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Comparison of Black–Scholes Formula with Fractional Black–Scholes Formula in the Foreign Exchange Option Market with Changing Volatility

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Abstract In this paper, the authors discuss the fractional option pricing with Black–Scholes formula, deduce the Fractional Black–Scholes formula, show the empirical results by using China merchants bank foreign exchange call option price, and find when the volatility is smaller, the asymptotic mean squared error of Fractional Black–Scholes is bigger than the Traditional Black–Scholes', while the volatility is bigger—the market mechanism has a full play, the result is reverse. Namely when the market mechanism is given a full scope, the estimating effect of Fractional Black–Scholes is better than Traditional Black–Scholes'.

Keywords Fractional Black–Scholes · Volatility · AMSE · Foreign exchange option

1 Introduction

In 1973, Black and Scholes modeled the famous Black–Scholes formula which has been influencing the option pricing since then. However, this classical formula is challenged endlessly. Many empirical data reveals that the stock price distribution usually has some properties like 'fat tail', 'self-similarity', 'long-range dependence', etc. which contradict to the Traditional Black–Scholes assumption. From Mandelbrot (1963) and Mandelbrot and Taylor (1967) who found the fractional character of the stock market, then Peters (1989) provided the fractional market, to Duncan et al. (2000), and Hu and Oksendal (2003) improved the fractional Black–Scholes formula by a stochastic integration with the fractional Brownian motion and Wick-product at initial time, many scholars make outstanding contribution in this field. In 2004, Ciprian Necula used Fourier transform to obtain an explicit fractional

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Black–Scholes formula for the price of an option for every $t \in [0, T]$, where T is the time of maturity. Then the fractional option price can be easy to calculate as the traditional one.

However, foreign exchange market is correspondingly special relatived to the stock market, because of more than one country's interest rates involved. In 1983, Garman and Kohlhagen firstly introduced two different states rates into the option pricing formula, and presented the first foreign exchange option pricing formula (Garman and Kohlhagen 1983). As a significant portion in the financial market, the foreign exchange market equally has obvious fractional properties (Yong-jian 2002). The fractional foreign exchange pricing formula also can be deduced as the same as the traditional one. Recently, Chinese scholars Xiumei Wang (2005), and Fu et al. (2008) have done much progress on this field.

Meanwhile, as a key parameter for the option pricing, the volatility should influence the option price a lot, even if it fluctuated or was modified a little. Back to the end of last century, some scholars have put forward the conception of the stochastic volatility, and then this conception developed into the ones discussed in the fractional condition (Mendes and Oliveira 2008; Bayraktar and Poor 2005; Chan and Ng 2006; and Bhansali 2007) shows the relationship between volatility and carry by empirical approach. Volatility is critical for option pricing or for the financial market.

There have been a few articles concerning advantages of Traditional Black–Scholes (BS) and Fractional Black–Scholes (FBS), respectively. Cajueiro and Barbachan (2003) used the data of the Telemar PN to estimate the option prices by BS and FBS, and get a result that for any H > 1/2, the price calculated by FBS is larger than BS's, and the relationship among FBS's (C), BS's ($C11$) and the sample price ($C1$)is $C > C11 > C1$, but they didn't analyze this phenomenon from a reasoned point of view. Fu et al. (2008) deduced the fractional foreign exchange BS, and used the Asymptotic Mean Squared Error (AMSE) as the evaluation criterion to judge of the two kinds of option price formulas. Using the data from China merchants bank of a certain period of 07, they found that FBS AMSE > BS AMSE, namely FBS is better than BS.

In this paper, based on the constant volatility, a choice between BS and FBS has been made by AMSE. The experimental data is gotten from China Merchants Bank (CMB), and the comparative results is shown by MATLAB graphs. Then we find that FBS AMSE < BS AMSE of certain kinds of foreign exchange; the comparative result between two formulas changes while the volatility fluctuates—the bigger the volatility is, the smaller the FBS AMSE is, and we can guess that FBS is better than BS when the option market mechanism is given a full scope; and based on our sample data, the price estimated by FBS is smaller than the BS's.

2 The Research on the Fractional Option Pricing Formula

2.1 Traditional Foreign Exchange Black–Scholes Formula

The traditional foreign exchange BS formula is similar to the classical BS formula, to some extent BS is the specific form to the foreign exchange BS. Both of them are based on the assumption of efficient market, that is: The stock price in the market can reflect the relative information effectively and timely; the future price only can be determined by the new information; the asset price obeys the Brownian motion.

The price at every $t \in [0, T]$ of an European call option with strike price $X$and maturity T is given by:

$$C(t, S(t)) = S(t)e^{-r^f(T-t)}N(d_1) - Ke^{-r^d(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r^d - r^f)(T-t) + \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r^d - r^f)(T-t) - \frac{\sigma^2}{2}(T-t)}{\sigma\sqrt{(T-t)}},$$

Here C is the observed price for the option, S is the asset price, r is the risk-free rate, $\sigma$is the volatility for the asset price S, $r^d$and r^f are domestic and foreign rates, respectively, and $N(\bullet)$ is the cumulative probability of the standard normal distribution.

This formula is easy for calculating the option price. But, increasing researches show that it is unscientific. Then Hu and Oksendal (2003), and Necula (2004) provide the fractional pricing formula as given in the following.

2.2 Fractional Black–Scholes Formula

First, we present the conception of fractional Brownian motion simply.

If $0 < H < 1$ the fractional Brownian motion with Hurst parameter H is the continuous Gaussian process $\{B_H(t), t \in R\}$, $B_H(t) = 0$ with mean $E[B_H(t)] = 0$ and covariance:

$$C_H(t, s) = E[B_H(t) B_H(t)] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

When $H = \frac{1}{2}$, $B_H(t)$ coincides with the standard Brownian motion $B(t)$.

The fractional Brownian $B_H(t)$ is a self-similar process meaning that for $\forall\alpha > 0$ and $H \in (0, 1)$ we have:

$$B_H(\alpha t) = \alpha^H B_H(t) \text{ in the sense of law.}$$

Then we present the outline for FBS:

FBS has a much more complex proof procedure than BS. It involves fractional Ito lemma, Wick product, Fourier transform, etc.

Lemma (Ito): *Let the target asset price (S) obeys fractional Ito process such that:*

$$dS = S\mu dt + S\sigma dB_H,$$

then we have: $f(t, S)$ also is a Fractional Ito process, where $f(t, S)$is continuously differentiable with t, and twice continuously differentiable with S. In the following, let the Hurst parameter H be such that $1/2 < H < 1$.

According to the no-arbitrage character in the fractional Black–Scholes market which has been shown by Hu and Oksendal (2003), if N_t is the value of the portfolio with risk-free, then we have:

$$dN_t = rN_t dt.$$

if $V(t, S(t))$is the price of a derivative on the stock price, then we have:

$$\frac{\partial V}{\partial t} + S^2\sigma^2 H t^{2H-1}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

(1) is the fractional Black–Scholes equation. From the equation above, we can see that the Fractional option pricing is also independent on the risky interest rate μ, but dependent on the riskless interest rate r. Then under the risk-neutral measure we have that:

$$dS = Srdt + S\sigma dB_H. \tag{2}$$

Then we need to find the fractional explicit expression of the option pricing.

Consider a fractional Black–Scholes market that has two investment possibilities:

1. risk-free asset (a money market account):

$$dM(t) = rM(t)dt, \quad M(0) = 1, 0 \le t \le T, \tag{3}$$

   where $M(t)$ represents the asset price, which moves with the time t, r is the riskless rate.
2. risky asset (a stock whose price satisfies the equation):

$$dS(t) = \mu S(t)dt + \sigma S(t)d\overline{B}_H(t), \quad S(0) = S > 0, 0 \le t \le T,$$

   where $\sigma$is the volatility of stock price, μ is the risky interest rate.

By the condition that the fractional market has on-arbitrage opportunity and is complete, we can obtain the fractional risk-neutral evaluation by the Clark-Ocone lemma (Necula 2004):

$$F(t) = e^{-r(T-t)}\widetilde{E}_t[F], \tag{4}$$

where $F(t)$ is the price of a replicating portfolio of the claim$(m(t), s(t))$, that is:

$$F(t) = m(t)M(t) + s(t)S(t),$$

$\widetilde{E}_t[.]$ is the quasi-conditional expectation under the risk-neutral measure which is given by the fractional Girsanov formula.

Finally, Using the lemma and theory presented above and the Fourier transform, we can get the fractional explicit formula for option pricing:

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{\left(T^{2H} - t^{2H}\right)}}$$

and

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{\left(T^{2H} - t^{2H}\right)}},$$

and H is the Hurst exponent. $N(\bullet)$ is the cumulative probability of the standard normal distribution.

2.3 Foreign Exchange Fractional Black–Scholes Formula

Based on the FBS, it is easy to obtain the foreign exchange FBS by a substitution (transfer r into $r^d - r^f$, where $rd\psi - rf\psi$ and rd stand for domestic and foreign rates, respectively).
So, (3) is transformed into $dM(t) = (r^d - r^f)M(t)dt$,
Hence, by risk-neutral measure, (2) and (4) can be changed into

$$dS(t) = (r^d - r^f)S(t)dt + \sigma S(t)dB_H(t)$$

and

$$F(t) = e^{-r^d(T-t)}\widetilde{E}_t[F].$$

Then we have

$$C(t, S(t)) = S(t)e^{-r^f(T-t)}N(d_1) - Ke^{-r^d(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r^d - r^f)(T - t) + \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{\left(T^{2H} - t^{2H}\right)}}$$

and

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + (r^d - r^f)(T - t) - \frac{\sigma^2}{2}(T^{2H} - t^{2H})}{\sigma\sqrt{\left(T^{2H} - t^{2H}\right)}}.$$

3 Empirical Analysis

3.1 Parameter Estimation

3.1.1 Hurst Parameter Estimation

In this paper, we estimate the Hurst parameter by using the traditional R/S analysis and the CMB foreign exchange prices of EURUSD, GBPUSD and AUDUSD from 01/01/2008 to 12/31/2008.

The Hurst parameter H is robust and efficient for fractional analysis. $1/2 < H \le 1$, that means the time sequence it has long range dependence or long memory, when $H = 1/2$, it is a standard Brownian motion, when $0 < H < 1/2$, it has the anti-permanence character.

By R/S analysis, we estimate the Hurst parameter H for EURUSD, GBPUSD and AUDUSD, respectively: 0.60016, 0.56983, and 0.62252.

3.1.2 Volatility Estimation

We utilize the historical volatility to estimate σ. The formula is

$$X_i = \ln\frac{P_{i+1}}{P_i},$$

$$\sigma = \sqrt{\frac{\sum\left(X_i - \bar{X}\right)^2}{N - 1}}, \quad \bar{X} = \frac{1}{N}\sum X_i.$$

Here P_i stand for the daily foreign exchange prices. Then we have the volatilities for EURUSD, GBPUSD and AUDUSD: 0.103552, 0.187317, and 0.16909.

3.2 Empirical Research of Foreign Exchange Option Pricing Between Traditional BS and Fractional BS (FBS)

3.2.1 Data

we use CMB foreign exchange call option closing prices of EURUSD, GBPUSD and AUDUSD from 12/04/2008 to 01/05/2009, 12/04/2008 to 01/08/2009 and 10/21/2008 to 12/31/2008 as the sample (see Table 1), and then compare it to the results coming from BS and FBS. Here the value of H is consistent with the Hurst parameters calculated in 3.1. And the risk-free interest rate for each country: USA-0.01(3 month national debt ratio), UK-0.0318(overnight interbank offered rate), EUR-0.03 (German overnight interbank offered rate), AUS-0.05(3 year national debt ratio). Calculating option pricing formulas (BS and FBS) with these parameters by MATLAB, we can get a time series—the call option prices and line graphs about the sample prices, BS prices and FBS prices.

3.2.2 Calculating Process

In this section, we use MATLAB as assistant tool to make calculation. We choose the data of EURUSD call option on 12/04/2008 to make an instance. Each parameter: $S(t) = 1.277$, $K = 1.26$, $T - t = 84/365 = 0.2301$(84 is the interval form 12/04/2008 to 02/26/09) and $\sqrt{Y - t} = 0.4797$, and t can be seen as zero (because the interval between 02/26/09 and 02/26/09 can be seen as zero), so we have that, $T^{2H} - t^{2H} = T^{2H}$ in the following calculation. $r^f = 0.0433$, $r^d = 0.0595$, $\sigma = 0.103552$, $H = 0.60016$. Then,

BS:

$$d_1 = \frac{\ln\left(\frac{1.277}{1.26}\right) + (0.0595 - 0.0433) \times 0.2301 + \frac{0.103552^2}{2} \times 0.2301}{0.103552\sqrt{0.2301}} = 0.20197,$$

$$d_2 = \frac{\ln\left(\frac{1.277}{1.26}\right) + (0.0595 - 0.0433) \times 0.2301 - \frac{0.103552^2}{2} \times 0.2301}{0.103552\sqrt{0.2301}} = 0.15229$$

$$C = 1.277 \times e^{-0.0433\times0.2301} N(0.20197) - 1.26 \times e^{-0.0595\times0.2301} N(0.15229)$$
$$= 3.5253$$

FBS:

$$d_1 = \frac{\ln\left(\frac{1.277}{1.26}\right) + (0.0595 - 0.0433) \times 0.2301 + \frac{0.103552^2}{2} \times 0.2301^{2\times0.60016}}{0.103552 \times 0.2301^{0.60016}}$$
$$= 0.22665$$

$$d_2 = \frac{\ln\left(\frac{1.277}{1.26}\right) + (0.0595 - 0.0433) \times 0.2301 - \frac{0.103552^2}{2} \times 0.2301^{2\times0.60016}}{0.103552 \times 0.2301^{0.60016}}$$
$$= 0.188377$$

Table 1 CMB foreign exchange call option closing prices of EURUSD, GBPUSD and AUDUSD

EURUSD (Mature time: 02/26/09, Strike price: 1.26$)			GBPUSD (Mature time: 02/26/09, Strike price:1.5$)				AUDUSD (Mature time: 01/08/09, Strike price: 0.63$)							
5.9776	5.3518	6.351	5.0516	4.7928	5.2464	4.5608	3.6943	3.6243	3.7834	2.023	1.5846	2.8535	3.9212	
6.2203	6.6095	9.0997	4.4656	5.7399	5.497	7.6965	4.0487	3.9421	3.906	5.1049	3.9363	3.3286	3.6948	
9.5653	12.5988	15.698	9.6059	9.0811	6.3546	5.3952	3.3058	2.6113	2.0194	3.3577	2.6384	2.5108	2.3567	
18.8919	17.9413	14.6664	4.7378	4.0737	3.998	4.0095	1.7617	0.9396	1.4436	2.0677	1.9384	2.0371	1.9598	
14.8049	14.4683	15.034	3.3201	2.9808	2.9533	3.5859	1.7417	1.0066	1.1874	1.1693	0.8932	0.9931	1.509	
15.0973	15.3195		3.2456	3.8448	4.582	5.3217	1.3169	1.1297	1.7115	1.5413	1.5661	3.265	4.058	3.1693
15.3122	16.3273		5.2797				2.4496	2.3474	1.6716	1.9339	1.8468	1.9799	2.4717	
14.8194	14.2628						2.5839							
11.8351														

Data resource: CMB foreign exchange trading software

Table 2 The sample data of option price compared with the BS and FBS model predictions

EURUSD			GBPUSD			AUDUSD		
$C1$	$C11$	C	$C1$	$C11$	C	$C1$	$C11$	C
5.9776	3.9545	3.5253	5.0516	5.2552	4.533	3.6943	2.6178	2.6178
5.3518	3.5326	3.0997	4.7928	5.2405	4.5163	3.6243	1.8441	1.8441
6.351	5.0942	4.6898	5.2464	6.3722	5.6045	3.7834	2.2144	2.2144
6.2203	5.1637	4.7619	4.5608	5.5258	4.7807	2.023	0.00058747	8.2437e-005
6.6095	5.9961	5.6237	4.4656	5.7437	4.989	1.5846	4.4569e-012	1.7845e-016
......	......	......	......	......	......	......	......	......

$$\begin{aligned} C &= 1.277 \times e^{-0.0433\times 0.2301} N(0.22665) - 1.26 \times e^{-0.0595\times 0.2301} N(0.18377) \\ &= 3.9545 \end{aligned}$$

The main program:

The option pricing calculating (partial):

```
function [Pd1, Pd2, Pd11, Pd22, PC, PC11]=F_future_price111(coin, St, K, rf, rd, T,
delta, H, C1)
n=length(T);
   for i=1:n
   d1(i)=((1/(delta*sqrt(T(i)^(2*H))))*(log(St(i)/K)+(rd-rf)*T(i)+(delta^2)*
   (T(i)^(2*H))/2));
   d2(i)=((1/(delta*sqrt(T(i)^(2*H))))*(log(St(i)/K)+(rd-rf)*T(i)-(delta^2)*
   (T(i)^(2*H))/2));
   ......
   N1(i)=cdf('Normal',d1(i),0,1);
   N2(i)=cdf('Normal',d2(i),0,1);
   ......
   C(i)=St(i)*(exp(-rf*(T(i))))*N1(i)-K*(exp(-rd*(T(i))))*N2(i);
   C(i)=100*C(i)*St(i);
   ......
   end
```

3.2.3 Analysis

Table 2 and Fig.1 shows the comparative among the sample data, estimated prices from BS and FBS.

According to Table 2 and Fig. 1, one can observe some phenomenon as following:

- On the whole, the prices from BS and FBS fluctuate along with the sample price basically. As we know, the CMB option price is calculated by BS, so there is no

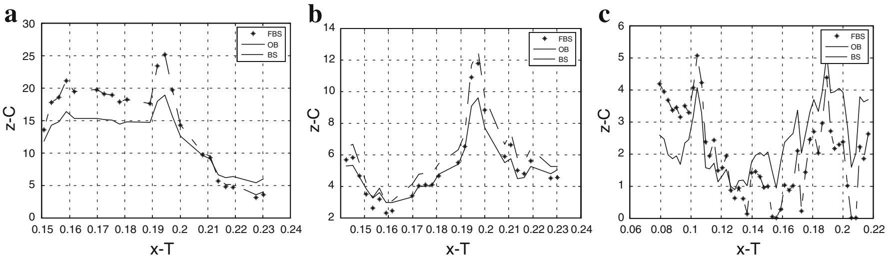


Fig. 1 Comparison among values of FBS, BS and observed value for different foreign exchanges. *denotes the estimated price by FBS, –denotes the estimated price by BS, —denotes the sample price

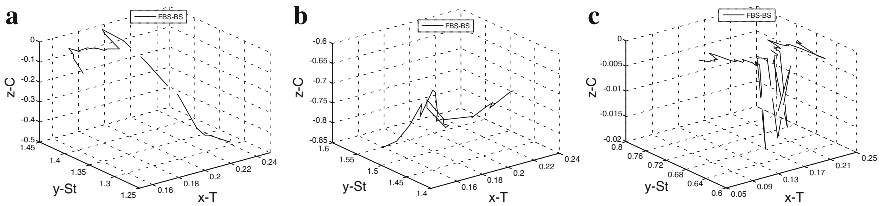


Fig. 2 FBS-BS $\times$ T $\times$ St for each foreign exchange

surprise to observe that the estimated prices and sample price have the same fluctuation when no big incident happened. And in Fig. 1a and c, BS is a little better than FBS, while FBS has a little advantage in Fig. 1b. Meanwhile, one can see that the price given by the FBS is smaller than the price given by BS, Fig. 2 which shows the value of FBS-BS would be more vivid to present the result. This is contrary to the result in the paper written by Daniel Oliveira Cajueiro José, Fajardo Barbachan (2003) this dues to the sample we choose.

- we still find there are some crossings in the three plots formed by BS price line and sample price line or FBS price line and sample price line, and sometimes before the crossing point BS and FBS prices are bigger than the sample price while after the point they are smaller than it, that means the price is influenced by the change of supply and demand. The theoretic price integrated with the market mechanism results in the final market price.

- Meanwhile, the error caused by sample selecting or computer calculating is avoidless. And the units of each CMB sample options are 100 ECU, 100£ and 100\$, respectively, but the unit assumed in the option pricing formulas is 1. So the value from BS and FBS need to be multiplied by 100, it results in the magnified error.

3.2.4 AMSE Analysis

Calculate the AMSE of prices from BS and FBS. It is given by:

$$\mathrm{AMSE} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{C1_i - C}{C1_i}\right)^2,$$

Table 3 AMSEs of FBS and BS

The sort of FX	EURUSD	GBPUSD	AUDUSD
FBS_AMSE	0.065614	0.013436	0.1953
BS_AMSE	0.057861	0.03845	0.19158

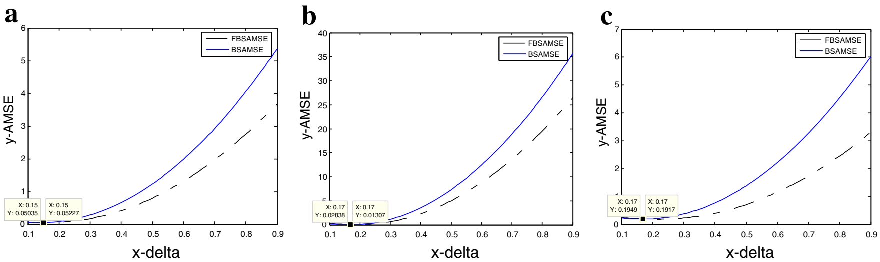


Fig. 3 AMSE × σ (from 0.1 to 0.9 by 0.01) for FBS and BS (**a**, **b**, **c** stand for EURUSD, GBPUSD, AUDUSD, respectively)

Table 4 Changes of AMSEs between FBS and BS with σ growing

AMSE	EURUSD	GBPUSD	AUDUSD
FBS >BS	[0.10, 0.15]	[0.10, 0.17]	[0.10, 0.17]
BS > FBS	(0.15, 0.90]	(0.17, 0.90]	(0.17, 0.90]

We just discuss in [0.10, 0.90], interval is 0.01

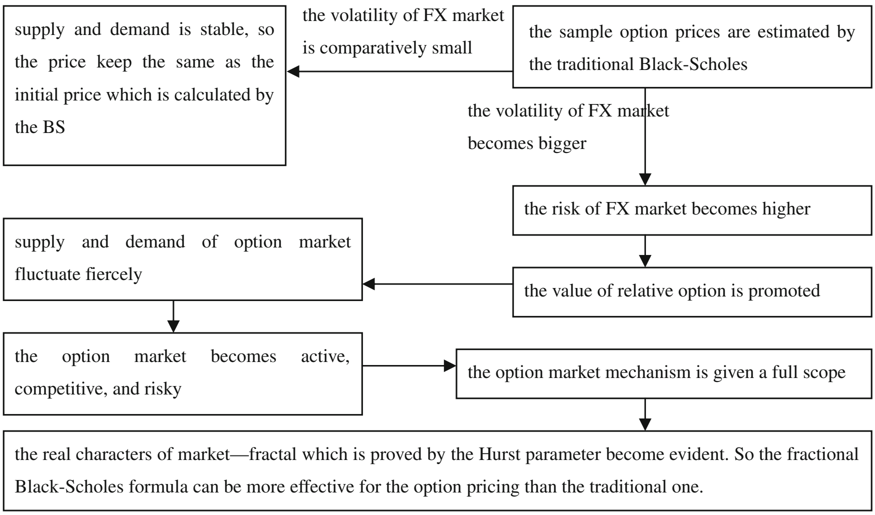
where $C1_i$ is the sample option price, C is the price from BS or FBS. One can see that the smaller the AMSE is, the better the pricing formula is.

Table 3 shows the AMSEs of BS and FBS (the values of parameters come from 3.1).

From the Table 3, we find that BS is better than FBS in EURUSD and AUDUSD while the result is contrary in GBPUSD. It's hard to judge the effect of two formulas from this result which confuses us much more, so we try to change the values of parameters. Then we find that the change of volatility can influence the comparison result (see Fig. 3).

One can see that, by the increasing of volatility, FBS_AMSE becomes smaller than BS_AMSE gradually. The change scale is shown in Table 4.

We find that FBS_AMSEs of three kinds of foreign exchange options all become smaller than their BS_AMSEs when $\sigma > 0.17$, so we can say that FBS is better than BS while the volatility is bigger ($\sigma > 0.17$). We conclude that when the market mechanism is given a full scope, the FBS would be more effective than the BS. The detailed reasoning process can be presented as the following:



Namely when the market mechanism is given a full scope, the estimating effect of FBS is better than BS'.

4 Conclusion

In this paper, we drive the fractional foreign exchange Black–Scholes formula. We show how to use the FBS to calculate the price of three kinds of foreign exchange CMB call option. By using the sample from CMB, we find that FBS_price > BS_price. And FBS is better than BS while the volatility is comparatively bigger ($\sigma > 0.17$). However, when the volatility is smaller ($\sigma < 0.17$), FBS is not always better than BS, and according to some consequence we can guess that FBS is better than BS when the option market mechanism is given a full scope. This may be relative to the pricing method (BS).

The shortage of this paper: We don't have enough sample data; the calculation is not accurate enough. We can collect more sample data to do experience, and try to reduce the error.

References

Bayraktar, E., & Poor, H. V. (2005). Arbitrage in fractal modulated Black–Scholes models when the volatility is stochastic. *International Journal of Theoretical and Applied Finance, 01* (http://adsabs.harvard.edu//abs/2005cs........1054B).

Bhansali, V. (2007). Volatility and the carry trade. *The Journal of Fixed Income, 17*(3), 72–85.

Cajueiro, D. O., & Barbachan, J. F. (2003). Volatility estimation and option pricing with fractional Brownian motion. Financlab Working Paper, FLWP, 06.

Chan, N. H., & Ng, C. T. (2006). Fractional constant elasticity of variance model. *Time Series and Related Topics, 52*, 149–164.

Duncan, T. E., Hu, Y. Z., & Pasik-Duncan, B. (2000). Stochastic calculus for fractional Brownian motion. *SIAM Journal on Control and Optimization, 38*, 582–612.

Fu, Q., Wang, K., & Liu, X. (2008). The foreign exchange option pricing based on the fractional Black–Scholes model and valuation. *Price Monthly, 369*, 68–70.

Garman, M. B., & Kohlhagen, S. W. (1983). Foreign currency option values. *Journal of International Money and Finance, 2*(12), 231–237.

Hu, Y., & Oksendal, B. (2003). *Fractional white noise calculus and applications to finance.* Mathematics Subject Classifications, 10.

Mandelbrot, B. B. (1963). The variation of certain speculative prices. *Journal of Business, 36*(1963), 394–419.

Mandelbrot. B., & Taylor, H. M. (1967). On the distribution of stock price differences. *Operations Research, 15*(6), 1057–1062.

Mendes, R. V., & Oliveira, M. J. (2008). A data-reconstructed fractional volatility model. *Economics Discussion paper, 22*. (http://www.economics-ejournal.org/economics/discussionpapers)

Necula, C. (2004). Option pricing in a fractional Brownian motion environment. *Mathematical Report, 6*(3).

Peters, E. E. (1989). Fractal structure in the capital market. *Financial Analyst, 1989*(7), 434–453.

Wang, X. (2005). Analysis of mathematical model about the pricing of foreign currency option, the paper for master degree, Xi'an Electronic Technology University.

Yong-jian, H. (2002). The efficiency of the foreign exchange market and Fractal market analysis. *The Theory and Practice of Finance and Economics, 23*, 75–79.