

# Monte Carlo Option Pricing for Tempered Stable (CGMY) Processes

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**Abstract** Lévy processes are popular models for stock price behavior since they allow to take into account jump risk and reproduce the implied volatility smile. In this paper, we focus on the tempered stable (also known as CGMY) processes, which form a flexible 6-parameter family of Lévy processes with infinite jump intensity. It is shown that under an appropriate equivalent probability measure a tempered stable process becomes a stable process whose increments can be simulated exactly. This provides a fast Monte Carlo algorithm for computing the expectation of any functional of tempered stable process. We use our method to price European options and compare the results to a recent approximate simulation method for tempered stable process by Madan and Yor (CGMY and Meixner Subordinators are absolutely continuous with respect to one sided stable subordinators, 2005).

**Keywords** Monte Carlo · Option pricing · Lévy process · Tempered stable process · CGMY model

## 1 Introduction

Lévy processes are stochastic processes with stationary and independent increments. That is to say, if  $(X_t)_{t \geq 0}$  is a Lévy process, then  $X_t - X_s$  with  $t > s$  is

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independent of the history of the process up to time  $s$ , and its law only depends on  $t - s$ .

Exponentials of Lévy processes constitute the simplest model of stock price behavior allowing to take into account price discontinuities and the implied volatility smile phenomenon (Cont and Tankov 2004). Despite this simplicity, unlike the Black-Scholes model, closed option pricing formulas are not available in exponential Lévy models and one must use either deterministic numerical methods (partial integro-differential equations (Cont and Voltchkova 2005) or Fourier methods (Carr and Madan 1999)) or Monte Carlo methods which form the object of the present paper.

For Monte Carlo methods, finite-intensity (compound Poisson) Lévy processes can be easily simulated on a computer, but for infinite intensity processes the situation is more difficult: explicit simulation methods are only available for stable processes (Chambers et al. 1976) and a few other parametric classes (see Cont and Tankov (2004) for a comprehensive survey). For other processes, approximate methods are available (Asmussen and Rosiński 2001; Rosiński 2001), which are basically reduced to truncating small jumps and replacing them with something easy to simulate. These methods are not adapted for Monte Carlo pricing of European options and other options with discrete observation of stock price trajectory since they require a complete simulation of the process trajectory even if only the terminal value is used for the final computation.

For the tempered stable process (also known as CGMY model), an approximate simulation method based on Brownian subordination and a rejection method by Rosiński (2001) was recently proposed by Madan and Yor (2005) but, to the best of our knowledge, no exact simulation method for the increments of this process is known. In Sect. 3 of this paper, we propose a method for Monte Carlo evaluation of any functional of the tempered stable process which avoids direct simulation of the increments of this process. Instead, we construct an equivalent probability measure under which the original tempered stable process becomes a stable process. Since the method for direct simulation of stable random variables is well-known (Chambers et al. 1976) and the measure change is explicit, this provides the desired algorithm.

After describing the theoretical framework of our method in Sect. 3, we apply it to the pricing of European options in Sect. 4 and compare the results with the ones obtained using the approximate simulation method by Madan and Yor. A strong point of the latter method is that it is easily generalizable to multiple dimensions via Brownian subordination, whereas in our setting a multidimensional extension is not straightforward. On the other hand, an important advantage of our method is that it provides unbiased estimators of option prices, while the approximate method introduces a bias, due to the truncation of small jumps, which is not easy to quantify. More importantly, for the parameter sets used in this paper, our method required 16 to 50 times less computer time to achieve the same precision than the approximate method.

## 2 Basic Tools and Definitions

In this section we review important properties of Lévy processes, stable processes and tempered stable processes that are used in the sequel. Unless otherwise mentioned, all proofs can be found in [Sato \(1999\)](#). We concentrate on the one-dimensional case.

### 2.1 Lévy-Khintchine Representation

Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be a measurable function such that for every  $z$ ,  $\int |e^{izx} - 1 - izh(x)|\nu(dx) < \infty$ . Such a function  $h$  is called a truncation function. The Levy-Khintchine representation with the truncation function  $h$  takes the form:

$$E[e^{izX_t}] = e^{t\psi(z)}, \quad z \in \mathbb{R}, \tag{2.1}$$

$$\psi(z) = -\frac{1}{2}zAz + i\gamma_h z + \int_{\mathbb{R}^d} (e^{izx} - 1 - izh(x))\nu(dx). \tag{2.2}$$

$A$  and  $\nu$  do not depend on the choice of  $h$  but  $\gamma_h$  depends on this choice. If  $\gamma$  is the value of  $\gamma_h$  for the standard truncation function  $h(x) = x\mathbf{1}_{|x|\leq 1}$ ,  $\gamma_h$  for arbitrary  $h$  can be computed with

$$\gamma_h = \gamma + \int_{-\infty}^{\infty} (h(x) - x\mathbf{1}_{|x|\leq 1})\nu(dx).$$

In the sequel, the value of  $\gamma_h$  corresponding to the truncation function  $h \equiv 0$  (drift) will be denoted by  $\gamma_0$  and the value corresponding to  $h \equiv x$  (center) will be denoted by  $\gamma_c$ .

In the sequel, we will use the following result on exponential moments of Lévy processes (see [Sato 1999](#), Theorem 25.17).

**Proposition 2.1** *Let  $X$  be a Lévy process with characteristic triplet  $(A, \nu, \gamma_h)$  with respect to a truncation function  $h$  and let  $\lambda \in \mathbb{R}$ . Then  $E[e^{\lambda X_t}] < \infty$  for some  $t > 0$  or equivalently for all  $t > 0$  if and only if*

$$\int_{|x|\geq 1} e^{\lambda x} \nu(dx) < \infty.$$

*In this case,  $E[e^{\lambda X_t}] = e^{t\psi(-i\lambda)}$  with  $\psi$  as in Eq. 2.2.*

### 2.2 Equivalent Measure Changes for Lévy Processes

**Proposition 2.2** (see [Sato 1999](#), Theorems 33.1 and 33.2) *Let  $(X_t, P)$  and  $(X_t, P')$  be two Lévy processes on  $\mathbb{R}$  with characteristics triplets  $(A, \nu, \gamma)$  and  $(A', \nu', \gamma')$ . Then  $P|_{\mathcal{F}_t}$  and  $P'|_{\mathcal{F}_t}$  are equivalent for all  $t$  (or equivalently for one  $t > 0$ ) if and only if the three following conditions are satisfied:*

1.  $A = A'$
2. The Lévy measures are equivalent with

$$\int_{-\infty}^{\infty} (e^{\Phi(x)/2} - 1)^2 \nu(dx) < \infty$$

where  $\Phi(x) = \ln\left(\frac{dv'}{d\nu}\right)$

3. If  $A = 0$  then we must in addition have

$$\gamma'_h = \gamma_h + \int_{-\infty}^{\infty} h(x)(\nu' - \nu)(dx)$$

When  $P$  and  $P'$  are equivalent, the Radon-Nikodym derivative is

$$\frac{dP'}{dP} \Big|_{\mathcal{F}_t} = e^{U_t}$$

with

$$U_t = \eta X_t^c - \frac{\eta^2 A t}{2} - \eta \gamma_h t + \lim_{\epsilon \rightarrow 0} \left( \sum_{|\Delta X_s| > \epsilon} \Phi(\Delta X_s) - t \int_{|x| > \epsilon} (e^{\Phi(x)} - 1) \nu(dx) \right)$$

Here  $(X_t^c)$  is the continuous part of  $(X_t)$ , i.e. the Lévy process with generating triplet  $(A, 0, \gamma)$  and  $\eta$  is such that

$$\gamma'_h - \gamma_h - \int_{-\infty}^{\infty} h(x)(\nu' - \nu)(dx) = A\eta \quad \text{if } A > 0$$

and zero if  $A = 0$ .

$U_t$  is a Lévy process with characteristic triplet  $(A_U, \nu_U, \gamma_U)$  given by:

$$\begin{aligned} A_U &= \eta^2 \sigma^2 \\ \nu_U &= \nu \circ \Phi^{-1} \\ \gamma_U &= -\frac{1}{2} A \eta^2 - \int_{-\infty}^{\infty} (e^y - 1 - h(y)) (\nu \Phi^{-1})(dy) \end{aligned}$$

### 2.3 Stable Processes

Real valued  $\alpha$ -stable processes with  $0 < \alpha < 2$  are Lévy processes with no continuous martingale part and Lévy measure of the form

$$\nu(x) = \frac{A}{|x|^{\alpha+1}} 1_{x>0} + \frac{B}{|x|^{\alpha+1}} 1_{x<0}.$$

If  $1 < \alpha < 2$ , the process has finite mean and its characteristic function has the following form:

$$\Phi_{X_t}(z) = \exp t \left\{ i\mu z + \int_{\mathbb{R}} (e^{izx} - 1 - izx) \nu(dx) \right\}$$

for some center  $\mu \in \mathbb{R}$ . In the case  $0 < \alpha < 1$ , the process has finite variation and we can write

$$\Phi_{X_t}(z) = \exp t \left\{ i\mu z + \int_{\mathbb{R}} (e^{izx} - 1) \nu(dx) \right\},$$

i.e.,  $\mu$  is now the drift. The characteristic function of a stable process may also be expressed as (see Sato 1999)

$$\begin{aligned} \Phi_{X_t}(z) &= \exp \left\{ -\sigma^\alpha |z|^\alpha t \left( 1 - i\beta \operatorname{sgn} z \tan \frac{\pi\alpha}{2} \right) + i\mu z t \right\}, \quad \text{if } \alpha \neq 1, \\ \Phi_{X_t}(z) &= \exp \left\{ -\sigma |z| t \left( 1 + i\beta \frac{2}{\pi} \operatorname{sgn} z \log |z| \right) + i\mu z t \right\}, \quad \text{if } \alpha = 1, \end{aligned} \quad (2.3)$$

where  $\alpha \in (0, 2]$ ,  $\sigma \geq 0$ ,  $\beta \in [-1, 1]$  and  $\mu \in \mathbb{R}$ . A stable law with parameters  $\alpha, \sigma, \beta, \nu$  is denoted by  $S_\alpha(\sigma, \beta, \nu)$ .

In the case  $\alpha < 2$  the two parametrizations are linked by the following relations:

$$\begin{aligned} \sigma &= \left[ -(A + B) \Gamma(-\alpha) \cos \left( \frac{\pi\alpha}{2} \right) \right]^{\frac{1}{\alpha}} \quad \text{when } \alpha \neq 1, \\ \sigma &= \frac{\pi}{2} (A + B) \quad \text{when } \alpha = 1, \\ \beta &= \frac{A - B}{A + B}. \end{aligned}$$

Furthermore, when  $\alpha \neq 1$  and  $\gamma$  is the third parameter of the characteristic triplet of  $X$  for the standard truncation function  $h(x) = x1_{|x| \leq 1}$  then

$$\mu = \gamma + \frac{B - A}{1 - \alpha}.$$

### 2.4 Simulation of Stable Random Variables

**Proposition 2.3** (see Chambers et al. 1976) *Let  $\Theta$  and  $W$  be independent with  $\Theta$  uniformly distributed on  $(-\frac{\pi}{2}, \frac{\pi}{2})$  and  $W$  exponentially distributed with mean 1, and let  $0 < \alpha \leq 2$ .*

- *The symmetric random variable*

$$Z = \begin{cases} \frac{\sin(\alpha\Theta)}{(\cos(\Theta))^{1/\alpha}} \left[ \frac{\cos(\alpha-1)\Theta}{W} \right]^{(1-\alpha)/\alpha}, & \alpha \neq 1 \\ \tan \Theta, & \alpha = 1 \end{cases}$$

has a  $S_\alpha(1, 0, 0)$  distribution.

- *In the nonsymmetric case, for any  $-1 \leq \beta \leq 1$ , define*

$$\Theta_0 = \arctan(\beta \tan(\pi\alpha/2))/\alpha$$

when  $\alpha \neq 1$ . Then

$$Z = \begin{cases} \frac{\sin(\alpha(\Theta + \Theta_0))}{(\cos(\alpha\Theta_0) \cos(\Theta))^{1/\alpha}} \left[ \frac{\cos(\alpha\Theta_0 + (\alpha - 1)\Theta)}{W} \right]^{(1-\alpha)/\alpha}, & \alpha \neq 1 \\ \frac{\pi}{2} \left[ \left( \frac{\pi}{2} + \beta\Theta \right) \tan \Theta - \beta \log \left( \frac{\frac{\pi}{2} W \cos(\Theta)}{\frac{\pi}{2} + \beta\Theta} \right) \right], & \alpha = 1, \end{cases}$$

has a  $S_\alpha(1, \beta, 0)$  distribution.

Stable random variates with  $\sigma \neq 1$  and  $\mu \neq 0$  may be obtained from  $S_\alpha(1, \beta, 0)$  by scaling and translation.

### 2.5 Tempered Stable Processes

A one-dimensional *tempered stable process* is obtained by taking a one-dimensional stable process and multiplying the Lévy measure with a decreasing exponential on each half of the real axis. Thus, a tempered stable process is a Lévy process on  $\mathbb{R}$  with no Gaussian component and Lévy density of the form:

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0},$$

with parameters satisfying  $c_- > 0, c_+ > 0, \lambda_- > 0, \lambda_+ > 0$  and  $0 < \alpha < 2$ . For greater generality one can allow different values of  $\alpha$  for positive and negative half-lines and speak of a *generalized tempered stable process* with Lévy measure of the form

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} 1_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} 1_{x<0}.$$

Tempered stable processes have been studied by many authors including Boyarchenko and Levendorskiĭ (2002), Carr et al. (2002), Cont et al. (1997), and Koponen (1995) under different names. In particular, the version when

$c_+ = c_-$  and  $\alpha_+ = \alpha_-$  was studied in Carr et al. (2002) under the name CGMY process with Lévy measure

$$\nu_{\text{CGMY}}(x) = C \left[ \frac{e^{-Mx}}{x^{1+Y}} \mathbf{1}_{x>0} + \frac{e^{-G|x|}}{|x|^{1+Y}} \mathbf{1}_{x<0} \right].$$

A multidimensional generalization of tempered stable processes is proposed in Rosiński (2004).

Because of the exponential tempering, in the case of the tempered stable process big jumps need not be truncated and one can use the truncation function  $h(x) = x$ . In the general case ( $\alpha_{\pm} \neq 1$  and  $\alpha_{\pm} \neq 0$ ) the characteristic exponent  $\psi(u) = t^{-1} \log E[e^{iuX_t}]$  then becomes

$$\begin{aligned} \psi(u) = iu\gamma_c + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ & \left\{ \left(1 - \frac{iu}{\lambda_+}\right)^{\alpha_+} - 1 + \frac{iu\alpha_+}{\lambda_+} \right\} \\ + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_- & \left\{ \left(1 + \frac{iu}{\lambda_-}\right)^{\alpha_-} - 1 - \frac{iu\alpha_-}{\lambda_-} \right\}. \end{aligned} \tag{2.4}$$

In the sequel we will also need the characteristic exponent of the tempered stable process for  $0 < \alpha < 1$  with  $h \equiv 0$  as the truncation function:

$$\psi(u) = iu\gamma_0 + \Gamma(-\alpha) [c_+((\lambda_+ - iu)^\alpha - \lambda_+^\alpha) + c_-((\lambda_- + iu)^\alpha - (\lambda_-)^\alpha)]. \tag{2.5}$$

### 3 Monte Carlo Evaluation of Functionals of CGMY Process Using Measure Change

The following theorem is the main result of this paper and shows that under an appropriate change of measure the tempered stable process becomes a sum of two one-sided stable processes.

**Theorem 3.1** *Let  $(X_t)$  be a (generalized) tempered stable Lévy process on the probability space  $(\Omega, \mathcal{F}, P)$  with Lévy density*

$$\nu(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} \mathbf{1}_{x>0} + \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} \mathbf{1}_{x<0},$$

and let  $(X_t^+)$  and  $(X_t^-)$  be tempered stable Lévy processes such that  $X = X^+ + X^-$  with characteristic triplets  $(0, \nu_+, \gamma^+)$  and  $(0, \nu_-, \gamma^-)$  where

$$\nu_+(x) = \frac{c_+ e^{-\lambda_+ x}}{x^{1+\alpha_+}} \mathbf{1}_{x>0} \quad \text{and} \quad \nu_-(x) = \frac{c_- e^{-\lambda_- |x|}}{|x|^{1+\alpha_-}} \mathbf{1}_{x<0}.$$

Then the following holds:

1. There exists a unique constant  $c$  such that  $e^{U_t}$  is a  $P$ -martingale, where  $U_t = \lambda_+ X_t^+ - \lambda_- X_t^- + ct$ .  $c$  is given by<sup>1</sup>

- for  $0 < \alpha < 1$ ,

$$c = -\lambda_+ \gamma_0^+ + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ + \lambda_- \gamma_0^- + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_-.$$

- for  $1 < \alpha < 2$ ,

$$c = -\lambda_+ \gamma_c^+ + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ (\alpha_+ - 1) + \lambda_- \gamma_c^- + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_- (\alpha_- - 1).$$

2. One can define a probability measure  $Q$  on  $(\Omega, \mathcal{F})$  such that  $Q|_{\mathcal{F}_t} \sim P|_{\mathcal{F}_t}$  for every  $t$  by  $\frac{dQ}{dP}|_{\mathcal{F}_t} = e^{U_t}$ .

3. Under  $Q$ , the processes  $(X_t^+)$  and  $(X_t^-)$  are stable processes with characteristic triplets  $(0, \tilde{\nu}_+, \tilde{\gamma}_+)$  and  $(0, \tilde{\nu}_-, \tilde{\gamma}_-)$  where

$$\tilde{\nu}_+(x) = \frac{c_+}{x^{1+\alpha_+}} 1_{x>0} \quad \text{and} \quad \tilde{\nu}_-(x) = \frac{c_-}{|x|^{1+\alpha_-}} 1_{x<0},$$

and

- for  $0 < \alpha < 1$

$$\tilde{\gamma}_0^+ = \gamma_0^+ \tag{3.1}$$

$$\tilde{\gamma}_0^- = \gamma_0^- \tag{3.2}$$

- for  $1 < \alpha < 2$

$$\tilde{\gamma}_c^+ = \gamma_c^+ + c_+ \int_0^\infty \frac{1 - e^{-\lambda_+ x}}{x^{\alpha_+}} dx = \gamma_c^+ - c_+ \lambda_+^{\alpha_+-1} \Gamma(1 - \alpha_+), \tag{3.3}$$

$$\tilde{\gamma}_c^- = \gamma_c^- + c_- \int_{-\infty}^0 \frac{1 - e^{-\lambda_- |x|}}{|x|^{\alpha_-}} dx = \gamma_c^- - c_- \lambda_-^{\alpha_--1} \Gamma(1 - \alpha_-). \tag{3.4}$$

*Remark 3.1* The processes  $X^+$  and  $X^-$  are unique up to a linear function: we can freely choose a constant  $k$  and put  $X^{+'} = X^+ + kt$  and  $X^{-'} = X^- - kt$ . It is easy to see that the value of  $-\lambda_+ X_t^+ + \lambda_- X_t^- - ct$  does not depend on  $k$  and therefore, the quantities that we want to compute, that is, expectations of functionals of  $X$ , do not depend on  $k$ .

*Proof* 1. By Proposition 2.1,  $E[e^{\lambda_+ X_1^+}] < \infty$ ,  $E[e^{-\lambda_- X_1^-}] < \infty$  and, since  $X^+$  and  $X^-$  are independent,

$$E[e^{\lambda_+ X_1^+ - \lambda_- X_1^-}] = E[e^{\lambda_+ X_1^+}] E[e^{-\lambda_- X_1^-}] < \infty.$$

<sup>1</sup> Here and below  $0 < \alpha < 1$  means that both  $0 < \alpha_+ < 1$  and  $0 < \alpha_- < 1$  and similarly for  $1 < \alpha < 2$ . The formulas for the other cases when, say  $0 < \alpha_+ < 1$  and  $1 < \alpha_- < 2$  are straightforward generalizations.



Let

$$c = -\ln E[e^{\lambda_+ X_1^+ - \lambda_- X_1^-}] \equiv -(\ln E[e^{\lambda_+ X_1^+}] + \ln E[e^{-\lambda_- X_1^-}]).$$

Since  $\lambda_+ X_1^+ - \lambda_- X_1^-$  is a Lévy process,

$$e^{\lambda_+ X_t^+ - \lambda_- X_t^- + ct} = \frac{e^{\lambda_+ X_t^+ - \lambda_- X_t^-}}{E[\lambda_+ X_t^+ - \lambda_- X_t^-]}$$

is a martingale. The constant  $c$  can be computed as follows:

- If  $0 < \alpha < 1$  we can use the truncation function  $h \equiv 0$  and Eq. 2.5 yields

$$c = -\lambda_+ \gamma_0^+ + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ + \lambda_- \gamma_0^- + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_-$$

- If  $1 < \alpha < 2$  we can use the truncation function  $h(x) = x$  and Eq. 2.4 yields:

$$c = -\lambda_+ \gamma_c^+ + \Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ (\alpha_+ - 1) + \lambda_- \gamma_c^- + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_- (\alpha_- - 1)$$

2. Let

$$\Phi(x) = \lambda_+ x \mathbf{1}_{x>0} - \lambda_- x \mathbf{1}_{x<0}.$$

Since, as is easily seen,  $\int_{-\infty}^{\infty} (e^{\Phi(x)/2} - 1)^2 \nu(dx) < \infty$ , by Proposition 2.2, we can define a new probability  $Q \sim P$  by:

$$\frac{dQ}{dP} \Big|_{\mathcal{F}_t} = e^{U_t}$$

with

$$\begin{aligned} U_t &= \lim_{\epsilon \rightarrow 0} \left( \sum_{|\Delta X_s| > \epsilon} \Phi(\Delta X_s) - t \int_{|x| > \epsilon} (e^{\Phi(x)} - 1) \nu(dx) \right) \\ &= \lim_{\epsilon \rightarrow 0} \left( \sum_{|\Delta X_s| > \epsilon} (\lambda_+ \Delta X_s \mathbf{1}_{\Delta X_s > 0} - \lambda_- \Delta X_s \mathbf{1}_{\Delta X_s < 0}) \right. \\ &\quad \left. - t \int_{|x| > \epsilon} (e^{\lambda_+ x \mathbf{1}_{x>0} - \lambda_- x \mathbf{1}_{x<0}} - 1) \nu(dx) \right) \\ &= \lim_{\epsilon \rightarrow 0} \left( \sum_{\Delta X_s > \epsilon} \lambda_+ \Delta X_s - t \int_{x > \epsilon} (e^{\lambda_+ x} - 1) \nu(dx) \right) \\ &\quad + \lim_{\epsilon \rightarrow 0} \left( \sum_{\Delta X_s < -\epsilon} (-\lambda_-) \Delta X_s - t \int_{x < -\epsilon} (e^{-\lambda_- x} - 1) \nu(dx) \right). \end{aligned} \tag{3.5}$$

The first term in the right-hand side can be rewritten as follows:

$$\lim_{\epsilon \rightarrow 0} \lambda_+ \left( \sum_{\epsilon \leq \Delta X_s} \Delta X_s - t \int_{\epsilon < x} h(x) \nu(dx) \right) - \lim_{\epsilon \rightarrow 0} \left( t \int_{x > \epsilon} (e^{\lambda_+ x} - 1 - \lambda_+ h(x)) \nu(dx) \right).$$

By the Lévy-Itô decomposition, the first term above equals

$$\lambda_+ X_t^+ - \lambda_+ \gamma_h^+ t,$$

and by the Lévy-Khintchine representation (2.1), the second term gives

$$\ln E[e^{\lambda_+ X_1^+}] - \lambda_+ \gamma_h^+.$$

Treating the second term on the same way the right-hand side of (3.5) in the same way and assembling all terms, we finally get

$$U_t = \lambda_+ X_t^+ - \lambda_- X_t^- + ct.$$

3. By Proposition 2.2, we can express the Lévy densities of  $X^+$  and  $X^-$  under  $Q$  as follows:

$$\begin{aligned} \tilde{\nu}_+(dx) &= e^{\Phi(x)} \nu_+(dx) = e^{\lambda_+ x} \nu_+(dx) = \frac{c_+}{x^{1+\alpha_+}} 1_{x>0}, \\ \tilde{\nu}_-(dx) &= e^{\Phi(x)} \nu_-(dx) = e^{-\lambda_- x} \nu_-(dx) = \frac{c_-}{x^{1+\alpha_-}} 1_{x<0}. \end{aligned}$$

Moreover, once again by Proposition (2.2),

$$\begin{aligned} \tilde{\gamma}_h^+ - \gamma_h^+ &= \int_{-\infty}^{\infty} h(x)(\tilde{\nu}_+ - \nu_+)(dx), \\ \tilde{\gamma}_h^- - \gamma_h^- &= \int_{-\infty}^{\infty} h(x)(\tilde{\nu}_- - \nu_-)(dx). \end{aligned}$$

Treating the cases  $0 < \alpha < 1$  and  $1 < \alpha < 2$ , separately, we obtain Eqs. 3.1–3.4. □

The expectation of any  $\mathcal{F}_T$ -measurable random variable  $H_T$  can be evaluated via

$$E^P[H_T] = E^Q[H_T e^{-\lambda_+ X_T^+ + \lambda_- X_T^- - cT}].$$

In particular, if  $H_T = f(X_T)$ , then

$$E^P[H_T] = E^P[f(X_T)] = E^Q[f(X_T)e^{-\lambda_+X_T^+ + \lambda_-X_T^- - cT}].$$

The Monte Carlo estimator  $\bar{H}$  of  $E[H_T]$  is given by

$$\bar{H} = \frac{1}{N} \sum_{i=1}^N H_T^i \exp(-\lambda_+X_T^{+,i} + \lambda_-X_T^{-,i} - cT),$$

where  $(X_T^i)$  for  $i = 1, \dots, N$  are independent realizations of  $X_T$  under  $Q$  and  $H_T^i$  are corresponding realizations of  $H_T$ .

We conclude this section with an upper bound for the variance of  $\bar{H}$ .

**Proposition 3.2** *Suppose that in Theorem 3.1,  $(X_t)$  is such that  $\alpha_+ \neq 1$  and  $\alpha_- \neq 1$  and that the random variable  $H_T$  is bounded:  $|H_T| \leq K$ . Then:*

$$\begin{aligned} \sqrt{\text{Var}^Q \bar{H}} &= \sqrt{\frac{1}{N} \text{Var}^Q (H_T e^{-\lambda_+X_T^+ + \lambda_-X_T^- - cT})} \\ &\leq \frac{K}{\sqrt{N}} \exp[\Gamma(-\alpha_+) \lambda_+^{\alpha_+} c_+ T (2^{\alpha_+ - 1} - 1) \\ &\quad + \Gamma(-\alpha_-) \lambda_-^{\alpha_-} c_- T (2^{\alpha_- - 1} - 1)]. \end{aligned}$$

*Proof* We have that

$$\begin{aligned} \text{Var}^Q H_T e^{-\lambda_+X_T^+ + \lambda_-X_T^- - cT} &\leq E^Q \left[ f(X_T)^2 \exp(-2\lambda_+X_T^+ + 2\lambda_-X_T^- - 2cT) \right] \\ &\leq K^2 E^Q [\exp(-2\lambda_+X_T^+ + 2\lambda_-X_T^- - 2cT)]. \end{aligned}$$

But since  $X_T^+$  and  $X_T^-$  are independent:

$$E^Q [\exp(-2\lambda_+X_T^+ + 2\lambda_-X_T^- - 2cT)] = e^{-2cT} E^Q [\exp(-2\lambda_+X_T^+)] E^Q [\exp(2\lambda_-X_T^-)].$$

Moreover,

$$\begin{aligned} E^Q [\exp(-2\lambda_+X_T^+)] &= E^P [\exp(-\lambda_+X_T^+ + \lambda_-X_T^- + cT)] \\ &= e^{cT} E^P [\exp(-\lambda_+X_T^+)] E^P [\exp(-\lambda_-X_T^-)] \end{aligned}$$

and

$$E^Q [\exp(2\lambda_-X_T^-)] = e^{cT} E^P [\exp(\lambda_-X_T^-)] E^P [\exp(\lambda_+X_T^+)].$$

Thus

$$\begin{aligned}
 & E^Q[\exp(-2\lambda_+X_T^+ + 2\lambda_-X_T^- - 2cT)] \\
 &= E^P[\exp(-\lambda_+X_T^+)]E^P[\exp(\lambda_-X_T^-)]E^P[\exp(-\lambda_-X_T^-)]E^P[\exp(\lambda_+X_T^+)] \\
 &= \exp(T(\psi_+(i\lambda_+) + \psi_+(-i\lambda_+) + \psi_-(i\lambda_-) + \psi_-(-i\lambda_-))) \\
 &= \exp(\Gamma(-\alpha_+)\lambda_+^{\alpha_+}c_+T(2^{\alpha_+} - 2) + \Gamma(-\alpha_-)\lambda_-^{\alpha_-}c_-T(2^{\alpha_-} - 2)). \quad \square
 \end{aligned}$$

### 4 Numerical Results

#### 4.1 Overview of the Direct Simulation Method

Let us briefly review the approximate simulation method for the tempered stable process, introduced in Madan and Yor (2005). This method applies to the 4-parameter CGMY family (tempered stable process with  $c_- = c_+ = c$  and  $\alpha_- = \alpha_+ = \alpha$ ). It is based on the fact that such a process can be represented as a time-changed Brownian motion with drift. Since some of the constants are not given explicitly in Madan and Yor (2005), we provide a short proof of this result here.

**Proposition 4.1** *Let  $c > 0$ ,  $\alpha \in (0, 2)$ ,  $\gamma_+ > 0$  and  $\gamma_- > 0$ , and  $Z$  be a subordinator with zero drift and Lévy density*

$$\nu_Z(t) = \frac{ce^{\frac{t}{2}A^2 - \frac{t}{4}B^2}D_{-\alpha}(B\sqrt{t})}{t^{\frac{\alpha}{2}+1}},$$

where  $D$  is the parabolic cylinder function (see Gradshetyn and Ryzhik 1995),  $A = \frac{\lambda_- - \lambda_+}{2}$  and  $B = \frac{\lambda_+ + \lambda_-}{2}$ . Then the process

$$X_t = AZ_t + W(Z_t),$$

where  $W$  is a standard Brownian motion, is a tempered stable process with Lévy density

$$\nu(x) = \frac{ce^{-\lambda_+x}1_{x>0} + ce^{-\lambda_-|x|}1_{x<0}}{|x|^{1+\alpha}} \tag{4.1}$$

and center

$$\gamma_c = \int_{\mathbb{R}} x(1 - e^{-Ax})\nu(x)dx.$$

In the case  $0 < \alpha < 1$ , the drift of  $X$  is equal to zero.

**Remark 4.1** In the case  $\alpha \neq 1$  the center is given explicitly by

$$\gamma_c = c\Gamma(1 - \alpha)(\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}).$$

*Proof* By Sato (1999, Theorem 30.1), the Lévy density of  $X$  is given by

$$\nu_X(x) = \int_0^\infty p_t(x) \nu_Z(t) dt$$

where

$$p_t(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-At)^2}{2t}}.$$

This integral corresponds to integral 7.728 in (Gradshteyn and Ryzhik, p. 837) and is equal to (4.1).

Once again, by Sato (1999, Theorem 30.1), the center of  $X$  (third component of the characteristic triplet with respect to the truncation function  $h(x) \equiv x$ ) is given by

$$\int_0^\infty \nu_Z(t) dt \int_{\mathbb{R}} xp_t(x) dx = A \int_0^\infty t \nu_Z(t) dt.$$

On the other hand, by Fubini’s theorem,

$$\begin{aligned} \int_{\mathbb{R}} x(1 - e^{-Ax}) \nu(x) dx &= \int_{\mathbb{R}} x(1 - e^{-Ax}) \int_0^\infty p_t(x) \nu_Z(t) dt dx \\ &= \int_0^\infty \nu_Z(t) dt \int_{\mathbb{R}} x(1 - e^{-Ax}) p_t(x) dx = A \int_0^\infty t \nu_Z(t) dt. \end{aligned}$$

The last statement of the theorem follows directly from Sato (1999, Theorem 30.1). □

The paper (Madan and Yor 2005) further shows that the density of the time change  $Z$  can be written as

$$\nu_Z(t) = f(t) \nu^0(t),$$

where  $f(t) \leq 1$  and  $\nu^0(t)$  is the Lévy density of an  $\frac{\alpha}{2}$ -stable subordinator:

$$\begin{aligned} f(t) &= \frac{2^{\frac{\alpha}{2}} \Gamma(\frac{\alpha}{2} + \frac{1}{2}) e^{\frac{t}{2} A^2 - \frac{t}{4} B^2}}{\sqrt{\pi}} D_{-\alpha}(B\sqrt{t}), \\ \nu_0(t) &= 2^{-\frac{\alpha}{2}} \sqrt{\pi} \frac{\lambda_+^{\alpha-2} + \lambda_-^{\alpha-2}}{\Gamma(\frac{\alpha}{2} + \frac{1}{2})} \frac{c}{t^{1+\frac{\alpha}{2}}} 1_{t>0} \equiv \frac{K}{t^{1+\frac{\alpha}{2}}} 1_{t>0}. \end{aligned}$$

This allows us to simulate paths of  $Z$  on the interval  $[0, T]$  using the Rosiński (2001) rejection method as follows.

### 4.2 Description of the Algorithm

1. Approximate a trajectory of a stable subordinator with Lévy density  $\nu_0$  on  $[0, T]$ . This approximation consists of replacing this subordinator with a compound Poisson subordinator with Lévy density

$$\frac{K}{t^{1+\frac{\alpha}{2}}} 1_{t>\varepsilon}$$

and drift

$$d = \int_0^\varepsilon y \frac{K}{y^{\frac{\alpha}{2}+1}} dy = \frac{K\varepsilon^{1-\frac{\alpha}{2}}}{1-\frac{\alpha}{2}}.$$

This amounts to replacing small jumps with their expectation. Note that an approximation bias is introduced at this stage. Denote by  $(T_i)$  the jump times and by  $(y_i)$  the jump sizes of the compound Poisson approximation (we do not give details since their simulation is straightforward).

2. Approximate the trajectory of the CGMY time change  $Z$  using the rejection method. In this setting this amounts to accept every jump  $y_i$  of the stable subordinator for which  $f(y_i)$  is greater than an independent uniform random variable on  $[0, 1]$ . The approximated time change is given by

$$\hat{Z}_t = td + \sum_i y_i 1_{\Gamma_i \leq t} 1_{f(y_i) > U_i},$$

where  $(U_i)$  is an independent sequence of uniforms on  $[0, 1]$ .

3. Approximate the CGMY process by

$$\hat{X}_t = A\hat{Z}_t + W(\hat{Z}_t),$$

where  $W$  is a standard Brownian motion.

### 4.3 Financial Model Setup

To compare the performance of the algorithm introduced in Sect. 3 with that of the approximate simulation algorithm of Madan and Yor (2005), we use both methods to compute the prices of European put options in an exponential Lévy model driven by a tempered stable process. The European options were chosen because their reference prices can be computed analytically, which allows to estimate the bias in the approximate simulation method. In this model, the stock price is given by

$$S_t = S_0 e^{rt+X_t},$$

where  $X$  is a tempered stable process and  $r$  is the interest rate supposed constant. The European put option is a contract which pays to its holder  $(K - S_T)^+$  at a future date  $T$ , where  $K$  is a fixed amount (strike). By the fundamental theorem of asset pricing, absence of arbitrage in a financial market is equivalent to the existence of a probability  $Q$  equivalent to the historical probability  $P$ , such that discounted prices of all assets are  $Q$ -martingales. In this paper we place ourselves directly under the probability  $Q$ . The process  $X$  must therefore satisfy the martingale condition  $E[e^{X_t}] = 1$  and the price of a European put option at time  $t = 0$  is

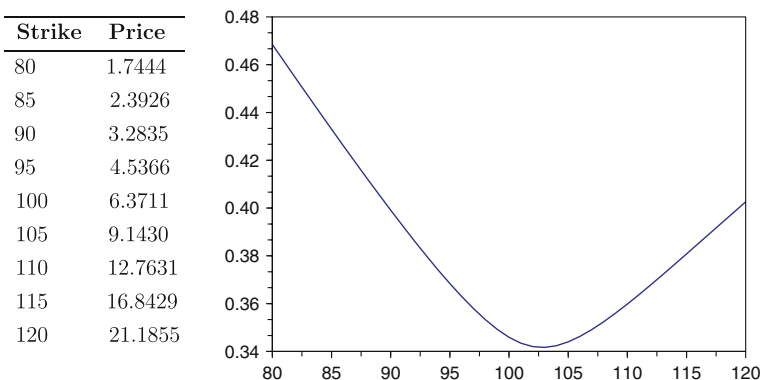
$$P(T, K) = e^{-rT} E^Q[(K - S_T)^+].$$

### 4.4 Results and Comparisons

For numerical computations, we used the following artificial parameters of the tempered stable process:  $\alpha = 0.5$ ,  $c = 0.5$ ,  $\lambda_+ = 3.5$  and  $\lambda_- = 2$ . The initial stock price was taken equal to 100, the option's maturity was  $T = 0.25$ , and the interest rate was 4%. The first step was to compute the reference prices using the Fourier transform method, described in [Cont and Tankov \(2004\)](#). The prices are given in Table 1. The implied volatility smile corresponding to these prices is shown in the graph next to this table. This is a typical shape of implied volatility observed for options on stocks.

Next we study the bias of the approximate simulation method. To reduce the number of simulations, we only treat the strike  $K = 100$ . The results are shown in Table 2. We see that for the parameter values used, the bias decreases rapidly to zero as  $\varepsilon \rightarrow 0$ . Moreover, since the computational complexity is proportional to  $\frac{1}{\varepsilon^{\alpha/2}}$ , decreasing  $\varepsilon$  by a factor of 10 reduces the bias by roughly the same factor but will only increase the complexity by a factor of  $10^{1/4} \approx 1.77$ . Therefore, in

**Table 1** Reference option prices computed using the Fourier transform method and the corresponding implied volatilities



**Table 2** Bias of the approximate simulation method

$\varepsilon$	Price (std.dev)	Bias (std.dev)	Time ( $10^7$ traj.)
$10^{-1}$	8.4608 (0.004)	2.090 (0.004)	61 s
$10^{-2}$	6.5834 (0.004)	0.2123 (0.004)	91 s
$10^{-3}$	6.3784 (0.004)	0.0073 (0.004)	144 s
$10^{-4}$	6.3658 (0.004)	-0.0053 (0.004)	233 s

**Table 3** Option prices computed using the approximate simulation method. The computation time is about 23 s for each price

Strike	Price	Std. dev.
80	1.745	0.007
85	2.378	0.008
90	3.278	0.009
95	4.544	0.011
100	6.349	0.012
105	9.131	0.013
110	12.765	0.014
115	16.855	0.015
120	21.192	0.016

this case it is sufficient to take  $\varepsilon = 10^{-4}$  and the Monte Carlo error will play the dominant role.

Armed with this knowledge, we set  $\varepsilon = 10^{-4}$ , the level at which the bias is insignificant compared to Monte Carlo error for  $10^6$  trajectories, and compute the option prices using the approximate simulation method over  $10^6$  trajectories. This number of trajectories allows to estimate option prices with a precision of about one cent. The prices, standard deviations, and computation times are shown in Table 3.

Next, we use the Monte Carlo method described in Sect. 3 to price the same options. With  $10^6$  trajectories we obtain roughly the same variance for strike  $K = 100$  as with the approximate method, whereas the computational time is reduced by a factor of 16. The results are shown in Table 4. We also compute the theoretical bound on variance, given by Proposition 3.2. The theoretical bound is quite far from the true variance since the option pay-off is estimated from above by the strike value (a very rough estimate).

As our last example, we compare the two algorithms in an infinite variation model with parameters  $\alpha = 1.5$ ,  $c = 0.1$ ,  $\lambda_+ = 3.5$ ,  $\gamma_- = 2$ . This corresponds to roughly the same unit variance as in the previous case. We fix the strike equal to 100 and leave all other parameters unchanged. The true price of the option (computed by Fourier transform) is equal to 8.3014 in this case.

First we analyze the bias of the approximate simulation algorithm. The results are shown in Table 5. In this case, the computational time is much more sensitive to  $\varepsilon$  and the bias is, roughly, inversely proportional to the computational effort.



**Table 4** Option prices computed using the method of Sect. 3. The theoretical bound is that given in Proposition 3.2. The computation time for one price is about 1.43 s. The errors are all of the same sign compared to Table 1 because they have been computed over the same set of trajectories

Strike	Price	Std. dev.	Theor. bound
80	1.751	0.005	0.12
85	2.403	0.006	0.13
90	3.299	0.008	0.14
95	4.560	0.010	0.15
100	6.397	0.012	0.15
105	9.168	0.015	0.16
110	12.789	0.019	0.17
115	16.867	0.023	0.18
120	21.207	0.028	0.18

**Table 5** Approximation bias, infinite variation case

$\varepsilon$	Price (std. dev)	Bias (std. dev)	Time ( $10^6$ traj.)
$10^{-1}$	10.081 (0.013)	1.780 (0.013)	1.5 sec
$10^{-2}$	8.683 (0.012)	0.381 (0.012)	5.17 sec
$10^{-3}$	8.368 (0.011)	0.067 (0.011)	24 sec
$10^{-4}$	8.304 (0.011)	0.002 (0.011)	132 sec

With the method of Sect. 3, the standard deviation of about 0.011 is obtained for  $1.8 \times 10^6$  trajectories, which requires about 2.62 s of machine time. Therefore, in this case our algorithm is about 50 times faster than approximate simulation, for the same precision.

To conclude, let us sum up the advantages and drawbacks of the two algorithms. First of all, we want to emphasize the fundamental difference between the two approaches: whereas the method of Madan and Yor (2005) is a simulation algorithm, our method is an algorithm for evaluating expectations, it cannot be used to, say, plot a trajectory of the tempered stable process for given parameter values.

The approximate simulation method by Madan and Yor (2005)

- Can be easily generalized to multiple dimensions by taking a multidimensional Brownian motion.
- Provides access to the entire trajectory of the process.
- Involves an approximation bias which may be important and is difficult to quantify.

On the other hand, our algorithm presented in Sect. 3

- Is more general (in one dimension) in the sense that we allow different values of  $\alpha_+$  and  $\alpha_-$  and different values of  $c_+$  and  $c_-$ , which is not possible with Brownian subordination.
- Provides unbiased Monte Carlo estimates.

- Runs 16 to 50 times faster for the parameter values used in this paper (for pricing European options).

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