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Free Algebras of Hilbert Automorphic Forms

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Abstract. Let $d > 0$ be a square-free integer, and let L_d be the corresponding Hilbert lattice. Suppose given a finite-index subgroup Γ of $O^+(L_d)$ generated by reflections and containing $-\,\mathrm{id}$ and let $A(\Gamma)$ be the algebra of Γ-automorphic forms. It is proved that if the algebra $A(\Gamma)$ is free, then $d \in \{2, 3, 5, 6, 13, 21\}$.

Key words: automorphic form.

1. Basic Definitions

Let A be a ring of principal ideals. A *quadratic A-module* is a free A-module of finite rank equipped with a nondegenerate symmetric bilinear form $(\,\cdot\,,\,\cdot\,)$ taking values in the ring A. The module A^n, $n \in \mathbb{N}$, in which inner product is specified by a Gram matrix S is denoted by (S). A quadratic module over a field is called a *quadratic* (*vector*) *space* and over the ring $\mathbb{Z}$, a *lattice*.

Let V be a quadratic space over the field of rational numbers. Then $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ and $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ are quadratic spaces over the fields $\mathbb{R}$ and $\mathbb{C}$, respectively. Suppose that the signature of the quadratic space $V_{\mathbb{R}}$ equals $(2, n)$, where $n \geqslant 2$.

In the projective space $PV_{\mathbb{C}}$ consider the domain

$$\widetilde{D}_n = \{[z] \in PV_{\mathbb{C}} : (z, z) = 0,\ (z, \bar{z}) > 0\},$$

which consists of two connected components. Let D_n denote any of these components. Its complex dimension equals n. The domain D_n is a Hermitian symmetric space of type IV (in E. Cartan's classification). The domain D_1 is biholomorphic to the upper half-plane H_+, and D_2 is biholomorphic to the product of two such half-planes.

Let $G = O(V_{\mathbb{R}})$ denote the orthogonal group of the quadratic space $V_{\mathbb{R}}$, and let G^+ be its subgroup of index 2 preserving the domain D_n. The group G^+ acts on D_n ineffectively with ineffectiveness kernel ± 1. As is known [7], G^+ is the full group of holomorphic automorphisms of the domain D_n, and its action on D_n is transitive.

1.1. An arithmetic group $\Gamma \subset G^+$ acting discretely on D_n. Choose a lattice L with $\mathrm{rk}_{\mathbb{Z}} L = \dim_{\mathbb{Q}} V$ in the space V. Let $O(L)$ denote the orthogonal group of the lattice $L \subset V$, and let $\Gamma = O(L) \cap G^+$. Such a group Γ is called an *arithmetic group.* It acts on the domain D_n discretely (but possibly ineffectively). The group Γ always has a finite-index normal subgroup Γ_1, which acts effectively. As is known, the volume of the quotient D_n/Γ_1 (with respect to any G^+-invariant volume form on D_n) is finite. This volume is called the *covolume* of the group Γ_1 (with respect to the chosen volume form on D_n) and denoted by $\mathrm{Covol}(\Gamma_1)$. The covolume of the group Γ is defined as $\mathrm{Covol}(\Gamma_1)/[\Gamma : \Gamma_1]$ (we denote it by $\mathrm{Covol}(\Gamma)$).

1.2. The algebra $A(\Gamma)$ of Γ-automorphic forms. Let $D_n^\bullet$ denote the cone over the domain D_n in the space $V_{\mathbb{C}}$. In what follows, it is assumed that if $n = 2$, then the Witt index of V is less than 2.

Definition 1.1. An *automorphic form of weight k with character $\chi\colon \Gamma \to \mathbb{C}^*$ with respect to the group* Γ is a holomorphic function f on $D_n^\bullet$ satisfying the following conditions:

(i) $f(tz) = t^{-k} f(z)$ for $t \in \mathbb{C}^*$;

(ii) $f(g(z)) = \chi(g) f(z)$ for $g \in \Gamma$.

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Remark 1.2. Appropriately choosing a nonzero section of the tautological bundle over the domain D_n, we can establish a one-to-one correspondence between the homogeneous Γ-invariant holomorphic functions on the cone $D_n^\bullet$ and the holomorphic functions f on D_n such that $f(g(z)) = a(g,z)\chi(g)f(z)$, where $g \in \Gamma$ and $a(g,z)$ is the automorphism factor for Γ.

It is known that if Γ is an arithmetic group, then the Γ-automorphic forms of all nonnegative weights with trivial character form a finitely generated graded algebra, which we denote by $A(\Gamma)$.

In what follows (unless otherwise specified) we use the notation introduced above.

2. Results

Suppose that $\dim_{\mathbb{Q}} V = 4$ (i.e., $n = 2$) and the space V is isotropic but has no two-dimensional isotropic subspaces, i.e., the Witt index of the space V equals 1.

We choose a square-free positive integer $d > 1$ and consider the lattice $L = L_d = U \oplus B_d$, where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B_d = \begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & (1-d)/2 \end{pmatrix} & \text{for } d \equiv 1 \pmod 4, \\ \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix} & \text{for } d \equiv 2, 3 \pmod 4). \end{cases}$$

We denote the group Γ constructed for the lattice L_d by Γ_d. It is not by accident that we have chosen this series of arithmetic subgroups. Recall that the real quadratic extension $\mathscr{K} = \mathbb{Q}(\sqrt{d})$ with ring of integers $\mathscr{O}_d$ is associated with the extended Hilbert modular group $\widetilde{PSL}(2, \mathscr{O}_d) = Gal(\mathscr{K}/\mathbb{Q}) \ltimes PSL(2, \mathscr{O}_d)$, which naturally acts on D_2 as a discrete arithmetic group. The group $\widetilde{PSL}(2, \mathscr{O}_d)$ is embedded in Γ_d [3]. As is known, Γ_d is a maximal discrete subgroup of G^+ (see Section 1.1) containing $\widetilde{PSL}(2, \mathscr{O}_d)$ [12].

At present, a number of examples of arithmetic groups Γ for domains D_n which have free algebras of automorphic forms have been constructed ([17], [14], [15], [23], [5]). In this paper we answer the following related question: What are values of d for which Γ_d can have a finite-index subgroup Γ' such that the algebra $A(\Gamma')$ is free (for $n = 2$, this means that $A(\Gamma')$ is an algebra of polynomials in three variables)?

Theorem 2.1. *Let $\Gamma' \subset \Gamma_d$ be a finite-index subgroup in Γ_d containing the element $-\mathrm{Id}$. If the algebra $A(\Gamma')$ is free, then $d \in \{2, 3, 5, 6, 13, 21\}$.*

Corollary 2.2. *The algebra $A(\Gamma_d)$ is free if and only if $d = 2$ or 5.*

Proof. In [4] it was shown, in particular, that, for the algebra $A(\Gamma)$ (see Section 1.2) to be free, it is necessary that the stabilizer Γ_v of each isotropic vector $v \in V$ in the group Γ be generated by reflections whose mirrors contain this vector. But, as communicated to the author by E. B. Vinberg, this is not so for the groups Γ_d with $d \in \{3, 6, 13, 21\}$.

Consider the cases $d = 2$ and $d = 5$. The group $\widetilde{PSL}(2, \mathscr{O}_d)$ is embedded in Γ_d but does not contain $-\mathrm{Id}$. Adjoining $-\mathrm{Id}$ to the subgroup $\widetilde{PSL}(2, \mathscr{O}_d)$, we obtain the entire group Γ_d. Indeed, the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{5})$ are one-class and contain a fundamental unit of norm -1; therefore, the group $\widetilde{PSL}(2, \mathscr{O}_d)$ is a maximal arithmetic group acting on the product of two upper planes [3]. But in [15] and [14] it was proved that the algebra of automorphic forms of even weight is free for the groups $\widetilde{PSL}(2, \mathscr{O}_d)$ with $d = 2$ and $d = 5$. □

3. Methods of Proof

3.1. Reflections. Each nonisotropic vector $e \in V_{\mathbb{R}}$ determines an orthogonal transformation r_e in the group $G = O(V_{\mathbb{R}})$, namely, the reflection in the hyperplane $e^\perp$, which acts on the vector space $V_{\mathbb{C}}$ by the rule $r_e(x) = x - \frac{2(x,e)}{(e,e)}e$. The reflection r_e preserves the domain D_n only if $(e,e) < 0$. In this case, the projectivization $[e^\perp \otimes \mathbb{C}]$ of the hyperplane $e^\perp \otimes \mathbb{C}$ (which is customarily called the *mirror* of the reflection) intersects D_n in a domain D_{n-1} embedded as a totally geodesic complex submanifold of codimension 1. We refer to this intersection as the *mirror* of the reflection r_e in the domain D_n and denote it by $\pi(r_e)$.

If a reflection r_e preserves a lattice L, then for the vector e we can take a primitive vector of this lattice. If, in addition, $(e,e) = -k$, $k \in \mathbb{N}$, then the vector e, which is determined up to sign, is called a k-root of the lattice L.

3.2. The Poincaré–Lelong–Bruinier formula. The complex quadric $Q_n = \{[z] \in PV_{\mathbb{C}} : (z,z) = 0\}$ contains a Cartan domain D_n and is a compact symmetric space dual to it.

Consider the restriction to the quadric Q_n of the tautological bundle over $PV_{\mathbb{C}}$. Let $\widetilde{\Omega}$ be a nondegenerate 2-form representing the first Chern class of this linear bundle over the quadric Q_n. The cohomology class of the restriction of the form $\widetilde{\Omega}$ to the domain D_n contains a G^+-invariant 2-form Ω. This form Ω can be written explicitly: the domain D_n can be realized as the future cylinder, i.e., D_n is isomorphic to a connected component of the domain $\{Z = X + iY \in \mathbb{C}^n \mid Q(Y) > 0\}$, where Q is a quadratic form of signature $(1, n-1)$; then $\Omega = -dd^c \log(Q(Y))$.

Let $\omega_n = \Omega^n$ and $\tilde{\omega}_n = \widetilde{\Omega}^n$ be the corresponding volume forms. It is easy to verify that the restriction of the form Ω^{n-1} to the totally geodesic complex hypersurface D_{n-1} is the volume form ω_{n-1}.

The ratio $\int_{D_n/\Gamma} \omega_n / \int_{Q_n} \tilde{\omega}_n$ is called the Mumford–Hirzebruch volume of the quotient D_n/Γ, or the Mumford–Hirzebruch covolume of the group Γ.

In what follows, we denote the Mumford–Hirzebruch covolume of the group Γ by $\operatorname{Covol}(\Gamma)$.

The main tool in the proof of Theorem 1 is the Poincaré–Lelong–Bruinier formula, which we write below in the generality needed for our purposes, using the notions and notation of Section 1.

In [10] this formula was obtained under the assumption that the group $P\Gamma$ acts freely on the domain D_n (for cocompact groups, this follows from the Poincaré–Lelong formula [21]). However, it can easily be generalized to the case of any group Γ (see, e.g., [20], where the generalized Poincaré–Lelong–Bruinier formula was essentially obtained in a special case). Namely, let F be an automorphic form (with a character), and let $\operatorname{div}(F)$ be a divisor of its zeros on the quotient D_n/Γ. We set $\operatorname{div}(F) = \sum_i m_i C_i$, where each C_i is an irreducible divisor. If the weight of the form F equals K, then

$$\sum_i \frac{m_i}{n_i} \int_{C_i} \Omega^{n-1} = K \int_{D_n/\Gamma} \Omega^n, \tag{1}$$

where each n_i is the ramification index over the divisor C_i in the ramified covering $D_n \to D_n/\Gamma$.

Suppose that the divisor of zeros of the form F on the domain D_n is the linear combination of mirrors π_i of reflections in Γ with coefficients m_i. We denote the stabilizer of the mirror π_i in Γ by Γ_{π_i}. In this special case, formula (1) can be rewritten in the following form convenient for calculations (this was proved, e.g., in [13]):

$$\sum_{i=1}^{k} m_i \operatorname{Covol}(\Gamma_{\pi_i}) = K \operatorname{Covol}(\Gamma).$$

The summation on the left-hand side is over the representatives $\{\pi_1, \dots, \pi_k\}$ of the Γ-equivalence classes of the mirrors of reflections in Γ. The following theorem, which plays an important role in our considerations, was proved in [9].

Theorem 3.1. *If the algebra $A(\Gamma)$ is free and $f_1, \dots, f_{n+1}$ are its generators of weights $k_1, \dots, k_{n+1}$, then there exists a unique* (*up to proportionality*) *Γ-automorphic form F of weight $n + \sum_{i=1}^{n+1} k_i$* (*with a character*) *such that the divisor of its zeros on the domain D_n is the sum of* *all mirrors of reflections in* Γ *with multiplicity* 1; *namely*,

$$F = \begin{vmatrix} k_1\tilde{f}_1 & k_2\tilde{f}_2 & \dots & k_{n+1}\tilde{f}_{n+1} \\ \partial\tilde{f}_1/\partial z_1 & \partial\tilde{f}_2/\partial z_1 & \dots & \partial\tilde{f}_{n+1}/\partial z_1 \\ \vdots & \vdots & \ddots & \vdots \\ \partial\tilde{f}_1/\partial z_n & \partial\tilde{f}_2/\partial z_n & \dots & \partial\tilde{f}_{n+1}/\partial z_n \end{vmatrix},$$

where the $\tilde{f}_i$ *are the images of the generators* f_i *under the bijection mentioned in Remark* 1.2 *and* $z_1, \dots, z_n$ *are coordinates in the domain* D_n.

Corollary 3.2. *If the group* Γ *contains* $-\mathrm{Id}$ *and the algebra* $A(\Gamma)$ *is free*, *then the weight* K *of the automorphic form* F *in Theorem* 3.1 *is at least* $3n+2$.

Proof. The algebra $A(\Gamma)$ contains no automorphic forms of odd weight. Indeed, if f is an automorphic form of odd weight, then condition (ii) in the definition of an automorphic form implies $f(-z) = f(z)$, and condition (i) implies $f(-z) = -f(z)$. Therefore, the algebra $A(\Gamma)$ has $n+1$ generators of even weight. Hence $K = n + \sum_{i=1}^{n+1} k_i \geqslant n + 2(n+1) = 3n+2$. □

If the algebra $A(\Gamma)$ is free, then, for the form F in the preceding theorem, the Poincaré–Lelong–Bruinier formula takes the form

$$\sum_{i=1}^{k} \mathrm{Covol}(\Gamma_{\pi_i}) = \left(n + \sum_{i=1}^{n+1} k_i\right) \mathrm{Covol}(\Gamma). \tag{2}$$

3.3. The number $K(\Gamma)$. Let $\Gamma' \subset \Gamma$ be a subgroup of finite index in the group Γ introduced in Section 1.1 (in particular, Γ' may coincide with Γ). With every such subgroup we associate the number

$$K(\Gamma') := \frac{\sum_{[\pi]} \mathrm{Covol}(\Gamma'_\pi)}{\mathrm{Covol}(\Gamma')}, \tag{3}$$

where Γ'_π is the stabilizer of the mirror π in Γ' and the summation in the numerator is over all Γ'-equivalence classes of the mirrors of all reflections in the group Γ'. Thus, if the algebra $A(\Gamma')$ is free and $k_1, \dots, k_{n+1}$ are the weights of its generators, then formula (2) takes the form

$$K(\Gamma') = n + \sum_{i=1}^{n+1} k_i.$$

O. V. Shvartsman obtained the following important result.

Lemma 3.3. $K(\Gamma) \geqslant K(\Gamma')$.

Proof. Let $\sum_{[\pi]}$ denote summation over the Γ-equivalence classes of the mirrors of all reflections in the group Γ, and let $\sum_{[\pi']}$ denote summation over the Γ'-equivalence classes of the mirrors of all reflections in Γ. Suppose that the orbit of a mirror π under the action of Γ is partitioned into $s(\pi)$ orbits under the action of Γ'. We denote representatives of these orbits by $\pi_1, \dots, \pi_{s(\pi)}$. Let us show that if all reflections in Γ belong to the group Γ', then $K(\Gamma') = K(\Gamma)$. Clearly, this implies the lemma. We have $\mathrm{Covol}(\Gamma') = \mathrm{Covol}(\Gamma) \cdot [\Gamma : \Gamma']$. Next,

$$\begin{aligned} \sum_{[\pi']} \mathrm{Covol}(\Gamma'_{\pi'}) &= \sum_{[\pi']} \mathrm{Covol}(\Gamma_{\pi'})[\Gamma_{\pi'} : \Gamma'_{\pi'}] = \sum_{[\pi]} \sum_{i=1}^{s(\pi)} \mathrm{Covol}(\Gamma_\pi)[\Gamma_{\pi_i} : \Gamma'_{\pi_i}] \\ &= \sum_{[\pi]} \mathrm{Covol}(\Gamma_\pi) \sum_{i=1}^{s(\pi)} [\Gamma_{\pi_i} : \Gamma'_{\pi_i}] = [\Gamma : \Gamma'] \sum_{[\pi]} \mathrm{Covol}(\Gamma_\pi). \end{aligned}$$

We have used a general lemma about actions of groups on sets, according to which $[\Gamma : \Gamma'] = \sum_{i=1}^{s(\pi)} [\Gamma_{\pi_i} : \Gamma'_{\pi_i}]$ (see, e.g., Theorem 5.2 in [1]). Comparing the numerators and denominators of the fractions $K(\Gamma)$ and $K(\Gamma')$, we obtain the required inequality. □

3.4. The strategy of the proof of Theorem 2.1. Applying Corollary 3.2 to the group Γ', we see that the weight K' of the automorphic form F' is at least 8 (in the case under consideration, $n = 2$). This means that $K(\Gamma_d) \geqslant K(\Gamma') = K' \geqslant 8$. Then, estimating the numerator of the fraction $K(\Gamma_d)$ from above and the denominator from below, we conclude that the denominator grows faster as a function of d. Therefore, for large d, we have $K(\Gamma_d) < 8$. By virtue of Corollary 3.2, the algebra $A(\Gamma')$ cannot be free for such d.

4. Proofs

4.1. Preparatory lemmas.

Kneser's Theorem [19]**.** *Suppose that a lattice L satisfies the following conditions*:

(i) $L \otimes \mathbb{R}$ *is an isotropic quadratic space of dimension higher than or equal to* 3;

(ii) *for each prime p, the quadratic module $L \otimes \mathbb{Z}_p$ has a multiple invariant factor. Then the class of the lattice L coincides with its genus.*

Let $d(L)$ denote the discriminant of the lattice L, and let $\epsilon_p(L)$ be the Hasse symbol of L over the ring $\mathbb{Z}_p$ of p-adic integers. An arbitrary lattice L admits a Jordan decomposition over the ring $\mathbb{Z}_p$: $L \otimes \mathbb{Z}_p = L^0 \oplus pL^1 \oplus p^2L^2 \oplus \dots$, where the L^i are unimodular $\mathbb{Z}_p$-modules.

Theorem 4.1. *If an even lattice L of signature $(2, 2)$ is such that $d(L) = d(L_d)$ and $\epsilon_p(L) = \epsilon_p(L_d)$ for all prime p, then it is isomorphic to the lattice L_d.*

Proof. The invariant factors of the lattice L are 1, 1, 1, and d if $d(L_d) \equiv 1 \pmod 4$ and 1, 1, 2, and $2d$ if $d(L_d) \equiv 0 \pmod 4$. This means that the class of the lattice L coincides with its genus (by Kneser's theorem). Therefore, it suffices to examine the local equivalence of the lattices L and L_d. For each odd prime p not dividing $d(L)$, the lattice $L \otimes \mathbb{Z}_p$ is unimodular and, therefore, equivalent to any other unimodular lattice with the same rank and discriminant. For any odd number p dividing $d(L)$, the lattice $L \otimes \mathbb{Z}_p$ has a three-dimensional unimodular component and a one-dimensional p-component in the Jordan decomposition. There exist precisely four different lattices over $\mathbb{Z}_p$ with these properties: $(1) \oplus (1) \oplus (1) \oplus (p)$, $(1) \oplus (1) \oplus (1) \oplus (rp)$, $(1) \oplus (1) \oplus (r) \oplus (p)$, and $(1) \oplus (1) \oplus (r) \oplus (rp)$, where r is a fixed quadratic nonresidue modulo p. For the first two lattices, we have $\epsilon_p(L) = 1$, and for the other two, $\epsilon_p(L) = -1$. Since the Hasse symbols are fixed, there remains only one of the two pairs of lattices. Since the lattices in the same pair have different discriminants, each pair contains only one lattice satisfying the required conditions.

It remains to check equivalence over $\mathbb{Z}_2$.

Let $d(L) \equiv 1 \pmod 4$. The lattice L is even; therefore, the lattice $L \otimes \mathbb{Z}_2$ has the form $U \oplus U$ or $U \oplus \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. The discriminants of these lattices differ modulo 8.

Let $d(L) \equiv 0 \pmod 4$. Since $d(L) = d(L_d)$, it follows that $d(L) = 4d$, where $d \equiv 3 \pmod 4$ or $d \equiv 2 \pmod 4$. Consider the former case. The Jordan decomposition of L has the form $L \otimes \mathbb{Z}_2 = L^0 \oplus 2L^1$, where $\operatorname{rk} L^0 = \operatorname{rk} L^1 = 2$. The lattice L^1 cannot be even, because if it were, then we would have $d \equiv 1 \pmod 4$. This means that $L \otimes \mathbb{Z}_2$ has the form $U \oplus (2u_1) \oplus (2u_2)$, where $u_1, u_2 \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$ and $u_1u_2 \equiv 1 \pmod 4$, because the lattice $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus (2u_1') \oplus (2u_2')$, where $u_1', u_2' \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$ and $u_1'u_2' \equiv 1 \pmod 4$, is obtained from the lattice under consideration by applying sign walking ([11], [8]). Considering all possible pairs (u_1, u_2) modulo 8, we see that the lattices under examination have different discriminants and Hasse symbols.

Consider the latter case. Let $d(L) = 8d'$, where d' is an odd number. The lattice $L \otimes \mathbb{Z}_2$ can be of the form $U \oplus (2u) \oplus (-4ud')$ or of the form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus (2u') \oplus (12u'd')$, where $u, u' \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$. The equivalence of these lattices is verified by sign walking. Considering all possible values of u modulo 8 for each fixed d' modulo 8, we obtain two lattices with different Hasse symbols. The result is presented in Table 1.

Table 1

d' (mod 8)	2-adic symbol of L	$\epsilon_2(L)$
1	$1_{\mathrm{II}}^{+2}[2^+4^+]_0$	-1
	$1_{\mathrm{II}}^{+2}[2^-4^-]_0$	1
-1	$1_{\mathrm{II}}^{+2}[2^+4^+]_2$	1
	$1_{\mathrm{II}}^{+2}[2^+4^+]_{-2}$	-1
3	$1_{\mathrm{II}}^{+2}[2^+4^-]_{-2}$	-1
	$1_{\mathrm{II}}^{+2}[2^+4^-]_2$	1
-3	$1_{\mathrm{II}}^{+2}[2^+4^-]_4$	1
	$1_{\mathrm{II}}^{+2}[2^-4^+]_4$	-1

4.2. Reflections. It is well known [2] that a vector e is a primitive root of a lattice L if and only if it can be represented as a direct summand in this lattice or in its sublattice of index 2. In the former case, we refer to this root as a root of type I and in the latter, as a root of type II.

Let e be a primitive root of type II in a given lattice L_d. We set $(e) = \mathbb{Z}e$, $L^e = L_d \cap e^\perp$, and $\Gamma^e = O^+(L^e)$. An important role in what follows is played by the following theorem, whose proof will be postponed until Section 4.4.

Theorem 4.2. *Let e be a root of a lattice L_d. Then the lattice L^e is uniquely* (*up to isomorphism*) *determined by the type and the square of e.*

Given a lattice L_d, consider the set M of all lattices L in the pseudo-Euclidean space $E^{2,2}(\mathbb{R})$ of signature $(2,2)$ which contain the lattice $(e) \oplus L^e$ as a sublattice of index 2 and are isomorphic to L_d.

Let $L^\vee$ denote the lattice dual to a lattice $L \in M$, and let $pr_{e^\perp} L$ be the projection of L on the subspace $e^\perp$. Clearly, $L^e \subset pr_{e^\perp} L \subset (L^e)^\vee$. It is also easy to check that $pr_{e^\perp} L/L^e = pr_e L/(e) = \mathbb{Z}/2\mathbb{Z}$. We denote the subgroup of 2-torsion elements in the discriminant group $\mathrm{disc}(L) = L^\vee/L$ of the lattice L by $\mathrm{disc}(L)_{(2)}$.

Definition 4.3. We refer to the nontrivial element $[x_L]$ of the group $pr_{e^\perp} L/L^e$ as the *gluing element* (and denote it simply by $[x]$, when it is clear which lattice is considered).

Clearly, the gluing element belongs to the group $\mathrm{disc}(L^e)_{(2)}$ and uniquely determines the lattice L. Indeed, $L = x + \frac{1}{2}e + (e) \oplus L^e$, where x is any representative of $[x] \in \mathrm{disc}(L^e)_{(2)}$ in the lattice $pr_{e^\perp} L$. In what follows, given any such representative x, we refer to the vector $x + \frac{1}{2}e$ as a *gluing vector*.

Extending each element $\gamma \in \Gamma^e$ to an automorphism of the space $E^{2,2}(\mathbb{R})$ by setting $\gamma(e) = e$, we obtain a natural action of the group Γ^e on the set M. This action induces the action of the group Γ^e on the set of gluing elements $[x]$ in the group $\mathrm{disc}(L^e)_{(2)}$.

Lemma 4.4. *There exists a bijection between the Γ_d-equivalence classes of k-roots e of type* II *in the lattice L_d and the Γ^e-equivalence classes of gluing elements in the group* $\mathrm{disc}(L^e)_{(2)}$.

Proof. Let e and e' be two k-roots in the lattice L_d. By Theorem 4.2 the three-dimensional lattices L^e and $L^{e'}$ are isomorphic, and hence there exists an isomorphism $g \in O^+(E^{2,2}(\mathbb{R}))$ taking the direct sum $(e') \oplus L^{e'}$ to the direct sum $(e) \oplus L^e$. Thus, $gL_d = L \in M$. Clearly, the roots e and e' are Γ_d-equivalent in the lattice L_d if and only if there exists an element $\gamma \in \Gamma^e$ which takes the lattice gL_d to the lattice L_d, or, equivalently, the gluing element $[x_L]$ to $[x_{L_d}]$ (the composition $\gamma \circ g$ serves as an isomorphism conjugating the roots e and e' in the lattice L_d). □

Lemma 4.5. *Let $L \in M$. An automorphism $g \in \Gamma^e$ can be extended to an automorphism $g' \in O^+(L)$ if and only if it acts identically on the gluing element $[x_L]$ of the lattice L.*

Proof. Any automorphism $g' \in O^+(L)$ preserving the lattice L^e preserves also the element $[x_L]$. Conversely, the required automorphism $g' \in O^+(L)$ is defined by the rule $g'(e) = e$, $g'|_{L^e} = g$. □

Let N_k denote the number of gluing elements $[x_L]$ in the group $\operatorname{disc}(L^e)_{(2)}$, where e is a k-root of type II.

Remark 4.6. This number is well defined by virtue of Theorem 4.2.

Lemma 4.7. *The numerator of the fraction $K(\Gamma_d)$ equals*

$$\frac{1}{2} \sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type II}}} \operatorname{Covol}(\Gamma^e) \cdot N_k + \frac{1}{2} \sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type I}}} \operatorname{Covol}(\Gamma^e).$$

Proof. Note that if e is a root of type I in the lattice L_d and π_e is the corresponding mirror of reflections, then $\operatorname{Covol}(\Gamma^e) = 2\operatorname{Covol}((\Gamma_d)_{\pi_e})$ (because Γ^e is a subgroup of index 2 in the group $(\Gamma_d)_{\pi_e}$). Therefore,

$$\frac{1}{2} \sum_{\substack{k \\ (e,e)=-k \\ e \text{ type I}}} \operatorname{Covol}(\Gamma^e) = \sum_{\substack{[\pi_e] \\ e \text{ type I}}} \operatorname{Covol}((\Gamma_d)_{\pi_e}).$$

It remains to prove the lemma for roots of type II. Let e be a k-root of type II in the lattice L_d, and let x be the corresponding gluing element. Denoting summation over the Γ^e-equivalence classes of gluing elements in $\operatorname{disc}(L^e)_{(2)}$ by $\sum_{[x]}$, we write

$$\sum_{\substack{[\pi_e] \\ e \text{ type II}}} \operatorname{Covol}((\Gamma_d)_{\pi_e}) = \sum_{k} \sum_{\substack{[\pi_e] \\ (e,e)=-k \\ e \text{ type II}}} \operatorname{Covol}((\Gamma_d)_{\pi_e}) = \sum_{\substack{k \\ (e,e)=-k \\ e \text{ type II}}} \sum_{[x]} \operatorname{Covol}((\Gamma_d)_{\pi_e});$$

the second equality follows from Lemma 4.4.

Consider the map ϕ from $(\Gamma_d)_{\pi_e}$ to Γ^e which takes each element $g \in (\Gamma_d)_{\pi_e}$ to its restriction to the lattice L^e. Its image $\operatorname{im}(\phi((\Gamma_d)_{\pi_e}))$ consists of automorphisms which extend from Γ^e to Γ_d, and $\ker(\phi) = \mathbb{Z}/2\mathbb{Z}$ (the kernel consists of id and the reflection in the vector e). By Lemma 4.5 we have $\phi((\Gamma_d)_{\pi_e}) = \Gamma^e_x$, where Γ^e_x denotes the subgroup of Γ^e identically acting on the gluing element x. Thus, $[\Gamma^e : \phi((\Gamma_d)_{\pi_e})] = [\Gamma^e : \Gamma^e_x]$, and $[\Gamma^e : \Gamma^e_x]$ is the number of points in the orbit of the gluing element x under the action of the group Γ^e. Thus,

$$\sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type II}}} \sum_{[x]} \operatorname{Covol}((\Gamma_d)_{\pi_e}) = \sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type II}}} \sum_{[x]} \frac{[\Gamma^e : \Gamma^e_x]}{2} \operatorname{Covol}(\Gamma^e)$$

$$= \sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type II}}} \operatorname{Covol}(\Gamma^e) \sum_{[x]} \frac{[\Gamma^e : \Gamma^e_x]}{2} = \frac{1}{2} \sum_{\substack{k \\ (e,e)=-k \\ e \text{ is of type II}}} \operatorname{Covol}(\Gamma^e) \cdot N_k.$$

We have used the equality $N_k = \sum_{[x]} [\Gamma^e : \Gamma^e_x]$. Indeed, the right-hand side of this equality is the sum of the lengths of orbits of the group Γ^e over all orbits of gluing elements under the action of Γ^e. Clearly, this sum equals the number on the left-hand side, because N_k is the cardinality of the set of gluing elements. □

4.3. Possible types of the lattices L^e. In this section we describe all possible types of the lattices L^e. Let $\nu_2(L)$ denote the maximum power of 2 dividing the determinant $d(L)$ of the Gram matrix of L. We set $d_e := -d(L^e)/2^{\nu_2(L^e)}$.

Lemma 4.8. (a) *If $d \equiv 1 \pmod 4$, then $\nu_2(L^e) = 1$.*
(b) *If $d \equiv 3 \pmod 4$, then $\nu_2(L^e) \in \{1, 2, 3\}$.*
(c) *If $d \equiv 2 \pmod 4$, then $\nu_2(L^e) \in \{1, 2, 3, 4\}$.*

(d) *The number* $\nu_2((e))$ *is uniquely determined by* $d \pmod 4$ *and* $\nu_2(L^e)$.

Proof. Note that $d(L_d)\cdot[L_d : L^e \oplus (e)]^2 = d(L^e)\cdot(e,e)$. It follows that $\nu_2(L^e) = \nu_2(L_d) - \nu_2((e))$ for roots of type I and $\nu_2(L^e) = \nu_2(L_d) - \nu_2((e)) + 2$ for roots of type II. Using these relations, we complete Table 2, whose last row contains all values of $\nu_2(L^e)$ possible a priori.

Table 2

$d \pmod 4$	1		3				2					
$\nu_2(L_d)$	0		2				3					
Inv. set	1, 1, 1, d		1, 1, 2, 2d				1, 1, 2, 2d					
$\nu_2((e))$	1	1	1	1	2	2	1	1	2	2	3	3
Type of e	I	II	I	II	I	II	I	II	I	II	I	II
$\nu_2(L^e)$	-1	1	1	3	0	2	2	4	1	3	0	2

In completing the fourth row, we used the fact that (e,e) divides twice the maximum invariant factor of L_d.

The lattice L^e is even; therefore, $\nu_2(L^e) \geqslant 1$. This implies assertions (a), (b), and (c) of the lemma.

Let us prove (d). We must show that there exist no roots e for which $\nu_2((e)) = 3$. Note that such a root can exist only if $d \equiv 2 \pmod 4$. Suppose that $e = (x,y,z,t)$ is a primitive root of the lattice L_d and $\nu_2((e)) = 3$. Writing the condition $(e,e) = -4\frac{d}{d_e}$ in coordinates, we obtain $2xy + 2z^2 - 2dt^2 = -4\frac{d}{d_e}$. Moreover, $\frac{2e}{(e,e)} \in L_d^\vee$. Considering the inner products $(\frac{2e}{(e,e)}, e_i)$ with the basis vectors e_i of the lattice L_d, we obtain $\frac{2d}{d_e}|x$, $\frac{2d}{d_e}|y$, and $\frac{d}{d_e}|z$. Setting $x = \frac{2d}{d_e}x'$, $y = \frac{2d}{d_e}y'$, and $z = \frac{d}{d_e}z'$, substituting these expressions into the equation of the squared vector e, and reducing both sides by $\frac{2d}{d_e}$, we obtain $\frac{4d}{d_e}x'y' + \frac{d}{d_e}z'^2 - d_e t^2 = -2$. Since the number d_e is odd, it follows that $2\,|\,t$, i.e., all coordinates of the root e are even. This contradicts the assumption that the root e is primitive. □

Let e be a root of type II in the lattice L_d. All possible invariant factors of the lattices L^e and the corresponding discriminant groups $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)$ are listed in Table 3.

Table 3

$d \pmod 4$	1	3		2	
$\nu_2(L^e)$	1	2	3	3	4
Inv. set L^e	1, 1, $2d_e$	1, 1, $4d_e$	2, 2, $2d_e$	2, 2, $2d_e$	2, 2, $4d_e$
$\operatorname{disc}(L^e \otimes \mathbb{Z}_2)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$

Comments to Table 3. The invariant factors in the first two columns are found by using Lemma 4.8 and the fact that the lattice L^e is even. It also follows from this fact that the invariant factors in the third and fourth columns cannot be 1, 2, $4d_e$. If the invariant factors of L^e are 1, 1, $8d_e$, then the group $\operatorname{disc}(L^e \otimes \mathbb{Z}_2) \cong \mathbb{Z}/8\mathbb{Z}$ has a unique gluing element of order 2. This element corresponds to a gluing vector with noninteger square in the case $d \equiv 3 \pmod 4$ and with odd square in the case $d \equiv 2 \pmod 4$. The invariant factors in the fifth column cannot be 1, 4, $4d_e$ or 1, 2, $8d_e$, because the lattice L^e is even. Suppose that they are 1, 1, $16d_e$. This means that $\operatorname{disc}(L^e \otimes \mathbb{Z}_2) \cong \mathbb{Z}/16\mathbb{Z}$. We obtain a contradiction, because the only element of order 2 in this group corresponds to a gluing vector with noninteger square. The discriminant groups $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)$ are naturally determined by the sets of invariant factors of the lattices $L^e \otimes \mathbb{Z}_2$.

Remark 4.9. A similar argument proves that if e is a root of type I in the lattice L_d, then the invariant factors of the lattice L^e are 1, 1, $2^{\nu_2(L^e)}$.

4.4. Proof of Theorem 4.2. It follows from the proof of Lemma 4.8 that the quantity $d(L^e)$ is determined by the type and the square of the root e. The Hasse symbols of the lattice L^e

are determined from the well-known relation $\epsilon_p(L_d) = \epsilon_p(L^e) \cdot ((e,e), d(L^e))_p$ [6], where $(\cdot\,,\cdot)_p$ is the p-adic Hilbert symbol. The invariant factors of the lattices L^e are given in Table 3 and Remark 4.9. In all cases under consideration, the class of the lattice L^e coincides with its genus by virtue of Kneser's theorem. As in Theorem 4.1, it suffices to examine the equivalence of the 2-adic completions of the lattices L^e with fixed discriminants and Hasse symbols (see the proof of Theorem 4.1).

Let $\nu_2(L^e) = 1$. In this case, two lattices are possible: $U \oplus (2d_e)$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus (-6d_e)$; their equivalence is verified by using sign walking.

Let $\nu_2(L^e) = 2$. Again, there are two possible lattices, $U \oplus (4d_e)$ and $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \oplus (-12d_e)$, but they have different Hasse symbols.

In the case $\nu_2(L^e) = 3$, the two possible lattices $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e)$ and $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e)$ again have different Hasse symbols.

Finally, consider the case $\nu_2(L^e) = 4$. Suppose that the Gram matrix of the lattice $L^e \otimes \mathbb{Z}_2$ is twice the matrix of an even lattice, i.e., has the form $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (4d_e)$ or $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-12d_e)$. Applying sign walking, we see that these lattices are equivalent. Considering all elements of order 2 in the group $\mathrm{disc}(L^e \otimes \mathbb{Z}_2)$, we see that they correspond to a gluing vector with noninteger square.

Therefore, the Gram matrix of the lattice $L^e \otimes \mathbb{Z}_2$ is twice the matrix of an odd lattice. Thus, the lattice $L^e \otimes \mathbb{Z}_2$ has the form $(2u_1) \oplus (2u_2) \oplus (4d_e u_1 u_2)$, where $u_1, u_2 \in \mathbb{Z}_2^*/(\mathbb{Z}_2^*)^2$. Note that the Hasse symbol is uniquely determined by the triple (u_1, u_2, d_e) modulo 8. Considering all possible triples (u_1, u_2, d_e) modulo 8, we see that the lattice L^e is uniquely determined by the pair (d_e, ϵ_2). ☐

4.5. The number of gluing vectors for various types of the lattices L^e. In this section e denotes a k-root of type II in the lattice L_d.

Lemma 4.10. *Let e be a primitive root of the lattice L_d such that $\nu_2((e)) = 1$ and $\nu_2(L^e) = 3$. Then*

$$L^e \otimes \mathbb{Z}_2 \cong \begin{cases} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e) & \text{for } d_e \equiv 1 \pmod 4, \\ \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e) & \text{for } d_e \equiv 3 \pmod 4. \end{cases}$$

Proof. Note that

$$\epsilon_2(L_d) = (2, 2d)_2 = \begin{cases} 1 & \text{if } d \equiv -1 \pmod 8, \\ -1 & \text{if } d \equiv 3 \pmod 8. \end{cases}$$

On the other hand, $\epsilon_2(L_d) = \epsilon_2((e) \oplus L^e) = \epsilon_2(L^e) \cdot (-8d_e, (e,e))_2$. Considering all possible pairs $(d_e, \frac{1}{2}(e,e))$ modulo 8, we see that $\epsilon_2(L^e) = 1$. Calculating the Hasse symbols ϵ_2 of the lattices $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e)$ and $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e)$, we obtain the required expression. ☐

Let N'_k denote the upper bound for the number N_k of gluing vectors.

Lemma 4.11. *The following table contains all possible triples $(d \pmod 4), \nu_2(L^e), N'_k)$.*

$d \pmod 4$	1	3		2	
$\nu_2(L^e)$	1	2	3	3	4
N'_k	1	1	1 *or* 3	1 *or* 3	2

If $\nu_2(L^e) = 3$ and $L^e \otimes \mathbb{Z}_2 \cong \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e)$, then $N'_k = 1$; otherwise, $N'_k = 3$.

Proof. Clearly, $N_k \leqslant |\operatorname{disc}(L^e \otimes \mathbb{Z}_2)_{(2)}|$, but, generally, not all elements of $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)_{(2)}$ are gluing elements. The proof of the lemma reduces to considering all elements of the group $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)_{(2)}$ for lattices $L^e \otimes \mathbb{Z}_2$ of various types. All discriminant groups $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)$ are given in Table 3. For example, consider the third column of this table. According to Lemma 4.10, we have $L^e \otimes \mathbb{Z}_2 \cong \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e)$ if $d_e \equiv 1 \pmod 4$ and $L^e \otimes \mathbb{Z}_2 \cong \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e)$ if $d_e \equiv 3 \pmod 4$. Considering all elements of the group $\operatorname{disc}(L^e \otimes \mathbb{Z}_2)_{(2)}$, we see that the gluing elements are $[x] = (1,1,1) \in (\mathbb{Z}/2\mathbb{Z})^3$ if $d_e \equiv 1 \pmod 4$ and $[x] = (1,0,1)$, $[x] = (1,1,1)$, $[x] = (0,1,1) \in (\mathbb{Z}/2\mathbb{Z})^3$ if $d_e \equiv 3 \pmod 4$. The remaining cases are considered in a similar way. □

Remark 4.12. It follows from Theorem 4.1 that $N'_k = N_k$.

4.6. Calculation of covolumes. We use Siegel's formula for calculating the covolume $\operatorname{Covol}(O^+(L))$ for a given lattice L (which is the only one of its kind) of signature $(2, n)$ ([22], [13]):

$$\operatorname{Covol}(O^+(L)) = 4|d(L)|^{(n+3)/2} \prod_{k=1}^{n+2} \pi^{-k/2}\Gamma(k/2) \cdot \prod_p a_p(L)^{-1}, \qquad (4)$$

where the $a_p(L) := \frac{1}{2} \lim_{r\to\infty} p^{-r(n+2)(n+1)/2} |O(L \otimes \mathbb{Z}/p^r\mathbb{Z})|$ are the local volumes of the orthogonal group of the lattice L.

Let $L(s, D) = L(s, \chi_D) = \prod_p \bigl(1 - \frac{\chi_D(p)}{p^s}\bigr)^{-1}$ be the Dirichlet L-function with character $\chi_D(p) = \bigl(\frac{D}{p}\bigr)$, where $(\frac{\cdot}{\cdot})$ is the Kronecker–Jacobi symbol.

Lemma 4.13. *Let $\rho(d)$ be the number of odd prime divisors of the number d.*

(a) *If $d \equiv 1 \pmod 4$, then* $\operatorname{Covol}(\Gamma_d) = \frac{d^{3/2}}{2^{\rho(d)+4} \cdot 3\pi^2} L(2, d)$.

(b) *If $d \equiv 2, 3 \pmod 4$, then* $\operatorname{Covol}(\Gamma_d) = \frac{d^{3/2}}{2^{\rho(d)+2} \cdot 3\pi^2} L(2, 4d)$.

Proof. Formulas for calculating local volumes can be found, e.g., in [18]. The local volumes $a_p(L_d)$, $p \neq 2$, do not depend on d modulo 4; they are given by

$$a_p(L_d) = \begin{cases} 2p(1 - p^{-2}) & \text{if } p \mid d, \\ (1 - p^{-2})(1 - (\frac{d}{p})p^{-2}) & \text{if } p \nmid d. \end{cases}$$

The local volumes $a_2(L_d)$ are given by

$$a_2(L_d) = \begin{cases} 2^4(1 - 2^{-2})(1 - (\frac{d}{2})2^{-2}) & \text{if } d \equiv 1 \pmod 4, \\ 2^7(1 - 2^{-2}) & \text{if } d \equiv 3 \pmod 4, \\ 2^8(1 - 2^{-2}) & \text{if } d \equiv 2 \pmod 4. \end{cases}$$

Substituting the values of local volumes into (4), we obtain the required result. □

Lemma 4.14. *The covolume* $\operatorname{Covol}(\Gamma^e)$ *does not exceed* $C_1 \cdot \prod_{p \mid d_e} \frac{p+1}{2}$, *where $C_1 = 1/12$ if $\nu_2(L^e) = 1$ and $C_1 = 1/8$ if $\nu_2(L^e) \in \{2, 3, 4\}$.*

Proof. The local volumes $a_p(L^e)$, $p \neq 2$, do not depend on $\nu_2(L^e)$ and are given by

$$a_p(L^e) = \begin{cases} 2p(1 - p^{-2})(1 \pm p^{-1})^{-1} & \text{if } p \mid d_e, \\ (1 - p^{-2}) & \text{if } p \nmid d_e. \end{cases}$$

The local volumes $a_2(L^e)$ are given by

$$a_2(L^e) = \begin{cases} 2^4(1 - 2^{-2}) & \text{if } \nu_2(L^e) = 1, \\ 2^6(1 - 2^{-2})(1 \pm 2^{-1})^{-1} & \text{if } \nu_2(L^e) = 2, \\ 2^8(1 - 2^{-2})(1 \pm 2^{-1})^{-1} & \text{if } \nu_2(L^e) = 3, \\ 2^9, & \text{if } \nu_2(L^e) = 4. \end{cases}$$

The last equality follows from the isomorphism $L^e \otimes \mathbb{Z}_2 \cong (2u_1) \oplus (2u_2) \oplus (-4u_1u_2d_e)$ (which can be derived from the proof of Theorem 4.2).

Substituting the values of local volumes into (4), we obtain the required result (the covolume $\mathrm{Covol}(\Gamma^e)$ is maximum when all signs are pluses). □

Lemma 4.15. *Let I_e be the set of odd primes dividing $d_e = -d(L^e)/2^{\nu_2(L^e)}$ and such that, for each $p \in I_e$, the unimodular component L^0 of the Jordan decomposition of the lattice $L^e \otimes \mathbb{Z}_p$ is nonisomorphic to any lattice of the form $\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$, where $n \in \mathbb{N}$ (recall that $L^e \otimes \mathbb{Z}_p = L^0 \oplus pL^1$ and $\mathrm{rk}\, L^0 = 2$). Then $\mathrm{Covol}(\Gamma^e) \leqslant C_1 \prod_{p\in I_e} \frac{p-1}{2} \prod_{p\notin I_e} \frac{p+1}{2} = C_1 \prod_{p|d_e} \frac{p+1}{2} \prod_{p\in I_e} \frac{1-p^{-1}}{1+p^{-1}}$, where the constant C_1 is the same as in Lemma 4.14.*

Proof. We must choose the sign in the formula $a_p = 2p(1-p^{-2})(1 \pm p^{-1})^{-1}$ for $p \in I_e$. As shown in [18], the plus sign corresponds to the case where the lattice L^0 has the form $\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$ for some $n \in \mathbb{N}$. Thus, by assumption, we choose minus. Substituting the values of local volumes into (4), we obtain the required relation. □

Note that the bound for the covolume $\mathrm{Covol}(\Gamma^e)$ under the conditions of the preceding lemma differs from that in Lemma 4.14 by a factor of $S_e := \prod_{p\in I_e} \frac{1-p^{-1}}{1+p^{-1}}$. We will use this coefficient in Section 4.9.

Lemma 4.16. *If $d = 3 \pmod 4$, e is a primitive root of the lattice L_d such that $\nu_2((e)) = 1$ and $\nu_2(L^e) = 3$, and $d_e = 3 \pmod 4$, then $\mathrm{Covol}(\Gamma^e) \leqslant \frac{1}{24} \prod_{p|d_e} \frac{p+1}{2}$.*

Proof. Note that all local volumes of the lattice L^e for odd p are the same as in Lemma 4.14. It follows from Lemma 4.10 that $L^e \otimes \mathbb{Z}_2 \cong \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e)$. Applying formulas for calculating local volumes given in the book [18], we obtain $a_2(L) = 2^8(1-2^{-2})(1-2^{-1})^{-1}$. Substituting values of local volumes into (4), we obtain the required relation. □

4.7. Estimates for the numerator and denominator of $K(\Gamma_d)$. Suppose that $d = p_1 \cdots p_{\rho(d)}$ or $d = 2p_1 \cdots p_{\rho(d)} = 2d'$, where the p_i are different odd primes and $3 \leqslant p_1 < \cdots < p_{\rho(d)}$ (if $d = 2d'$, then the set $\{p_i\}_i$ may be empty).

Lemma 4.17. *The denominator of $K(\Gamma_d)$ is not smaller than $Cd^{3/2}$, where $C = \frac{1}{2^{\rho(d)+4}\cdot 45}$ if $d \equiv 1 \pmod 4$ and $C = \frac{1}{2^{\rho(d)+2}\cdot 45}$ if $d \equiv 2 \pmod 4$ or $d \equiv 3 \pmod 4$.*

Proof. Note that $L(s,\chi) = \prod_p \big(1 - \frac{\chi(p)}{p^s}\big)^{-1} \geqslant \prod_p (1+p^{-s})^{-1} = \prod_p (1-p^{-2s})^{-1}(1-p^{-s}) = \zeta(2s)/\zeta(s)$. The first inequality holds because $\chi(p) \in \{\pm 1, 0\}$ for all p and each multiplier attains its minimum at $\chi(p) = -1$. Thus, $L(2,\chi) \geqslant \zeta(4)/\zeta(2) = \pi^2/15$. The required assertion now follows from Lemma 4.13. □

Lemma 4.18. *The numerator of $K(\Gamma_d)$ is not greater than $C_2 \prod_{i=1}^{\rho(d)} \frac{p_i+3}{2}$, where $C_2 = 1/24$ if $d \equiv 1 \pmod 4$, $C_2 = 1/6$ if $d \equiv 3 \pmod 4$, and $C_2 = 7/24$ if $d \equiv 2 \pmod 4$.*

Proof. Let us fix the type of a root e and $\nu_2((e))$. Substituting the estimates of the covolume $\mathrm{Covol}(\Gamma^e)$ obtained in Lemma 4.14 into the numerator of $K(\Gamma_d)$ for each root e, we obtain

$$\sum_{\substack{k \\ (e,e)=-k}} \mathrm{Covol}(\Gamma^e) = C_1 \sum_{\substack{\{i_1,\dots,i_m\} \\ \subseteq\{1,\dots,\rho(d)\}}} \prod_{j=1}^{m} \frac{p_{i_j}+1}{2} = C_1 \prod_{i=1}^{\rho(d)} \left(\frac{p_i+1}{2} + 1\right) = C_1 \prod_{i=1}^{\rho(d)} \frac{p_i+3}{2}.$$

Substituting the bounds for the maximum number of gluing vectors given in Lemma 4.11 for all possible pairs $(e, d(L^e))$ and all d modulo 4 into the expression in Lemma 4.7, we obtain the required estimates.

Consider the case $d \equiv 3 \pmod 4$ in detail (the cases $d \equiv 1 \pmod 4$ and $d \equiv 2 \pmod 4$ are similar). If a root e is such that $\nu_2(L^e) = 3$, then there are two possible cases: $L^e \otimes \mathbb{Z}_2 \cong$

$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \oplus (2d_e)$ and $L^e \otimes \mathbb{Z}_2 \cong \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \oplus (-6d_e)$. By virtue of Lemma 4.11, we have $N'_k = 1$ in the former case and $N'_k = 3$ in the latter. But in the latter case, we also have $C_1 \leqslant 1/24$ (by Lemma 4.16). Therefore, the contribution of each of these terms to the numerator of $K(\Gamma_d)$ is at most $C_1 \cdot N'_k \prod_{p_j | d_e} \frac{p_j+1}{2} = \frac{1}{8} \prod_{p_j | d_e} \frac{p_j+1}{2}$. According to Lemma 4.7, the numerator of $K(\Gamma_d)$ does not exceed $\frac{1}{2}\bigl(\frac{1}{12} + \frac{1}{8} + \frac{1}{8}\bigr) \prod_{i=1}^{\rho(d)} \frac{p_i+3}{2} = \frac{1}{6} \prod_{i=1}^{\rho(d)} \frac{p_i+3}{2}$. □

4.8. Proof of Theorem 2.1. Let $\widetilde{K}(\Gamma_d)$ denote the upper bound for $K(\Gamma_d)$ obtained by using Lemmas 4.17 and 4.18. It is easy to check that the following assertions hold.

Remark 4.19. If $\widetilde{K}(\Gamma_d) < 8$ for $k = k_0$ and any k_0-tuple $(p_1, \dots, p_{k_0})$ of odd primes, then this inequality holds also for any $k > k_0$ and any k-tuple $(p_1, \dots, p_k)$ of odd primes.

Remark 4.20. If $\widetilde{K}(\Gamma_d) < 8$ for an m-tuple $(p_1, \dots, p_m)$, then this inequality remains valid under the replacement of one of the numbers p_i by $p_j > p_i$.

Proof of Theorem 2.1. Suppose that the number d is odd. It follows from Lemmas 4.17 and 4.18 that

$$K(\Gamma_d) \leqslant \widetilde{K}(\Gamma_d) = 30 \prod_{i=1}^{\rho(d)} (p_i + 3) \Big/ \prod_{i=1}^{\rho(d)} p_i^{3/2}.$$

We will consider k-tuples of increasing odd primes and show that $K(\Gamma_d) < 8$ for all $d \notin \{2, 3, 5, 6, 13, 21\}$ (see Section 3.4).

For $k = 4$, the least k-tuple is $(3, 5, 7, 11)$, for which we have $\widetilde{K}(\Gamma_d) < 8$. It follows from Remark 4.19 that $K(\Gamma_d) < 8$ for all d divisible by at least four different odd primes.

Suppose that $k = 3$. Note that $\widetilde{K}(\Gamma_d) < 8$ for the triples $(5, 7, 11)$, $(3, 7, 11)$, and $(3, 5, 17)$ and all triples majorizing them. The remaining triples have the form $(3, 5, x)$, where $x < 17$. Therefore, in the case $k = 3$, the inequality $K(\Gamma_d) \geqslant 8$ can hold only for $d \in \{105, 165, 195\}$.

In the case $k = 2$, for the pairs $(7, 11)$, $(5, 13)$, and $(3, 29)$ and all pairs majorizing them, we have $\widetilde{K}(\Gamma_d) < 8$. There remain the pairs $(5, 7)$, $(5, 11)$, and $(3, x)$, where $x \leqslant 23$. Thus, in the case $k = 2$, the inequality $K(\Gamma_d) \geqslant 8$ can hold only for $d \in \{15, 21, 33, 35, 39, 51, 55, 57, 69\}$.

Finally, consider the case $k = 1$. A simple exhaustive search shows that $\widetilde{K}(\Gamma_d) \geqslant 8$ only for $d \in \{3, 5, 7, 11, 13, 17\}$.

For even d, it can be verified in a similar way that $\widetilde{K}(\Gamma_d) \geqslant 8$ only for $d \in \{210, 66, 42, 30, 14, 10, 6, 2\}$.

Note that, instead of the bound of Lemma 4.17 (which is convenient for general calculations) for the denominator of $K(\Gamma_d)$, we can use the exact value given by Lemma 4.13. Substituting the values of the L-function $L(2, d)$ or $L(2, 4d)$ into the denominator for all remaining numbers d, we see that $\widetilde{K}(\Gamma_d) < 8$ for $d \in \{210, 195, 165, 105, 69, 66, 57, 55, 51, 42, 39, 33, 17, 14\}$. In the next section we show that $K(\Gamma_d) < 8$ for $d \in \{35, 30, 15, 11, 10, 7\}$, which completes the proof of the theorem.

4.9. Exceptional cases. In Lemma 4.18 we estimated the numerator of $K(\Gamma_d)$ under the assumption that all theoretically possible roots of the lattice L_d exist and each of them makes the maximum possible contribution to the numerator of the fraction $K(\Gamma_d)$. In this section we improve this estimate by refining the description of the roots and the corresponding covolumes.

Lemma 4.21. *Let $d \equiv 1 \pmod 4$. For a vector e to be a k-root of the lattice L_d, it is necessary that $(\frac{d_e}{p}) = 1$ for each odd p dividing k.*

Proof. Suppose that $e = (x, y, z, t)$ is a primitive root of the lattice L_d. Let $k = 2k'$, where k' is odd. Writing the condition $(e, e) = -2k'$ in coordinates, we obtain $2xy + 2z^2 + 2tz + \frac{1-d}{2}t^2 = -2k'$. Moreover, $\frac{2e}{(e,e)} \in L_d^\vee$. Considering the inner products $(\frac{2e}{(e,e)}, e_i)$ with the basis vectors e_i of the lattice L_d, we see that $k'|x$, $k'|y$, and $k'|(2z + t)$. Setting $x = k'x'$, $y = k'y'$, and $2z = k't' - t$, substituting these expressions into the equation for the squared vector e, reducing both sides by

k', and multiplying by 2, we obtain $4k'x'y' + k't'^2 - d_e t^2 = -4$. Considering this relation modulo each p dividing k', we obtain the required result. □

Remark 4.22. A similar argument shows that if e is a root of the lattice L_d and $d \equiv 2, 3 \pmod 4$, then the congruence $2^{\nu_2(d)} d_e t^2 \equiv \nu_2((e)) \pmod p$ has a solution for each prime p dividing the number $\frac{d}{2^{\nu_2(d)} d_e}$.

Lemma 4.23. *Let $d \equiv 3 \pmod 4$, and let e be a primitive k-root of type I in the lattice L_d. If $\frac{1}{2}k$ is odd, then $\frac{1}{2}k \equiv 3 \pmod 4$.*

Proof. Considering all pairs $(\frac{1}{2}k, d_e) \pmod 8$, we see that the Conway codes of the lattices $L_d \otimes \mathbb{Z}_2$ and $U \oplus (2d_e) \oplus (-k)$ coincide if and only if $\frac{1}{2}k \equiv 3 \pmod 4$. □

Lemma 4.24. *For $d \in \{35, 30, 15, 11, 10, 7\}$, $K(\Gamma_d) < 8$.*

Proof. In Lemma 4.18 we estimated the numerator of $K(\Gamma_d)$, assuming that the lattice L_d may have any root e whose square is a divisor of twice an invariant factor of L_d (except roots for which $\nu_2((e)) = 3$, whose nonexistence was shown in the proof of assertion (d) of Lemma 4.8)). However, Lemma 4.21 and Remark 4.22 imply that, for the values d in the first row of Table 4, there exist no roots e whose squares are given in the second row.

Table 4

d	35	30	21	15	11	10	6	3
(e,e)	$-10,\ -14,\ -28,\ -140$	$-30,\ -20,\ -12$	-14	$-60,\ -10,\ -6$	-44	-10	-6	-12

It follows from Lemma 4.23 that, for $d \in \{35, 15, 11, 7, 3\}$, the lattice L_d has no 2-roots of type I.

The absence of the roots e mentioned above means that the bound for the numerator of $K(\Gamma_d)$ can be decreased by the contribution of the corresponding summands. This proves the assertion of the Lemma for $d \in \{35, 30, 11, 10\}$.

Let us apply Lemma 4.15 for the remaining values d. Considering the divisor d_e for each pair $(d, (e, e))$, we check that the conditions of Lemma 4.15 hold and complete the third row of Table 5 (the fourth row is obtained directly from the third, because $S_e = \prod_{p \in I_e} \frac{1-p^{-1}}{1+p^{-1}}$).

Table 5

d	21	15			7	3
(e,e)	-2	-2	-20	-4	-4	-2
I_e	$\{3,7\}$	$\{3\}$	$\{3\}$	$\{5\}$	$\{7\}$	$\{3\}$
S_e	3/8	1/2	1/2	2/3	3/4	1/2

According to Lemma 4.15, for each pair $(d, (e, e))$, the summand $\mathrm{Covol}(\Gamma^e)$ is bounded above by the number $S_e C_1 \prod_{p|d_e} \frac{p+1}{2}$ (rather than by $C_1 \prod_{p|d_e} \frac{p+1}{2}$, as in Lemma 4.14). Decreasing the contribution of the corresponding summands to the numerator of $K(\Gamma_d)$, we obtain the assertion of the lemma for the remaining values d. □

Remark 4.25. The calculations performed in the proof of Lemma 4.24 yield an upper estimate of $K(\Gamma_d)$ for the remaining values d.

d	21	13	6	5	3	2
$K(\Gamma_d)$	$\leqslant 8$	$\leqslant 8$	$\leqslant 10$	$\leqslant 20$	$\leqslant 14$	$\leqslant 14$

In particular, if $d \in \{13, 21\}$ and Γ' is a subgroup of Γ_d not containing some mirrors of reflections in the group Γ_d, then the algebra $A(\Gamma')$ cannot be free.

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