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Understanding the Implied Volatility Surface for Options on a Diversified Index

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Abstract. This paper describes a two-factor model for a diversified index that attempts to explain both the leverage effect and the implied volatility skews that are characteristic of index options. Our formulation is based on an analysis of the growth optimal portfolio and a corresponding random market activity time where the discounted growth optimal portfolio is expressed as a time transformed squared Bessel process of dimension four. It turns out that for this index model an equivalent risk neutral martingale measure does not exist because the corresponding Radon-Nikodym derivative process is a strict local martingale. However, a consistent pricing and hedging framework is established by using the benchmark approach. The proposed model, which includes a random initial condition for market activity, generates implied volatility surfaces for European call and put options that are typically observed in real markets. The paper also examines the price differences of binary options for the proposed model and their Black-Scholes counterparts.

Key words: index derivatives, minimal market model, random scaling, growth optimal portfolio, fair pricing, binary options

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1. Introduction

Understanding the implied volatility skews that are characteristic of index options has challenged financial mathematicians for nearly three decades. Typically researchers seek to explain these effects by assuming that the return distribution for an index does not follow the lognormal diffusion dynamics that underpin the Black-Scholes model. For instance, volatility skews and smiles can be obtained from local volatility function models developed by Dupire (1992–1994). Other researchers have focused on stochastic volatility function models (see Derman and Kani, 1994), and jump diffusion models (see Kou, 2002). Some other contributions to equity index modeling are contained in Renault and Touzi (1996), Musiela and Rutkowski (1997), Rebonato (1999), Schönbucher (1999), Fouque et al. (2000), Lewis (2000), Rosenberg (2000), Heath et al. (2001), Balland (2002), Barndorff-Nielsen and Shephard (2002), Carr and Wu (2003) and Brigo et al. (2004).

A number of empirical studies have been undertaken on the dynamics of indices and corresponding implied volatility surfaces. Black (1976) documented empirically the leverage effect, which for an index states that volatility is negatively correlated to the level of the index. Rubinstein (2000) examined observed S&P500 index implied volatilities and discovered that the implied volatility surface was nearly flat for periods before the 1987 stock market crash, but has shown consistent negative skews for the decade after the crash. In a recent paper (Heath and Platen, 2004) found a pronounced negative skew with small smile features for S&P500 index call options. Other interesting empirical studies have been undertaken by Franks and Schwartz (1991), Heynen (1993), Heynen et al. (1994), Bakshi et al. (1997), Dumas et al. (1997), Das and Sundaram (1999), Skiadopolous et al. (2000), Tompkins (2001) and Cont and da Fonseca (2002).

It is widely accepted that the negative volatility skew observed for index options is related to the leverage effect. This itself can at least be partly explained by the so-called wealth effect. That is, if the level of an index declines wealth is reduced leading to greater risk aversion and thus higher volatility. Other economic reasons that have been cited to provide justification for the leverage effect include for example what (Rubinstein, 1985) calls the diversification effect, where correlation among index constituents increases for falling index values, leading to greater index volatility.

In this paper we present a two-factor model for a diversified index that provides a structural explanation as to why large indices show negative implied volatility skews. Our formulation is established using global properties of the market and avoids specific assumptions regarding the return distributions. Rather, these follow as a consequence of other more fundamental analysis. The framework established in this paper is important because it provides not only intuition on the causes of the leverage effect, but also on how to quantify this effect and why the implied volatility patterns observed for index options can be different from those for stock and FX markets.

Our treatment starts with an examination of the properties of the *growth optimal portfolio* (GOP) introduced by Kelly (1956). The GOP is the portfolio that maximizes expected log-utility. Using basic diffusion dynamics for securities and the GOP optimality criterion, it is possible to characterize the GOP. It turns out that with respect to a certain time transformation, the GOP can be expressed precisely as a squared Bessel process of dimension four. The corresponding random time transformation can be obtained directly from the quadratic variation of the square root of the discounted GOP.

It is then argued that this analysis can be extended to any diversified portfolio because by applying results in Platen (2004) it follows that such a portfolio approximates the GOP. Combining these two arguments means that the underlying dynamics for a diversified index are visible and can be directly calibrated from observed market data. This formulation is therefore based on an underlying squared Bessel process and provides a precise quantification of both the degree of leverage and corresponding implied volatility skews that can be expected for a diversified index and its derivatives.

Principal component analysis indicates that a one-factor model can capture approximately 75–95% of the movements of an index and its implied volatilities, see for example (Cont and da Fonseca, 2002). However, a two-factor model is required to increase these percentages to say 90–99%. Also, at present it is not possible to implement a model containing more than two factors with reliable, fast and accurate pricing and hedging tools (see Brigo and Mercurio, 2001). Therefore, choosing a two-factor model represents a reasonable compromise between accuracy and tractability. In this paper our proposed two-factor model is based on the above GOP analysis and is a generalization of the *minimal market model* (MMM) described in Platen (2001, 2002). For this model the factors are the GOP together with a random scaling component constructed from the random time transformation. These factors, together with the short rate, determine the overall dynamics of the GOP and thus, by extension, of a diversified index.

For most current index models the existence of an equivalent risk neutral martingale measure is assumed. However, for the dynamics considered here such a measure does not exist, because the corresponding Radon-Nikodym derivative is a strict local martingale. This is related to the fact that for the MMM, volatilities become unbounded as the index approaches zero. Consequently, in this paper the existence of an equivalent risk neutral martingale measure is not assumed. Rather, to obtain a consistent derivative pricing and hedging framework, we employ the benchmark approach described in Platen (2002, 2004). The benchmark approach is characterized by choosing the GOP as the numeraire portfolio. This provides a natural choice because this portfolio achieves optimal long term growth and is independent of the denomination. Furthermore, it leads to pricing formulae with respect to the real world probability measure. Also as the benchmark approach extends risk neutral pricing, it can accommodate a wider class of model formulations including the MMM.

This paper is organized as follows: In Section 2 the GOP and benchmark approach are described. Section 3 considers the MMM with a random scaling factor. Using the benchmark approach the pricing of zero coupon bonds and European put options are examined in Section 4. These calculations are used to compute implied volatility term structures and to compare them to surfaces observed in real markets. At the end of Section 4 a brief account of the pricing of path dependent binary options is given with emphasis on the differences between these prices and corresponding Black-Scholes prices.

2. A Benchmark Model

2.1. PRIMARY SECURITY ACCOUNTS

Consider a financial market with $d + 1$ primary assets with a corresponding set $S^{(0)}, S^{(1)}, \ldots, S^{(d)}$ of *primary security account processes*. The primary assets could be shares, indices, commodities, foreign currencies or even derivatives. It is assumed that accrued income or loss from holding a primary asset is always reinvested. Let $S_t^{(j)}$ be the price of the jth primary security account, $j \in \{0, 1, \ldots, d\}$, denominated in units of the domestic currency at time $t \in [0, T]$ for some finite time horizon $T \in (0, \infty)$. It is assumed that $S_t^{(j)}$ is the strong, unique solution of the stochastic differential equation (SDE)

$$dS_t^{(j)} = S_t^{(j)} \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right) \tag{2.1}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$ with $S_0^{(j)} > 0$. Here the vector $W = \{W_t = (W_t^1, \ldots, W_t^d), t \in [0, T]\}$ of independent Wiener processes is defined on a filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$ that satisfies the usual conditions, see (Protter, 1990). The filtration $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \in [0,T]}$ is the P-augmentation of the natural filtration $\mathcal{A}^W$ under P, where P is the real world probability measure.

The jth *appreciation rate* $a^j = \{a_t^j, t \in [0, T]\}$ and (j, k)th *volatility* $b^{j,k} = \{b_t^{j,k}, t \in [0, T]\}$ are considered to be $\underline{\mathcal{A}}$-adapted stochastic processes for $j \in \{0, 1, \ldots, d\}$ and $k \in \{1, 2, \ldots, d\}$, see (Protter, 1990). We set $a_t^0 = r_t$ and $b_t^{0,k} = 0$ for $k \in \{1, 2, \ldots, d\}$ so that $S_t^{(0)}$ is the value of the savings account at time t, where r_t is the short term interest rate at time t. Furthermore, it is assumed that the *volatility matrix* $b_t = [b_t^{j,k}]_{j,k=1}^d$ is for Lebesgue-almost-every $t \in [0, T]$ *invertible*.

By introducing the appreciation rate vector $a_t = (a_t^1, \ldots, a_t^d)^\top$, where $A^\top$ denotes the transpose of a vector or matrix A, and the unit vector $\mathbf{1} = (1, \ldots, 1)^\top$, the *market price for risk* vector is given by

$$\theta_t = \left(\theta_t^1, \ldots, \theta_t^d\right)^\top = b_t^{-1} \left[a_t - r_t \, \mathbf{1}\right] \tag{2.2}$$

for $t \in [0, T]$. This expression allows us to rewrite the SDE (2.1) in the form

$$dS_t^{(j)} = S_t^{(j)} \left(r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \left[\theta_t^k \, dt + dW_t^k\right] \right) \tag{2.3}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$.

2.2. STRATEGIES

Let us now consider portfolios of primary assets. We say that a predictable stochastic process $\delta = \{\delta(t) = (\delta^{(0)}(t), \delta^{(1)}(t), \ldots, \delta^{(d)}(t))^\top, t \in [0, T]\}$ is a *strategy* if δ is S-integrable, see (Protter, 1990). Here the jth component $\delta^{(j)}(t)$ denotes the number of units of the jth primary security account that are held at time $t \in [0, T]$ in the corresponding portfolio, $j \in \{0, 1, \ldots, d\}$. For a strategy δ we denote by $S_t^{(\delta)}$ the value of the corresponding portfolio at time t, measured in units of the domestic currency. This means that

$$S_t^{(\delta)} = \sum_{j=0}^{d} \delta^{(j)}(t)\, S_t^{(j)} \tag{2.4}$$

for $t \in [0, T]$. A strategy δ and the corresponding portfolio process $S^{(\delta)}$ are called *self-financing* if

$$dS_t^{(\delta)} = \sum_{j=0}^{d} \delta^{(j)}(t)\, dS_t^{(j)} \tag{2.5}$$

for all $t \in [0, T]$. For a self-financing strategy no outflow or inflow of funds occur in the corresponding portfolio. That is, all changes in the value of the portfolio are due to corresponding gains from trade resulting from random movements of the primary security accounts. In what follows, only self-financing strategies and corresponding self-financing portfolios are considered. Therefore, from now on we omit the phrase "self-financing".

Using (2.5) and (2.3) for a given strategy δ, the corresponding portfolio value $S_t^{(\delta)}$ satisfies the SDE

$$dS_t^{(\delta)} = S_t^{(\delta)}\, r_t\, dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \delta^{(j)}(t)\, S_t^{(j)}\, b_t^{j,k} \left(\theta_t^k\, dt + dW_t^k\right) \tag{2.6}$$

for $t \in [0, T]$.

2.3. GROWTH OPTIMAL PORTFOLIO

We now introduce the *growth optimal portfolio* (GOP) (see Kelly, 1956; Long, 1990; Karatzas and Shreve, 1998; Platen, 2002). This is defined as the portfolio that maximizes the growth rate, that is the drift of $\ln(S_t^{(\delta)})$, for all $t \in [0, T]$. The optimal strategy $\delta_* = \{\delta_*(t) = (\delta_*^{(0)}(t), \delta_*^{(1)}(t), \ldots, \delta_*^{(d)}(t))^\top,\ t \in [0, T]\}$ follows in a straightforward manner by solving the first order conditions for the quadratic growth rate maximization problem (see Karatzas and Shreve, 1998; Platen, 2004). It can be shown that the GOP satisfies the SDE

$$dS_t^{(\delta_*)} = S_t^{(\delta_*)} \left(r_t\, dt + \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k\right) \right) \tag{2.7}$$

for $t \in [0, T]$ with

$$S_0^{(\delta_*)} > 0. \tag{2.8}$$

Note that the GOP dynamics are fully characterized by the market price for risk vector process $\theta = \{\theta_t,\ t \in [0, T]\}$ and the short rate process $r = \{r_t,\ t \in [0, T]\}$. In particular, inspection of (2.7) reveals that the GOP volatilities are determined by the market price of risk process.

Let the discounted GOP value $\bar{S}_t^{(\delta_*)}$ be defined by

$$\bar{S}_t^{(\delta_*)} = \frac{S_t^{(\delta_*)}}{S_t^{(0)}}. \tag{2.9}$$

By application of the Itô formula and using (2.7) and (2.1), the discounted GOP value $\bar{S}_t^{(\delta_*)}$ satisfies the SDE

$$d\bar{S}_t^{(\delta_*)} = \bar{S}_t^{(\delta_*)} \sum_{k=1}^{d} \theta_t^k \big(\theta_t^k\, dt + dW_t^k\big) \tag{2.10}$$

for $t \in [0, T]$. Let the *total market price for risk* $|\theta_t|$ be given by the expression

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} \big(\theta_t^k\big)^2} \tag{2.11}$$

for $t \in [0, T]$. Combining (2.10) and (2.11) yields the SDE

$$d\bar{S}_t^{(\delta_*)} = \bar{S}_t^{(\delta_*)}\, |\theta_t| \,\big(|\theta_t|\, dt + d\hat{W}_t\big) \tag{2.12}$$

with

$$d\hat{W}_t = \frac{1}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k\, dW_t^k \tag{2.13}$$

for $t \in [0, T]$ (see Platen, 2002). Using Levy's theorem (see Karatzas and Shreve, 1991), it can be shown that $\hat{W} = \{\hat{W}_t,\ t \in [0, T]\}$ is a standard Wiener process on $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$. We have therefore established in (2.12) a simple but powerful characterization of the discounted GOP dynamics.

Let us now introduce a new parameter process $\alpha = \{\alpha_t,\ t \in [0, T]\}$, called the *discounted GOP drift*, with

$$\alpha_t = \bar{S}_t^{(\delta_*)}\, |\theta_t|^2 \tag{2.14}$$

for $t \in [0, T]$ so that

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{(\delta_*)}}}. \tag{2.15}$$

By (2.14) and (2.15) the SDE (2.12) for the discounted GOP can be rewritten in the form

$$d\bar{S}_t^{(\delta_*)} = \alpha_t\, dt + \sqrt{\alpha_t\, \bar{S}_t^{(\delta_*)}}\, d\hat{W}_t \tag{2.16}$$

for $t \in [0, T]$.

Applying the Itô formula using (2.16) yields

$$d\sqrt{\bar{S}_t^{(\delta_*)}} = \frac{3\,\alpha_t}{8\sqrt{\bar{S}_t^{(\delta_*)}}}\, dt + \frac{1}{2}\sqrt{\alpha_t}\, d\hat{W}_t \tag{2.17}$$

for $t \in [0, T]$. The quadratic variation of $\sqrt{\bar{S}_t^{(\delta_*)}}$ can be directly computed from (2.17) as

$$\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t = \frac{1}{4}\int_0^t \alpha_s\, ds \tag{2.18}$$

for $t \in [0, T]$. Hence the discounted GOP drift takes the form

$$\alpha_t = 4\,\frac{d}{dt}\left\langle \sqrt{\bar{S}^{(\delta_*)}} \right\rangle_t \tag{2.19}$$

for $t \in [0, T]$. Therefore, we now have a rather complete characterization of the dynamics for the discounted GOP via Equations (2.16) and (2.19). A precise quantification of the GOP risk premium or expected excess return $|\theta_t|^2$ is given in (2.7) together with (2.15). Furthermore, the parameter process α can be directly observed from market data using (2.19). Note that usually drift parameters can only be reliably estimated on the basis of unrealistically long observation periods. However, (2.19) reduces the problem to one of estimating a quadratic variation, which can be done even over very short time periods if high-frequency data is used.

To see how the corresponding GOP time process φ can be formulated, let $\varphi = \{\varphi_t,\ t \in [0, T]\}$ be a random time transformation given by

$$\varphi_t = \frac{1}{4}\int_0^t \alpha_s\, ds \tag{2.20}$$

with

$$X_{\varphi_t} = \bar{S}_t^{(\delta_*)} \tag{2.21}$$

and

$$d\hat{W}^*_{\varphi_t} = \sqrt{\alpha_t}\, d\hat{W}_t \tag{2.22}$$

for $t \in [0, T]$. It can be shown by application of Levy's theorem (see Karatzas and Shreve, 1991), that in φ-time $\hat{W}^*$ is a Wiener process. The corresponding SDE for the process X in φ-time is then

$$dX_\varphi = d\varphi + \sqrt{X_\varphi}\, d\hat{W}^*_\varphi \tag{2.23}$$

for $\varphi \in [0, \varphi_T]$. Therefore, viewed from the perspective of φ-time, the process X is a squared Bessel process of dimension four (see Revuz and Yor, 1999).

As is shown in Platen (2004) the dynamics for any diversified portfolio will approximate that of the GOP. This important result allows us to use the above structure to model any well diversified index.

2.4. BENCHMARKED PRICES

Throughout the following we use the GOP as *numeraire* or *benchmark* and call prices, expressed in units of $S_t^{(\delta_*)}$, *benchmarked prices*. The jth *benchmarked primary security account price* at time t is then

$$\hat{S}_t^{(j)} = \frac{S_t^{(j)}}{S_t^{(\delta_*)}} \tag{2.24}$$

for $t \in [0, T]$ and $j \in \{0, 1, \ldots, d\}$. The *benchmarked value* of a portfolio $S^{(\delta)}$ is similarly given by

$$\hat{S}_t^{(\delta)} = \frac{S_t^{(\delta)}}{S_t^{(\delta_*)}} \tag{2.25}$$

for $t \in [0, T]$. By application of the Itô formula using (2.6) and (2.7), the SDE for $\hat{S}_t^{(\delta)}$ is given by

$$d\hat{S}_t^{(\delta)} = \sum_{k=1}^{d} \sum_{j=0}^{d} \delta^{(j)}(t)\, \hat{S}_t^{(j)} \big(b_t^{j,k} - \theta_t^k\big)\, dW_t^k \tag{2.26}$$

for $t \in [0, T]$. The right hand side of (2.26) is driftless and therefore $\hat{S}^{(\delta)}$ is an $(\underline{\mathcal{A}}, P)$-local martingale.

2.5. FAIR PRICING

As explained previously, for the model considered in this paper, there is no equivalent risk neutral martingale measure. Hence we use the benchmark approach described in (Platen, 2002), which employs the concept of *fair pricing*.

A value process $U = \{U(t),\ t \in [0, T]\}$, with

$$E\left(\frac{|U(t)|}{S_t^{(\delta_*)}}\right) < \infty$$

for $t \in [0, T]$, is called *fair* if the corresponding benchmarked value process $\hat{U} = \{\hat{U}(t) = \frac{U(t)}{S_t^{(\delta_*)}},\ t \in [0, T]\}$ is an $(\underline{\mathcal{A}}, P)$-martingale, that is

$$\hat{U}(t) = E\left(\hat{U}(\bar{T}) \,|\, \mathcal{A}_t\right) \tag{2.27}$$

for $0 \le t \le \bar{T} \le T$.

We define a *contingent claim* $H_{\bar{T}}$ that matures at a stopping time $\bar{T} \in [0, T]$ as an $\mathcal{A}_{\bar{T}}$-measurable random variable with

$$E\left(\frac{|H_{\bar{T}}|}{S_{\bar{T}}^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right) < \infty \tag{2.28}$$

for all $t \in [0, \bar{T}]$. The corresponding fair price process $U_{H_{\bar{T}}} = \{U_{H_{\bar{T}}}(t),\ t \in [0, \bar{T}]\}$ for this contingent claim satisfies the payoff condition

$$U_{H_{\bar{T}}}(\bar{T}) = H_{\bar{T}}. \tag{2.29}$$

Thus, the corresponding fair derivative price process, when benchmarked, is an $(\underline{\mathcal{A}}, P)$-martingale, see (2.27). Consequently, its benchmarked value $\hat{U}_{H_{\bar{T}}}(t)$ is at time $t \in [0, \bar{T}]$ given by the conditional expectation

$$\hat{U}_{H_{\bar{T}}}(t) = \frac{U_{H_{\bar{T}}}(t)}{S_t^{(\delta_*)}} = E\left(\hat{U}_{H_{\bar{T}}}(\bar{T}) \,\middle|\, \mathcal{A}_t\right). \tag{2.30}$$

Therefore, the fair contingent claim price $U_{H_{\bar{T}}}(t)$ at time t, when expressed in units of the domestic currency, is given by the *fair pricing formula*

$$U_{H_{\bar{T}}}(t) = S_t^{(\delta_*)}\, E\left(\frac{H_{\bar{T}}}{S_{\bar{T}}^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right) \tag{2.31}$$

for $t \in [0, \bar{T}]$. We point out that in formula (2.31) pricing is performed under the real world probability measure by using the GOP $S^{(\delta_*)}$ as numeraire. It is straightforward to show that if an equivalent risk neutral martingale measure exists, then the fair pricing formula coincides with the well-known risk neutral pricing formula (see Platen, 2002).

3. MMM with Random Scaling

3.1. MODEL FORMULATION

The version of the MMM described here is governed by a particular choice of the parameter process α, which characterizes the discounted GOP drift and equals by (2.20) four times the slope of the GOP time. By (2.19) it can be seen that α_t can be observed and, as shown in Platen (2004), is a non-decreasing slowly varying stochastic process. If we take α_t to be deterministic, then we are forced to a squared Bessel process of dimension $\nu = 4$. One could interpret this as an ideal market dynamics, where α_t expresses the average discounted wealth that is transferred per unit of time into the stock market. In order to capture perturbed market dynamics we allow a generalized dynamics with dimension ν above two. This is achieved by defining the process $Z = \{Z_t,\ t \in [0, T]\}$ via the power transformation

$$Z_t = \left(\bar{S}_t^{(\delta_*)}\right)^{\frac{2}{\nu-2}} \tag{3.1}$$

for $t \in [0, T]$ and $\nu \in (2, \infty)$. The Itô formula applied to (2.16) and (3.1) yields

$$dZ_t = \frac{\nu}{4}\,\gamma_t\,dt + \sqrt{\gamma_t\,Z_t}\,d\hat{W}_t \tag{3.2}$$

where the *scaling process* $\gamma = \{\gamma_t,\ t \in [0, T]\}$ is given by

$$\gamma_t = \alpha_t \left(\frac{2}{\nu - 2}\right)^2 (Z_t)^{2-\frac{\nu}{2}} \tag{3.3}$$

for $t \in [0, T]$. This means that Z is a time transformed squared Bessel process of dimension ν. As we will see in what follows, structuring the model equations in this form has the advantage that the reduced one-factor formulation with a deterministic γ_t can capture via the dimension ν different degrees of slope in the implied volatility surface for European call and put options. Note that for $\nu = 4$ the scaling γ_t does not depend on Z_t.

Using the Itô formula together with (2.1), (3.1) and (3.2), the GOP $S_t^{(\delta_*)}$ can be shown to satisfy the SDE

$$dS_t^{(\delta_*)} = S_t^{(\delta_*)}\left(\left[r_t + \left(\frac{\nu}{2} - 1\right)^2 \gamma_t \left(\frac{S_t^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{2}{2-\nu}}\right] dt + \left(\frac{\nu}{2} - 1\right)\sqrt{\gamma_t}\left(\frac{S_t^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{1}{2-\nu}} d\hat{W}_t\right) \tag{3.4}$$

for $t \in [0, T]$. The GOP volatility or total market price for risk is therefore by (2.12) of the form

$$|\theta_t| = \left(\frac{\nu}{2} - 1\right)\sqrt{\gamma_t}\left(\frac{S_t^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{1}{2-\nu}} \tag{3.5}$$

for $t \in [0, T]$. This means that the index volatility is stochastic and depends at time t on both the level of the index $S_t^{(\delta_*)}$ and the scaling γ_t.

Furthermore, the discounted GOP drift is by (2.14) and (3.5) given by

$$\alpha_t = \left(\frac{\nu}{2} - 1\right)^2 \gamma_t \left(\bar{S}_t^{(\delta_*)}\right)^{\frac{2}{2-\nu}+1} \tag{3.6}$$

for $t \in [0, T]$.

3.2. RANDOM SCALING

Motivated by empirical findings in Breymann et al. (2004) on the observable scaling or market activity, let us assume that the scaling process $\gamma = \{\gamma_t,\ t \in [0, T]\}$ is a nonnegative, adapted stochastic process that satisfies the SDE

$$d\gamma_t = a(t, \gamma_t)\,dt + b(t, \gamma_t)\left(\varrho_t\,d\hat{W}_t + \sqrt{1 - \varrho_t^2}\,d\tilde{W}_t\right) \tag{3.7}$$

for $t \in [0, T]$ with a random initial value $\gamma_0 > 0$. Here $\tilde{W}$ is a Wiener process that models trading activity and is independent from $W^1, \ldots, W^d$ and therefore also $\hat{W}$. The scaling correlation $\varrho = \{\varrho_t,\ t \in [0, T]\}$ is assumed to be a given deterministic function of time. The scaling drift function $a(\cdot, \cdot)$ and scaling diffusion function $b(\cdot, \cdot)$ are given functions of time t and scaling level γ_t. This formulation for the dynamics of the random scaling as a diffusion process is chosen to provide the freedom to match empirical evidence (see Breymann et al., 2004).

For this two-factor model the main stochastic volatility effect is produced by the time transformed squared Bessel process Z of dimension ν, see (3.1) and (3.2). However, derivative security prices and corresponding implied volatilities are also influenced by the scaling process γ_t and its random initial value $\gamma_0 > 0$. The effects from random scaling, which captures random trading activity, are seen mainly in short term derivative prices. As we will see below for European call and put options, increased stochastic scaling increases the curvature of the implied volatility surface for short dated options. As shown in Breymann et al. (2004), the scaling models short term fluctuations in the volatility of the GOP dynamics.

To be specific, let us provide an example that matches closely the intraday empirical results obtained in Breymann et al. (2004) for the denomination of the GOP in US Dollars. There it was found that the scaling can be modeled by a product of the type

$$\gamma_t = \xi_t\, m_t \tag{3.8}$$

with

$$\xi_t = \xi_0 \exp\left\{\int_0^t \eta_s\, ds\right\} \tag{3.9}$$

for $t \in [0, T]$, where $\xi_0 > 0$ is some constant. The parameter process $\eta = \{\eta_t,\ t \in [0, T]\}$ is called the *net market growth rate* process and reflects the average long term growth rate of the discounted GOP. Here, for simplicity, it is assumed to be deterministic. The *market activity process* $m = \{m_t,\ t \in [0, T]\}$ is designed to model normalized trading activity. The market activity process is assumed to be a nonnegative ergodic process that satisfies the SDE

$$dm_t = A(m_t)\, \beta_t^2\, dt + \beta_t\, m_t \big(\varrho_t\, d\hat{W}_t + \sqrt{1 - \varrho_t^2}\, d\tilde{W}_t\big) \tag{3.10}$$

for $t \in [0, T]$ with $m_0 \geq 0$. Empirical support for such a model is presented in Breymann et al. (2004). In this SDE we have used multiplicative noise, where β_t is the deterministic time dependent activity volatility, which also appears in the drift of (3.10). The function $A(\cdot)$ controls the feedback behavior or mean reversion of market activity. Breymann et al. (2004) find that a good choice for this function is of the form

$$A(m) = \left(\frac{p}{2} - \frac{g}{2}\, m\right) m \tag{3.11}$$

with speed of adjustment parameter g and reference level p. These parameters are set so that the expected value of market activity is about one. This market activity process has a stationary density of the form

$$p_m(y) = \frac{g^{p-1}}{\Gamma(p-1)}\, y^{p-2} \exp\{-g\, y\} \tag{3.12}$$

for $y \in [0, \infty)$, where $\Gamma(\cdot)$ is the gamma function. This is a gamma density with mean $\frac{p-1}{g}$ and variance $\frac{1}{g}$ for parameters $p > 1$ and $g > 0$.

Figure 1 shows the stationary density (3.12) for different values of y and g with $p = g + 1$ to ensure that the mean of the displayed family of stationary densities always equals one.

Applying the Itô formula and using (3.8)–(3.10), the drift and diffusion functions $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ appearing in (3.7) take the form

$$a(t, \gamma) = \xi_t\, \beta_t^2\, A\left(\frac{\gamma}{\xi_t}\right) + \gamma\, \eta_t \tag{3.13}$$

and

$$b(t, \gamma) = \beta_t\, \gamma, \tag{3.14}$$

respectively, for $t \in [0, T]$.

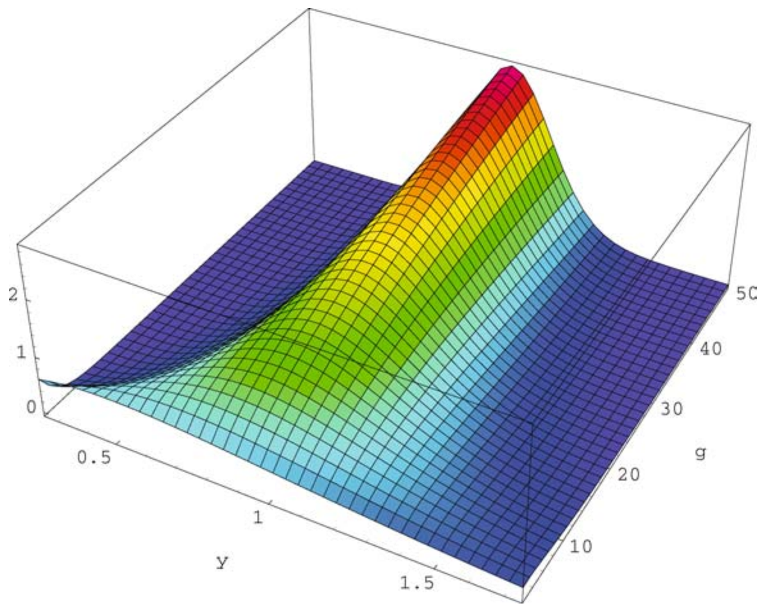

Figure 1. Stationary density as a function of market activity y and speed of adjustment parameter g.

Note that at any time $t \in [0, T]$ the actual value of the market activity m_t, and thus the random scaling γ_t, cannot be directly observed, since it changes very rapidly and can only be estimated after sufficient time has passed. However, its stationary density can be observed (see Breymann et al., 2004). Therefore, the initial value m_0 of the market activity is modeled as a random variable with the stationary density (3.12) as its probability density.

4. Derivative Pricing

4.1. ZERO COUPON BONDS

In Platen (2004) it has been shown under general assumptions that any globally diversified portfolio approximates for increasing number of securities the GOP. A globally diversified portfolio is one where the fractions invested in each primary security decline sufficiently fast as the number of securities increases. Based on these results we will henceforth use the GOP as a proxy for the index. To study index derivatives we consider first a zero coupon bond that pays one unit of the domestic currency at the maturity date $\bar{T} \in [0, T]$. In a risk neutral framework one would not consider this to be an index derivative. However, we will see that since an equivalent risk neutral martingale measure does not exist, a zero coupon bond turns out to be a genuine index derivative. More precisely, according to (2.30) and by using (3.1) the fair benchmarked price $\hat{P}_{\bar{T}}(t, Z_t, \gamma_t)$ for a zero coupon bond at time t with maturity $\bar{T}$ is given by the conditional expectation

$$\hat{P}_{\bar{T}}(t, Z_t, \gamma_t) = E\left(\frac{1}{S_{\bar{T}}^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{1}{S_{\bar{T}}^{(0)}(Z_{\bar{T}})^{\frac{\nu}{2}-1}} \,\middle|\, \mathcal{A}_t\right) \tag{4.1}$$

for $t \in [0, \bar{T}]$. Hence the corresponding price $P_{\bar{T}}(t, Z_t, \gamma_t)$ in domestic currency is

$$P_{\bar{T}}(t, Z_t, \gamma_t) = S_t^{(\delta_*)}\, \hat{P}_{\bar{T}}(t, Z_t, \gamma_t) = S_t^{(0)}\,(Z_t)^{\frac{\nu}{2}-1}\, \hat{P}_{\bar{T}}(t, Z_t, \gamma_t) \tag{4.2}$$

for $t \in [0, \bar{T}]$. Let us introduce the diffusion operator $\mathcal{L}^0$, which when applied to a sufficiently smooth function $f : (0, \bar{T}) \times (0, \infty)^2 \to \Re$ gives

$$\begin{aligned}\mathcal{L}^0 f(t, Z, \gamma) = \Bigg[& \frac{\partial}{\partial t} + \frac{\nu\gamma}{4}\frac{\partial}{\partial Z} + a(t, \gamma)\frac{\partial}{\partial \gamma} + \frac{1}{2}\gamma Z \frac{\partial^2}{\partial Z^2} \\ & + \varrho_t b(t, \gamma)\gamma^{\frac{1}{2}} Z^{\frac{1}{2}} \frac{\partial^2}{\partial Z \partial \gamma} + \frac{1}{2} b(t, \gamma)^2 \gamma \frac{\partial^2}{\partial \gamma^2}\Bigg] f(t, Z, \gamma)\end{aligned} \tag{4.3}$$

for $(t, Z, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$. Using (3.2) and (3.7), the benchmarked pricing function $\hat{P}_{\bar{T}}(\cdot, \cdot, \cdot)$ satisfies the Kolmogorov backward equation

$$\mathcal{L}^0\, \hat{P}_{\bar{T}}(t, Z, \gamma) = 0 \tag{4.4}$$

for $(t, Z, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$ with boundary condition

$$\hat{P}_{\bar{T}}(\bar{T}, Z, \gamma) = \frac{1}{S_{\bar{T}}^{(0)}\, Z^{\frac{\nu}{2}-1}} \tag{4.5}$$

for $(Z, \gamma) \in (0, \infty)^2$.

For the above two-factor model define the forward rate for maturity $\bar{T} \in [0, T]$ at time $t \in [0, \bar{T}]$ by the formula

$$f_{\bar{T}}(t, Z_t, \gamma_t) = -\frac{\partial}{\partial \bar{T}} \ln\left(P_{\bar{T}}(t, Z_t, \gamma_t)\right). \tag{4.6}$$

Figure 2 shows the forward rate surface at time $t = 0$ as a function of $Z_0 \in [50, 150]$ and $\bar{T} \in [0.25, 10]$. For this and subsequent plots the default parameter values used were: $\nu = 4$, $r_t = 0.05$, $\varrho_t = 0$, $\eta_t = 0.048$, $\xi = 10$, $p = 3$ and $g = 2$. Note that the forward rates are not constant and are always greater than the short rate. These results together with those described in the remaining part of this paper were obtained using PDE finite difference methods. It turns out that the initial density for m_0 can be conveniently approximated by a two-point distributed random variable with mean $\frac{p-1}{g}$ and variance $\frac{1}{g}$. The fact that the realistically hump shaped forward

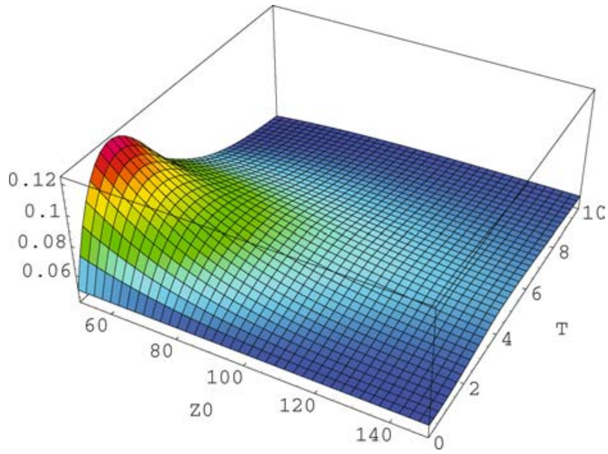

Figure 2. Forward rates as a function of Z_0 and $\bar{T}$.

rates, shown in Figure 2, are greater than the short rate demonstrates that the process $\hat{S}^{(0)}$, see (3.3), is a strict $(\underline{\mathcal{A}}, P)$-local martingale (see Revuz and Yor, 1999). It is well-known that in the complete market case, when using the GOP as numeraire, Λ characterizes the unique candidate risk neutral measure. This means that the Radon-Nikodym derivative process $\Lambda = \{\Lambda_t = \frac{\hat{S}^{(0)}_t}{\hat{S}^{(0)}_0},\ t \in [0, T]\}$ for such candidate risk neutral measure Q, where $\frac{dQ}{dP}\,|_{\mathcal{A}_t} = \Lambda_t$, is *not* an $(\underline{\mathcal{A}}, P)$-martingale. Consequently, for the proposed model there does not exist an equivalent risk neutral martingale measure. As is shown in Platen (2004), empirical evidence suggests that the inverse of the benchmarked savings account, which is the Radon-Nikodym derivative, is in reality not a true martingale and only a strict supermartingale. This forces us to consider models of the above kind where no equivalent risk neutral martingale measure exists. For additional commentary on these and related issues we refer to Platen (2002).

4.2. EUROPEAN OPTIONS

Consider a European put option on the index $S^{(\delta_*)}$ with strike K and maturity date $\bar{T} \in [0, T]$. Using the fair pricing formula (2.31), the option price $p_{\bar{T},K}(t, Z_t, \gamma_t)$ is given by

$$p_{\bar{T},K}(t, Z_t, \gamma_t) = S^{(0)}_t \,(Z_t)^{\frac{\nu}{2}-1}\, E\left(\left(\frac{K}{S^{(0)}_{\bar{T}}\, Z_{\bar{T}}^{\frac{\nu}{2}-1}} - 1\right)^+ \,\middle|\, \mathcal{A}_t\right) \tag{4.7}$$

for $t \in [0, \bar{T}]$, in terms of the domestic currency.

To see the effect on implied volatilities, Figure 3 displays a term structure of implied volatilities for European puts as a function of the maturity date $\bar{T}$ and the strike K. These results were obtained using the fair zero coupon bond price, see

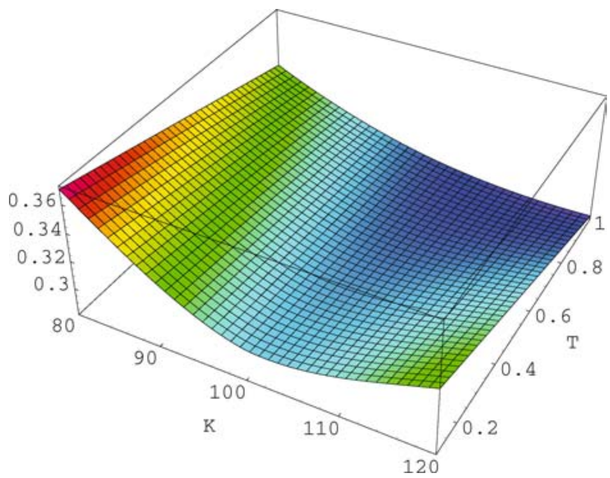

Figure 3. Implied volatilities for put options as a function of K and $\bar{T}$.

(4.2), to infer the discount factor used in the Black-Scholes formula. The implied volatilities shown in Figure 3 are very close to those observed for European index options in real markets, see, for example (Cont and da Fonseca, 2002). Note that the curvature arises from the random scaling for short dated options. More precisely, this curvature is mainly generated by the random initial value m_0 of the market activity process. If a fixed initial value m_0 were used, then much of the curvature for the short dated implied volatility surface would disappear.

It is well-known that smile and skew patterns for implied volatility surfaces, as shown in Figure 3, can be obtained by various models (see Carr and Wu, 2003; Brigo et al., 2004). However, most of these models have problems to calibrate with the same parameters exotic derivatives. It will be demonstrated in Section 4.3 that our parsimonious model does not have such problems. Furthermore, the real world dynamics of an index under our proposed model is close to empirically observed index dynamics, which has been documented in Breymann et al. (2004).

Our formulation of the MMM with random scaling allows for the Wiener processes driving the components Z_t and γ_t to be correlated. Figure 4 shows implied volatilities for European put index options as a function of the strike K and constant correlation $\varrho_t = \varrho$ for a fixed maturity date $\bar{T} = 0.25$. It can be seen that by increasing ϱ the slope of the implied volatility curve, as a function of the strike K, also increases. A strong negative correlation produces a strongly negatively skewed implied volatility curve, whereas a strong positive correlation generates a strongly positively skewed implied volatility curve. In this context it should be noted that changing the dimension ν also affects the slope of the implied volatility surface, similarly to Figure 4, when plotting the surface against ν instead of ϱ. That is, for fixed ϱ, lowering the dimension ν produces a stronger negative skew for the implied volatility surface.

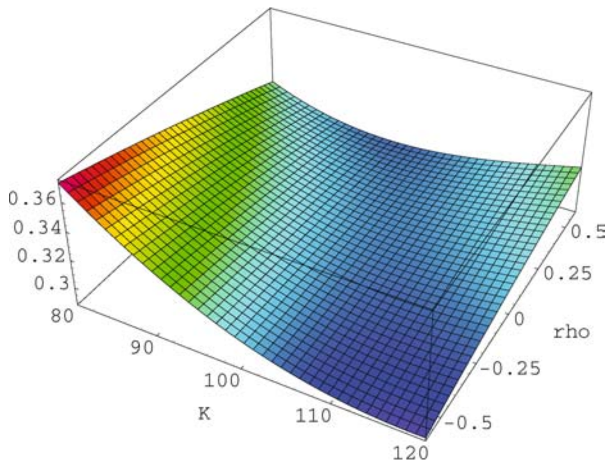

Figure 4. Implied volatilities for put options as a function of K and ϱ.

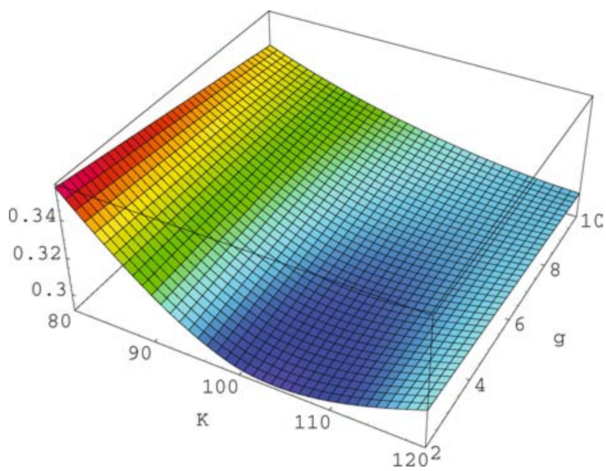

Figure 5. Implied volatilities for call options as a function of K and g.

To demonstrate the effect of making the scaling process γ stochastic, Figure 5 shows implied volatilities for European puts as a function of the strike K and the speed of adjustment parameter g for a fixed maturity date $\bar{T} = 0.25$ and with $p = g + 1$. The figure indicates that an increase in g decreases the curvature of the implied volatility curve, viewed as a function of the strike K. For different values of g the corresponding initial random density for m_0 is also adjusted to match the mean and variance of the corresponding stationary distribution.

Figure 6 shows the term structure of implied volatilities for long dated put options with maturities ranging between one and ten years. Note that a remarkably sustained increase in overall implied volatilities occurs for longer maturities. This

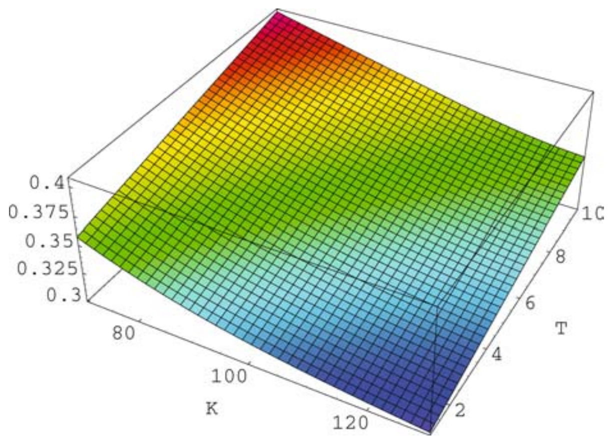

Figure 6. Implied volatilities for long term put options as a function of K and $\bar{T}$.

is not usually obtained from a stochastic volatility model with constant parameters, where an equivalent risk neutral martingale measure exists. Furthermore, it can be observed that the impact of using random scaling, which is mainly reflected in the curvature of implied volatilities for short dated options, is not so prominent for longer maturities.

The fair pricing formula (2.31) can also be used to compute the fair price of a European call option. Thus, for a European call on the index with strike K and maturity date $\bar{T} \in [0, T]$ the fair price in domestic currency, see (2.1), takes the form

$$c_{\bar{T},K}(t, Z_t, \gamma_t) = S_t^{(0)} (Z_t)^{\frac{\nu}{2}-1} E\left(\left(1 - \frac{K}{S_{\bar{T}}^{(0)} (Z_{\bar{T}})^{\frac{\nu}{2}-1}}\right)^+ \Bigg| \mathcal{A}_t\right) \tag{4.8}$$

for $t \in [0, \bar{T}]$. Using (4.2), (4.7) and (4.8), the put-call parity relation within the current framework takes the form

$$c_{\bar{T},K}(t, Z_t, \gamma_t) = p_{\bar{T},K}(t, Z_t, \gamma_t) + S_t^{(0)} (Z_t)^{\frac{\nu}{2}-1} - K\, P_{\bar{T}}(t, Z_t, \gamma_t) \tag{4.9}$$

for $t \in [0, \bar{T}]$. By using this result it is evident that if the fair price of a zero coupon bond is used as the discount factor in the Black-Scholes formula, then the same implied volatilities will be returned for both European put and call options.

4.3. BINARY OPTIONS

As an example of an important class of path-dependent contingent claims we consider the pricing of up-and-out binary options on the index under the proposed MMM with random scaling.

For a maturity date $\bar{T} \in (0, T]$ and level $U > S_t^{(\delta_*)}$ let τ^U be the stopping time given by

$$\tau^U = \inf\left\{t \geq 0 : \left(t, S_t^{(\delta_*)}\right) \notin [0, \bar{T}) \times (0, U)\right\}. \tag{4.10}$$

Using the benchmarked fair pricing formula (2.31), the benchmarked fair price for an up-and-out binary option with level U is then

$$\widehat{\mathrm{bin}}_{\bar{T},U}(t, Z_t, \gamma_t) = E\left(\frac{\mathbf{1}_{\{\tau^U = \bar{T}\}}}{S_{\bar{T}}^{(\delta_*)}} \,\middle|\, \mathcal{A}_t\right) \tag{4.11}$$

for $t \in [0, \bar{T}]$.

Using (3.4) and (3.7), the benchmarked fair pricing function $\widehat{\mathrm{bin}}_{\bar{T},U}(t, S^{(\delta_*)}, \gamma)$ satisfies the PDE

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + S^{(\delta_*)}\left(r_t + \left(\frac{\nu}{2} - 1\right)^2 \gamma \left(\frac{S^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{2}{2-\nu}}\right)\frac{\partial}{\partial S^{(\delta_*)}}\right.\\
&\quad + a(t,\gamma)\frac{\partial}{\partial \gamma} + \frac{1}{2}\left(\frac{\nu}{2} - 1\right)^2 \gamma \left(S^{(\delta_*)}\right)^2 \left(\frac{S^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{2}{2-\nu}} \frac{\partial^2}{\partial (S^{(\delta_*)})^2}\\
&\quad + \left(\frac{\nu}{2} - 1\right) b(t,\gamma) \varrho_t \gamma^{\frac{1}{2}}\, S^{(\delta_*)} \left(\frac{S^{(\delta_*)}}{S_t^{(0)}}\right)^{\frac{1}{2-\nu}} \frac{\partial^2}{\partial S^{(\delta_*)}\,\partial \gamma}\\
&\quad \left. + \frac{1}{2}\,(b(t,\gamma))^2 \frac{\partial^2}{\partial \gamma^2}\right] \widehat{\mathrm{bin}}_{\bar{T},U}\left(t, S^{(\delta_*)}, \gamma\right) = 0
\end{aligned} \tag{4.12}$$

for $(t, S^{(\delta_*)}, \gamma) \in (0, \bar{T}) \times (0, \infty)^2$ with boundary conditions

$$\widehat{\mathrm{bin}}_{\bar{T},U}\left(\bar{T}, S^{(\delta_*)}, \gamma\right) = \frac{1}{S^{(\delta_*)}} \tag{4.13}$$

for $(S^{(\delta_*)}, \gamma) \in (0, \infty)^2$ and

$$\widehat{\mathrm{bin}}_{\bar{T},U}\,(t, U, \gamma) = 0 \tag{4.14}$$

for $(t, \gamma) \in (0, \bar{T}) \times (0, \infty)$. The corresponding Black-Scholes up-and-out binary option price formula is given by

$$\mathrm{bin}^{\mathrm{BS}}_{\bar{T},U}\left(t, S_t^{(\delta_*)}, \sigma\right) = e^{-r(\bar{T}-t)}\left(N(-d_2(t)) - \left(\frac{U}{S_t^{(\delta_*)}}\right)^{\frac{2r}{\sigma^2}-1} N(-d_2(t))\right), \tag{4.15}$$

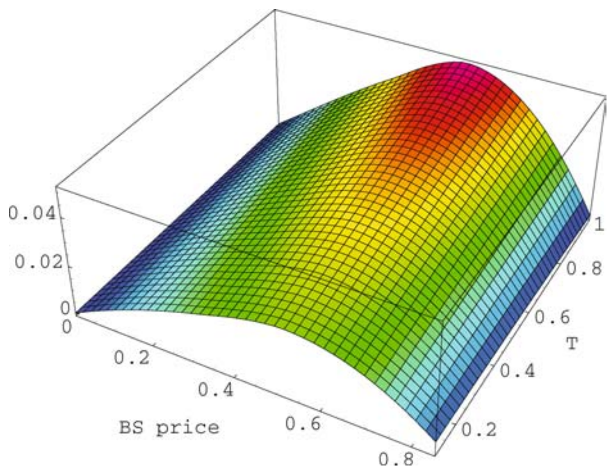

Figure 7. Differences between MMM and Black-Scholes prices for up-and-out binary options for different maturity values $\bar{T}$.

where

$$d_2(t) = \frac{\ln\left(\frac{S_t^{(\delta_*)}}{U}\right) + \left(r - \frac{\sigma^2}{2}\right)(\bar{T} - t)}{\sigma\sqrt{\bar{T} - t}}$$

for $t \in [0, \bar{T}]$ (see Rubinstein and Reiner, 1991). Here $N(\cdot)$ is the standard Gaussian distribution function and σ denotes the Black-Scholes volatility.

For the binary option example described here, the corresponding Black-Scholes binary option prices are obtained by choosing implied volatilities such that the prices of an at-the-money forward European put option for the MMM with random scaling and the Black-Scholes model coincide. Here an at-the-money forward European put option is one that for $t = 0$ has a strike $\frac{K}{P_{\bar{T}}(0, Z_0, \gamma_0)}$ for fixed $K = 1.0$.

Different Black-Scholes binary option prices are obtained by varying the level U and the maturity date $\bar{T}$ for fixed model parameters and initial value $S_0^{(\delta_*)} = 100$ and subject to the constraint $U > S_0^{(\delta_*)}$. Figure 7 displays price differences obtained for up-and-out binary options as a function of the Black-Scholes price and maturity date. It can be seen that price differences of the order of 3–5% are obtained as the maturity date increases from one month to one year. By increasing the value of the speed of adjustment parameter g these price differences decrease. The incorporation of a random initial condition for m_0 ensures that these price differences are maintained for short dated options. The presented price differences are very similar to those observed in real markets.

Consider now the impact of non-zero correlation between the index $S_t^{(\delta_*)}$ and the scaling factor γ_t. Figure 8 shows the price differences obtained between the MMM with random scaling and corresponding prices for Black-Scholes up-and-out binary options, as a function of the Black-Scholes binary option price and a

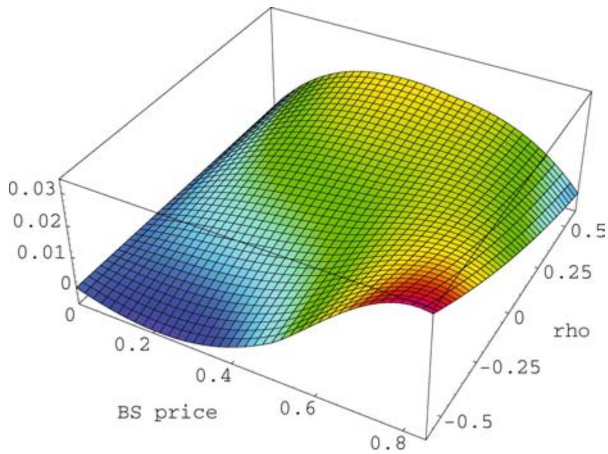

Figure 8. Differences between MMM and Black-Scholes prices for up-and-out binary options for different correlation values ϱ.

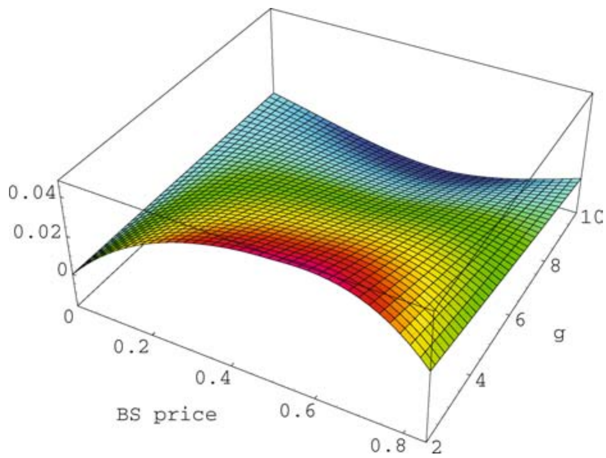

Figure 9. Differences between MMM and Black-Scholes prices for up-and-out binaries for different speed of adjustment values g.

constant correlation $\varrho_t = \varrho$. These results were produced using a fixed maturity date $\bar{T} = 0.25$ and speed of adjustment parameter $g = 2$.

Finally, we display in Figure 9 price differences between the MMM with random scaling and corresponding Black-Scholes prices for up-and-out binary options, as a function of the Black-Scholes binary option price and the speed of adjustment parameter g. We also keep the maturity date fixed at $\bar{T} = 0.25$ and maintain a constant zero correlation. Inspection of Figure 9 demonstrates that, in general, increasing the parameter g decreases the corresponding price differences. For $g = 10$ the hump-shaped price difference curve becomes inverted to some degree.

We point out that hedging can be performed under the given model if three securities are involved in a hedge portfolio. The corresponding PDE operator for the pricing function is shown in (4.3). By using the fair pricing formula (2.31) one can directly obtain via the Itô formula a martingale representation for a given contingent claim. The integrands in the martingale representation yield in a straightforward manner the hedge ratios. For more details we refer to Platen (2002).

Conclusion

The minimal market model with random scaling can be used to price a range of short and long dated European and path-dependent contingent claims on a diversified index. The pricing system applied is based on the benchmark approach for which the reference unit chosen is the growth optimal portfolio. This pricing methodology is more general than risk neutral pricing. In fact, for the model under consideration there exists no equivalent risk neutral martingale measure. Numerical results are presented which document the type of implied volatility term structures that are obtained. Of particular interest are the resulting implied volatilities for long term put and call options. Price differences between the proposed model and corresponding Black-Scholes prices, for a class of up-and-out binary options, are similar to those actually observed. This means that the proposed two-factor model naturally generates patterns typically observed in real markets for forward rates, standard European options and binary options.

References

Bakshi, G., Cao, C. and Chen, Z. (1997), Empirical performance of alternative option pricing models, *J. Finance* **LII**, 2003–2049.

Balland, P. (2002), Deterministic implied volatility surfaces, *Quant. Finance.* **2**, 31–44.

Barndorff-Nielsen, O. and Shephard, N. (2002), Economic analysis of realised volatility and its use in estimating stochastic volatility models, *J. Roy. Statist. Soc. Ser.* B **64**(2), 253–280.

Black, F. (1976), Studies in stock price volatility changes. In *Proceedings of the 1976 Business Meeting of the Business and Economic Statistics Section, American Statistical Association*, pp. 177–181.

Breymann, W., Kelly, L. and Platen, E. (2004), Intraday empirical analysis and modeling of diversified world stock indices. Technical report, University of Technology, Sydney. QFRC Research Paper 125.

Brigo, D. and Mercurio, F. (2001), *Interest Rate Models - Theory and Practice*. Springer Finance.

Brigo, D., Mercurio, F. and Rapisarda (2004), Smile at the uncertainty, *Risk* **17**(5), 97–101.

Carr, P. and Wu, L. (2003), Finite moment log stable processes and option pricing, *J. Finance* **58**, 753–777.

Cont, R. and da Fonseca, J. (2002), Dynamics of implied volatility surfaces, *Quant. Finance.* **2**, 45–60.

Das, S. R. and Sundaram, R. K. (1999), Of smiles and smirks: A term-structure perspective. **34**, 211–240.

Derman, E. and Kani, I. (1994), The volatility smile and its implied tree, *Goldman Sachs Quantitative Strategies Research Notes*.

Dumas, B., Fleming, J. and Whaley, R. (1997), Implied volatility functions: Empirical tests, *J. Finance* **53**, 2059–2106.

Dupire, B. (1992), Arbitrage pricing with stochastic volatility. In *Proceedings of AFFI Conference, Paris*.
Dupire, B. (1993), Model art, *Risk* **6**, 118–124.
Dupire, B. (1994), Pricing with a smile, *Risk* **7**, 18–20.
Fouque, J. P., Papanicolau, G. and Sircar, K. R. (2000), *Derivatives in Markets with Stochastic Volatility*. Cambridge University Press.
Franks, J. R. and Schwartz, E. J. (1991), The stochastic behaviour of market variance implied in the price of index options, *Econometrica* **101**, 1460–1475.
Heath, D., Hurst, S. R. and Platen, E. (2001), Modelling the stochastic dynamics of volatility for equity indices, *Asia-Pacific Financial Markets* **8**, 179–195.
Heath, D. and Platen, E. (2004), Local volatility function models under a benchmark approach. Technical report, University of Technology, Sydney. QFRC Research Paper 124, 2004.
Heynen, R. (1993), An empirical investigation of observed smile patterns, *Rev. Futures Markets* **13**, 317–353.
Heynen, R., Kemna, A. and Vorst, T. (1994), Analysis of the term structure of implied volatilities, *J. Financial and Quantitative Analysis* **29**, 31–56.
Karatzas, I. and Shreve, S. E. (1991), *Brownian Motion and Stochastic Calculus* (2nd ed.). Springer.
Karatzas, I. and Shreve, S. E. (1998), *Methods of Mathematical Finance*, Volume 39 of *Appl. Math.* Springer.
Kelly, J. R. (1956), A new interpretation of information rate, *Bell Syst. Techn. J.* **35**, 917–926.
Kou, S. G. (2002), A jump diffusion model for option pricing, *Management Science* **48**, 1086–1101.
Lewis, A. L. (2000), *Option Valuation Under Stochastic Volatility*. Finance Press, Newport Beach.
Long, J. B. (1990), The numeraire portfolio, *J. Financial Economics* **26**, 29–69.
Musiela, M. and Rutkowski, M. (1997), *Martingale Methods in Financial Modelling. Theory and Applications*, Volume 36 of *Appl. Math.* Springer.
Platen, E. (2001), A minimal financial market model. In *Trends in Mathematics*, pp. 293–301. Birkhäuser.
Platen, E. (2002), Arbitrage in continuous complete markets, *Adv. in Appl. Probab.* **34**(3), 540–558.
Platen, E. (2004), Modeling the volatility and expected value of a diversified world index, *Int. J. Theor. Appl. Finance* **7**(4), 511–529.
Protter, P. (1990), *Stochastic Integration and Differential Equations*. Springer.
Rebonato, R. (1999), *Volatility and Correlation in the Pricing of Equity, FX and Interest Rate Options*. New York: Wiley.
Renault, E. and Touzi, N. (1996), Option hedging and implied volatilities in a stochastic volatility model, *Math. Finance* **6**, 279–302.
Revuz, D. and Yor, M. (1999), *Continuous Martingales and Brownian Motion* (3rd ed.). Springer.
Rosenberg, J. V. (2000), Implied volatility functions: A reprise, *J. Derivatives* **7**.
Rubinstein, M. (1985), Nonparametric tests of alternative option pricing models using all reported trades and quotes on the 30 most active CBOE option classes from August 23, 1976 through August 31, 1978, *J. Finance* **11**, 455–480.
Rubinstein, M. (2000), Comments on the 1987 stock market crash: Eleven years later. In *Risks in Investment Accumulation Products of Financial Institutions*. Society of Actuaries. Chapter 2.
Rubinstein, M. and Reiner, E. (1991), Unscrambling the binary code, *Risk* **4**(8), 75–83.
Schönbucher, P. J. (1999), A market model for stochastic implied volatility, *Philos. Trans. Roy. Soc. London Ser.* A **357**, 2071–2092.
Skiadopolous, G., Hodges, S. and Clewlow, L. (2000), Dynamics of the S&P500 implied volatility surface, *Rev. Derivatives Res.* **2**, 263–282.
Tompkins, R. (2001), Stock index futures markets: Stochastic volatility models and smiles, *J. Futures Markets* **21**, 4378.