Functional Analysis and Its Applications, Vol. 52, *No.* 3, *pp.* 163–177, 2018
Translated from Funktsional′nyi Analiz i Ego Prilozheniya, Vol. 52, *No.* 3, *pp.* 3–21, 2018

The Absolute of Finitely Generated Groups: II. The Laplacian and Degenerate Parts*

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Received May 21, 2018

Abstract. The article continues a series of papers on the absolute of finitely generated groups. The absolute of a group with a fixed system of generators is defined as the set of ergodic Markov measures for which the system of *cotransition probabilities* is the same as for the simple (right) random walk generated by the uniform distribution on the generators. The absolute is a new boundary of a group, generated by random walks on the group.

We divide the absolute into two parts, Laplacian and degenerate, and describe the connection between the absolute, homogeneous Markov processes, and the Laplace operator; prove that the Laplacian part is preserved under taking certain central extensions of groups; reduce the computation of the Laplacian part of the absolute of a nilpotent group to that of its abelianization; consider a number of fundamental examples (free groups, commutative groups, the discrete Heisenberg group).

Key words: absolute, Laplace operator, dynamic Cayley graph, nilpotent groups, Laplacian part of absolute.

1. Introduction

In this article, we continue to study the notion of the absolute of discrete groups and semigroups with a fixed system of generators (see the previous papers [1]–[3], [5], [16], and [17]). It appeared as a natural generalization of the well-known notions of Poisson–Furstenberg boundary, Martin boundary, etc. for random walks on groups and semigroups and the exit boundary of the corresponding Markov chains. Unlike the Poisson–Furstenberg boundary, which is defined as a measure space, the absolute, as well as the Martin boundary, is a topological boundary. On the other hand, the absolute of a group is a special case of the general notion of the absolute of a graded graph (or branching graph, or Bratteli diagram, which are different names for the same notion), i.e., the set of ergodic central measures on the Cantor-like set of infinite paths in the so-called dynamic Cayley graph. All these terms are defined later on; here, in the introduction, we explain the purpose of the suggested theory.

The theory of random walks on groups and the asymptotic theory of trajectories of these random walks have always been closely related to the theory of homogeneous Markov chains and harmonic analysis on groups, in particular, to the study of Laplace operators on groups, which are, in some sense or other, generators of Markov chains. The correspondence between the set of harmonic functions on a group with respect to a given Laplacian and the Poisson–Furstenberg (PF) boundary of the Markov chain is now well known; for the most part, it has been studied relatively thoroughly (see [2], [6], [13]–[15], and the references therein). However, this correspondence, for all its importance, reflects the asymptotic properties of the walk only very roughly. First, for many groups, the PF boundary, i.e., the set of harmonic positive functions, consists of only the constants; this is the case for Abelian, nilpotent, and some other groups. Second, the correspondence itself between the Laplacian and the asymptotic behavior of trajectories of the random walk is sometimes not defined: for many Markov chains, the notion of a generator needs to be generalized and the traditional Laplace operator has not yet been appropriately generalized. Our key notion—of the absolute of a random walk on a group or semigroup with given generators—is defined not in terms of classes of functions on groups related to the Laplacian but in terms of Markov measures (i.e.,

*Supported by the RSF grant 17-71-20153.

random walks on groups) having *the same cotransition probabilities* as the simple random walk determined by the set of generators. The PF boundary is only a part of the Laplacian absolute; namely, this is a space with a (harmonic) measure which realizes the decomposition of the Markov measure determined by the simple random walk into ergodic components. The whole absolute and, in particular, its Laplacian part contain much more information on the asymptotic properties of the group. E. B. Dynkin was probably the first to state, in some special cases, the problem of describing the Markov measures having the same cotransition probabilities as a given Markov measure.

We retain the term "Laplacian absolute" for a certain part of the absolute, namely, for the part consisting of nondegenerate measures; on this part, our generalization of classical theory reduces to simply considering simultaneously not only harmonic functions, as usual, but also all minimal positive eigenfunctions of the Laplace operator. The main results of the paper concern primarily the Laplacian part of the absolute. In the relevant papers [7], [8], and [18] known to us this problem was posed both in the general setting [8] and specifically for nilpotent groups ([7], [18]). It turns out (see Theorem 4.2) that there are precise geometric conditions on a group epimorphism that guarantee the preservation of the Laplacian part of the absolute under taking the corresponding quotient. This provides a generalization and a new proof of Margulis' well-known theorem on nilpotent groups [8]. Theorem 4.2 implies that the Laplacian part of the absolute of a nilpotent group coincides with that of its abelianization.

The general problem of describing even the Laplacian part of the absolute of an arbitrary discrete finitely generated group is difficult, and it is not yet clear how wide is the class of groups for which it can be reduced to finding the PF boundary. In [3] we demonstrated such a reduction for free groups, for which the Laplacian part of the absolute is the product of the PF boundary and a half-open interval. We cite this result in the section containing examples. We also mention the paper [17], which is devoted to the problem of describing the absolute for commutative groups. Together with Theorem 4.2 of the present paper, results of [17] partially solve the problem in the case of nilpotent groups, but only for the Laplacian part of the absolute.

By definition, the degenerate part of the absolute consists of those central ergodic measures for which some cylinder sets have zero measure. Another way to define degeneracy is to say that the support of the measure is a proper ideal in the path space of the graph (see below). A proper ideal of a dynamic Cayley graph is not necessarily the dynamic graph of a group; hence the description of degenerate central measures reduces, in some way or other, to finding the central measures for an arbitrary branching graph. Nevertheless, the study of the degenerate part of the absolute for the Heisenberg group shows that the approach using geometric group theory remains useful in this case, too: the problems under consideration are closely related to group-geometric problems (the theory of geodesics in groups), as well as to the theory of filtrations of σ-algebras and, of course, to harmonic analysis on groups.

The paper is organized as follows.

Section 2 contains numerous definitions and results related to the notion of absolute, but primarily those needed for what follows. For more details, see the references cited above.

In Section 3 we study the correspondence between central measures and positive eigenfunctions of the Laplace operator. We prove that every measure from the Laplacian absolute is *homogeneous* in an appropriate sense (see Lemma 3.2) and establish a bijective correspondence between the homogeneous nondegenerate central measures and the classes of proportional positive eigenfunctions of the Laplace operator (see assertion 2 of Theorem 3.4). As a consequence, we obtain one of the main results of the paper, namely, assertion 1 of Theorem 3.4, which says that the measures from the Laplacian absolute are in one-to-one correspondence with the classes of proportional minimal positive eigenfunctions of the Laplace operator.

In Section 4 we introduce the notion of a totally distorted subgroup and prove that taking the quotient of a group by a totally distorted subgroup all of whose elements have finite conjugacy classes in the original group changes neither the set of eigenfunctions of the Laplacian (Lemma 4.3) nor the Laplacian part of the absolute (Theorem 4.2). Theorem 4.2 implies Corollary 4.4, which says that the Laplacian part of the absolute of a nilpotent group coincides with that of its abelianization.

Section 5 contains results on the degenerate part of the absolute. The main result of this section is Theorem 5.1, which says that, for any finitely generated group and any choice of a set of generators, all degenerate central measures are concentrated on paths whose projections on the Cayley graph contain no cycles. It follows that, in many interesting cases, all degenerate central measures are concentrated on paths whose projections on the Cayley graph are geodesics (Corollary 5.4).

In Section 6 we give descriptions of the absolute for the cases of free groups, the discrete Heisenberg group, commutative groups, and finite groups. For free groups and homogeneous trees, the absolute was described in our paper [3]. Here we briefly reproduce the main result of [3] and give its illustrative formulation. For the discrete Heisenberg group, we only state the result; its proof will be given elsewhere. The case of commutative groups was considered in [17].

A detailed study of the absolute (including its degenerate part) for nilpotent and other classes of groups will be the subject of our future papers.

2. Necessary Definitions

We give only definitions of (old and new) notions used in what follows. For more detailed information on branching graphs and absolutes, not necessarily related to groups, see the papers cited in the introduction.

2.1. Graphs, branching graphs, and the Cayley and dynamic graph of a group. By a *graph* we mean a locally finite directed graph with a marked vertex. Loops and multiple edges are allowed. A *path* in a graph is a (finite or infinite) sequence of alternating vertices and edges of the form

$$v_0, e_1, v_1, e_2, \dots, e_n, v_n,$$

where each e_k is an edge with initial vertex v_{k-1} and terminal vertex v_k (both vertices and edges may repeat). We consider graphs in which every path can be extended to an infinite path. A special class of graphs is that of branching graphs. A *branching graph* is a graph in which the set of paths from the marked vertex v_0 to every vertex v is nonempty (in this case, one says that v is *reachable* from v_0) and all these paths have the same length. On the vertex set of a branching graph there is a natural grading by the distance to the marked vertex. Such graphs are also called (locally finite) $\mathbb{N}$-*graded graphs*, or *Bratteli diagrams*.

To each graph Γ with a marked vertex v_0 a *dynamic graph* $\mathrm{D}_{v_0}(\Gamma)$ is canonically associated. This is the branching graph constructed as follows. The nth level of $\mathrm{D}_{v_0}(\Gamma)$ is a copy of the set of vertices of Γ connected with the marked vertex v_0 by paths of length n. Two vertices v_1 and v_2 in $\mathrm{D}_{v_0}(\Gamma)$ are connected by exactly k edges directed from v_1 to v_2 if and only if the level of $\mathrm{D}_{v_0}(\Gamma)$ containing v_2 is higher by 1 than that containing v_1 and the number of edges from the vertex w_1 of Γ that corresponds to v_1 to the vertex w_2 of Γ that corresponds to v_2 is precisely k. Note that a graph Γ coincides with its dynamic graph $\mathrm{D}_{v_0}(\Gamma)$ if and only if Γ is a branching graph.

Let us apply the above definitions to groups. The notion of the Cayley graph of a (semi)group with a chosen set of generators is well known. By a set of generators in a group G we mean a subset of G generating it as a semigroup. It is not necessarily symmetric and does not necessarily contain the identity. Note that the results of this paper extend automatically to the case of sets of generators with multiplicities or weights. The *dynamic* (*Cayley*) *graph* of a group G with a chosen set of generators S, denoted by $\mathrm{D}(G,S)$ in what follows, is the dynamic graph (in the sense of the above definition) constructed from the Cayley graph of the pair (G,S).

2.2. The path space of a branching graph, central measures, ergodicity, and the absolute. Let D be a branching graph, and let $T(\mathrm{D})$ denote the set of all infinite paths in D from the marked vertex. It is equipped with the weak (projective limit) topology, defined in a natural way, in which $T(\mathrm{D})$ is compact. Let $\mathscr{M}(\mathrm{D})$ denote the set of Borel probability measures on this space. A measure ν on $T(\mathrm{D})$ is said to be *central* if, for almost every (with respect to ν) path t, the conditional measure on the set of paths that differ from t at finitely many places is uniform. In another terminology, this means that the tail equivalence relation is *semihomogeneous*. An equivalent definition of centrality is as follows: a measure on $T(\mathrm{D})$ is central if, for every vertex v of D, the probabilities of all finite paths from the marked vertex to v corresponding to this measure are equal. One can easily show that all central measures are Markov. The central measures constitute a convex compact set $\mathscr{C}(\mathrm{D})$, which is a simplex (see [2]) in $\mathscr{M}(\mathrm{D})$. The simplex $\mathscr{C}(\mathrm{D})$ is the projective limit of a sequence of finite-dimensional simplices of measures on finite paths; see [2]. Here we do not use this fact.

A central measure is called *ergodic* (or *regular*) if it is an extreme point of the simplex $\mathscr{C}(\mathrm{D})$. The *absolute* of a branching graph is the set of all ergodic central measures on the compact set $T(\mathrm{D})$ of infinite paths starting at the marked point.

Now we will apply all these notions to the dynamic graph of a group.

Definition 2.1. The *absolute of a finitely generated group with a fixed finite set of generators* is the absolute of the corresponding dynamic graph.

The above definitions are equivalent to the following definition in terms close to the theory of random walks. The absolute of a group G is the set of Markov measures generated by ergodic random walks on the Cayley graph of G and satisfying the following condition: for every n and any element g of G representable as a product of n generators, the conditional measure on the representations of g as a product of n generators is uniform. For stationary Markov measures, the notion of a central measure coincides with the well-known notion of a measure of maximal entropy. Note that the ergodicity of a measure, which is defined above in terms of the impossibility of writing this measure as a nontrivial convex combination of other central measures, can be defined directly in terms of the intersection of the past σ-algebras (see Section 2.5).

The absolute of a group G with a set of generators S is denoted by $\mathscr{A}(G,S)$.

2.3. The Laplace operator and its eigenfunctions. Given a group G and a set of generators S, we define the *Laplace operator* acting on the space of all functions on G as the linear operator given by the formula

$$(\Delta_S f)(g) := \frac{\sum_{s\in S} f(gs)}{|S|}.$$

We will be interested in the action of Δ_S on nonnegative functions and its eigenfunctions corresponding to nonnegative eigenvalues.

A nonnegative eigenfunction f of an operator A is called *minimal* if every nonnegative eigenfunction of A having the same eigenvalue and dominated by f is proportional to f.

The relationships between the operator Δ_S, its eigenfunctions, Markov measures, and random walks on groups will be studied in more detail in the next section.

2.4. The nondegenerate and degenerate parts of the absolute. We say that a measure ν on the path space of a branching graph is *nondegenerate* if the probability of every finite path (i.e., every cylinder set) is nonzero. The *main part* of the absolute is its subset consisting of nondegenerate measures. In the group case, the main part of the absolute is called the *Laplacian part* of the absolute, or simply the *Laplacian absolute*. The set of degenerate ergodic central measures will be called the *degenerate part* of the absolute. It is related to the geometry of groups and the geometry of paths in Cayley graphs.

Every graded graph determines a partial order on its vertices. An *ideal* of this partial order is a subset of vertices that contains, along with every vertex v, all vertices smaller than v. One can consider the path space of a given ideal as a closed subset in the space of all paths. It is easy to see that the definition of degenerate and nondegenerate central measures can be reformulated by using the notion of ideal as follows. The support of every central measure (as a closed set in the path space) is always the space of all paths of some ideal. If this ideal coincides with the whole path space, then the measure is nondegenerate; if the ideal is proper, then the measure is degenerate.

Hence, in order to find the whole absolute, one has to find the absolutes of some proper subgraphs of the dynamic graph. The problem of describing ideals for the subgraphs of a given graph is of independent interest.

2.5. Additional structures and comments.

A filtration structure. We consider the tail equivalence relation on the path space of a graph (for instance, a branching graph). Here it is convenient to use the framework of the theory of filtrations (see [4]). The *tail filtration on the path space of a graph*, or, in short, the *tail filtration*, is defined as follows. Two paths are said to be *n-equivalent* if they coincide beginning with the nth vertex and *tail equivalent* if they are n-equivalent for some n. Let $\mathfrak{A}_n$, $n \in \mathbb{N}_0$, be the σ-algebra of those Borel subsets in the compact set of infinite paths which contain, along with every path, all n-equivalent paths. The decreasing sequence

$$\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \dots$$

of σ-subalgebras of Borel sets is called the *tail filtration on the path space*. The equivalence classes described above are also called *blocks*, or *elements*, of the filtration. The filtration is said to be *ergodic* if the intersection of the above σ-algebras is trivial. An *automorphism of the filtration* is an automorphism of the measure space preserving the tail equivalence classes. In these terms, a *central measure* is a measure invariant under all automorphisms of the tail filtration.

Generalizations: equipped multigraphs. In the present paper, we consider primarily simple walks related to the uniform distribution on the set of generators, but one can also consider the absolute in a more general setting depending on a cocycle (for more details on equipped multigraphs and their cocycles, see Sections 3.3 and 3.5 in [4]). The generalization to the case of multigraphs will be needed in Section 6.2.

The compact space of infinite words. An alternative approach to the theory of absolutes of groups and semigroups is provided by the algebraic structure: the dynamic graph of a group is the Cayley graph of a graded semigroup, and the path space of the dynamic graph is canonically isomorphic to the space of infinite words in the alphabet of generators (see [10]). In this paper we use this approach in the example of Section 5 and in Section 6.4, which considers the discrete Heisenberg group. Within this approach, when considering finite and infinite sequences of elements of a set, we say that this set is an *alphabet*, its elements are *symbols*, or *letters*, and sequences of elements are *words* in this alphabet. For presentations of groups, we use alphabets consisting of pairs of letters of the form $\{a, a^{-1}\}$; such letters are called *inverse* to each other. A *word without inverse letters* is a word in which any (not necessarily neighboring) two letters are not inverse to each other. Words are written without commas, a block of n successive letters a is denoted by a^n, and a block of n successive letters a^{-1} is denoted by a^{-n}. The notation $(a^{-1})^{-1}$ is interpreted as a, and a^0 stands for the empty (sub)word. The *inverse* word of $w_1 \dots w_n$ is $w_n^{-1} \dots w_1^{-1}$. The notation $a^{+\infty}$ ($a^{-\infty}$) is used for the right-infinite word whose every letter is a (respectively, a^{-1}).

3. Homogeneous Measures, the Laplace Operator, and the Laplacian Absolute

In this section we establish a connection between homogeneous Markov chains on a group G (more precisely, on the Cayley and dynamic graphs) and the functions associated with the Laplace operator on G. This connection is well known for harmonic functions, i.e., functions invariant under the Laplace operator; it is the basis of harmonic analysis on groups. For eigenfunctions, the connection is less known and less studied; perhaps, one of the first papers on the subject, using a somewhat different viewpoint, was Molchanov's paper [8]. We treat the subject systematically and refine the correspondence between properties of eigenfunctions and their linear combinations (positivity, minimality, etc.) on the one hand and properties of Markov chains (homogeneity, centrality, ergodicity, etc.) on the other hand. It is this correspondence which provides a link between the nondegenerate part of the absolute and the theory of the Laplace operator. Note that it is the absence of such a correspondence for degenerate Markov chains which causes difficulties in the study of the degenerate part of the absolute.

Before proving the main theorem, we discuss the notion of homogeneity. Let G be a finitely generated group, and let S be a finite set of generators of G. We construct the dynamic graph $\mathrm{D}(G,S)$ corresponding to the pair (G,S) and consider central measures on the compact set of infinite paths of this graph. It is easy to verify (see the introduction) that the random processes corresponding to central measures, both on the dynamic graph and on the Cayley graph, are Markov.

Definition 3.1. We refer to measures on the path space of the dynamic graph that give rise to time-homogeneous Markov chains on the Cayley graph as *homogeneous* measures. Recall that a Markov chain is called *time-homogeneous* if its transition probabilities do not depend on time. (This terminology causes no confusion, since there are no time-inhomogeneous Markov processes on dynamic graphs.)

Lemma 3.2 (on homogeneity)**.** *All points of the Laplacian absolute of a group*, *i.e.*, *all nondegenerate ergodic central measures on the path space of the dynamic graph*, *are homogeneous. In other words*, *every such measure is generated by a homogeneous Markov measure on the group.*

Proof. Let v be a vertex of the dynamic graph $\mathrm{D} = \mathrm{D}(G,S)$, and let D_v be the subgraph in D formed by the vertices and edges reachable from v. If a central measure μ does not vanish on the set of paths passing through v (for example, if μ is nondegenerate), then we can consider its restriction μ_v to the path space of D_v. The group structure gives rise to a *translation*, i.e., a canonical natural isomorphism α between D_v and D. If v represents a central element (for example, the identity) of the group, then, for every vertex x in D, the vertex $\alpha^{-1}(x)$ is, obviously, reachable from x. It follows that the measure μ dominates the measure $\alpha_*(\mu_v)$. Clearly, the measures μ_v and $\alpha_*(\mu_v)$ are central. Therefore, if μ is ergodic, then $\alpha_*(\mu_v) = \mu$.

Applying the above argument to the vertices of the dynamic graph that project to the identity of the group, we see that the transition probabilities of the Markov chain corresponding to a nondegenerate ergodic central measure are invariant under the time shift by k, provided that the Cayley graph of the pair (G,S) contains a cycle of length k. However, in the group case, the Markov chain corresponding to a nondegenerate central measure is *indecomposable* in the sense of the theory of Markov chains, and it is well known (see, e.g., Theorem 3.3 in [9]) that in an indecomposable chain the time shift by the period of the chain can be written as the composition of the (direct and inverse) shifts by the lengths of some cycles, which implies the desired result. □

Remark 3.3. The second assertion of Theorem 3.4 proved below shows that, as a rule, only a small part of central measures are homogeneous.

Now we proceed to a detailed description of the relationship between central measures and the Laplace operator. In particular, we will explain the connection between this operator and the Laplacian absolute. A Markov measure on the paths of the dynamic graph is determined by a collection of transition probabilities. Our aim is to construct a system of transition probabilities, i.e., define some measure, from the eigenfunctions of the Laplace operator. Given a positive function $f\colon G \to \mathbb{R}$, we define a transition probability $p(g, gs)$, where $g \in G$ and $s \in S$, by the rule

$$p(g, gs) := f(gs) \Big/ \sum_{t \in S} f(gt). \tag{1}$$

Let $\mathscr{D}$ denote the mapping that sends a function f to the set of transition probabilities on the Cayley graph given by (1) and, thereby, to a Markov measure ν_f on the path space of the dynamic graph. Recall that the classical theory establishes a correspondence between the minimal positive harmonic functions and the ergodic central measures of a certain form, or the points of the Poisson–Furstenberg boundary. The following theorem extends this correspondence.

Theorem 3.4. 1. *The mapping $\mathscr{D}$ induces a bijection between the set of classes of proportional positive* minimal *eigenfunctions of the Laplacian and the Laplacian absolute*, *i.e.*, *the set of all nondegenerate central* ergodic *measures on the path space of the graph* $\mathrm{D}(G,S)$.

2. *The mapping $\mathscr{D}$ induces a bijection between the set of classes of proportional positive (not necessarily minimal) eigenfunctions of the Laplacian and the set of all* homogeneous *nondegenerate central (not necessarily ergodic) measures on the path space of the graph* $\mathrm{D}(G, S)$.

The first assertion of Theorem 3.4 is the main result of this section.

To prove Theorem 3.4, we will need the following lemma, which introduces the notion of *characteristic.*

Lemma 3.5 (on the existence of a characteristic)**.** *Let ν be a homogeneous nondegenerate central measure on the path space of the graph $\mathrm{D}(G, S)$. Then there exists a unique number $A_\nu \in \mathbb{R}$ (called the* characteristic *of ν) and a unique function $f_\nu\colon G \to \mathbb{R}$ such that, for every finite path P in the dynamic graph,*

$$\nu(P) = f_\nu(g) \cdot A_\nu^{|P|}, \tag{2}$$

where $\nu(P)$ is the measure of the cylinder set of paths that begin with P, $|P|$ is the length of P, and g is the element of G corresponding to the terminal vertex of P.

Proof. Choose an arbitrary path Z in $\mathrm{D}(G, S)$ of nonzero length $|Z|$ that represents the identity of G, and let $A_\nu := \nu(Z)^{1/|Z|}$. The value A_ν does not depend on the choice of Z. Indeed, let Y be another path representing the identity. Consider the path $Z^{|Y|}$ whose projection on the Cayley graph is the $|Y|$-fold repetition of the projection of Z. Since the paths $Z^{|Y|}$ and $Y^{|Z|}$ have the same length, it follows from centrality that

$$\nu(Z)^{|Y|} = \nu(Z^{|Y|}) \overset{\text{centrality}}{=} \nu(Y^{|Z|}) = \nu(Y)^{|Z|}. \tag{3}$$

Further, for each element $g \in G$, choose an arbitrary path P_g representing this element in the dynamic graph and let

$$f_\nu(g) := \nu(P_g) \cdot A_\nu^{-|P_g|}. \tag{4}$$

Then f_ν does not depend on the choice of a path: if, for some path Q_g representing the same element g, the values $\nu(Q_g) \cdot A_\nu^{-|Q_g|}$ and $\nu(P_g) \cdot A_\nu^{-|P_g|}$ differed, then, extending P_g and Q_g by paths with the same projections on the Cayley graph to paths representing the identity, we would obtain a contradiction with (3).

To see the uniqueness of A_ν, it suffices to consider the paths representing the identity of G. By (2), this immediately implies the uniqueness of f_ν. □

Remarks 3.6. 1. The notion of characteristic is parallel to the notion of eigenvalue for eigenfunctions of the Laplacian on the semigroup $\mathbb{N}_0 := \{0, 1, 2, \dots\}$; this can be seen by regarding the dynamic graph as a subset in the direct product of the Cayley graph of some group and the Cayley graph of the semigroup $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and using the canonical bijection between the Laplacian absolute and the set of harmonic functions on the dynamic graph: the harmonic functions corresponding to homogeneous measures can be decomposed into products of eigenfunctions of the Laplacian on the original group and on the semigroup $\mathbb{N}_0$. (The problem of decomposing minimal harmonic functions into such products was discussed in [8].)

2. It is clear from the proof of Lemma 3.5 that (in terms of this lemma) $\log A_\nu$ is given by the formula

$$\frac{\log \nu(P_1) - \log \nu(P_2)}{|P_1| - |P_2|}, \tag{5}$$

where P_1 and P_2 are arbitrary paths of different lengths in the Cayley graph for which both initial and terminal vertices coincide, $|P_i|$ is the length of P_i, and $\nu(P_i)$ is the product of the transition probabilities corresponding to the measure ν over all edges of P_i (counting multiplicities).

Proof of assertion 2 of Theorem 3.4. If f is an eigenfunction of the Laplacian with eigenvalue α, then the defining formula (1) implies that the value of the measure $\nu_f = \mathscr{D}(f)$ on the cylinder set of paths that begin with a given path P of length k representing an element g is given by the formula

$$\nu_f(P) = \frac{f(g)}{f(1_G)} \cdot (\alpha \cdot |S|)^{-k}. \tag{6}$$

This proves the centrality of the measure ν_f; the homogeneity of this measure follows from the definition.

Conversely, let ν be a homogeneous nondegenerate central measure on the path space of the graph $\mathrm{D}(G, S)$. Then the corresponding function f_ν in Lemma 3.5 is an eigenfunction of the Laplacian with eigenvalue $A_\nu^{-1} \cdot |S|^{-1}$ (this follows from (4)). Comparing formulas (4) and (6), we see that the composition of $f \mapsto \nu_f$ and $\nu \mapsto f_\nu$ is the identity mapping on the sets under consideration (we distinguish the subset of normalized functions in the set of eigenfunctions). □

Proof of assertion 1 of Theorem 3.4. Let us show that the bijection in the second assertion of the theorem proved above sends ergodic measures to minimal functions. (This will immediately imply the desired result by Lemma 3.2 on homogeneity.) It follows from the formula (4) defining the function f_ν that the correspondence $\nu \mapsto f_\nu$ sends the simplex of measures with characteristic A to the simplex of positive eigenfunctions (taking the value 1 at the identity of the group) of the Laplacian with eigenvalue $A^{-1} \cdot |S|^{-1}$ and preserves the affine structure. Since the minimality of an eigenfunction is defined with respect to functions having the same eigenvalue, it follows that the correspondence $\nu \mapsto f_\nu$ yields an embedding of the main part of the absolute into the set of classes of proportional minimal positive eigenfunctions of the Laplacian. To check that this embedding is bijective, we must verify that the decomposition of a homogeneous nondegenerate central measure ν into ergodic components involves neither degenerate measures nor measures with characteristics different from that of ν. To see this, in the Cayley graph choose a cycle of nonzero length with endpoints at the identity of the group and observe that, by (5), the values of homogeneous measures with different characteristics at the powers of this cycle produce exponentials with different bases, while every degenerate measure vanishes at these powers (see Theorem 5.1). □

Remarks 3.7. 1 (on the algebra of eigenfunctions). Apparently, the Laplacian absolute can be described and studied by analogy with the theory of harmonic functions, in which a nontrivial commutative multiplication of such functions is introduced so that, on the one hand, the Gelfand spectrum of the corresponding Banach algebra coincides with the set of minimal positive harmonic functions and, on the other hand, this is exactly the exit boundary of the simple walk. Such a construction of the Laplacian absolute would, first, clarify the situation and, second, simplify the proofs given above (see [14], [11], and the literature on harmonic analysis). It is also of interest to establish direct analytic links between the asymptotics of the typical trajectories of ergodic random walks on a group and the corresponding positive minimal eigenfunctions of the Laplace operator.

2 (on minimal eigenfunctions). The results of this section remain valid in a more general context: in particular, the main part of the absolute can be described in terms of minimal eigenfunctions of the Laplacian for a wide class of graphs that do not have symmetries of Cayley graphs of groups (see [8]). We also mention that the collection of minimal eigenfunctions itself was studied in the literature and is well known for some cases (see [18] and a description of minimal eigenfunctions for the product of graphs in [8]).

3 (on the structure of the Laplacian absolute). In the investigated cases, there is a bijection between the spaces of positive minimal eigenfunctions on a group for any two nonextreme eigenvalues. Accordingly, in these cases, the Laplacian absolute (with the "extreme" fiber excluded) is the product of some space and an interval. The authors do not know examples of groups for which this is not so. In any case, the class of groups with this property is of interest.

4 (on the group action on the Laplacian absolute). Identifying the Laplacian absolute with the set of minimal eigenfunctions of the Laplacian (see Theorem 3.4), we obtain an action of the group on its Laplacian absolute. It is of interest to describe the class of groups for which this action is trivial. This is exactly the class of groups for which every nondegenerate ergodic central measure gives rise to a Markov chain with independent identically distributed increments (the transition probabilities are the same at all vertices of the graph and depend only on the generators labeling the

edges). This class contains all finitely generated nilpotent groups (see Section 6.3 and Corollary 4.4 below).

4. Preservation of the Laplacian Absolute under Extensions

In this section we introduce the notion of a totally distorted subgroup and prove that taking the quotient of a group by a totally distorted subgroup all of whose elements have finite conjugacy classes in the original group changes neither the eigenfunctions of the Laplacian (Lemma 4.3) nor the Laplacian part of the absolute (Theorem 4.2). These results are a natural generalization of Margulis' theorem which arises when looking at the latter from the viewpoint of geometric group theory. A crucial role here is played by the notion of a distorted subgroup in this theory. As we will see, in the case of nilpotent groups, the existence of a central totally distorted subgroup is typical. This gives Corollary 4.4, which says that the Laplacian absolute of a nilpotent group coincides with that of its abelianization.

Definition 4.1. A finitely generated subgroup H of a finitely generated group G is called *distorted* if the identity embedding $H \to G$ is not a quasi-isometry with respect to word metrics; the property of being distorted does not depend on the choice of a set of generators (see [12]). We say that a subgroup K in a group G is *totally distorted* if all infinite cyclic subgroups of K are distorted in G.

In the case of an infinite cyclic subgroup, we have the following equivalent definition of distortion: the infinite cyclic subgroup $\langle g\rangle$ generated by an element $g \in G$ is distorted if and only if $|g^k|_G = o(k)$, where $|\cdot|_G$ is the length of an element in the word metric of the group G. Thus, a subgroup K in G is totally distorted if and only if $|g^k|_G = o(k)$ for every element $g \in K$.

Theorem 4.2. *Let $\phi\colon G \to G/K$ be an epimorphism of finitely generated groups such that the kernel K is a totally distorted subgroup and every element of K has finite conjugacy class in G. Let S be a set of generators of G, and assume that ϕ is injective on S. There arises a natural isomorphism ϕ_* between the path spaces of the graphs $\mathrm{D}(G,S)$ and $\mathrm{D}(G/K,\phi(S))$. This isomorphism ϕ_* induces an isomorphism of the Laplacian absolutes of the pairs (G,S) and $(G/K,\phi(S))$.*

The main fact used in the proof of Theorem 4.2 is that ϕ_* sends central measures to central measures. If K is not totally distorted, this is not always true. To prove Theorem 4.2, we will need the following lemma related to Margulis' theorem on harmonic functions on nilpotent groups [7] (see also [18]). This lemma can be directly extended to Laplacians of arbitrary measures whose supports generate the whole group; however, here we restrict ourselves to the main case we are interested in.

Lemma 4.3. *Let G be a finitely generated group. Assume that a subgroup H in G is totally distorted and every element of K has finite conjugacy class in G. Let S be a set of generators in G. Then every positive eigenfunction of the Laplacian constructed from S is constant on the right cosets of H.*

Proof. Consider a left-invariant word metric d on G (for example, the one corresponding to the set S of generators). To each element g of G we associate the function

$$\phi_g\colon G \to \mathbb{R}, \qquad x \mapsto d(x,gx) = d(1_G, x^{-1}gx),$$

sending an element $x \in G$ to the distance $d(x,gx)$ (in the chosen metric) between x and gx. If the conjugacy class of g in G is finite, then, obviously, the function ϕ_g takes only finitely many values (since the element $x^{-1}gx$ is conjugate to g). Using Harnack's inequality (see, e.g., [18, p. 262]), we see that, for every positive eigenfunction f of the Laplacian, there is a positive constant ε such that $f(x) > \varepsilon \cdot f(gx)$ for all $x \in G$. If f is minimal and, thus, cannot dominate eigenfunctions not proportional to f, then it follows that there is a positive $t \in \mathbb{R}$ such that $f(gx) = t \cdot f(x)$ for all $x \in G$. But $f(g^n) = t^n \cdot f(1_G)$; hence if the subgroup $\langle g\rangle$ generated by g is finite or distorted, then, again using Harnack's inequality but this time applying it to elements of the sequence $f(g^n)$, we see that $t = 1$. It remains to observe that every eigenfunction of the Laplacian can be decomposed into a sum of minimal eigenfunctions. □

Proof of Theorem 4.2. Observe that, for an arbitrary group epimorphism $G_1 \to G_2$ preserving the set of generators, every eigenfunction of the Laplacian on G_2 can be lifted to an eigenfunction of the Laplacian on G_1, but in the general case, the Laplacian on G_1 may have eigenfunctions that do not correspond to any eigenfunctions of the Laplacian on G_2. If the kernel of the epimorphism satisfies the conditions of Lemma 4.3, then the lemma guarantees that every positive eigenfunction of the Laplacian on G_2 is the lifting of an eigenfunction of the Laplacian on G_1. Thus, an epimorphism with such kernel induces a bijection at the level of positive eigenfunctions of the Laplacians. The proof is completed by passing from eigenfunctions of the Laplacians to the Laplacian absolutes (see Theorem 3.4). □

Corollary 4.4. *Let N be a finitely generated nilpotent group, and let $\phi\colon N \to \mathrm{Ab}(N)$ be the abelianization homomorphism. Choose a set S of generators in N and assume that ϕ is injective on S. There arises a natural isomorphism ϕ_* between the path spaces of the graphs* $\mathrm{D}(N, S)$ *and* $\mathrm{D}(\mathrm{Ab}(N), \phi(S))$. *This isomorphism ϕ_* induces an isomorphism of the Laplacian absolutes of the pairs (N, S) and* $(\mathrm{Ab}(N), \phi(S))$.

Proof. In an arbitrary group the subgroup generated by the commutators lying in the center of the group is totally distorted; this can easily be deduced from the fact that if a commutator $[a, b]$ lies in the center, then $[a^m, b^n] = [a, b]^{mn}$ for arbitrary integers m and n.

In a nilpotent group N of class s, the subgroup N_{s-1} (where $N_i = [N, N_{i-1}]$ and $N_0 := N$) is generated by the commutators lying in the center of N and, consequently, totally distorted; moreover, the quotient N/N_{s-1} is a nilpotent group of class $s - 1$, and $\mathrm{Ab}(N) = \mathrm{Ab}(N/N_{s-1})$. Therefore, $\mathrm{Ab}(N)$ is obtained from N by taking a sequence of quotients by central totally distorted subgroups, and the required assertion follows from Theorem 4.2. □

Remark 4.5. Theorem 4.2 and Lemma 4.3 generalize Margulis' important theorem [7] that every positive eigenfunction of the Laplacian of a nilpotent group is constant on the cosets of the commutant in the sense that nilpotency is replaced by the conditions in Theorem 4.2; see also [18].

5. The Degenerate Part of the Absolute and Geodesics on the Group

In this section we prove a number of statements concerning the degenerate part of the absolute. As in the previous sections, G is a finitely generated group with a finite set of generators S. We construct the dynamic graph $\mathrm{D}(G, S)$ corresponding to the pair (G, S) and consider central measures on the space of infinite paths of this graph.

Theorem 5.1. *All degenerate central measures on the path space of the dynamic graph are concentrated on paths whose projections on the Cayley graph do not contain cycles.*

Proof. It suffices to consider the case of an ergodic (degenerate central) measure. Assume that the probability of traversing a finite path P whose projection on the Cayley graph contains a cycle does not vanish. Since we deal with the dynamic graph of a group, there is a path P' in $\mathrm{D}(G, S)$ that leads to the same vertex as P and begins with a path P'' whose projection on the Cayley graph is a cycle. We have $\nu(P'') \geqslant \nu(P') = \nu(P) > 0$, because the measure is central and additive. Applying the argument from the proof* of Lemma 3.2 on homogeneity, we conclude that the conditional measure on the subgraph D_v for the terminal vertex v of P'' is isomorphic to the original measure ν. It follows that the original measure ν takes nonzero values not only at P'' but also at all paths corresponding to the powers of the cycle given by the projection of P''. Thus, the original measure takes nonzero values at arbitrarily long paths that project to cycles. But it follows from standard results of the theory of Markov chains (see Theorems 3.2 and 3.3 in [9]) that, for every path Q in the Cayley graph, there is an $N \in \mathbb{N}$ such that, for every $n > N$, the path Q can

*Lemma 3.2 deals with the case of a nondegenerate measure, but the argument in its proof applies also to a degenerate measure, provided that we consider paths on which the measure is concentrated.

be extended to a cycle of length n, provided that cycles of length n exist. It remains to use the centrality of the measure to obtain a contradiction with the assumed degeneracy. □

Since the measures under consideration are central, it follows that Theorem 5.1 implies Corollary 5.3 that degenerate central measures are concentrated on paths whose projections on the Cayley graph are close to *geodesics*. To rigorously define the notion of "being close to geodesics," we introduce the following notion of the *defect* of a path.

Definition 5.2. The *defect* of a finite path in the Cayley graph of a group is the difference between the length of this path and the length of the shortest path with the same initial and terminal vertices. The *defect* of an infinite path is the supremum of the defects of its finite parts. Paths with zero defect are called *geodesics*. The *defect* of a (finite or infinite) path in the dynamic graph of a group is the defect of its projection on the Cayley graph.

Corollary 5.3. *Let G be a finitely generated group, and let S be a finite set of generators of G. Then there exists an $N \geqslant 0$ such that all degenerate central measures on the path space of the graph* $\mathrm{D}(G,S)$ *are concentrated on paths with defect at most N.*

Proof. Let D be the greatest common divisor of all lengths of cycles in the Cayley graph of the pair (G,S). Then, as we know from the combinatorics of Markov chains (see Theorem 3.3 in [9]), there exists a $J \in \mathbb{N}_0$ such that, for every $j \geqslant J$, the Cayley graph contains a cycle of length jD. Note also that the defect of every path is divisible by D (to see this, it suffices to extend the path and the corresponding geodesic to cycles by the same path). Let P be a finite path (beginning at the marked vertex) with defect $\operatorname{def}(P) > JD$ in the dynamic graph, and let Q be a geodesic path in the Cayley graph connecting the same vertices as the projection of P. Then, since $\operatorname{def}(P) > JD$, the Cayley graph contains a cycle Z of length $|Z| = \operatorname{def}(P)$. We may assume without loss of generality that Z starts and ends at the identity of the group. Then the path consisting of the cycle Z and the path Q has the same length and connects the same vertices as the projection of P. Since the measure is central, it follows by Theorem 5.1 that degenerate measures vanish on the path P. It remains to set $N := JD$. □

It turns out that one can easily give not too restrictive sufficient conditions on G and S under which all degenerate central measures are concentrated on paths whose projections on the Cayley graph are geodesics.

Corollary 5.4. *Let G be a finitely generated group, and let S be a finite set of generators of G. Assume that the Cayley graph Γ of the pair (G,S) contains a cycle with length equal to the greatest common divisor of all lengths of cycles in Γ* (*this condition is satisfied, for example, if S contains the identity of the group or if S is symmetric and all cycles in Γ are even*). *Then all degenerate central measures on the path space of the graph* $\mathrm{D}(G,S)$ *are concentrated on paths whose projections on Γ are geodesics.*

Proof. The assumptions of the corollary correspond to the case $N = J = 0$ in the proof of Corollary 5.3. □

The spectrum of cases in which degenerate central measures are concentrated on geodesics is far from exhausted by Corollary 5.4. For example, in a commutative group this is so for any finite set of generators (see [17]). Nevertheless, the following example shows that, in some cases, geodesics are not sufficient to describe the degenerate part of the absolute: a degenerate ergodic central measure may be neither concentrated on paths with zero defect nor homogeneous.

Example. Consider the Baumslag–Solitar group $BS(2,1)$ (this is the group with two generators a and b and one relation $ab^2a^{-1} = b$) with the set of generators $S := \{a, b, a^{-1}, b^{-1}\}$. One can easily check that the three infinite paths corresponding to the three infinite words

$$abbba^{+\infty},\ baabba^{+\infty},\ bbab^{-1}a^{+\infty}$$

constitute a tail equivalence class; hence the uniform measure on these three paths is (central and) ergodic. At the same time, as is easy to check, the projections of these three paths on the Cayley graph are not geodesics, and the transition probabilities of the corresponding Markov process at the point $g = abb = ba$ are different for the third and the fourth step.

6. Example Descriptions of the Absolute

6.1. The absolute of free groups and homogeneous trees. Although against the natural order of exposition, we begin with a more complicated and illustrative example of computing the absolute of a free group, after which we consider the problem of computing the absolute in the order of increasing complexity of the group structure. This example clearly demonstrates the instructiveness of the transition from the Poisson–Furstenberg boundary to the absolute.

Consider the homogeneous tree T_{q+1} in which the valences of all vertices equal $q+1$. The set $\mathscr{H}_{\min}$ of minimal positive harmonic (i.e., invariant under the Laplace operator) functions on T_{q+1} coincides with the family of functions of the form $q^{-h(v)}$, where $h(v)$ is an arbitrary horofunction on T_{q+1}. One can easily check that, for every (real or complex) number α, the power $(q^{-h(v)})^\alpha = q^{-\alpha h(v)}$ of any minimal harmonic function $q^{-h(v)}$ is an eigenfunction of the Laplace operator with eigenvalue

$$s_\alpha = \frac{q^\alpha + q^{1-\alpha}}{q+1}\,.$$

A complete description of the absolute (see Theorem 2.1 in [3]) is as follows: the absolute of the free group with respect to the natural generators is the direct product of the boundary of the free group and a closed interval:

$$\mathscr{A}(T_{q+1}) = \partial T_{q+1} \times [1/2, 1].$$

To obtain the main part of the absolute, one should consider the same product with the half-open interval $[1/2, 1)$.

Let us explain this formula. A pair $\omega \times r$ in $\partial T_{q+1} \times [1/2, 1]$ is interpreted as the Markov measure on the dynamic graph of T_{q+1} that corresponds to the Markov measure on paths in the tree T_{q+1} (which is the Cayley graph of the free group) such that the probability $p(g, gw)$ of the transition from an arbitrary vertex g to the vertex gw, where w is a generator, is equal to r if the edge leading to ω is labeled by w and $(1-r)/q$ otherwise.

It is natural to regard the probability r, the eigenvalue corresponding to the eigenfunction, and the rate defined below as functions of the number α parametrizing the eigenfunctions of the Laplace operator. Then we obtain the following three characteristics of the Markov chain:

- r_α, the above probability;
- s_α, the eigenvalue;
- $v_\alpha := 2r_\alpha - 1$, the rate at which the point moves toward a chosen point ω at infinity.

These characteristics are given by the formulas

$$s_\alpha = \frac{q^\alpha + q^{1-\alpha}}{q+1},$$

$$r_\alpha = \frac{q^\alpha}{q^\alpha + q^{1-\alpha}} = \frac{1}{1 + q^{1-2\alpha}},$$

$$v_\alpha = \frac{2}{1 + q^{1-2\alpha}} - 1 = \frac{1 - q^{1-2\alpha}}{1 + q^{1-2\alpha}} = \frac{q^\alpha - q^{1-\alpha}}{q^\alpha + q^{1-\alpha}}\,.$$

Using these formulas, we easily derive direct relationships between the parameters.

The table below shows the correspondences between the above parameters at the critical points.

The value $\alpha = 1/2$ corresponds to the critical point at which the Markov chain loses ergodicity (phase transition). At this point, the eigenfunctions cease to be minimal. More precisely, for every eigenvalue greater than $2\sqrt{q}/(q+1)$, there are two sets of eigenfunctions: one of them, corresponding to $\alpha > 1/2$, consists of minimal eigenfunctions, and the other consists of eigenfunctions that are not minimal. The difference between them is well illustrated by the change of the rate v_α. For $\alpha > 1/2$ (which is the same as $r_\alpha > 1/2$), almost every trajectory of the random walk approaches

Parameters			Simple random walk	Critical point: loss of ergodicity	Harmonic measure (values of parameters)		
α	$-\infty$	$\cdots$	0	$\frac{1}{2}$	1	$\cdots$	$+\infty$
r_α	0	$\cdots$	$\frac{1}{q+1}$	$\frac{1}{2}$	$\frac{q}{q+1}$	$\cdots$	1
v_α	-1	$\cdots$	$\frac{1-q}{q+1}$	0	$\frac{q-1}{q+1}$	$\cdots$	1
s_α	$+\infty$	$\cdots$	1	$\frac{2\sqrt{q}}{q+1}$	1	$\cdots$	$+\infty$

the corresponding point of the boundary at a linear rate v_α (drift). For $\alpha = 1/2 = r_{1/2}$, there is no such convergence. Negative rates correspond to moving away from the point at infinity.

The value $\alpha = 1$ corresponds to ergodic Markov measures associated with minimal harmonic functions. The value $\alpha = 0$ corresponds to the nonergodic Markov chain that is generated by the Laplace operator; more precisely, the transition probabilities of this chain determine the corresponding Laplace operator. It is its decomposition into ergodic components which determines the harmonic measure on the fiber $\partial T_{q+1} \times q/(q+1)$.

6.2. The absolute of finite groups.

Theorem 6.1. *The absolute of any finite group with any set of generators S consists of a single point, which is the uniform measure on the paths of equal length. The transition probabilities of the corresponding Markov process on the Cayley graph are equal to $1/|S|$ for all edges of the graph. This measure belongs to the Laplacian part, and the degenerate part of the absolute is empty.*

Proof. Let us introduce the notion of a *system of generators*: by a *system of generators* of a group G we mean a (finite in our case) set S and a mapping $S \to G$ (not necessarily an embedding) whose image generates G as a semigroup. It is natural to interpret systems of generators as sets of generators with multiplicities. All constructions of this paper can be extended in an obvious way to the case of a group with a chosen system of generators, including the notions of Cayley graph (in the case under consideration, it may have multiple edges), dynamic graph, central measure, absolute, etc. Obviously, for the one-element group with a system of generators of any cardinality, the whole simplex of central measures consists of one measure, which is described in the statement of the theorem. Considering the epimorphism from a finite group to the one-element group and applying Theorem 4.2 (more precisely, its direct generalization to the case of sets of generators with multiplicities), we see that the Laplacian absolute of a finite group consists of this measure. The degenerate part of the absolute of a finite group is empty by Theorem 5.1, since every infinite path in a finite graph contains cycles. □

6.3. The absolute of commutative groups. The following results were obtained in [17].

Theorem 6.2 (on the absolute of commutative groups and semigroups)**.** *For every commutative semigroup and an arbitrary finite set of generators, the set of ergodic central measures (i.e., the absolute) coincides with the set of those central measures which give rise to Markov chains with independent identically distributed increments. Thus, there is a one-to-one correspondence between the absolute and the collection of all measures on the set of generators that determine Markov chains with this centrality property.*

Theorem 6.3 (on the topology of the absolute of commutative groups)**.** *The absolute of a finitely generated commutative group with respect to any finite set of generators is homeomorphic to a closed disk with dimension equal to the rank of the group. The Laplacian part of the absolute corresponds to the interior of the disk.*

6.4. The absolute of the discrete Heisenberg group. The problem of describing the Laplacian part of the absolute of any finitely generated nilpotent group reduces to the already solved problem of describing the Laplacian part of the absolute of its abelianization (see Corollary 4.4). But the degenerate part of the absolute of a nilpotent group may differ significantly from the degenerate part of the absolute of its abelianization, and describing it in the general case is a difficult problem. In a paper under preparation, we describe the structure of the absolute for the Heisenberg group with the standard set of generators. In the following theorem we give a description of the absolute of the Heisenberg group from that paper in terms of measures on the space of infinite words in the alphabet of generators (this space S^∞ is in one-to-one correspondence with the space of infinite paths in the dynamic graph).

Theorem 6.4. 1. *The absolute* $\mathscr{A}(N_2, S)$ *of the discrete Heisenberg group*

$$N_2 = \langle x, y \mid [[x, y], x] = [[x, y], y] = 1\rangle$$

with the set of generators $S = \{x, y, x^{-1}, y^{-1}\}$ *is the union of a countable set of atomic measures and the set* $\mathscr{B}$ *of Bernoulli measures* μ^∞ *on* S^∞ *for which the generating measure* μ *on* S *satisfies the condition* $\mu(x) \cdot \mu(x^{-1}) = \mu(y) \cdot \mu(y^{-1})$.

2. *There is a natural bijection between the set* $\mathscr{C}$ *of atomic measures in* $\mathscr{A}(N_2, S)$ *and the set* $\mathscr{W}$ *of words of the form* $a^m b^n a b^{+\infty}$, *where* $\{a, b\}$ *is a pair of elements of* S *not inverse to each other and* $(m, n) \in \mathbb{N}_0 \times \mathbb{N}_0$; *namely, for every measure* ν *in* $\mathscr{C}$, *there is a unique word* W *in* $\mathscr{W}$ *such that* ν *is the uniform measure on the tail equivalence class of* W; *moreover, the tail equivalence class in* N_2 *of every word in* $\mathscr{W}$ *is finite, and the uniform measure on this class belongs to* $\mathscr{C}$.

References

[1] A. M. Vershik, "The problem of describing central measures on the path spaces of graded graphs," Funkts. Anal. Prilozhen., **48**:4 (2014), 26–46; English transl.: Functional Anal. Appl., **48**:4 (2014), 256–271.

[2] A. M. Vershik, "Equipped graded graphs, projective limits of simplices, and their boundaries," Zap. Nauchn. Sem. POMI, **432** (2015), 83–104; English transl.: J. Math. Sci. (N. Y.), **209**:6 (2015), 860–873.

[3] A. M. Vershik and A. V. Malyutin, "Phase transition in the exit boundary problem for random walks on groups," Funkts. Anal. Prilozhen., **49**:2 (2015), 7–20; English transl.: Functional Anal. Appl., **49**:2 (2015), 86–96.

[4] A. M. Vershik, "The theory of filtrations of subalgebras, standardness, and independence," Uspekhi Mat. Nauk, **72**:2(434) (2017), 67–146; English transl.: Russian Math. Surveys, **72**:2 (2017), 257–333.

[5] A. M. Vershik and A. V. Malyutin, "Infinite geodesics in the discrete Heisenberg group," Zap. Nauchn. Sem. POMI, **462** (2017), 39–51; English transl.: J. Math. Sci. (N. Y.), **232**:2 (2017), 121–128.

[6] E. B. Dynkin, "The space of exits of a Markov process," Uspekhi Mat. Nauk, **24**:4(148) (1969), 89–152; English transl.: Russian Math. Surveys, **24**:4 (1969), 89–157.

[7] G. A. Margulis, "Positive harmonic functions on nilpotent groups," Dokl. Akad. Nauk SSSR, **166**:5 (1966), 1054–1057; English transl.: Soviet Math. Dokl., **7** (1966), 241–244.

[8] S. A. Molchanov, "On Martin boundaries for the direct product of Markov chains," Teor. Veroyatn. Primen., **12**:2 (1967), 353–358; English transl.: Theor. Probab. Appl., **12** (1967), 310–314.

[9] Kai Lai Chung, Markov Chains with Stationary Transition Probabilities, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1960.

[10] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, and W. P. Thurston, Word Processing in Groups, Jones and Bartlett Publishers, Boston, MA, 1992.

[11] G. B. Folland, Harmonic Analysis in Phase Space, Annals of Mathematics Studies, vol. 122, Princeton Univ. Press, Princeton, NJ, 1989.

[12] M. Gromov, Geometric Group Theory, Vol. 2: Asymptotic Invariants of Infinite Groups, London Math. Soc. Lecture Note Series, 182, Cambridge University Press, Cambridge, 1993.

[13] Y. Guivarc'h and A. Raugi, "Frontière de Furstenberg, propriétés de contraction et théorème de convergence," Z. Wahrsch. Verw. Gebiete, **69**:2 (1985), 187–242.

[14] V. A. Kaimanovich and A. M. Vershik, "Random walks on discrete groups: boundary and entropy," Ann. Probab., **11**:3 (1983), 457–490.
[15] V. A. Kaimanovich and W. Woess, "Boundary and entropy of space homogeneous Markov chains," Ann. Probab., **30**:1 (2002), 323–363.
[16] A. M. Vershik, "Intrinsic metric on graded graphs, standardness, and invariant measures," Zap. Nauchn. Sem. POMI, **421**:6 (2014), 58–67; English transl.: J. Math. Sci. (N. Y.), **200**:6 (2014), 677–681.
[17] A. M. Vershik and A. V. Malyutin, "The absolute of finitely generated groups: I. Commutative (semi)groups," to appear in Eur. J. Math.
[18] W. Woess, Random Walks on Infinite Graphs and Groups, Cambridge Tracts in Math., vol. 138, Cambridge Univ. Press, Cambridge, 2000.

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