $\theta'(\varphi) \ll 1$. The calculations were organized as follows: using the known values of ξ_r for the current layer of φ, we calculated ψ for the next layer with the help of the sweep method, and then on the next level we found ξ_r using the method of trapezoids.

As an example, in Fig. 3 we present the calculation results for the free surface ψ and the leading edge ξ_r for the flow from an axisymmetric source over the substantially non-axisymmetric surface $\theta(\varphi) = \pi/3 + (\pi/12)\cos(3\varphi)$, which has three crests and three canyons. For the example considered, the maximum value $\theta' \approx 0.8$. The calculations showed that in this case the shape of the leading flow front, obtained from the solution of the complete system of equations, differs only slightly ($\sim 10\%$) from the corresponding analytical solution (2.5) obtained by neglecting the azimuthal fluid flow (Fig. 3*b*). The example considered confirms the possibility of applying self-similar solution (2.5) even to lava flows over substantially non-axisymmetric conical surfaces.

4. STEADY-STATE SOLUTIONS

Let us consider the steady-state solutions of the equations for the free-surface shape. In the case of small $\theta'(\varphi)$, Eq. (2.2) has the steady-state solution for the layer thickness $h = Cx^{-1/3}$, where the constant C depends parametrically on φ. With account for the azimuthal distribution of the mass supply rate (2.4), we obtain the analytical steady-state solution for the free-surface shape in the form:

$$h = \left(\frac{6Q_s(\varphi)}{x\sin 2\theta(\varphi)}\right)^{1/3} \tag{4.1}$$

Comparing this solution with solution [5], we conclude that, for flow over a conical surface with smoothly varying properties, for fixed φ the layer thickness depends on x as in the case of a circular cone with the generatrix inclination angle $\theta(\varphi)$ and the dimensional source flow rate $QQ_s(\varphi)$.

In the case of a steady flow at finite $\theta'(\varphi)$, Eq. (3.3) can be written in the form:

$$\frac{\partial h_1}{\partial x} + \frac{\partial h_1}{\partial \varphi}\left(\frac{\theta'\cos\theta}{xB^2\sin\theta}\right) = -\frac{h_1}{x}\left[1 + \frac{1}{B\sin\theta}\left(\frac{\theta'\cos\theta}{B}\right)'\right] \tag{4.2}$$

Here, $h_1 = h^3$, and the expression for $B(\varphi)$ is given by (3.8). We now write Eq. (4.2) in the characteristic form

$$\frac{dh_1}{dx} = -\frac{h_1}{x}\left[1 + \frac{1}{B\sin\theta}\left(\frac{\theta'\cos\theta}{B}\right)'\right], \qquad \frac{d\varphi}{dx} = \frac{\theta'\cos\theta}{xB^2\sin\theta} \tag{4.3}$$

The initial conditions for this system of equations follow from the expression for the azimuthal distribution of the source rate (3.5)

$$x = x_0 \ll 1: \ \ h_1 = \frac{3Q_s(\varphi_0)}{x_0 B(\varphi_0)\sin\theta(\varphi_0)}, \quad \varphi = \varphi_0 \in [0,\ 2\pi) \tag{4.4}$$

In Eq. (4.3) the variables can be separated, which makes it possible to integrate these equations analytically in quadratures. In view of initial conditions (4.4), we obtain the solution along the characteristics $x(\varphi)$

$$\begin{aligned} h &= \left[\frac{3Q_s(\varphi_0)}{x_0 B(\varphi_0)\sin\theta(\varphi_0)}\left(\frac{\theta'(\varphi_0)B(\varphi)}{\theta'(\varphi)B(\varphi_0)}\right)\right]^{1/3} \exp\left(-\int\limits_{\varphi_0}^{\varphi}\frac{\sin 2\theta(\tau)}{6\theta'(\tau)}\,d\tau\right) \\ x(\varphi) &= x_0\frac{\cos\theta(\varphi_0)}{\cos\theta(\varphi)}\exp\left(\int\limits_{\varphi_0}^{\varphi}\frac{\sin 2\theta(\tau)}{2\theta'(\tau)}\,d\tau\right) \end{aligned} \tag{4.5}$$

In the general case, in order to construct the steady-state solution over the entire flow domain it is necessary to separate the curve of initial data $x = x_0$ into two (disconnected) domains with respect to φ, in which θ' is small and finite, respectively. In the first domain the solution is constructed along the characteristics in accordance with formula (4.1), while in the second domain the solution is given by formulas (4.5).

5. THE STRUCTURE OF THE SHOCK AT THE LEADING FLOW FRONT

Each self-similar solution of the free-surface shape among those examined above contains a shock at the leading flow front. The law of leading-front motion can be obtained without resolving this singularity. However, in order to obtain a more detailed flow pattern in the vicinity of the flow front it is necessary to construct a continuous solution satisfying the zero-thickness condition at the leading edge. For this purpose, on the basis of Eq. (3.2) describing the general case of a non-axisymmetric flow we will construct a two-scale expansion for the film thickness, which will make it possible to obtain a continuous solution.

For finite θ, the dynamical equation in projection on the generatrix (see (3.2)) contains only the derivative of the shear stress and the projection of the gravity force and hence the evolution equation for the film thickness is hyperbolic, whereas for small θ the dynamical equation contains the longitudinal pressure gradient, the derivative of the shear stress, and the gravity force [5]. Thus, in contrast to the case of finite θ, the equation for the film thickness is parabolic, which makes it possible to construct a continuous solution.

The asymptotic analysis of the original Eq. (1.1) showed that in the case of finite θ the entire flow domain can be separated into three asymptotic regions: (i) an outer region, where the length and the thickness are of the order of 1 and $\sqrt{\varepsilon}$, respectively, and thin-layer equations (3.2) hold true; (ii) an intermediate region, where both scales are of the same order $\sqrt{\varepsilon}$ and the complete Stokes equations with account for the gravity force are valid; (iii) a small vicinity of the contact line, where the length and thickness scales are both of the order of ε and the complete Stokes equations for the flow in a right-angled corner without account for the gravity force are valid.

We assume that, in the general case, in the outer region flow with an arbitrary law of mass supply takes place, and hence the flow is non-self-similar. On the basis of Eq. (3.2), we will construct a two-scale expansion for the required solution, which describes the free-surface shape in both the outer and intermediate regions. For this purpose, in Eq. (3.2) written on the outer scale it is sufficient to take into account the leading higher-order term with the longitudinal pressure gradient. We note that the inner solution does not satisfy the complete Stokes equations in the intermediate region, but only represents the asymptotics of the solution of the thin-layer equations with account for the longitudinal pressure gradient.

Taking into account the higher-order term with the longitudinal pressure gradient in the dynamical equation in projection on the generatrix (3.2), we obtain

$$\sqrt{\varepsilon}\,\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial \eta^2} + \sin\theta$$

The other equations and boundary conditions of (3.2) remain unchanged. Integrating the momentum and the continuity equations with account for the above-mentioned term, we obtain the equation

$$\frac{\partial h}{\partial t} + \frac{1}{3x}\frac{\partial}{\partial x}\left[xh^3\left(\sin\theta - \sqrt{\varepsilon}\,\frac{\partial h}{\partial x}\frac{\cos^2\theta}{\sqrt{\theta'^2 + \cos^2\theta}}\right)\right] + \frac{1}{\sqrt{\theta'^2 + \cos^2\theta}}\frac{\partial}{\partial \varphi}\left(\frac{h^3\theta'\cos\theta}{\sqrt{\theta'^2 + \cos^2\theta}}\right) = 0, \qquad x = x_r:\;\; h = 0 \quad (5.1)$$

Here, x_r is the flow front coordinate. This is a second-order parabolic partial differential equation with a small coefficient $\sqrt{\varepsilon}$ in the leading spatial derivative. In the limit $\varepsilon \to 0$, we obtain Eq. (3.3), which is valid in the outer region. We will now construct the inner expansion for the layer thickness in the vicinity of the leading front, where the second spatial derivative should be taken into account. We introduce the new variables

$$t = t_1, \quad x = x_r(t,\ \varphi) + \sqrt{\varepsilon}\,\xi, \quad \varphi = \varphi_1, \quad h(t,\ x) = h_1(t_1,\ \xi,\ \varphi)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} - \frac{u_r}{\sqrt{\varepsilon}}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{1}{\sqrt{\varepsilon}}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi_1} - \frac{w_r}{\sqrt{\varepsilon}}\frac{\partial}{\partial \xi}$$

In these formulas, $u_r = \partial x_r/\partial t$ and $w_r = \partial x_r/\partial\varphi$ are the projections of the flow front velocity on the x and φ axes (x_r, u_r, and w_r are known from the outer solution). Substituting these expressions in (5.1) and retaining only the leading terms, as $\varepsilon \to 0$, we obtain

$$\frac{\partial}{\partial \xi}\left[x_r h_1^3\left(\sin\theta - \frac{\partial h_1}{\partial \xi}\frac{\cos^2\theta}{\sqrt{\theta'^2 + \cos^2\theta}}\right) - 3x_r u_r h_1 - \frac{h_1^3 w_r \theta'\cos\theta}{\theta'^2 + \cos^2\theta}\right] = 0$$

$$\xi = 0:\;\; h_1 = 0, \qquad \xi \to -\infty:\;\; h_1 = h_0 \quad (5.2)$$

This equation contains only derivatives with respect to ξ, and hence the solution depends on t and φ parametrically, so the ξ-derivative can be regarded as an ordinary derivative. The last condition of (5.2) is the condition of asymptotic matching [11] with the outer solution, where $h_0(t,\ x)$ is the layer thickness at the leading flow front obtained from the outer solution. Integrating Eq. (5.2) once, we find that the expression under the sign of the derivative with respect to ξ is a certain function of time and φ. From the condition of matching with the outer solution, we find that this function is identically equal to zero. We introduce the new variables

$$\eta = \sqrt{a}\,\eta_s, \qquad h_1 = \sqrt{a}\,h_s, \qquad \xi = \sqrt{a}\,b\zeta$$

$$a(t,\ \varphi) = \frac{3u_r x_r(\theta'^2 + \cos^2\theta)}{x_r\sin\theta(\theta'^2 + \cos^2\theta) - \theta' w_r\cos\theta}$$

$$b(t,\ \varphi) = \frac{x_r\cos^2\theta\sqrt{\theta'^2 + \cos^2\theta}}{x_r\sin\theta(\theta'^2 + \cos^2\theta) - \theta' w_r\cos\theta}$$

In these variables, the derived equation for $h_s(\zeta)$ is independent of time and the angle

$$\frac{dh_s}{d\zeta} = 1 - \frac{1}{h_s^2}, \qquad \zeta = 0: \;\; h_s = 0 \tag{5.3}$$

Integrating this equation, we obtain[1]

$$h_s + \frac{1}{2}\ln\left(\frac{1-h_s}{1+h_s}\right) = \zeta \tag{5.4}$$

The asymptotics of this solution near the contact line are as follows:

$$\zeta \to 0: \;\; h_s = (3|\zeta|)^{1/3} + O(|\zeta|^{5/3})$$

Solution (5.4) is presented in Fig. 4. We note that, in the general case, the solution derived describes the shock structure at the leading front of an arbitrary flow which is not necessarily self-similar.

In particular, solution (5.4) describes the free-surface shape in the vicinity of the leading front of the axisymmetric flow over a circular cone. In this case, from (2.2) we one can derive the velocity components in the stretched variables, which do not depend explicitly on θ:

$$\begin{aligned} u &= \left(\eta h - \frac{\eta^2}{2}\right)\left(1 - \sqrt{\varepsilon}\frac{\partial h}{\partial x}\right) \\ v &= -\frac{\eta^2}{2x}\left(h - \frac{\eta}{3}\right)\left(1 - \sqrt{\varepsilon}\frac{\partial h}{\partial x}\right) - \frac{\partial}{\partial x}\left[\frac{\eta^2}{2}\left(h - \frac{\eta}{3}\right)\left(1 - \sqrt{\varepsilon}\frac{\partial h}{\partial x}\right)\right] \end{aligned} \tag{5.5}$$

In order to investigate the streamline pattern, we rewrite the expressions for the velocity components (5.5) in the new variables

$$(u - u_r)/u_r = u_s(\zeta, \chi), \quad v/3u_r = v_s(\zeta, \chi), \quad \chi = \eta_s/h_s$$

In the asymptotic limit $\varepsilon \to 0$, we obtain the formulas

$$u_s = -\frac{3}{2}\chi^2 + 3\chi - 1, \qquad v_s = \frac{1-h_s^2}{6h_s^2}(2\chi - 3)\chi^2 \tag{5.6}$$

It turns out that, on passage into the inner region, the asymptotic order of the transverse velocity component changes and becomes equal to unity. Thus, the longitudinal and transverse velocity components have the same order in the $\sqrt{\varepsilon}$-vicinity of the leading front. The streamlines satisfy the ordinary differential equation $d\chi/d\zeta = v_s/u_s$, which in view of (5.6) can be written in the form:

$$\frac{d\chi}{d\zeta} = \frac{1-h_s^2}{3h_s^2}\,\frac{\chi^2(2\chi-3)}{3\chi(2-\chi)-2} \tag{5.7}$$

Separating the variables and expressing h_s in terms of $dh_s/d\zeta$ using Eq. (5.3), we obtain the analytical solution in the form:

$$h_s + \frac{1}{6}\ln\left(\frac{3-2\chi}{2\chi^{28}}\right) + \frac{2}{\chi} = \text{const} \tag{5.8}$$

The streamlines described by analytical solution (5.8) are shown in Fig. 4 in the variables ζ and η_s. The broken curve is the separating line $h_d(\zeta)$, on which the longitudinal velocity component changes the sign. Above this curve $u_s > 0$, and below $u_s < 0$. Moreover, over the entire flow domain $v_s < 0$, except for

[1]This result was first obtained in the study by A. A. Osiptsov, *Asymptotic Models of Lava Flows Over Curvilinear Substrate Surfaces* [in Russian] // MS's Thesis, Mech. and Math. Faculty, Lomonosov Moscow State University, 1–112 (May 2005).

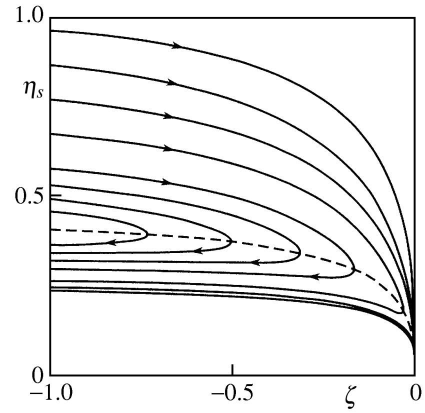

Fig. 4. Local analytical solution for the free-surface shape and the streamline pattern near the leading flow front. The arrows denote the flow direction. On the broken curve, the longitudinal velocity component is zero

the lower boundary, where $v_s = 0$. Equation (5.8) makes it possible to find an explicit expression for the separating curve from the condition that, on this curve, the denominator of the formula for $d\chi/d\zeta$ is equal to zero. Thus, we obtain

$$h_d = \left(1 - \frac{1}{\sqrt{3}}\right) h_s$$

Solution (5.4) also describes the structure of the shock at the leading front in the case of flow over an inclined plane. For this flow, the asymptotics of the self-similar solution for the free-surface shape, which coincides with (5.4), were indicated in [4]. However, in [4] the velocity field and the streamline pattern in the vicinity of the flow front were not investigated.[2]

Using a finite-difference method, Eq. (5.1) was solved numerically on a rectangular grid using an implicit two-layer scheme on a six-point template. The left boundary condition was specified in the form (2.5) (written in the axisymmetric case). On the right boundary, the condition of zero spatial derivative was used. The scheme is first-order accurate. At the initial instant of time, a thin layer of finite thickness (precursor) was specified over the entire flow domain, except for the vicinity of the left margin, where the thickness suddenly increases to the value prescribed by (2.5). The calculation results are presented in Fig. 5. Analytical solution (5.4) describes the free-surface shape shown in Fig. 5 in a small vicinity of the leading edge, where the term with the longitudinal pressure gradient in the dynamical equation is of the order of unity.

Summary. In the case of a 3D isothermal flow from a non-axisym- metric source located at the apex of a non-axisymmetric conical surface with smoothly varying properties in the azimuthal direction, for the free-surface shape a 2D first-order hyperbolic partial differential equation is obtained. This equation contains the azimuthal angle only as a parameter. For a constant mass supply, the steady-state solution for the free-surface shape is obtained analytically. For a power-law time dependence of the mass supply, an analytical self-similar solution for the law of flow front propagation is found.

In the case of the flow over a substantially non-axisymmetric conical surface, a 3D hyperbolic equation for the free-surface shape which takes into account the fluid flow in the azimuthal direction is obtained. The steady-state solution of this equation is found analytically in quadratures. For a power-law time dependence of the total fluid volume, a self-similar solution is obtained. It is shown that law of flow-front propagation with time is the same for all conical surfaces. On the basis of the numerical calculations, it is shown that the

[2]During the preparation of this paper for publication, the author became acquainted with the paper by M. Dragoni, I. Borsari, and A. Tallarico, "A model for the shape of lava flow fronts" // J. Geophys. Res., 2005, V. 110, B09203, in which an analytical solution was obtained for the velocity field and the free-surface shape in the vicinity of the leading front of a 2D steady flow over an inclined plane. Note that the solution (5.4) obtained in the present study is more general, since it describes an arbitrary unsteady 3D flow over a substantially non-axisymmetric conical surface.

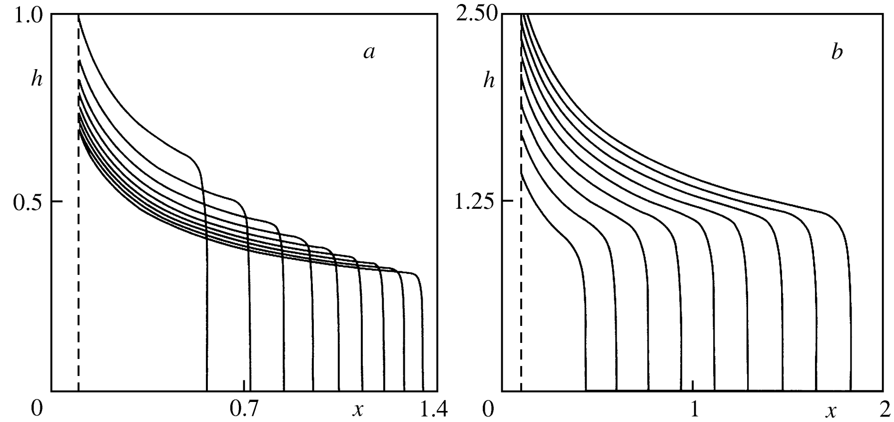

Fig. 5. Free-surface shape found numerically for the flow over a cone with account for the longitudinal pressure gradient. The time interval is constant, $\gamma = 0.5$ (*a*) and (*b*)

analytical self-similar solution for a conical surface with smoothly varying properties can also be used for the approximate description of the flows over substantially non-axisymmetric conical surfaces, for which the derivative of the generatrix inclination angle with respect to the azimuthal angle reaches finite values.

Within the framework of the thin-layer equations, with account for the leading higher-order term with the longitudinal pressure gradient, an analytical solution is found for the free-surface shape and the streamline pattern near the leading flow front. The solution constructed describes the flow structure in the flow front vicinity in the general case of an arbitrary non-self-similar flow over a substantially non-axisymmetric conical surface.

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