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Behavior of a Cylindrical Drop under Multi-Frequency Vibration

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Abstract — The behavior of an inviscid-fluid drop surrounded by a different fluid under the action of multi-frequency vibration is investigated. The second-order effects in the vibration amplitude are studied. A superharmonic resonance is registered. The stability of the forced oscillations with respect to small perturbations is studied. The condition of the onset of parametric resonance is found. An average drop shape is investigated. The two-frequency case is considered as a particular case of multi-frequency vibration.

Keywords: cylindrical drop, multi-frequency vibration, superharmonic resonance, parametric resonance.

Vibration is one of the most widely used means of influencing the behavior of nonuniform hydrodynamic systems. The non-trivial behavior of the system under the action of vibration makes the theoretical and experimental investigation of these phenomena topical. The need for research is also conditioned by the fact that vibration may either result from the action of external sources or be used for controlling engineering processes.

We have investigated the effect of vibration on an interface. This subject was first explored by Faraday, who considered the effect of one-frequency vertical vibrations on the horizontal interface between a liquid and a gas. He discovered experimentally and described the excitation of parametric oscillations of the interface (so-called "Faraday ripple" or "Faraday waves") [1]. A theoretical explanation of this phenomenon for the system considered was given in [2], using a linear stability analysis for a potential flow.

In recent years, the action of vibration on a curved interface has been comprehensively researched. The behavior of a spherical drop of incompressible fluid surrounded by another fluid and subjected to high-frequency vibrations was investigated in [3]. The main effect is an average contraction of the drop along the vibration axis. This effect takes place for both a heavy and a light-weight drop. A similar result was obtained for the average shape of a cylindrical drop of incompressible inviscid fluid located between two rigid planes in the presence of monochromatic vibration perpendicular to the drop symmetry axis [4]. In the case of the acoustic levitation of a spherical drop [5, 6], when the compressibility of the media is important, similar phenomena were observed: the drop was compressed along the vertical axis and eventually took the shape of an apple ("dog-bone" shape); however this shape was unstable. As in [6], in [7] the behavior of liquid drops in a one-frequency acoustic field was investigated experimentally. Attention was focused on the variation of the equilibrium drop shape with variation of the acoustic pressure and on drop fragmentation in intense acoustic fields.

In [8], the behavior of a compressed cylindrical drop was studied experimentally in the case of small ratios of the layer thickness to the equilibrium drop radius. It was found that, on average, heavy and light-weight drops are, respectively, stretched and compressed along the vibration axis.

The parametric resonance registered in [1] takes place when the vibration frequency ω is twice the natural frequency $\omega = 2\Omega$ (here, Ω is the natural frequency). In [9], unlike most other studies, the case of single-frequency vibration in which the surface and volume oscillation amplitudes are of the order of the bubble size was considered. The study was aimed mainly at investigating the interaction between the bubble volume and bubble shape oscillations due to the initial nonequilibrium bubble shape. For spherical bubbles, the 2:1 resonance condition was considered, and for non-spherical bubbles, the 1:1 and 2:1 conditions, i.e. $\Omega_0 = \Omega_m$ or $\Omega_0 = 2\Omega_m$ (predicted in the small-amplitude approximation), where ω_0 is the volume-oscillation natural frequency, and Ω_m is the natural frequency of the m-th shape oscillation mode. Parametric instability of the forced oscillations of the shape of a spherical and a cylindrical drop is manifested when the synchronism condition $\omega = \Omega_m + \Omega_{m+1}$ is satisfied [10, 11] (here, ω is the vibration frequency, and Ω_m and Ω_{m+1} are the frequencies of two neighboring natural vibration modes).

In study [12] of the forced finite-amplitude waves on the surface of an inviscid fluid in an infinitely deep channel, the linear $\omega = \Omega$ and subharmonic $\omega = 3\Omega$ resonances were considered. The linear resonance proved to be more dangerous than the subharmonic one.

Similar phenomena are also observed in nonlinear mechanical systems with different numbers of degrees of freedom (see reviews [13, 14]). In [15], the cases of 1:2, 1:3, and 1:4 internal resonance were considered for a two-degree-of-freedom system with hysteresis. In [16], the parametric $\omega = 2\Omega_1$ and subharmonic $\omega = 3\Omega$ resonances were considered in studying the stability of a nonlinear single-degree-of-freedom system. It was shown that the parametric resonance is the more dangerous.

In [17], with reference to the example of the Duffing equation, the stability of a system with two equilibrium states under the action of two-frequency vibration at low and high frequencies was investigated. In a system with high friction subjected to high-frequency vibration, specific resonances (so-called vibrational resonances) are possible, although the ordinary linear resonance is absent. In this case, the parametric resonance is only weakly expressed.

A nonlinear oscillator under multi-frequency vibration was studied in [18]. Three cases were considered: (i) the external frequencies differ significantly from each other and from the natural oscillator frequency, (ii) the external frequencies are similar but differ significantly from the natural frequency, and (iii) the external and natural frequencies are similar. It was found that, in the first case, the free natural oscillations are either damped or have a constant value. In the other cases, depending on the vibration frequencies, the oscillations may grow.

Study [19] was devoted to the basic forms of nonlinear oscillations of an oscillatory system with a single degree of freedom and second-, third-, and fourth-order nonlinearities in the presence of harmonic multi-frequency vibration. Linear, parametric, and superharmonic resonances were considered. The effect of nonlinearity on the system behavior was studied. It was demonstrated that, in the case of synchronized linear and superharmonic resonances, the oscillation amplitude can be controlled by adding new external frequencies.

In [4], a nonlinear resonance, which appears when the vibration frequency is equal to one-half the frequency of the basic natural-oscillation mode of a cylindrical drop, was detected. This superharmonic-resonance phenomenon appearing when $2\omega = \Omega_m$ was described in [14, 20], which dealt with mechanical systems. For the stability of a nonlinear mechanical system with a single degree of freedom, the superharmonic resonance $2\omega = \Omega_m$ is more dangerous than the linear and parametric resonances $\omega = 2\Omega_1$ [16].

In [21], the scattering of a modulated acoustic wave with carrier frequency ω_i and modulation frequency ω_p on the surface of a gas bubble was used for measuring the bubble size and density. When the frequency ω_p is close to the natural oscillation frequency of the bubble, resonant growth of the amplitude of the bubble surface oscillations with the frequency $\omega_i \pm \omega_p$ occurs (linear resonance). Parametric resonance in a similar system at the frequency $\omega_i \pm \omega_p/2$ was considered in [22].

In this study, we will consider the behavior of a cylindrical drop located between two plane walls under the action of multi-frequency vibration.

1. FORMULATION OF THE PROBLEM

We will consider the behavior of a drop of inviscid fluid with the density ρ_i^* surrounded by another fluid with the density ρ_e^*. The entire system is bounded by two parallel rigid planes (Fig. 1). In the absence of vibration, the drop is cylindrical with radius R. The contact angle between the lateral surface of the drop

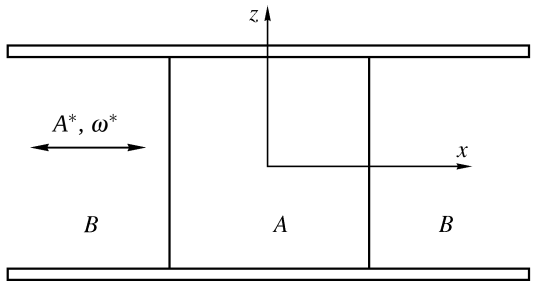

Fig. 1. Geometry of the problem: A — drop, B — surrounding fluid

and the rigid walls is equal to $\pi/2$ and does not change in the course of the motion.

The flow around the drop is oscillating. The center of mass of the drop travels relative to the laboratory (fixed) reference frame along the x axis at the velocity $U(t)$, which has to be determined. Far away from the drop, the fluid velocity is directed along the x axis and can be represented as

$$\mathbf{v} = \mathbf{j}\frac{d}{dt}f(t), \qquad f(t) = \sum_{k=1}^{N} A_k^* e^{i\omega_k^* t} + \text{c.c.}$$

where A_k^* and ω_k^* are the amplitude and the frequency of the k-th vibration component, and $\mathbf{j}$ is the unit vector of the x axis. Here and in what follows, the symbol "c.c." signifies the complex conjugate terms. The external-vibration amplitudes A_k^* are much smaller than the equilibrium radius R.

We assume that the vibration frequencies are sufficiently high to neglect the viscosity, but are low enough to neglect the compressibility: $l = \sqrt{\nu_{i,e}/\omega_k^*} \ll R$ and $\omega_k^* R \ll c$, where l is the thickness of the viscous boundary layer, ν is the kinematic viscosity, c is the sonic velocity, and the subscripts "i" and "e" denote the parameters inside and outside the drop.

We will consider only a two-dimensional flow in the plane $(x,\ y)$. It is convenient to introduce the polar coordinates $(r,\ \alpha)$, in which the drop surface is described by the equation $r = R + \zeta(\alpha,\ t)$, where $\zeta(\alpha,\ t)$ is the deviation of the lateral surface of the drop from the equilibrium state, and the angle α is measured from the x axis. In [4], it was demonstrated that, in the leading order of the expansion in a small vibration amplitude, the drop is displaced as a whole, with the drop shape remaining unchanged. Accordingly, it is convenient to go over to the coordinate system fitted to the center of mass. The conditions of its fixity and of drop volume conservation can be written as

$$\int_0^{2\pi} (R + \zeta)^3 \cos\alpha \, d\alpha = 0, \qquad \int_0^{2\pi} (R + \zeta)\, d\alpha = 2\pi R^2$$

In the center of mass system, the Euler equation takes the form:

$$\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho^*}\nabla p - \mathbf{j}\frac{\partial}{\partial t}U$$

Far away from the drop, the following condition should be satisfied:

$$r \to \infty: \qquad \mathbf{v} \to \left(\frac{d f(t)}{dt} - U\right)\mathbf{j}$$

On the drop surface, the condition of continuity of the normal velocity, the kinematic condition, and the balance condition for the normal stresses should hold:

$$[v_n] = 0, \quad \frac{\partial}{\partial t}\zeta = v\nabla F, \quad [p] = -\sigma\,\mathrm{div}\,\mathbf{n}, \quad F = r - R - \zeta, \quad \mathbf{n} = \nabla F/|\nabla F|$$

Here, **n** is the unit vector of the outward normal to the drop lateral surface and σ is the surface tension. Square brackets denote the jump in the bracketed quantity on passage from the external to the internal fluid.

We will consider a potential flow and introduce the velocity potentials $\mathbf{v} = \nabla\varphi$. The time, length, surface deviation, velocity, density, and pressure scales are: t_0, R, A_0 (vibration amplitude), v_0, ρ_0, and p_0, where

$$t_0 = \sqrt{\frac{(\rho_i^* + \rho_e^*)R^3}{2\sigma}}, \qquad A_0 = \frac{1}{N}\sum_{k=1}^{N}|A_k^*|, \qquad v_0 = A_0\sqrt{\frac{2\sigma}{(\rho_e^* + \rho_i^*)R^3}},$$

$$\rho_0 = \frac{1}{2}(\rho_i^* + \rho_e^*), \qquad p_0 = \frac{A_0\sigma}{R^2}$$

In the nondimensional variables, the problem formulation takes the form:

$$\Delta\varphi_{i,e} = 0, \qquad p = -\rho\varphi_t - \frac{1}{2}\varepsilon\rho(\nabla\varphi)^2 - \rho U_t r\cos\alpha \tag{1.1}$$

$$r \to \infty: \ \varphi \to (f_t - U)r\cos\alpha \tag{1.2}$$

$$r = 1 + \varepsilon\zeta: \ [\varphi_n] = 0, \quad \zeta_t = \boldsymbol{\nabla}\varphi\boldsymbol{\nabla}F, \quad \varepsilon[p] = -\operatorname{div}\mathbf{n} \tag{1.3}$$

$$\int_0^{2\pi}(1 + \varepsilon\zeta)^3\cos\alpha\, d\alpha = 0, \qquad \int_0^{2\pi}\zeta\, d\alpha = -\frac{1}{2}\varepsilon\int_0^{2\pi}\zeta^2\, d\alpha \tag{1.4}$$

The subscripts “t”, “r”, and “α” signify differentiation with respect to the corresponding variables. The problem contains the following nondimensional parameters: the relative amplitudes a_k, the vibration frequencies ω_k, the mean vibration amplitude ε, the ambient-fluid density ρ_e, and the drop fluid density ρ_i, where

$$a_k = \frac{A_k^*}{A_0}, \qquad \sum_{k=1}^{N} a_k = N, \qquad \omega_k = \omega_k^*\sqrt{\frac{(\rho_i^* + \rho_e^*)R^3}{2\sigma}}, \qquad \varepsilon = \frac{A}{R}$$

$$\rho_e = \frac{2\rho_e^*}{\rho_i^* + \rho_e^*}, \qquad \rho_i = \frac{2\rho_i^*}{\rho_i^* + \rho_e^*}, \qquad \rho_i + \rho_e = 2$$

2. FORCED OSCILLATIONS

We will seek the solution of system (1.1)–(1.4) in the form of power series in ε:

$$\varphi = \varphi^{(0)} + \varepsilon\varphi^{(1)} + \ldots, \qquad p = p^{(0)} + \varepsilon p^{(1)} + \ldots, \qquad \zeta = \zeta^{(0)} + \varepsilon\zeta^{(1)} + \ldots$$

In the zero-order approximation, we obtain the problem:

$$\delta\varphi^{(0)} = 0, \qquad p^{(0)} = -\rho\varphi_t^{(0)} - \rho U_t r\cos\alpha$$

$$r \to \infty: \ \varphi_e^{(0)} \to (f_t - U)r\cos\alpha$$

$$r = 1: \ [\varphi_e^{(0)}] = 0, \qquad \zeta_t^{(0)} = \varphi_r^{(0)}, \qquad [p^{(0)}] = \zeta^{(0)} + \zeta_{\alpha\alpha}^{(0)}$$

The solution of this problem is

$$\zeta^{(0)}, \quad \varphi_i^{(0)} = 0, \quad \varphi_e^{(0)} = \Pi f_t\left(r + \frac{1}{r}\right)\cos\alpha, \quad \Pi = \frac{1}{2}(\rho_i - \rho_e)$$
$$U(t) = \rho_e f_t \tag{2.1}$$

The conditions of stationarity of the center of mass and constancy of the volume (1.4) are automatically satisfied. For a very heavy drop, the velocity U of the center of mass relative to the laboratory reference frame vanishes; for $\rho_i = \rho_i = 1$, U coincides with the ambient-fluid velocity; and, for a light-weight drop (bubble) U is greater than the fluid velocity at infinity. This agrees with the results of [4].

In the first approximation, we obtain the problem:

$$\Delta\varphi^{(1)} = 0, \qquad p^{(1)} = -\rho\varphi_t^{(1)} - \frac{1}{2}\rho(\nabla\varphi^{(0)})^2$$

$$r \to \infty: \qquad \varphi_e^{(1)} \to 0$$

$$r = 1: \ [\varphi_r^{(1)}] = 0, \qquad \zeta_t^{(1)}\varphi_r^{(1)}, \qquad [p^{(1)}] = \zeta^{(1)} + \zeta_{\alpha\alpha}^{(1)}$$

The solution of this problem is

$$\varphi_i^{(1)} = \frac{1}{2}Br^2\cos 2\alpha, \qquad \varphi_e^{(1)} = -\frac{1}{2}B_t\frac{1}{r^2}\cos 2\alpha \tag{2.2}$$

$$\zeta^{(1)} = B\cos 2\alpha \tag{2.3}$$

The function $B(t)$ is found from the solution of the differential equation

$$B_{tt} + 3B = -\rho_e\Pi^2 f_t^2 \tag{2.4}$$

The solution of the previous order (2.1) contains only terms with $\exp(\pm i\omega_k t)$. In the first approximation, as is clear from the form of the nonuniform term in Eq. (2.4), solution (2.2)–(2.3) depends on pair combinations (sum or difference) of the vibration frequencies $\exp(\pm i(\omega_k \pm \omega_l)t)$. In the units used, the natural frequencies of the drop oscillations are given by the formula [23]

$$\Omega_n^2 = 1/2n(n^2 - 1), \qquad n \geq 2 \tag{2.5}$$

The natural frequency of the general solution of the uniform equation (2.4) is equal to the quadrupole mode frequency ($n = 2$) of the natural drop oscillations. If, in the nonuniform term in (2.4), some combination of the external frequencies is close to this frequency, a resonance appears. In the case of monochromatic vibration, the resonant growth of the oscillation amplitude occurs at a frequency which is one-half the minimum natural-oscillation frequency (superharmonic or, in the terminology of [20], nonlinear resonance). In the case of multi-frequency vibration, as is clear from (2.4), forced oscillation resonance may also appear when the sum or difference of two vibration frequencies is equal to the quadrupole mode frequency.

The time-independent part of the function ζ describes the variation of the mean drop shape:

$$\langle\zeta\rangle = -(1/3)\rho_e\Pi^2\langle f_t^2\rangle\cos 2\alpha \tag{2.6}$$

Here, the parentheses signify the time averaging of the corresponding quantity. As is clear from (2.6), no matter what the density ratio, the drop compression along the vibration axis takes place. This agrees with the results of [4]. In the experimental study [8], the compression of a light-weight drop and the stretching of a heavy drop were registered. This is apparently attributable to the nonstationary contact angle dynamics. In any case, in [8] the drop radius was much greater than the layer thickness. This means that the contact line phenomena become decisive. The contact-angle dynamics, taken into account in [24] for the degenerate case of an equilibrium angle equal to $\pi/2$, demonstrated the fundamental possibility of stretching a heavy drop.

3. STABILITY OF THE FORCED OSCILLATIONS

To study the stability of the forced oscillations, we will introduce the main-flow perturbation:

$$\varphi' = \varphi + \psi, \qquad \zeta' = \zeta + \xi, \qquad p' = p + q \tag{3.1}$$

Here, φ', ζ', p' are the perturbed fields, φ, ζ, and p denote the main flow, and ψ, ξ, and q are small unsteady perturbations. After the introduction of perturbed fields (3.1) into the original system (1.1)–(1.4) and linearization, for the perturbations we obtain the system of equations and boundary conditions

$$\Delta\psi = 0, \qquad q = -\rho(\psi_t + \nabla\varphi\cdot\nabla\psi)$$
$$r \to \infty: \ \psi \to 0$$
$$r = 1: \ [\psi_n] = 0, \qquad \xi_t = \nabla\psi\cdot\nabla F - \nabla\varphi\cdot\nabla\xi$$
$$[q] = \xi^{(0)} + \xi^{(0)}_{\alpha\alpha} - \varepsilon(\xi^{(1)} + \xi^{(1)}_{\alpha\alpha}) + \ldots$$
$$\int_0^{2\pi}(1 + 2\varepsilon\zeta + \varepsilon^2\zeta^2)\zeta\cos\alpha\, d\alpha = 0, \qquad \int_0^{2\pi}\xi\, d\alpha = -\varepsilon\int_0^{2\pi}\zeta\xi\, d\alpha$$

We will seek the solution of this problem using the multiple-scale method:

$$\psi_i = \sum_m C_1^{(m)} r^m e^{im\alpha}, \qquad \psi_e = \sum_m C_2^{(m)}\frac{1}{r^m}e^{im\alpha}, \qquad \xi = \sum_m T_m(t)e^{im\alpha}$$
$$C_{1,2}^{(m)} = C_{1,2}^{(m,0)} + \varepsilon C_{1,2}^{(m,1)} + \ldots, \qquad T_m = T_m^{(0)} + \varepsilon T_m^{(1)} + \ldots$$
$$\frac{d}{dt} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + \ldots = D_0 + \varepsilon D_1 + \ldots$$

where m is the natural-oscillation mode number.

As mentioned above, in the presence of monochromatic vibration the parametric instability for the principal resonances appears when the synchronism condition $\omega = \Omega_m + \Omega_{m+1}$ is satisfied. This condition also applies in the case of multi-frequency vibration, because it is associated only with the fact that for the forced oscillations the fundamental mode is translational while for the natural oscillations it is quadrupole. For a quantitative description of the proximity of the external frequencies to the sum $\Omega_m + \Omega_{m+1}$, we introduce the detuning parameter $\omega_k - (\Omega_m + \Omega_{m+1}) = \gamma = \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \ldots$

Thus, in the zero-order expansion, we obtain the problem

$$\Delta\psi^{(0)} = 0, \qquad q^{(0)} = -\rho\,\psi_t^{(0)}$$
$$r \to \infty: \ \psi^{(0)} \to 0$$
$$r = 1: \ [\psi_r^{(0)}] = 0, \qquad \xi_t^{(0)} = \psi_r^{(0)}, \qquad [q^{(0)}] = \xi^{(0)} + \xi^{(0)}_{\alpha\alpha}$$

The solution of this problem is

$$\psi_i^{(0)} = \sum_{m=2} i\frac{\Omega_m}{m}B_m^{(0)}(t_1,\ldots)r^m e^{i\Omega_m t_0}e^{im\alpha} + \text{c.c.}$$
$$\psi_e^{(0)} = -\sum_{m=2} i\frac{\Omega_m}{m}B_m^{(0)}(t_1,\ldots)\frac{1}{r^m}e^{i\Omega_m t_0}e^{im\alpha} + \text{c.c.}$$
$$\xi^{(0)} = \sum_{m=2} B_m^0(t_1,\ldots)e^{i\Omega_m t_0}e^{im\alpha} + \text{c.c.}$$

In the first order, we have the following problem:

$$\Delta\psi^{(1)} = 0, \qquad q^{(1)} = -\rho(\psi^{(1)}_{t_0} + \psi^{(0)}_{t_1}) - \rho\nabla\psi^{(0)}\cdot\nabla\varphi^{(0)}$$

$$r \to \infty: \ \psi^{(1)} \to 0$$

$$r = 1: \ [\psi^{(1)}_r] + \xi^{(0)}\varphi^{(0)}_{err} - \xi^{(0)}_\alpha\varphi^{(0)}_{e\alpha} = 0, \qquad \xi^{(1)}_{t_0} + \xi^{(0)}_{t_1} = \psi^{(1)}_{ir}$$

$$[q^{(1)}] + \xi^{(0)}[p^{(0)}_r] = \xi^{(1)} + \xi^{(1)}_{\alpha\alpha}$$

Eliminating the secular terms gives the differential equations for the amplitudes

$$i\Omega_m D_1 B^{(0)}_m e^{i\Omega_m t_0} - m\rho_e\Pi\overline{B}^{(0)}_{m+1}e^{-i\Omega_{m+1}t_0}(D_0^2 f_1 - i\Omega_{m+1}D_0 f_1) = 0 \tag{3.2}$$

$$\Omega_{m+1}D_1\overline{B}^{(0)}_{m+1}e^{-i\Omega_{m+1}t_0} - (m+1)\rho_e\Pi\Omega_m B^{(0)}_m e^{i\Omega_m t_0}D_0\overline{f}_1 = 0$$

$$f_1(t) = \sum_{k=1}^{N} a_k e^{i\omega_k t} \tag{3.3}$$

We introduce the function $f_1(t)$ into system (3.2)–(3.3) and take the frequency detuning γ into account:

$$D_1 B^{(0)}_m + i\gamma_1 B^{(0)}_m = \frac{m\rho_e\Pi\omega_k a_k}{2i\Omega_m}(\omega_k - \Omega_{m+1})\overline{B}^{(0)}_{m+1}$$

$$D_1\overline{B}^{(0)}_{m+1} - i\gamma_1\overline{B}^{(0)}_{m+1} = \frac{(m+1)\rho_e\Pi\omega_k\Omega_m\overline{a}_k}{2i\Omega_{m+1}}B^{(0)}_m$$

Representing the solution for the amplitudes $B^{(0)}_m$, $B^{(0)}_{m+1}$ in the form $\exp(\lambda t_1)$, we obtain the equation for the growth rate λ:

$$\lambda^2 = \frac{m(m+1)\rho_e^2\Pi^2\omega_k^2\Omega_m|a_k|^2}{4\Omega_{m+1}} - \gamma_1^2 \tag{3.4}$$

If the natural frequencies Ω_m and Ω_{m+1} have different signs, then the right side of Eq. (3.4) is negative and the disturbances do not grow. As a result, resonance is possible only for the sum of the frequencies of two neighboring modes, i.e. when the frequencies are of the same sign. For fairly large vibration amplitudes, we have $\lambda^2 > 0$ and, hence, growing disturbances exist. From this equation, we find the threshold vibration amplitude as a function of the detuning parameter γ_1. The form of the neutral curve $\varepsilon(\omega_k)$ is given by the expression

$$\omega_k = \Omega_m + \Omega_{m+1} \pm \varepsilon\sqrt{\frac{m(m+1)\rho_e^2\Pi^2\omega_k^2|a_k|^2\Omega_m}{4\Omega_{m+1}}} \tag{3.5}$$

4. TWO-FREQUENCY VIBRATION

We will consider the main effects obtained for a specific case of external action. We represent the function $f(t)$ in the form

$$f(t) = A_1 e^{i\omega_1^* t} + A_2 e^{i\omega_2^* t} + \text{c.c.}$$

The velocity of the center of mass relative to the laboratory reference frame (2.1) is

$$U(t) = \rho_e(i\omega_1 a_1 e^{i\omega_1 t} + i\omega_2 a_2 e^{i\omega_2 t} + \text{c.c.})$$

In the first-order expansion, solution (2.2)–(2.3) takes the form:

$$\varphi^{(1)}_i = (i\omega_1 A e^{2i\omega_1 t} + i\omega_2 B e^{2i\omega_2 t} - 1/2i(\omega_2 - \omega_1)Ce^{i(\omega_2-\omega_1)t} + 1/2i(\omega_2 + \omega_1)De^{i(\omega_2+\omega_1)t} + \text{c.c.})r^2\cos 2\alpha$$

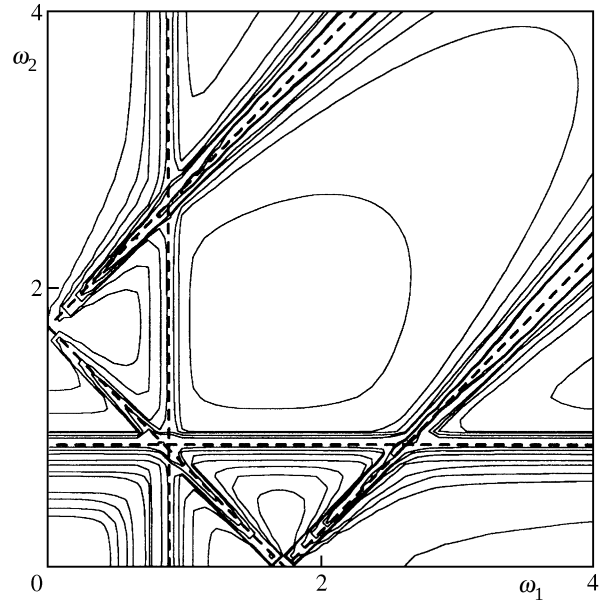

Fig. 2. Isolines of the oscillation energy dependence on the external frequencies ($\Omega_2^2 = 3$). The maxima, shown by the broken lines, correspond to the resonances $\omega_1 \pm \omega_2 = \pm\Omega_2$, $\omega_1 = \omega_2/2$, $\omega_2 = \omega_2/2$

$$\varphi_e^{(1)} = -(i\omega_1 A e^{2i\omega_1 t} + i\omega_2 B e^{2i\omega_2 t} - 1/2i(\omega_2 - \omega_1) C e^{i(\omega_2-\omega_1)t} + 1/2i(\omega_2 + \omega_1) D e^{i(\omega_2+\omega_1)t} + \text{c.c.}) r^{-2}\cos 2\alpha$$

$$\zeta^{(1)} = (A e^{2i\omega_1 t} + B e^{2i\omega_2 t} - C e^{i(\omega_2-\omega_1)t} + D e^{i(\omega_2+\omega_1)t} + \text{c.c.})\cos 2\alpha - (1/3\rho_e \Pi^2(|a_1|^2\omega_1^2 + |a_2|^2\omega_2^2))\cos 2\alpha$$

$$A = \frac{\rho_e \Pi^2 a_1^2 \omega_1^2}{3 - 4\omega_1^2}, \quad B = \frac{\rho_e \Pi^2 a_2^2 \omega_2^2}{3 - 4\omega_2^2}, \quad C = \frac{\rho_e \Pi^2 \bar{a}_1 a_2 \omega_1 \omega_2}{3 - (\omega_2 - \omega_1)^2}, \quad D = \frac{\rho_e \Pi^2 a_1 a_2 \omega_1 \omega_2}{3 - (\omega_2 + \omega_1)^2}$$

The time-independent part of the function ζ describes the variation of the mean drop shape. No matter what the density ratio, the drop is compressed in the vibration axis direction.

The time-dependent part of the function ζ is resonant in nature: the drop oscillation amplitude grows without bound as the external frequencies approach the resonant values. Resonant growth of the oscillation amplitude may occur not only when the vibration frequency is equal to one-half the frequency of the natural-oscillation quadrupole mode (superharmonic or nonlinear resonance) but also when the sum or the difference of the vibration frequencies is equal to this frequency. We recall that, in the monochromatic case, the terms contain only the double frequency of the external vibrations.

Figure 2 shows the isolines of the oscillation energy as a function of the vibration frequencies. As the frequencies approach the resonant values, the oscillation amplitude grows without bound. This is attributable to the absence of dissipation. The broken curves correspond to the resonances $\omega_1 \pm \omega_2 = \pm\Omega_2$, $\omega_1 = \Omega_2/2$, and $\omega_2 = \Omega_2/2$. Parametric resonance takes place when the synchronism conditions are satisfied: $\omega = \Omega_m + \Omega_{m+1}$, where the vibration frequency $\omega = \{\omega_1, \omega_2\}$. In the case considered, the neutral curve (4.5) takes the form:

$$\omega = \omega_m + \Omega_{m+1} \pm \varepsilon\sqrt{\frac{m(m+1)\rho_e^2\Pi^2\omega^2|b|^2\Omega_m}{4\Omega_{m+1}}}$$

where $b = \{a_1, a_2\}$. Figure 3 shows the first two parametric-resonance regions.

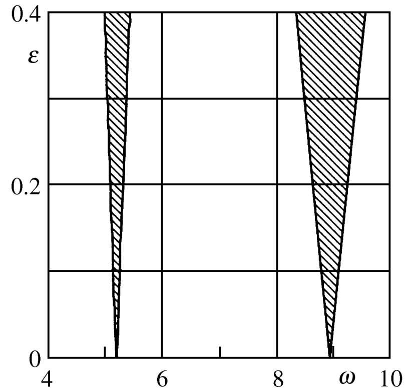

Fig. 3. Regions of parametric instability (shaded)

Summary. The oscillations and mean deformation of a cylindrical drop of inviscid fluid surrounded by another inviscid fluid and located between two rigid surfaces are studied. The drop is immersed in a multi-frequency pulsating flow. It is assumed that the mean distortion of the drop shape is small. The contact angle is constant and equal to $\pi/2$.

The phenomenon of superharmonic (nonlinear) resonance is registered when the vibration frequencies are equal to one-half the frequency of the natural-oscillation quadrupole mode or when the sum/difference of the vibration frequencies is equal to the natural frequency. In the monochromatic case, resonance occurs when the external-vibration frequency is equal to half the natural frequency.

The stability of the forced oscillations with respect to small disturbances is investigated. It is shown that parametric resonance appears when the synchronism condition is satisfied, i.e. when one of the vibration frequencies is equal to the sum of the frequencies of the neighboring modes of the natural oscillations. The detuning parameter is determined and the regions of instability are found.

The deviation of the mean drop shape from cylindrical is proportional to the square of the vibration amplitude. Compression of the drop in the direction of the axis of vibration is observed for an arbitrary density ratio.

REFERENCES

1. M. Faraday,"On a peculiar class of acoustic figures," *Phyl. Trans. Roy. Soc. London* **52**, 299–340 (1831).
2. T. B. Benjamin and F. Ursell,"The stability of a plane free surface of a liquid in vertical periodic motion," *Proc. Roy. Soc. A. London*, **225**, 505–515 (1954).
3. D. V. Lyubimov, A. A. Cherepanov, T. P. Lyubimova, and B. Roux, "Deformation of gas or drop inclusion in high frequency vibration field," *Microgravity Quart.*, **6**, No. 2–3, 125–130 (1996).
4. A. A. Alabuzhev, V. V. Konovalov, and D. V. Lyubimov, "Drop deformation and nonlinear resonance in a vibrational field," in: *Vibration Effects in Hydrodynamics* [in Russian], **1**, Perm' (1998), pp. 7–16.
5. P. L. Marston, "Shape oscillation and static deformation of drops and bubbles driven by modulated radiation stresses. Theory," *J. Acoust. Soc. Amer.*, **67**, No. 1, 15–26 (1980).
6. P. L. Marston and R. E. Apfel, "Quadrupole resonance of drops driven by modulated acoustic radiation pressure. Experimental properties," *J. Acoustic. Soc. Amer.*, **67**, No. 1, 27–37 (1980).
7. A. V. Anikumar, C. P. Lee, and T. G. Wang, "Stability of an acoustically levitated and flattened drop: An experimental study," *Phys. Fluids*, **5**, No. 11, 2763–2774 (1993).
8. S. V. Zorin, A. A. Ivanova, and V. G. Kozlov, "Experimental study of the shape of phase inclusions in a vibrational field," in: *Vibration Effects in Hydrodynamics* [in Russian], **1**, Perm' (1998), pp. 109–120.
9. N. K. McDougald and L. G. Leal, "Numerical study of the oscillations of a non-spherical bubble in an inviscid, incompressible liquid. Part 1: free oscillation from non-equilibrium initial conditions," *Int. J. Multiphase Flow*, **25**, No. 5, 887–919 (1999).
10. D. V. Luibimov, T. P. Lyubimova, A. A. Cherepanov, et al., "Equilibrium and stability of a drop in a vibrational field," *Prikl. Matem. Mekh.*, **51**, No. 4, 593–599 (1987).

11. D. V. Lyubimov and A. A. Alabuzhev, “Resonant vibrational actions on a cylindrical drop,” in: *Abstr. 12th Intern. Winter School on Continuum Mechanics* [in Russian], Perm’ (1999), p. 217.
12. S. V. Nesterov, “The Cauchy-Poisson problem for forced finite-amplitude waves,” *Izv. Ross. Akad. Nauk, Mekh. Zhidk. Gaza*, No. 4, 116–121 (1995).
13. G. Schmidt, *Parameter Erregte Schwingungen*, WEB Deutscher Verlag der Wissenschaften, Berlin (1975).
14. A. H. Nayfeh, *Introduction to Perturbation Methods*, Wiley and Sons, New York (1981).
15. R. Masiani, D. Capecchi, and F. Vestroni, “Resonant and coupled response of hysteric two-degree-of-freedom systems using harmonic balance method,” *J. Non-Linear Mech.*, **37**, No. 8, 1421–1434.
16. A.F. El-Bassiony and M. Eissa, “Dynamics of a single-degree-of-freedom structure with quadratic, cubic, and quartic non-linearities to a harmonic resonance,” *Appl. Math. Comput.*, **139**, No. 1, 1–21 (2003).
17. I. I. Blekhman and P. S. Landa, “Conjugate resonances and bifurcations in nonlinear systems under biharmonical excitation,” *J. Non-Linear Mech.*, **39**, No. 3, 421–426 (2004).
18. A. Maccari, “Non-linear oscillations with multiple resonant or non-resonant forcing terms,” *J. Non-Linear Mech.*, **34**, No. 1, 27–34 (1999).
19. H. M. Abdelhafez, “Resonance of multiple frequency excited systems with quadratic, cubic, and quartic non-linearity,” *Math. Comput. Simul.*, **61**, No. 1, 17–34 (2002).
20. L. D. Landau and E. M. Lifshits, *Theoretical Physics. V. 1. Mechanics* [in Russian], Nauka, Moscow (1988).
21. P. M. Shankar, J. Y. Chapelon, and V. L. Newhouse, “Fluid pressure measurement using bubbles insonified by two frequencies,” *Ultrasonics*, **24**, No. 6, 333–336 (1986).
22. A. O. Maksimov, “On the subharmonic emission of gas bubbles under two-frequency excitation,” *Ultrasonics*, **35**, No. 6, 79–86 (1997).
23. H. Lamb, *Hydrodynamics*, Dover, New York (1945).
24. A. Alabuzhev and D. Lyubimov, “Deformation of a cylindrical drop in a vibrational field,” in: *Proc. 19th Summer School “Advanced Problems in Mechanics”, St. Petersburg, 2002. V.1*, St. Petersburg (2002), pp. 3–10.