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On Some Free Algebras of Automorphic Forms*

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Abstract. It is proved that, for $n = 8, 9, 10$, the natural algebra of automorphic forms of the group $O^+_{2,n}(\mathbb{Z})$ acting on the n-dimensional symmetric domain of type IV is free, and the weights of generators are found. This extends results obtained in the author's previous paper for $n \leqslant 7$. On the other hand, as proved in a recent joint paper of the author and O. V. Shvartsman, similar algebras of automorphic forms cannot be free for $n > 10$.

Key words: symmetric domain, automorphic form, reflection group, $K3$-surface, moduli space, period map.

1. Introduction

1.1. The celebrated Shephard–Todd–Chevalley theorem ([1], [2]) asserts that the algebra of invariants of a finite linear group Γ acting on a complex vector space V is free if and only if the group Γ is generated by (complex) reflections.

A natural infinite counterpart of a finite linear group is a discrete automorphism group Γ of a complex symmetric domain $\mathscr{D}$ with fundamental domain of finite volume acting on an equivariant $\mathbb{C}^*$-bundle over $\mathscr{D}$. Here the domain $\mathscr{D}$ is the counterpart of the projective space PV and the total space of the $\mathbb{C}^*$-bundle is the counterpart of the punctured vector space V. The counterpart of the polynomial invariants of a finite linear group are the automorphic forms (with respect to the $\mathbb{C}^*$-bundle under consideration).

A simple topological argument (see [3] and Proposition 8.3 in [4]), which applies equally well to finite linear groups and discrete groups of holomorphic transformations, shows that, for the algebra of automorphic forms to be free, it is necessary that the group Γ be generated by reflections. It is easy to see that reflections exist only in two series of symmetric domains: complex balls and symmetric domains of type IV in Cartan's classification.

Determining the structure of the algebra of automorphic forms is generally a difficult problem. Until recently, this was done for only few groups (apart from those in dimension 1), mainly of dimension 2. In the author's preceding paper [5] it was proved that, for $n = 4, 5, 6, 7$, the algebra of automorphic forms of the group $O^+_{2,n}(\mathbb{Z})$ acting on the n-dimensional symmetric domain of type IV is free and the weights of its generators were found. In this paper, similar results are obtained for $n = 8, 9, 10$.

1.2. We proceed to precise statements.

Let $\mathbb{R}^{2,n}$ be the pseudo-Euclidean vector space of signature $(2, n)$ with inner product

$$(x, y) = x_1y_1 + x_2y_2 - x_3y_3 - \cdots - x_{n+2}y_{n+2}.$$

We set $\mathbb{C}^{2,n} = \mathbb{R}^{2,n} \otimes \mathbb{C}$ and consider the cone

$$\tilde{\mathscr{L}}_n = \{z \in \mathbb{C}^{2,n} : (z, z) = 0,\ (z, \bar{z}) > 0\}.$$

It has two complex conjugate connected components. Let $\mathscr{L}_n$ be one of them, and let $\mathscr{D}_n \subset \mathbb{C}P^n$ be its projectivization.

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The natural action of the pseudo-orthogonal group $O_{2,n}$ on $\mathbb{C}^{2,n}$ preserves the cone $\tilde{\mathscr{L}}_n$. Its subgroup $O^+_{2,n}$ of index 2, which preserves the component $\mathscr{L}_n$, acts transitively on $\mathscr{D}_n$. This action identifies the domain $\mathscr{D}_n$ with the Hermitian symmetric space $O^+_{2,n}/(SO_2 \times O_n)$, that is, the symmetric domain of type IV. Simultaneously, we obtain the natural $O^+_{2,n}$-equivariant holomorphic $\mathbb{C}^*$-bundle

$$\pi\colon \mathscr{L}_n \to \mathscr{D}_n.$$

Suppose that $n \geqslant 3$, and let $\Gamma \subset O^+_{2,n}$ be a discrete subgroup of finite covolume. For each $k \in \mathbb{Z}_+$, Γ-invariant holomorphic functions on $\mathscr{L}_n$ homogeneous of degree $-k$ on each fiber of the bundle π are called *automorphic forms* of weight k with respect to the group Γ. They form a finite-dimensional vector space $A(\mathscr{D}_n,\Gamma)_k$. The algebra

$$A(\mathscr{D}_n,\Gamma) = \bigoplus_{k=0}^{\infty} A(\mathscr{D}_n,\Gamma)_k$$

is called the *natural algebra of automorphic forms* on $\mathscr{D}_n$ with respect to the group Γ.

As is known, $A(\mathscr{D}_n,\Gamma)$ is a normal finitely generated graded algebra, and $A(\mathscr{D}_n,\Gamma)_0 = \mathbb{C}$. The affine algebraic variety $\operatorname{Spec} A(\mathscr{D}_n,\Gamma)$ contains the analytic quotient $\mathscr{L}_n/\Gamma$ as a Zariski open subset with boundary of dimension at most 2. The projective algebraic variety $\operatorname{Proj} A(\mathscr{D}_n,\Gamma)$ is the so-called Satake–Baily–Borel compactification of the variety $\mathscr{D}_n/\Gamma$ [6].

This paper is devoted to the algebras of automorphic forms of the groups $\Gamma_n = O^+_{2,n}(\mathbb{Z})$ consisting of all integer matrices in $O^+_{2,n}$.

In the 1962 paper [7] Igusa proved that the algebra of even Siegel modular forms of genus 2 is freely generated by forms of weights 4, 6, 10, and 12. It can be shown [8] that this algebra is naturally isomorphic to the algebra $A(\mathscr{D}_3,\Gamma)$ for some group $\Gamma \subset O^+_{2,3}$ commensurable with Γ_3 (although the algebra $A(\mathscr{D}_3,\Gamma_3)$ itself is not free; see [5, Theorem 8]).

Interpreting the quotient $\mathscr{D}_n/\Gamma_n$ as the moduli spaces of appropriately multipolarized $K3$-surfaces, in the previous paper [5] the author has succeeded in proving that the algebra $A(\mathscr{D}_n,\Gamma_n)$ is free and finding the weights of its generators for $n = 4,5,6,7$. In this paper, these results are extended to $n = 8,9,10$. For the reader's convenience, we summarize the results of these two papers in one theorem.

Theorem 1. *For* $4 \leqslant n \leqslant 10$, *the algebra* $A(\mathscr{D}_n,\Gamma_n)$ *is freely generated by forms of weights tabulated below*:

n	weights
4	$4,6,8,10,12$
5	$4,6,8,10,12,18$
6	$4,6,8,10,12,16,18$
7	$4,6,8,10,12,14,16,18$
8	$4,6,8,10,12,12,14,16,18$
9	$4,6,8,10,10,12,12,14,16,18$
10	$4,6,8,8,10,10,12,12,14,16,18$

Note that, for $n > 10$, the algebra $A(\mathscr{D}_n,\Gamma)$ can be free for no arithmetic group $\Gamma \subset O^+_{2,n}$ with noncompact factor [9].

As proved in [5, Theorem 3], for any $n \geqslant 3$, the natural embeddings $\mathscr{L}_n \hookrightarrow \mathscr{L}_{n+1}$ and $\Gamma_n \hookrightarrow \Gamma_{n+1}$ induce epimorphisms of graded algebras

$$A(\mathscr{D}_{n+1},\Gamma_{n+1}) \to A(\mathscr{D}_n,\Gamma_n).$$

It is clear from the preceding considerations that, for $n = 4,5,6,7,8,9$, the kernel of this epimorphism is the principal ideal of $A(\mathscr{D}_{n+1},\Gamma_{n+1})$ generated by the generator of weight 18, 16, 14, 12, 10, and 8, respectively.

In [5] the moduli spaces of multipolarized $K3$-surfaces were explicitly described by using their projective models as quartics in $\mathbb{C}P^3$. Unfortunately, the $K3$-surfaces considered in this paper have no such models, and we have to work with their projective models in $\mathbb{C}P^5$, which are not complete intersections.

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2. Certain Quadratic Lattices and Their Roots

2.1. In this section we introduce notation and recall certain facts related to quadratic lattices. For more details, see [5, Sec. 1].

By $I_{k,l}$ we denote the quadratic lattice of rank $n = k + l$ with orthogonal basis $\{e_1, \dots, e_n\}$ for which

$$(e_1, e_1) = \dots = (e_k, e_k) = 1, \qquad (e_{k+1}, e_{k+1}) = \dots = (e_n, e_n) = -1.$$

This is an odd unimodular quadratic lattice. For $k, l > 0$, any odd unimodular quadratic lattice of signature (k, l) is isomorphic to $I_{k,l}$.

By $D_{k,l}$ we denote the even sublattice of $I_{k,l}$ (it consists of all vectors with even sum of coordinates). In particular, $D_{n,0}$ is the root lattice of type D_n.

If $k \equiv l \pmod 8$, then the group generated by the lattice $D_{k,l}$ and the vector

$$\sigma = \tfrac{1}{2}(e_1 + \dots + e_n) \in I_{k,l} \otimes \mathbb{Q}$$

is an even unimodular quadratic lattice. We denote it by $J_{k,l}$. If $k, l > 0$, then any even unimodular quadratic lattice of signature (k, l) is isomorphic to $J_{k,l}$. The lattice $J_{8,0}$ is the root lattice of type E_8.

If $k - l \not\equiv 4 \pmod 8$, then the lattice $I_{k,l}$ is the unique odd extension of the lattice $D_{k,l}$; therefore, the groups of automorphisms (orthogonal transformations) of these lattices coincide. If $k \equiv l \pmod 8$, then the lattice $D_{k,l}$ has a unique even extension (isomorphic to $J_{k,l}$).

In the case where $p \equiv q \pmod 8$, $k < p$, and $l < q$, we have

$$I_{k,l} \oplus I_{p-k,q-l} = I_{p,q}$$

up to a permutation of the basis vectors. Therefore,

$$D_{k,l} \oplus D_{p-k,q-l} \subset D_{p,q} \subset J_{p,q},$$

which gives an embedding of the lattice $D_{k,l}$ in $J_{p,q}$ as a primitive sublattice. This embedding is unique up to an automorphism of the lattice $J_{p,q}$ ([10, Theorem 1.14.4], [5, Proposition 5]).

2.2. A primitive vector α of a quadratic lattice L is called a *k-root* if $(\alpha, \alpha) = -k < 0$ and the reflection

$$R_\alpha \colon x \mapsto x + \frac{2(\alpha, x)}{k}\,\alpha$$

preserves the lattice L. The latter condition holds automatically if $k = 1$ or 2. If the lattice L is unimodular, then so are all of its roots.

In this paper we consider only 2-roots and refer to them simply as roots. We denote the group generated by the corresponding reflections by $W(L)$. This is a normal subgroup of the group $O(L)$ of all automorphisms of the lattice L.

A quadratic lattice of signature $(1, m)$ is said to be *hyperbolic*. Given a hyperbolic lattice L, we set

$$\mathbb{R}^{1,m} = L \otimes \mathbb{R}, \qquad \widetilde{C}_m = \{x \in \mathbb{R}^{1,m} : (x, x) > 0\}$$

and denote one of the connected components of the cone $\widetilde{C}_m$ (the "future cone") by C_m. The hyperboloid

$$H^m = \{x \in C_m : (x, x) = 1\}$$

is a model of m-dimensional Lobachevsky space. The motion group of the space H^m in this model is the subgroup $O^+_{1,m}$ of index 2 in the Lorentz group $O_{1,m}$ that preserves the cone C_m. The group

$$O^+(L) = O(L) \cap O^+_{1,m}$$

is a discrete motion group of H^m with fundamental domain of finite volume.

The group $W(L)$ acting on H^m is generated by reflections in the sense of Lobachevsky geometry. Let $P(L)$ be a fundamental polyhedron of this group in H^m. The cone over it is a fundamental cone for the action of the group $W(L)$ on C_m. We denote its closure in Minkowski space $\mathbb{R}^{1,m}$ by $A(L)$. The roots orthogonal to its walls and directed outward are called the *simple* 2-*roots* of the lattice L, and the corresponding reflections are called the *simple reflections*.

2.3. To prove the main theorem of this paper, we need the simple roots of the lattice $D_{1,m}$ with $m = 9, 10, 11$. They can easily be found by the algorithm described in [11].

First, it follows from the above considerations that if $m \not\equiv 5 \pmod 8$ (all m considered in what follows satisfy this condition), then $O(D_{1,m}) = O(I_{1,m})$ and hence $W(D_{1,m}) = W(I_{1,m})$.

Let $\{e_0, e_1, \dots, e_m\}$ be the standard basis of the lattice $I_{1,m}$, and let $x_0, x_1, \dots, x_m$ be the coordinates with respect to this basis. For the base point of the algorithm we take e_0. The stabilizer of this point in $W(I_{1,m})$ is the finite reflection group of type D_m permuting the coordinates $x_1, \dots, x_m$ and multiplying an even number of these coordinates by -1. For the fundamental cone we can take the cone defined by the inequalities

$$x_1 \geqslant \dots \geqslant x_{m-1} \geqslant |x_m|.$$

The corresponding simple roots are

$$\alpha_i = -e_i + e_{i+1} \quad (i = 1, \dots, m-1), \qquad \alpha_m = -e_{m-1} - e_m.$$

At the next step of the algorithm we must choose a root α_{m+1} satisfying the conditions

$$(\alpha_{m+1}, e_0) \geqslant 0, \quad (\alpha_{m+1}, \alpha_i) \geqslant 0 \qquad (i = 1, \dots, m)$$

and minimizing the inner product (α_{m+1}, e_0) (i.e., the coordinate x_0). Such a root is

$$\alpha_{m+1} = e_0 + e_1 + e_2 + e_3.$$

The Coxeter diagram of the found roots is shown in Fig. 1. Analyzing it, we see that, for $m = 9$, it determines a simplex of finite volume with one infinite vertex of type $\widetilde{D}_8$ and two infinite vertices of type $\tilde{E}_8$, so that, in this case, the simple roots are exhausted by those already found. It can be shown that, for $m = 10$, there are infinitely many other simple roots (i.e., a fundamental polyhedron of the group $W(I_{1,10})$ has infinite volume), and for $m = 11$, there are two more simple roots. However, the roots already found are sufficient for our purposes.

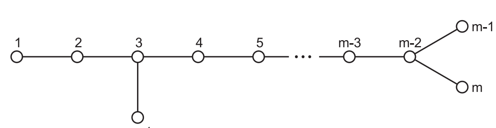

Fig. 1

3. Multipolarized $K3$-surfaces

3.1. Recall that the homology group $H_2(X,\mathbb{Z})$ of any $K3$-surface X is an even unimodular quadratic lattice isomorphic to the lattice $J_{3,19}$. The *lattice* $S(X)$ *of algebraic cycles* (the Picard group) of a surface X is a primitive hyperbolic sublattice in $H_2(X,\mathbb{Z})$ of signature $(1,m)$. Its orthogonal complement $T(X)$ in $H_2(X,\mathbb{Z})$, called the *lattice of transcendental cycles*, has signature $(2,n)$, where $n = 19 - m$.

Under a suitable choice of the future cone C_m and the fundamental polyhedron $P(S(X))$ of the group $W(S(X))$ in the space H^m, the simple roots correspond to smooth rational curves on the surface X, and the primitive isotropic vectors of the lattice $S(X)$ on the boundary of the cone $A(S(X))$ (which represent the vertices at infinity of the polyhedron $P(S(X))$) correspond to the elliptic sheaves on X.

In what follows, we write for brevity $W(X)$, $P(X)$, and $A(X)$ instead of $W(S(X))$, $P(S(X))$, and $A(S(X))$.

Let $h \in A(X)$ be a primitive vector of $S(X)$ satisfying the condition $(h,h) = d > 0$ and the additional condition

$(*)$ there exist no isotropic vectors u in $S(X) \cap A(X)$ for which $(h,u) = 1$ or 2.

Then the corresponding linear system determines a birational morphism

$$\varphi_h\colon X \to Y \subset \mathbb{C}P^g, \quad \text{where } 2g-2 = d,$$

onto a normal projective variety Y of degree d; this φ_h contracts all smooth rational curves corresponding to simple roots orthogonal to h to a point and is an isomorphism outside them [13]. For this reason, we refer to the cone $A(X)$ as the *cone of ample divisors* (although not all divisor classes in this cone are ample).

We also recall that, on each $K3$-surface X, there is a unique (up to proportionality) nowhere vanishing regular differential 2-form ω (a *symplectic form*). We refer to the pair (X,ω) as a *normed $K3$-surface.*

The *period map* (integration over transcendental cycles) allows us to consider the form ω as an element of the space $\mathbb{C}^{2,n} = T(X)\otimes\mathbb{C}$. Moreover, for an appropriately chosen connected component $\mathscr{L}_n$ of the cone $\tilde{\mathscr{L}}_n \subset \mathbb{C}^{2,n}$, we have $\omega \in \mathscr{L}_n$ (see the notation in Section 1.2).

3.2. Let us fix a primitive hyperbolic sublattice $S_0 \subset J_{3,19}$ of signature $(1,m)$ and a primitive vector $h_0 \in S_0$ for which $(h_0,h_0) = d > 0$. We denote the orthogonal complement of S_0 in $J_{3,19}$ by T_0.

By a *multipolarization of type* (S_0,h_0) of a $K3$-surface X we mean a pair of a sublattice $S \subset S(X)$ and a vector $h \in S\cap A(X)$ that are mapped to S_0 and h_0, respectively, by an isomorphism $\varphi\colon H_2(X,\mathbb{Z}) \to J_{3,19}$. Such an isomorphism φ (if it exists) is determined uniquely up to left multiplication by a transformation in $O^+(J_{3,19})$ preserving S_0 and h_0 (or, equivalently, T_0 and h_0). We denote the group of the restrictions of such transformations to T_0 by $O^+(T_0,h_0)$.

The isomorphism φ takes the symplectic form ω on a multipolarized $K3$-surface X to a vector in $T_0\otimes\mathbb{C}$ determined up to the action of the group $O^+(T_0,h_0)$.

We say that the type (S_0,h_0) of multipolarization is *admissible* if the following condition holds:

$(**)$ none of the hyperbolic sublattices $S \subset J_{3,19}$ containing S_0 has an isotropic vector u for which (h_0,u) equals 1 or 2.

In this case, the moduli space of normed multipolarized $K3$-surfaces of type (S_0,h_0) is the quotient $\mathscr{L}_n/O^+(T_0,h_0)$, and, accordingly, the moduli space of $K3$-surfaces of type (S_0,h_0) is the quotient $\mathscr{D}_n/O^+(T_0,h_0)$ [5]. We refer to the group $O^+(T_0,h_0)$ as the *modular group* of type (S_0,h_0).

A multipolarized $K3$-surface X for which $S = S(X)$ is said to be *irrational.* In the moduli space the set of such surfaces is the complement of a dense union of countably many algebraic hypersurfaces.

Remark 1. In the definition of multipolarization the vector h_0 is determined only up to an automorphism of S_0 which can be extended to an automorphism of $J_{3,19}$ being the identity on the orthogonal complement. In particular, if the lattice S_0 is 2-elementary, then, multiplying by -1, we can bring the vector h_0 to a chosen future cone. In any case, if h_0 belongs to the future cone, then, by means of reflections in the lattice S_0, we can move it to a chosen fundamental cone $A(S_0)$ of the group $W(S_0)$.

4. Choice of Multipolarization

4.1. Let $D_{1,9} \subset J_{3,19}$ be the standard embedding described in Section 1.1, and let

$$h_0 = 4e_0 + e_1 + \cdots + e_8 \in D_{1,9}. \tag{1}$$

We have $(h_0, h_0) = 8$. The inner products of the vector h_0 and the simple roots are shown in Fig. 2 (the absence of a mark means zero).

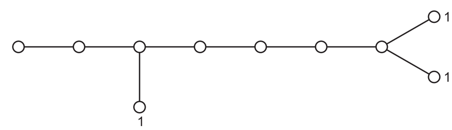

Fig. 2

Lemma 2 (cf. [5, Proposition 8])**.** *None of the hyperbolic sublattices $S \subset J_{3,19}$ containing $D_{1,9}$ has an isotropic vector u for which $(h_0, u) = 1$ or 2.*

Proof. First, we prove that there is no such vector in the lattice $D_{1,9}$ itself. According to Lemma 2 in [5], to prove this, it suffices to check the primitive isotropic vectors of the cone $A(D_{1,9})$. There are three such vectors. One of them corresponds to a parabolic subdiagram of type $\widetilde{D}_8$ of the Coxeter diagram, and the two others, to two symmetric parabolic subdiagrams of type $\tilde{E}_8$. We denote them by u_0, u_1, and u_2, respectively. The isotropic vector corresponding to a parabolic subdiagram is a linear combination of the simple roots in this subdiagram with coefficients equal to those in the linear dependence of the corresponding extended diagram of simple roots [12, Section 1.9]. Using this fact, we obtain

$$(h_0, u_0) = 3, \quad (h_0, u_1) = (h_0, u_2) = 4.$$

Next, let $u \in J_{3,19}$ be an isotropic vector not belonging to the lattice $D_{1,9}$ but belonging to a hyperbolic lattice $S \subset J_{3,19}$ containing $D_{1,9}$. Suppose that (h_0, u) equals 1 or 2, and let u' be the orthogonal projection of the vector u on $D_{1,9} \otimes \mathbb{Q}$. Then

$$(h_0, u') = (h_0, u), \quad (u', u') > 0.$$

Moreover, u' is contained in the lattice dual to $D_{1,9}$ and, therefore, $2u' \in D_{1,9}$, which implies

$$(u', u') \geqslant 1/2.$$

Consider the Gramian of the vectors h_0 and u'. By virtue of the inequalities obtained above, it is nonnegative. Since $D_{1,9}$ is a lattice of signature $(1, 9)$, it follows that this is possible only if the vectors h_0 and u are proportional and their Gramian vanishes. In the latter case, we have $(u', u') = 1/2$, so that $h_0 = 4u' \in 2D_{1,9}$; obviously, this is not true. $\square$

Thus, multipolarization of type $(D_{1,9}, h_0)$ is admissible. The orthogonal complement of the lattice $D_{1,9}$ in $J_{3,19}$ is $D_{2,10}$.

4.2. Now, let us find the modular group $O^+(D_{2,10}, h_0)$.

Proposition 1. $O^+(D_{2,10}, h_0) = O^+(D_{2,10}) (= O^+(I_{2,10}))$.

Proof. The automorphisms σ and ρ of the lattices $O(D_{2,10})$ and $O(D_{1,9})$ can be "glued together" into an automorphism of the lattice $J_{3,19}$ if and only if their actions on the common discriminant group $\mathfrak{D}$ of these lattices coincide [5, Proposition 4]. The group $\mathfrak{D}$ is the Klein four-group. Two of its nonzero elements, say δ_1 and δ_2, correspond to two (isomorphic) even extensions of the corresponding lattice, and the third nonzero element corresponds to the unique odd extension of this lattice. The automorphism σ can either act trivially on $\mathfrak{D}$ or transpose δ_1 and δ_2. In the former case, for ρ we can take the identity automorphism of $D_{1,9}$, and in the latter, the automorphism transposing the simple roots α_8 and α_9, i.e., multiplying e_9 by -1. In any case, we obtain $\rho(h_0) = h_0$. □

Thus, the moduli space of normed multipolarized $K3$-surfaces of type $(D_{1,9}, h_0)$ is the quotient $\mathscr{L}_{10}/\Gamma_{10}$, where $\Gamma_{10} = O^+_{2,10}(\mathbb{Z})$. This is a "large" Zariski open subset of the affine variety $\operatorname{Spec} A(\mathscr{D}_{10}, O^+_{2,10}(\mathbb{Z}))$ in the sense that its boundary contains no divisors (in fact, it is two-dimensional).

5. A Projective Model

5.1. First, we recall the necessary information about vector bundles over the projective line $\mathbb{C}P^1$. We treat $\mathbb{C}P^1$ as the homogeneous space SL_2/B, where $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$ and $B \subset \mathrm{SL}_2$ is the Borel subgroup consisting of triangular matrices.

We begin with describing line bundles. Any line bundle over $\mathbb{C}P^1$ is homogeneous (in the sense that the action of the group SL_2 on the base space can be lifted to an action by bundle automorphisms) and hence is determined by the character of B defining its action on the fiber over the base point. We denote the line bundle corresponding to the kth power of the first diagonal element by $E(k)$.

For $k > 0$, the regular functions on $E(k)$ linear on the fibers determine an SL_2-equivariant birational morphism Φ of the variety $E(k)$ onto a closed subvariety $\mathfrak{S}(k)$ of the space of the irreducible representation of SL_2 with highest weight k, namely, onto $\mathrm{SL}_2 H$, where H is the one-dimensional subspace spanned by the highest-weight vector of the representation. This morphism contracts the zero section of the bundle to a point and is an isomorphism outside this point. For $k = 0$, the bundle is the direct product $\mathbb{C}P^1 \times \mathbb{C}^1$ and Φ is the projection onto the second factor. For $k < 0$, there exist no nonzero regular functions linear on the fibers.

By Grothendieck's theorem any vector bundle over $\mathbb{C}P^1$ is a direct sum of line bundles. Consider the bundle $E(k_1, \dots, k_s) = E(k_1) \oplus \dots \oplus E(k_s)$, where $k_1, \dots, k_s \geqslant 0$ and not all $k_1, \dots, k_s$ are zero. The regular functions on $E(k_1, \dots, k_s)$ linear on the fibers determine an SL_2-equivariant birational morphism Φ of the variety $E(k_1, \dots, k_s)$ onto a closed subvariety $\mathfrak{S}(k_1, \dots, k_s)$ of the direct sum of the spaces of irreducible representations of SL_2 with highest weights $k_1, \dots, k_s$, namely, onto $\mathrm{SL}_2 H$, where H is the subspace spanned by the highest-weight vectors of these representations. The fibers of $E(k_1, \dots, k_s)$ are mapped to the s-dimensional subspaces gH ($g \in \mathrm{SL}_2$), whose intersection is the subspace of SL_2-invariant vectors of dimension equal to the number of zeros among $k_1, \dots, k_s$. This subspace is the set of singular points of the variety $\mathfrak{S}(k_1, \dots, k_s)$.

The projectivization $P\mathfrak{S}(k_1, \dots, k_s)$ of the variety $\mathfrak{S}(k_1, \dots, k_s)$ is called a *scroll* of type $(k_1, \dots, k_s)$. This is a normal s-dimensional projective variety of degree $k_1 + \dots + k_s$ in the projective space of dimension $k_1 + \dots + k_s + s - 1$; its ideal is generated by quadratic forms (see, e.g., [14]). We refer to the variety $\mathfrak{S}(k_1, \dots, k_s)$ itself as a *linear scroll*, as opposed to the *projective scroll* $P\mathfrak{S}(k_1, \dots, k_s)$, and to the subspaces gH ($g \in \mathrm{SL}_2$) (or to their projectivizations) as the *fibers of the scroll* $\mathfrak{S}(k_1, \dots, k_s)$ (respectively, of the scroll $P\mathfrak{S}(k_1, \dots, k_s)$).

5.2. Now, let (X, S, h) be a multipolarized $K3$-surface of type $(D_{1,9}, h_0)$, where $h_0 \in D_{1,9}$ is the vector given by (1). Somewhat abusing notation, we identify the lattice S with the sublattice

$D_{1,9} \subset J_{3,19}$ and the vector h with the vector h_0. According to what was said in Section 3.1, the linear system $|h|$ gives a birational morphism

$$\varphi_h \colon X \to Y \subset \mathbb{C}P^5$$

onto a normal projective variety Y of degree 8 having only simple singularities.

First, consider the case where the given multipolarization is irrational, i.e., $S = S(X)$.

By C_i $(i = 1, 2, \dots)$ we denote the smooth rational curve on X corresponding to the simple root α_i of the lattice S. The inner products (h, α_i) presented in Fig. 2 show that the curves $C_1, \dots, C_7$ contract to the unique singular point $o \in Y$ (of type A_7) of the surface Y and the curves C_8, C_9, and C_{10} are mapped to some three lines $l_1, l_2, l_3 \subset Y$ intersecting in o. The surface Y contains no other lines or conics.

The vertex at infinity of the polyhedron $P(S)$ of type $\widetilde{D}_8$ corresponds to the isotropic vector

$$u_0 = \alpha_{10} + \alpha_2 + 2(\alpha_3 + \dots + \alpha_7) + \alpha_8 + \alpha_9 \in S = e_0 + e_1.$$

The linear system $|u_0|$ determines the structure of an elliptic fibration on X with singular fiber of type $\widetilde{D}_8$, and the curve C_1 is a section of this fibration, because $(u_0, \alpha_1) = 1$.

Since $(h, u_0) = 3$, it follows that the generic fibers of this elliptic fibration are mapped to irreducible curves of degree 3 in $\mathbb{C}P^5$ (passing through the point o). From dimension considerations, these curves must be flat, i.e., belong to projectivizations of some three-dimensional subspaces of $\mathbb{C}^6$. It follows that the image of any fiber is contained in a plane. Thus, we have a three-dimensional vector bundle $E(k_1, k_2, k_3)$ over $\mathbb{C}P^1$ together with a morphism

$$\Phi \colon E(k_1, k_2, k_3) \to \mathfrak{S}(k_1, k_2, k_3) \subset \mathbb{C}^6$$

linear on fibers. The projectivization of the morphism Φ is an extension of the morphism φ_h. Therefore, $Y \subset P\mathfrak{S}(k_1, k_2, k_3)$, and the images of the fibers of the elliptic fibration on X under consideration are intersections of Y with the fibers of the scroll $P\mathfrak{S}(k_1, k_2, k_3)$.

By abuse of language, we speak about an elliptic fibration on Y whose fibers are the images of the fibers of an elliptic fibration on X (but bearing in mind that all of them intersect in the point o). Note that the generic fibers of this fibration are nonsingular cubics in the corresponding planes, because otherwise they would be rational.

The ideal of the surface $Y \subset \mathbb{C}P^5$ is generated by quadratic forms generating the ideal of the scroll and some cubic forms which do not identically vanish on the scroll [14]. If F is one of such cubic forms, then the equation $F = 0$ defines a surface of degree 9 in the scroll, which is the union of the surface Y and some "extra" plane Π depending on the choice of F.

5.3. Let us determine the type (k_1, k_2, k_3) of the scroll defined in Section 5.2.

Proposition 2. *Up to permutation,*

$$(k_1, k_2, k_3) = (0, 1, 2).$$

Proof. Since the surface Y is not contained in any hyperplane of the space $\mathbb{C}P^5$, it follows that the scroll $\mathfrak{S}(k_1, k_2, k_3)$ generates the space $\mathbb{C}^6$. This means that $k_1 + k_2 + k_3 = 3$.

Recall that the images of all fibers of an elliptic fibration contain the point o; therefore, all fibers of $\mathfrak{S}(k_1, k_2, k_3)$ contain the one-dimensional subspace corresponding to this point. Hence at least one of the numbers k_1, k_2, and k_3 is zero. There remain the possibilities

$$(k_1, k_2, k_3) = (0, 1, 2) \quad \text{and} \quad (k_1, k_2, k_3) = (0, 0, 3).$$

Let us show that, in fact, only the former case can occur.

Suppose that $Y \subset P\mathfrak{S}(0, 0, 3)$, i.e., the representation of the group SL_2 on the space $\mathbb{C}^6$ is the sum of two trivial one-dimensional representations and the representation on the space of cubic binary forms. A vector belongs to the linear scroll $\mathfrak{S}(0, 0, 3)$ if and only if its last component is the cube of a linear form. All fibers of $\mathfrak{S}(0, 0, 3)$ contain the sum of the one-dimensional components of the representation.

Consider the 5-dimensional subspace $U \subset \mathbb{C}^6$ defined by the condition that the last component of the vector (which is a cubic form on $\mathbb{C}^2$) vanishes at a fixed nonzero vector $v \in \mathbb{C}^2$. Its intersection with the scroll $\mathfrak{S}(0,0,3)$ is the three-dimensional subspace consisting of vectors whose last component is proportional to the cube of a fixed linear form (vanishing at v), i.e., this is a fiber of $\mathfrak{S}(0,0,3)$. Accordingly, the intersection $PU \cap P\mathfrak{S}(0,0,3)$ is a plane P being a fiber of the projective scroll, and the intersection $P \cap Y$ is a fiber of the elliptic fibration. Appropriately choosing v, we can assume that this is a generic fiber, i.e., $P \cap Y$ is an irreducible cubic.

Now let F be a cubic form in the ideal of Y which does not vanish identically on the scroll $P\mathfrak{S}(0,0,3)$. The equation $F = 0$ determines the union of the surface Y and some plane Π in the scroll. Appropriately choosing such a form F in the ideal, we can achieve that $\Pi \neq P$. Then in the plane P the equation $F = 0$ defines the union of the intersections $P \cap Y$ and $P \cap \Pi$. The former is an irreducible cubic in P, and the latter coincides with the intersection $PU \cap \Pi$ and hence is a line. This contradicts the form F being cubic. Thus, the case $(k_1, k_2, k_3) = (0, 0, 3)$ is impossible. □

5.4. Since the lattice S is 2-elementary, there is an antisymplectic involution σ of X which acts as the identity map on $S = S(X)$ and as multiplication by -1 on $T = T(X)$. It determines a projective involution of the surface Y, which we denote by the same letter.

Acting on X, the involution σ takes each smooth rational curve to itself. Therefore, the intersection point of these curves is fixed. Since any nontrivial involutive automorphism of the projective line has precisely two fixed points and the involution σ, being antisymplectic, cannot act trivially on two intersecting curves, it follows that σ acts trivially on the curves C_1, C_3, C_5, and C_7 and nontrivially on C_2, C_4, C_6, C_8, C_9, and C_{10}.

Let us endow the generic fibers of the elliptic fibration with a group structure, taking the intersection point with the curve C_1 for the identity element. Then σ acts on each fiber as an inversion and on its image in Y as a projective involution for which o is an isolated fixed point. Since the surface Y is not contained in any hyperplane of $\mathbb{C}P^5$, it follows that o is an isolated fixed point of the involution σ in the whole space $\mathbb{C}P^5$. In other words, σ is induced by the linear transformation with matrix $\operatorname{diag}(-1,1,1,1,1,1)$ in some basis such that the first basis vector corresponds to the point o.

5.5. Now, let us see what changes when the lattice $S(X)$ is strictly larger than $D_{1,9}$.

In any case, the stabilizer of the point e_0 in the group $W(X)$ contains a finite reflection group of type D_9 and, therefore, is the direct product of a group of type D_p ($p \geqslant 9$) containing D_9 and, possibly, some other finite reflection group. Moreover, the roots $\alpha_1, \dots, \alpha_8$ can be included in the set of simple roots of the group D_p (but α_9 is no longer a simple root for $p > 9$). The following simple root of the group D_p is orthogonal to the vectors $e_0, e_1, \dots, e_8$ and hence to h_0, and all of the remaining simple roots of the stabilizer are orthogonal to the entire lattice $D_{1,9}$. We denote the number of simple roots of the stabilizer of e_0 by q.

At the next step of the algorithm, in any case, we can choose the root $\alpha_{q+1} = e_0 + e_1 + e_2 + e_3$ (which equals α_{10} in the case of irrational multipolarization), because its inner products with all roots already chosen are nonnegative, and the inner product $(\alpha_{q+1}, e_0) = 1$ is the least possible.

For $p = 10, 11$, the Coxeter diagrams of the system of simple roots of the group D_p and the root α_{q+1}, together with their inner products with h_0, are shown in Figs. 3 and 4.

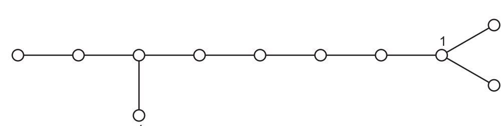

Fig. 3

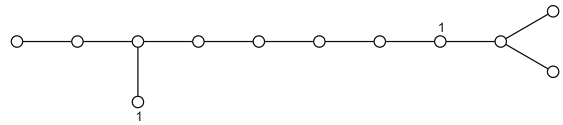

Fig. 4

A direct verification shows that the vector h_0 can be expressed linearly in terms of the simple roots of the group D_p and the root α_{q+1} with positive integer coefficients, namely,

$$h_0 = (3\alpha_1 + 6\alpha_2 + 9\alpha_3 + 8\alpha_4 + 7\alpha_5 + 6\alpha_6 + 5\alpha_7) \\ + 4(\alpha_8 + \dots + \alpha_{p-2}) + 2(\alpha_{p-1} + \alpha_p) + 4\alpha_{q+1}. \quad (2)$$

It follows that the inner products of this vector with all other simple roots are nonnegative as well. Under our identifications we can assume that h_0 is contained in the cone $A(X)$ of ample divisors of the surface X.

The isotropic vector $u_0 = e_0 + e_1$ is a positive integer linear combination of the roots $\alpha_2, \dots, \alpha_p$, α_{q+1}, which form an extended system of simple roots of type $\widetilde{D}_{p-1}$. Therefore, it is also contained in the cone $A(X)$. The system of all simple roots orthogonal to u_0 contains the subsystem $\widetilde{D}_{p-1}$ as an indecomposable component.

Suppose that there is another component Σ of the system of simple roots orthogonal to u_0. Then u_0 is a positive integer linear combination of roots in Σ [12, Sec. 1.9]. Since $(u_0, \alpha_1) = 1$, it follows that all roots of the system Σ, except a single root β which occurs in the decomposition of u_0 with coefficient 1, are orthogonal to α_1, and $(\beta, \alpha_1) = 1$. Since simple roots from different components are orthogonal, it follows from (2) that

$$(h_0, u_0) = (h_0, \beta) = 3.$$

This means that the singular fiber of the elliptic fibration of X, which corresponds to the component Σ, is mapped onto a singular irreducible plane cubic in Y.

In the general case, the antisymplectic involution σ of the surface X is defined in a different way. Namely, the involution σ_0 of $J_{3,19}$ which acts trivially on $S = D_{1,9}$ and as multiplication by -1 on the orthogonal complement preserves the lattice $S(X) \supset S$, but it does not generally preserve the cone $A(X)$. To return this cone back in its place, we must multiply σ_0 by a suitable element $w_0 \in W(X)$ trivially acting on S (in particular, belonging to the stabilizer of e_0). Then $\sigma = w_0\sigma_0$ is the required involution.

The involution σ_0 preserves the sublattice $D_p \subset D_{1,p}$; hence σ somehow permutes the simple roots $\alpha_1, \dots, \alpha_p$ of this lattice and the corresponding curves $C_1, \dots, C_p$. Clearly, in reality, only the roots α_{p-1} and α_p can be permuted.

The smooth rational curve on X corresponding to a simple root α is mapped to a line on Y if and only if $(h_0, \alpha) = 1$. For $p = 9$, only three simple roots among those occurring in the expression (2) of h_0 satisfy this condition, and for $p > 9$, there are only two such roots. They correspond to three lines $l_1, l_2, l_3 \subset Y$, which merge into two lines for $p > 9$. As seen from (2), the inner products of the vector h_0 with other simple roots either vanish or are at least 2. Hence there are no other lines on the surface Y.

Thus, in the general case, the projective model $\varphi_h(X) = Y \subset \mathbb{C}P^5$ has all those properties which we proved for the case of irrational polarization, except that the lines l_1 and l_2 may merge, singular points different from o may arise, the type of the singular fiber of the elliptic fibration may become more complex, and new singular fibers may emerge, which are represented by singular irreducible flat cubics in the projective model.

6. The Scroll $\mathfrak{S}(0,1,2)$

6.1. Let us study the scroll $\mathfrak{S} = \mathfrak{S}(0,1,2) \subset \mathbb{C}^6$ in detail.

The representation of the group SL_2 on the space $\mathbb{C}^6$ decomposes as

$$\mathbb{C}^6 = \mathbb{C} + \mathbb{C}^2 + S^2\mathbb{C}^2, \tag{3}$$

where SL_2 acts trivially on the first term and tautologically on $\mathbb{C}^2$. Let $\{v_1, v_2\}$ be a basis in $\mathbb{C}^2$. In the space $\mathbb{C}^6$ for basis vectors we take

$$1,\ v_1,\ v_2,\ v_1^2,\ v_2^2,\ 2v_1v_2.$$

We denote coordinates in this basis by

$$x_0,\ x_1,\ x_2,\ y_1,\ y_2,\ z.$$

A condition for a vector to belong to the scroll $\mathfrak{S}$ is the proportionality of its third component and squared second component. In coordinates this means that the matrices

$$\begin{pmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1 & z \\ z & y_2 \end{pmatrix}$$

are proportional, which is equivalent to the system of equations

$$x_2y_1 = x_1z, \quad x_1y_2 = x_2z, \quad y_1y_2 = z^2. \tag{4}$$

It is easy to see that any quadratic form in x_0, x_1, x_2, y_1, y_2, and z vanishing on $\mathfrak{S}$ is a linear combination of the forms $x_2y_1 - x_1z$, $x_1y_2 - x_2z$, and $y_1y_2 - z^2$. Therefore, these three forms generate the ideal of the scroll $\mathfrak{S}$, or, in other words, the equations (4) are defining relations of the algebra of polynomial functions on $\mathfrak{S}$. Using these relations, we can uniquely represent any polynomial function on $\mathfrak{S}$ as a linear combination of monomials not containing the products x_2y_1, x_1y_2, and y_1y_2. We refer to such representations as *reduced*.

6.2. Let us describe the 3-dimensional subspaces contained in the scroll $\mathfrak{S}$. First, these are the fibers of the scroll. In terms of the decomposition (3), these are the subspaces of the form

$$L(v) = \langle 1, v, v^2 \rangle \qquad (v \in \mathbb{C}^2,\ v \neq 0).$$

Moreover, $\mathfrak{S}$ contains also the 3-dimensional subspace

$$L(0) = \mathbb{C} + \mathbb{C}^2.$$

Lemma 3. *Any 3-dimensional subspace contained in $\mathfrak{S}$ is one of the subspaces $L(v)$ and $L(0)$ described above.*

Proof. Note that the projection of $\mathfrak{S}$ on the third term $S^2\mathbb{C}^2$ of the decomposition (3) is the quadratic cone determined by the equation $y_1y_2 = z^2$. It contains no 2-dimensional subspaces.

Now suppose that $L \subset \mathfrak{S}$ is a 3-dimensional subspace different from $L(0)$. Then its projection on $S^2\mathbb{C}^2$ is one-dimensional and has the form $\langle v^2 \rangle$, where $v \in \mathbb{C}^2$, $v \neq 0$. Therefore, the projection of L on the second term equals $\langle v \rangle$; this means that $L = L(v)$. $\square$

6.3. A projective model of a $K3$-surface X in a fixed scroll $P\mathfrak{S}$ is determined uniquely up to a projective automorphism of the scroll. Any such automorphism originates from a linear automorphism of the linear scroll $\mathfrak{S}$.

Obviously, the linear automorphism group $\operatorname{Aut}\mathfrak{S}$ of $\mathfrak{S}$ contains the group SL_2 and the 3-torus Z acting by scalar multiplication on each term in (3), and there are no other automorphisms preserving the decomposition (3).

In any case, every automorphism preserves the first term of the decomposition (3) (as the intersection of all fibers) and the sum of the first two terms (as a unique 3-dimensional subspace not being a fiber). Therefore, $Z \cdot \mathrm{SL}_2$ is a maximal reductive subgroup of the group $\operatorname{Aut}\mathfrak{S}$, and the unipotent radical of the group $\operatorname{Aut}\mathfrak{S}$ preserves the flag $\mathbb{C} \subset L(0) \subset \mathbb{C}^6$ and acts trivially on its factors. A direct verification shows that this radical consists of the products of the transformations

$$(x_0, x_1, x_2, y_1, y_2, z) \mapsto (x_0 + l(x_1, x_2, y_1, y_2, z), x_1, x_2, y_1, y_2, z), \tag{5}$$

where l is any linear form, and

$$(x_0, x_1, x_2, y_1, y_2, z) \mapsto (x_0, x_1 + \alpha y_1 + \beta z, x_2 + \beta y_2 + \alpha z, y_1, y_2, z), \tag{6}$$

The automorphisms of the scroll that preserve some of the subspaces $L(v)$ (a fiber of the scroll) form a Borel subgroup $B(v)$ of $\operatorname{Aut}\mathfrak{S}$, which is a semidirect product of the unipotent radical of $\operatorname{Aut}\mathfrak{S}$ and some Borel subgroup of $Z \cdot \mathrm{SL}_2$. In particular, $B(v_2)$ contains a Borel subgroup of the group SL_2, whose unipotent radical consists of all transformations of the form

$$(x_0, x_1, x_2, y_1, y_2, z) \mapsto (x_0, x_1, x_2 + \gamma x_1, y_1, y_2 + 2\gamma z + \gamma^2 y_1, z + \gamma y_1). \tag{7}$$

The following transformations constitute a maximal torus T of $Z \cdot \mathrm{SL}_2$ contained in $B(v_2)$:

$$(x_0, x_1, x_2, y_1, y_2, z) \mapsto (t_0 x_0, t_1 t x_1, t_1 t^{-1} x_2, t_2 t^2 y_1, t_2 t^{-2} y_2, t_2 z). \tag{8}$$

(The parameters $t_0, t_1, t_2, t \in \mathbb{C}^*$ are determined up to simultaneous multiplication of t and t_1 by -1.)

Proposition 3. *The group $B(v_2)$ acts transitively on the complement of $L(v_2) \cup L(0)$ in the scroll $\mathfrak{S}$.*

Proof. Let X be a vector in the complement mentioned in the statement, and let X_0, X_1, and X_2 be its projections on the components of the decomposition (2). Then, in particular, $X_2 \neq 0$. Using transformations of the form (2), we can achieve that $X_0, X_1 \neq 0$. Moreover, the component X_1 is not proportional to v_2, and using a transformation of the form (7), we can make it proportional to v_1. Then the component X_2 will become proportional to v_1^2. Finally, using a transformation of the form (8), we can map the obtained vector to $1 + v_1 + v_1^2$. □

6.4. Consider the rational differential 4-form

$$\Omega = \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dz}{x_2} \tag{9}$$

on the nonsingular points of the scroll $\mathfrak{S}$.

Proposition 4. *The form Ω is semi-invariant with respect to the group $B(v_2)$ with weight $(t_0 t_1 t_2 t)^{-1}$ (in the notation of (8)).*

Proof. Since the projection of the scroll on the sum of the last two components in (3) is three-dimensional, it follows that any four of the five differentials dx_1, dx_2, dy_1, dy_2, and dz are linearly dependent on $\mathfrak{S}$. Therefore, the form Ω is invariant with respect to the transformations (5).

Differentiating the first relation in (4), we obtain

$$y_1\, dx_2 + x_2\, dy_1 = z\, dx_1 + x_1\, dz, \tag{10}$$

which implies

$$dx_2 = \frac{z\, dx_1 + x_1\, dz - x_2\, dy_1}{y_1}.$$

Substituting this expression into the definition of the form Ω, we obtain the alternative representation

$$\Omega = -\frac{dx_0 \wedge dx_1 \wedge dy_1 \wedge dz}{y_1} \tag{11}$$

of this form. It easily follows that Ω is invariant with respect to the transformations (6) and (7).

Finally, the transformation (8) multiplies Ω by $(t_0 t_1 t_2 t)^{-1}$. □

Proposition 5. *The form Ω is regular; its zero divisor is the (one-fold) subspace $L(v_2)$.*

Proof. Since the form Ω semi-invariant with respect to the group $B(v_2)$, it follows that Ω is everywhere defined and nowhere vanishes on the open orbit of this group. It remains to examine its behavior on the subspaces $L(0)$ and $L(v_2)$.

Considering the Jacobian matrix of the relations (4), we easily see that x_0, x_1, x_2, and z form a coordinate system in a domain of $\mathfrak{S}$ which intersects $L(0)$; thus, it follows directly from the definition of the form Ω that the subspace $L(0)$ is not contained in its divisor.

Similarly, x_0, x_2, y_2, and z form a coordinate system in a domain of $\mathfrak{S}$ which intersects $L(v_2)$. In the same way as in the proof of Proposition 4, using the second relation in (4), we can obtain yet another representation of the form Ω:

$$\Omega = \frac{z dx_0 \wedge dx_2 \wedge dy_2 \wedge dz}{y_2^2}. \tag{12}$$

It is seen that the form Ω is defined and vanishes with multiplicity 1 on the subspace $L(v_2)$. □

6.5. The involution σ defined in Section 5.4 is conjugate in $\operatorname{Aut}\mathfrak{S}$ to an element of the maximal reductive subgroup $Z\cdot\mathrm{SL}_2$. Applying the corresponding projective automorphism of the scroll $P\mathfrak{S}$ to the surface Y, we can achieve that $\sigma \in Z\cdot\mathrm{SL}_2$. Obviously, σ then acts on $\mathbb{C}^6$ as the multiplication of the coordinate x_0 by -1:

$$\sigma\colon (x_0, x_1, x_2, y_1, y_2, z) \mapsto (-x_0, x_1, x_2, y_1, y_2, z). \tag{13}$$

Under this condition the projective model Y is determined uniquely up to an automorphism of $P\mathfrak{S}$ commuting with σ.

Let $\operatorname{Aut}(\mathfrak{S}, \sigma)$ denote the group of all linear automorphisms of $\mathfrak{S}$ commuting with σ defined by (13), i.e., preserving the sum of the last two terms in the decomposition (3). Obviously, this group contains $Z\cdot\mathrm{SL}_2$ as a maximal reductive subgroup, and its unipotent radical consists of transformations of the form (6).

7. The Canonical Equation

7.1. Let X be a multipolarized $K3$-surface of the type considered in Section 5, and let Y be its projective model in the scroll $P\mathfrak{S} = P\mathfrak{S}(0,1,2)$. There exists a homogeneous cubic polynomial F in x_0, x_1, x_2, y_1, y_2, and z such that the equation $F = 0$ in the scroll defines the union of the surface Y and some scroll fiber $PL(v)$.

We can assume the polynomial F to be reduced (see Section 6.1) and σ-invariant. The latter condition means that all terms of this polynomial contain x_0 to an even power.

Acting on Y by the group $\mathrm{SL}_2 \subset \operatorname{Aut}\mathfrak{S}$, we can achieve that the singular fiber of the elliptic fibration containing the lines l_1, l_2, and l_3 (which intersect in the point o) is contained in the plane $PL(v_2)$, i.e., that

$$Y \cap PL(v_2) = l_1 \cup l_2 \cup l_3. \tag{14}$$

Under this condition the surface Y is determined uniquely up to a transformation in the Borel subgroup $B(v_2)$ of the group $\operatorname{Aut}\mathfrak{S}$.

After that, appropriately choosing the polynomial F in the ideal of the surface Y, we can achieve that $v = v_2$, i.e., that the polynomial F has no terms depending only on x_0, x_2, and y_2.

To satisfy condition (14), we must move the extra plane $PL(v_2)$, which prevents us from seeing the intersection of the surface Y with this plane. This can be done by multiplying the polynomial F by x_2/x_1. Taking into account the relations

$$\frac{x_2}{x_1} = \frac{y_2}{z} = \frac{z}{y_1}, \tag{15}$$

which hold on the scroll according to (4), we obtain another cubic polynomial $\widetilde{F}$ in the ideal of the surface Y, which vanishes on the fiber $PL(v_1)$ rather than on $PL(v_2)$. We refer to this new polynomial as the *satellite* of the polynomial F. It follows from condition (14) that it does not contain the terms $x_0^2x_2$ and $x_0^2y_2$, or, equivalently, the initial polynomial F does not contain the terms $x_0^2x_1$ and x_0^2z.

Thus, if the polynomial F has a term containing x_0, then this term must be $x_0^2y_1$. On the other hand, such a term must be present, because otherwise the surface Y would be a cone, which is impossible.

Next, it follows from preceding considerations that the restriction of the polynomial F to the subspace $L(0) = \langle v_1, v_2\rangle$ is a cubic form $f(x_1, x_2)$, and hence in the plane $PL(0)$ the equation $F = 0$ defines three lines intersecting in o. One of them is the line $l = PL(0) \cap PL(v_2)$ defined by the equation $x_1 = 0$ in $PL(0)$. The other two lines must be contained in the surface Y and, therefore, in the same plane $P(v_2)$. Thus, the equation $F = 0$ defines a triple line l in the plane $PL(0)$. This means that $f(x_1, x_2) = Bx_1^3$, where $B \neq 0$.

7.2. Acting on the polynomial F by transformations (6) and (7) in the unipotent radical of the group $B(v_2) \cap \mathrm{Aut}(\mathfrak{S}, \sigma)$ and using the term x_1^3, we can kill the coefficients of $x_1^2y_1$, x_1^2z, and x_1x_2z. The parameters α, β, and γ of these transformations are determined uniquely.

As a result, we obtain

$$F = Ax_0^2y_1 + Bx_1^3 + Cx_2^2z + (ax_1 + by_1)y_1^2 + (a_1x_1y_1 + a_2x_2y_2 + b_1y_1^2 + b_2y_2^2)z + (f_1x_1 + f_2x_2 + g_1y_1 + g_2y_2 + hz)z^2 \qquad (A, B \neq 0). \tag{16}$$

This form of the polynomial F is determined uniquely up to the transformations (8) in the maximal torus of the group $B(v_2) \cap \mathrm{Aut}(\mathfrak{S}, \sigma)$.

The satellite of the polynomial (16) has the form

$$\widetilde{F} = Ax_0^2z + Bx_1^2x_2 + (Cx_2^2 + a_2x_2y_2 + b_2y_2^2)y_2 + (ax_1 + by_1)y_1z + (f_2x_2 + g_2y_2)y_2z + (a_1x_1 + f_1x_2 + b_1y_1 + hy_2 + g_1z)z^2. \tag{17}$$

It is easy to see that, at all points of the line $P\langle 1, v_2\rangle$ on the tangent space of the scroll $P\mathfrak{S}$, on the affine chart $x_0 = 1$ the relation $d\widetilde{F} = Cx_2^2\,dy_2$ holds, so that, for $C = 0$, all points of this line are singular points of the surface Y, which is impossible. Therefore, $C \neq 0$.

Let T° denote the 3-torus formed by the transformations (8) preserving the form Ω, i.e., satisfying the condition

$$t_0t_1t_2t = 1.$$

Applying transformations in the torus T°, we can make the coefficients A, B, and C equal to 1. Indeed, it suffices to solve the system of equations

$$t_0t_1t_2t = 1, \quad t_0^2t_2t^2 = A, \quad t_1^3t^3 = B, \quad t_1^2t_2t^{-2} = C.$$

It is easy to see that this system does have a solution. Moreover, its solution is unique up to the simultaneous multiplication of t_1 and t by -1, which does not change torus elements, and the simultaneous multiplication of t_0, t_1, and t_2 by a cubic root of 1, which does not change the action of torus elements on the space of cubic forms.

Thus, applying transformations in $B(v_2) \cap \mathrm{Aut}(\mathfrak{S}, \sigma)$ preserving the form Ω to the polynomial F, we can reduce F to the form

$$F = x_0^2y_1 + x_1^3 + x_2^2z + (ax_1 + by_1)y_1^2 + (a_1x_1y_1 + a_2x_2y_2 + b_1y_1^2 + b_2y_2^2)z + (f_1x_1 + f_2x_2 + g_1y_1 + g_2y_2 + hz)z^2, \tag{18}$$

and this form is determined uniquely. We refer to the polynomials F of this form as *canonical*.

7.3. Let R denote the set of all canonical polynomials (18). This is an 11-dimensional affine space. On this space, in addition to the torus T°, the 1-torus $T^1 \subset T$ preserving the monomials $x_0^2y_1$, x_1^3, and x_2^2z acts. It is easy to see that this torus consists of all transformations of the form

$$g(t)\colon (x_0, x_1, x_2, y_1, y_2, z) \mapsto (t^3x_0, x_1, t^2x_2, t^{-6}y_1, t^{-2}y_2, t^{-4}z). \tag{19}$$

A direct calculation shows that the transformation $g(t)$ multiplies the form Ω by t and the monomials occurring in the canonical polynomial (18) with indeterminate coefficients, by powers of t with exponents specified in the following table. (The first row of the table contains the notation of the corresponding coefficients.)

a	b	a_1	a_2	b_1	b_2	f_1	f_2	g_1	g_2	h
$x_1y_1^2$	y_1^3	x_1y_1z	x_2y_2z	y_1^2z	y_2^2z	x_1z^2	x_2z^2	y_1z^2	y_2z^2	z^3
12	18	10	4	16	8	8	6	14	10	12

8. Examination of the Canonical Surface

8.1. Consider the inverse problem. Let F be the canonical polynomial given by (18). Then the equation $F = 0$ defines the union of some surface Y of degree 8 and the plane $PL(v_2)$ in the scroll $P\mathfrak{S}$, and the equation $\widetilde{F} = 0$, where $\widetilde{F}$ is the satellite of F, defines the union of the surface Y and the plane $PL(v_1)$. The surface Y itself is defined by the system of equations

$$F = \widetilde{F} = 0 \tag{20}$$

in $P\mathfrak{S}$. Let us find sufficient conditions for the surface Y thus defined to be a $K3$-surface with simple singularities.

The intersection of Y with the plane $PL(v_2)$ is given in this plane by the equation

$$(x_2^2 + a_2x_2y_2 + b_2y_2^2)y_2 = 0; \tag{21}$$

this is a triple of lines l_1, l_2, and l_3 intersecting in the point o, the first two of which merge when

$$4b_2 = a_2^2. \tag{22}$$

Now let us find the intersections of the surface Y with the other fibers of the scroll $P\mathfrak{S}$. The fiber $L(v_1 + \tau v_2)$ of the scroll $\mathfrak{S}$ is given in $\mathbb{C}^6$ by the linear equations

$$x_2 = \tau x_1, \quad y_2 = \tau^2 y_1, \quad z = \tau y_1.$$

Substituting these expressions for x_2, y_2, and z into (18), we obtain a representation of the restriction of the polynomial F to $L(v_1 + \tau v_2)$ in Weierstrass' normal form:

$$F = x_0^2y_1 + x_1^3 + \tau^3x_1^2y_1 + \phi_2(\tau)x_1y_1^2 + \phi_3(\tau)y_1^3, \tag{23}$$

where

$$\phi_2(\tau) = a_2\tau^4 + f_2\tau^3 + f_1\tau^2 + a_1\tau + a, \tag{24}$$

$$\phi_3(\tau) = b_2\tau^5 + g_2\tau^4 + h\tau^3 + g_1\tau^2 + b_1\tau + b. \tag{25}$$

Thus, we see that the intersection

$$Y(\tau) = Y \cap PL(v_1 + \tau v_2)$$

is an irreducible cubic curve. Therefore, the surface Y itself is irreducible as well.

8.2. Let us now consider what singular points other than o the surface Y can have.

First, clearly, a point $p \in Y \cap PL(v)$ different from o can be a singular point of Y only if it is a singular point of the curve $Y \cap PL(v)$. As applied to $v = v_2$, this means that the surface Y can have singular points different from o in the fiber $PL(v_2)$ only under the condition (22), when the lines l_1 and l_2 merge, and these points can be only some points of this double line, which we denote by l_0.

To be more precise, under condition (22) the line l_0 is defined by the equation

$$2x_2 + a_2y_2 = 0$$

in $PL(v_2)$. At the points of this line on the tangent space of the scroll $\mathfrak{S}$, on the affine chart $x_0 = 1$, the relation

$$d\widetilde{F} = (1 + (f_2x_2 + g_2y_2)y_2)dz = (1 + (g_2 - \tfrac{1}{2}a_2f_2)y_2^2)\,dz$$

holds. Therefore, in addition to o, the surface Y has two singular points on the line l_0, which are defined by the condition

$$x_0^2 + (g_2 - \tfrac{1}{2}a_2f_2)y_2^2 = 0$$

in homogeneous coordinates; they merge into one singular point when

$$2g_2 = a_2f_2. \tag{26}$$

Moreover, the surface Y can have one singular point different from o on the curve $Y(\tau)$ if the discriminant $D(\tau)$ of the cubic polynomial

$$f = x^3 + \tau^3x^2 + \phi_2(\tau)x + \phi_3(\tau) \in \mathbb{C}[\tau][x] \tag{27}$$

vanishes at τ. It may happen that this discriminant identically vanishes. Let us show that this happens only rarely.

Let R denote the set of all canonical polynomials (18). This is an 11-dimensional affine space. The set R_0 of polynomials $F \in R$ for which the discriminant $D(\tau)$ identically vanishes is a closed subvariety in R.

Proposition 6. $\dim R_0 = 2$.

Proof. If the discriminant $D(\tau)$ identically vanishes, then the polynomial (27) has a multiple root in $\mathbb{C}[\tau]$ and, therefore, can be represented in the form

$$f = (x - p(\tau))^2(x - q(\tau)),$$

where p and q are some polynomials. The relations

$$2p(\tau) + q(\tau) = -\tau^3, \quad p(\tau)^2q(\tau) = -\phi_3(\tau)$$

imply that p and q are polynomials of degrees 1 and 3, respectively, and the polynomial q is uniquely determined by the polynomial p, which can be chosen arbitrarily. Thus, a polynomial $F \in R_0$ depends on two parameters. □

8.3. It follows from the above considerations that if $F \in R \setminus R_0$, then the surface Y has only finitely many singularities. Let us find sufficient conditions under which all of them are simple.

Proposition 7. *For any polynomial $F \in R \setminus R_0$, the surface Y has a simple singularity of type A at the point o.*

Proof. The surface Y is defined in $\mathbb{C}P^5$ by the three quadratic equations (4) and the two cubic equations (20). On the affine chart $x_0 = 1$ the coordinates y_1 and z in a neighborhood of o can be expressed from (20) in the form of formal power series in x_1, x_2, and y_2 with lower-order terms of degree 3. Thus, the singularity of the surface Y at o is determined by an ideal I of the ring $\mathbb{C}[[x_1, x_2, y_2]]$. In particular, substituting the expression for z found above into the equation $x_1y_2 = y_2z$, we obtain an element of I with lower-order term x_1y_2, which is a quadratic form of corank 1. Therefore, o is a simple singularity of type A [15]. □

Now suppose that $p \in Y(\tau_0)$ is a singular point of Y different from o. Then p is a singular point of the curve $Y(\tau_0)$. If this singularity is a simple self-intersection, then its 2-jet is nondegenerate, and hence the rank of its 2-jet as a singularity of Y is at least 2. Thus, in this case, p is a simple singularity of type A of the surface Y.

Finally, suppose that p is a cusp of the curve $Y(\tau_0)$, i.e., (27) is a perfect cube for $\tau = \tau_0$. Making the change $x = \bar{x} - \frac{1}{3}\tau^3$, we obtain

$$f = \bar{x}^3 + \bar{\phi}_2(\tau)\bar{x} + \bar{\phi}_3(\tau), \tag{28}$$

where

$$\bar\phi_2(\tau) = -\tfrac{1}{3}\tau^6 + \phi_2(\tau), \tag{29}$$

$$\bar\phi_3(\tau) = \tfrac{2}{27}\tau^9 - \tfrac{1}{3}\tau^3\phi_2(\tau) + \phi_3(\tau), \tag{30}$$

and $\bar\phi_2(\tau_0) = \bar\phi_3(\tau_0) = 0$. The singularity of the surface Y at the point p is equivalent to the singularity of the polynomial (28) as a function of $\bar x$ and τ at the point $\bar x = 0$, $\tau = \tau_0$.

Let us expand the polynomials $\bar\phi_2(\tau)$ and $\bar\phi_3(\tau)$ in powers of $\bar\tau = \tau - \tau_0$ and consider the lowest-order term of the obtained representation of f in the form of a polynomial in $\bar x$ and $\bar\tau$. This term cannot be of the first degree, because if it were, then the point p would not be a singular point of the surface Y. If this term is of the second or the third degree, then p is a simple singularity of type A_1, A_2, or D_4. If this is $\bar\tau^4$, $\bar\tau^3\bar x$, or $\bar\tau^5$, then p is a simple singularity of type E_6, E_7, or E_8, respectively.

Thus, the point p is not a simple singularity of the surface Y only if τ_0 is a root of multiplicity $\geqslant 4$ of the polynomial $\bar\phi_2$ and, simultaneously, a root of multiplicity $\geqslant 6$ of the polynomial $\bar\phi_3$.

Let us denote the set of those polynomials $F \in R$ for which there exists such a τ_0 by R_1. This is a closed algebraic subvariety in R.

Proposition 8. $\dim R_1 = 2$.

Proof. The condition under consideration means that the polynomials $\bar\phi_2$ and $\bar\phi_3$ can be represented in the form

$$\bar\phi_2(\tau)(= -\tfrac{1}{3}\tau^6 + \phi_2(\tau)) = -\tfrac{1}{3}(\tau^2 + p_1\tau + p_2)(\tau - \tau_0)^4, \tag{31}$$

$$\bar\phi_3(\tau)(= \tfrac{2}{27}\tau^9 - \tfrac{1}{3}\tau^3\phi_2(\tau) + \phi_3(\tau)) = \tfrac{2}{27}(\tau^3 + q_1\tau^2 + q_2\tau + q_3)(\tau - \tau_0)^6. \tag{32}$$

Adding the first equality multiplied by $\frac{1}{3}\tau^3$ to the second one, we obtain

$$-\tfrac{1}{27}\tau^9 + \phi_3(\tau) = \tfrac{2}{27}(\tau^3 + q_1\tau^2 + q_2\tau + q_3)(\tau - \tau_0)^6 - \tfrac{1}{9}\tau^3(\tau^2 + p_1\tau + p_2)(\tau - \tau_0)^4. \tag{33}$$

Comparing the coefficients of τ^5 in (31) and the coefficients of τ^8 in (32), we can express p_1 and q_1 in terms of τ_0, and comparing the coefficients of τ^7 and τ^6 in (33), we can express q_2 and q_3 in terms of τ_0 and p_2. The parameters τ_0 and p_2 can be chosen arbitrarily. Thus, a polynomial $F \in R_1$ depends on two parameters. □

Finally, consider the singular points on the double line $l_0 \subset Y \cap PL(v_2)$ (under condition (22)).

On the affine chart $y_2 = 1$ the scroll $P\mathfrak{S}$ is given by the equations

$$x_1 = x_2 z, \quad y_1 = z^2, \tag{34}$$

so that we can take x_0, x_2, and z for its internal coordinates. In these coordinates the line l_0 is defined by the equations

$$x_2 = -\tfrac{1}{2}a_2, \quad z = 0,$$

and the singular points on it are given by

$$x_0^2 = \tfrac{1}{2}a_2 f_2 - g_2.$$

In a neighborhood of the fiber $PL(v_2)$ of the scroll $P\mathfrak{S}$ punctured at o the surface Y is defined by the equation $\widetilde F = 0$. Taking into account relations (34), we can write the polynomial $\widetilde F$ in the internal coordinates on the scroll chosen above as

$$\begin{aligned}\widetilde F = x_0^2 z + x_2^2 z^3 + x_2^2 + a_2 x_2 + b + (a x_2 + b z) z^4 + (f_2 x_2 + g_2) z \\ + (a_1 x_2 + b_1 z) z^3 + (f_1 x_2 + h) z^2 + g_1 z^3.\end{aligned} \tag{35}$$

A direct verification shows that

$$\tfrac{1}{2} d^2 \widetilde F = 2x_0\, dx_0\, dz + (dx_2)^2 + f_2\, dx_2\, dz + f_1 x_2 (dz)^2$$

at the singular points of interest to us. For $x_0 \neq 0$, the rank of this quadratic form is at least 2, so that we have two simple singular points of type A. At $x_0 = 0$, where the singular points merge, the rank of this quadratic form equals 1 and its kernel is given by the linear equation $dx_2 + \frac{1}{2} f_2\, dz = 0$. Calculating the third differential, we obtain

$$\tfrac{1}{6} d^3 \widetilde{F} = (dx_0)^2 dz + (x_2^2 + a_1 x_2 + g_1)(dz)^3 + f_1\, dx_2 (dz)^2,$$

which, together with the constraint on the kernel of the second differential, gives the cubic form $(dx_0)^2 dz + k(dz)^3$, where k is some coefficient. This means that, in the case under consideration, we have a singular point of type D.

Thus, if $F \in R \setminus (R_0 \cup R_1)$, then the surface Y is irreducible and has only simple singularities.

8.4. Now we explicitly construct a symplectic form on the variety Y^{reg} of nonsingular points of Y under the assumption $F \in R \setminus (R_0 \cup R_1)$.

Let $\widehat{Y}$ denote the cone over Y, and let $\widehat{Y}^{\mathrm{reg}}$ denote the variety of its nonsingular points. At each point $v \in \widehat{Y}^{\mathrm{reg}} \setminus L(v_2)$ we "divide" the 4-form Ω constructed in Section 6 by dF, i.e., find a 3-form Φ such that

$$\Omega = dF \wedge \Phi.$$

This division is not unique, but the restriction of the form Φ to the tangent space of the cone is determined uniquely.

To define Φ at the points $v \in \widehat{Y}^{\mathrm{reg}} \cap L(v_2)$, we use the representation (12) of the form Ω, which implies

$$\frac{x_2}{x_1}\,\Omega = \tilde{\Omega} = \frac{dx_0 \wedge dx_2 \wedge dy_2 \wedge dz}{y_2}\,.$$

Thus, in this case, instead of dividing Ω by dF, we can divide $\tilde{\Omega}$ by $d\widetilde{F} = \frac{x_2}{x_1}\, dF$. This makes it possible to determine the form Φ on the entire variety $\widehat{Y}^{\mathrm{reg}}$. Obviously, it vanishes nowhere.

Further, substituting the radius vector of a point v into the form Φ as one of the arguments, we obtain a 2-form $\hat{\omega}$ on $\widehat{Y}^{\mathrm{reg}}$, whose kernel at each point is the one-dimensional subspace spanned by the radius vector of v.

Finally, we prove that the form $\hat{\omega}$ is invariant with respect to homothety. Indeed, the homothety with coefficient λ of $\mathbb{C}^6$ is implemented by an element of the torus T with $t_0 = t_1 = t_2 = \lambda$ and $t = 1$. This element multiplies the form Ω and the polynomial F by λ^3 and, therefore, preserves the form Φ. Since the field of radius vectors is invariant with respect to homothety, it follows from the invariance of Φ that $\hat{\omega}$ is invariant as well.

Thus, the form $\hat{\omega}$ determines a nowhere vanishing 2-form ω on $Y^{\mathrm{reg}} = P\widehat{Y}^{\mathrm{reg}}$. This means that Y is a projective model of some $K3$-surface in our family.

9. Proof of the Main Theorem

9.1. According to Torelli's theorem for $K3$-surfaces (see, e.g., [16]), integrating the form ω canonically constructed from F as above over the transcendental cycles of the surface Y, we obtain an isomorphism of analytic varieties

$$p\colon R \setminus (R_0 \cup R_1) \xrightarrow{\sim} \mathscr{L}_{10}/\Gamma_{10} \tag{36}$$

(the period map).

Proposition 9. *The map p is equivariant with respect to the action of the 1-torus T^1 introduced in Section 7.3 under the assumption that T^1 acts on $\mathscr{L}_{10}$ as multiplication by t^{-1}.*

Proof. Let $F \in R \setminus (R_0 \cup R_1)$, and let $Y \subset P\mathfrak{S}$ be the corresponding projective surface. Given an element $g(t) \in T^1$, the polynomial $g(t)F$ corresponds to the surface $g(t)Y$ isomorphic to Y, and we need only compare the image of the form ω on Y constructed from F by "dividing" the form Ω by dF (as in Section 8.4) with the corresponding form ω_t on the surface $g(t)Y$.

Clearly, the form $g(t)\omega$ can be obtained by dividing the form $g(t)\Omega = t\Omega$ by $g(t)\, dF = d(g(t)F)$. On the other hand, the form ω_t is obtained by dividing Ω by $d(g(t)F)$. Therefore, $\omega_t = t^{-1} g(t)\omega$. $\square$

The action of the torus T^1 on R described in Section 7.3 defines a grading of the algebra $\mathbb{C}[R]$, which is the algebra of polynomials in the coefficients $a, b, \dots, h$ of the canonical polynomial. The degrees of these variables in this grading are, respectively,

$$12,\ 18,\ 10,\ 4,\ 16,\ 8,\ 8,\ 6,\ 14,\ 10,\ 12.$$

Proposition 9 means that the map p induces an isomorphism of graded algebras

$$p*\colon A(\mathscr{D}_{10}, \Gamma_{10}) \to \mathbb{C}[R],$$

which implies the assertion of the main theorem for $n = 10$.

9.2. Consider the possible degeneracies. In the case where $S(X) \supset D_{1,10}$, the lines l_1 and l_2 on the surface Y merge (see Fig. 3), i.e., relation (22) holds. This means that one form of weight 8 becomes proportional to the squared generator of weight 4 and, thereby, falls out of the set of generators of the algebra of automorphic forms and. There cannot be any other relations from dimension considerations. This gives the assertion of the theorem for $n = 9$.

Next, if $S(X) \supset D_{1,11}$, then two singular points of type A_1 on the double line l_0 merge (see Fig. 4), i.e., relation (26) holds. This means that one of the generators of weight 10 becomes proportional to a product of the generators of weights 4 and 6. This implies the assertion of the theorem for $n = 8$.

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