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On Fourier Series in Generalized Eigenfunctions of a Discrete Sturm–Liouville Operator

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Abstract. For semicontinuous summation methods generated by $\Lambda = \{\lambda_n(h)\}$ $(n = 0, 1, 2, \dots;$ $h > 0)$ of Fourier series in eigenfunctions of a discrete Sturm–Liouville operator of class $\mathscr{B}$, some results on the uniform a.e. behavior of Λ-means are obtained. The results are based on strong- and weak-type estimates of maximal functions. As a consequence, some statements on the behavior of the summation methods generated by the exponential means $\lambda_n(h) = \exp(-u^\alpha(n)h)$ are obtained. An application to a generalized heat equation is given.

Key words: Fourier series, discrete operator, Sturm-Liouville operator, eigenfunctions, orthogonal polynomials, semicontinuous summation methods, generalized heat equation, Jacobi polynomials, Pollaczek polynomials, loaded Gegenbauer polynomials.

1. The Discrete Sturm–Liouville operator. Let

$$J = \begin{pmatrix} b_0 & a_1 & 0 & 0 & 0 & \dots \\ a_1 & b_1 & a_2 & 0 & 0 & \dots \\ 0 & a_2 & b_2 & a_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{1}$$

be an infinite Jacobian (tridiagonal) symmetric matrix with $a_{n+1} > 0$ and $b_n \in \mathbb{R}$. Let $\mathscr{L}$ be the discrete Sturm–Liouville operator generated by the differences

$$(Lu)_n = a_{n+1}u_{n+1} + b_n u_n + a_n u_{n-1} \qquad (n \in \mathbb{Z}_+,\ u_{-1} = 0), \tag{2}$$

where $u = \{u_n\}_{n=0}^\infty \in l^2$ ([1, Chap. VII, Sec. 1]). Solving the eigenvalue and eigenfunction problem, we obtain the set $\{p_n(x)\}_{n=0}^\infty$ of polynomials defined by the three-term recurrence relation

$$x p_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$
$$(n \in \mathbb{Z}_+,\ p_0(x) = c > 0,\ p_{-1}(x) = 0,\ a_0 = 0).$$

It is known ([1, Chap. VII, Sec. 1], [3, Chap. II, Sec. 8]) that if the entries of the Jacobian matrix (1) are bounded, then there exists a unique positive Borel measure μ such that $\operatorname{Supp}(\mu)$ is compact in $\mathbb{R}$ and the polynomials $p_n(x)$ $(n = 0, 1, 2, \dots)$ form an orthonormal system with respect to the measure μ.

Let us consider Nevai's class $\mathscr{M}$ of Jacobian matrices (1) for which

$$\lim_{n\to\infty} a_n = 1/2, \quad \lim_{n\to\infty} b_n = 0.$$

We have [3, Chap. II, Sec. 9] $\operatorname{Supp}(\mu) = [-1, 1] \cup S$, where S is a finite or countable set of real numbers outside $[-1, 1]$ without accumulation points other than -1 and 1. We say that a discrete Sturm–Liouville operator $\mathscr{L}$ belongs to the class $\mathscr{B}$ if the associated Jacobian matrix belongs to the class $\mathscr{M}$, $\operatorname{Supp}(\mu) = [-1, 1]$, and

$$\sum_{n=0}^{\infty}(|a_n - a_{n+1}| + |b_n - b_{n+1}|) < \infty.$$

In this case, the measure μ is absolutely continuous on the interval $(-1, 1)$ and the weight function $\mu'(x) = w(x)$ is continuous and positive for all $x \in (-1, 1)$ ([9], [11]).

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Examples of eigenfunction systems for operators in the class $\mathscr{B}$ ([4], [6], [10]).

1. The classical Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta>-1)$ orthonormal on $[-1,1]$ with respect to the weight function $w(x)=(1-x)^\alpha(1+x)^\beta$.

2. The Pollaczek polynomials $w^{(a,b)}(x)$ orthonormal on $[-1,1]$ with respect to a weight function $w^{(a,b)}(x)$ satisfying the conditions $\operatorname{Supp} w^{(a,b)}(x)=[-1,1]$ for $a,b\in\mathbb{R}$, $a>|b|$, and

$$w^{(a,b)}(\cos\theta)=2\exp\left\{\frac{\theta}{\sin\theta}(a\cos\theta+b)\right\}\left[1+\exp\left\{\frac{\pi}{\sin\theta}(a\cos\theta+b)\right\}\right]^{-1}$$

(these polynomials are the"singular case" of orthogonal polynomials, because the Pollaczek weight does not belong to the Szegö class).

3. The loaded Gegenbauer polynomials $q_n^{(\alpha)}(x)$ obtained by orthogonalizing the system $\{x^n\}_{n=0}^\infty$ with respect to the inner product

$$\langle f,g\rangle_\alpha=\int_{-1}^1 fg\,d\mu_\alpha,$$

$$d\mu_\alpha=\frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma^2(\alpha+1)}\,(1-x^2)^\alpha dx+L\delta(x-1)+M\delta(x+1)$$

(here $L,M\geqslant 0$, $\alpha\geqslant -1/2$, and $\delta(x)$ is the delta-function).

2. Semicontinuous summation methods for the Fourier series. To each function $f\in L_\mu^1[-1,1]$ we assign its Fourier series in the generalized eigenfunctions $p_n(x)$, $n=0,1,2,\dots$, namely,

$$S(f;x)\sim\sum_{n=0}^\infty c_n(f)p_n(x),\quad c_n(f)=\langle f,p_n\rangle=\int_{-1}^1 fp_n\,d\mu\qquad(n=0,1,2,\dots),$$
$$d\mu=w(x)\,dx+L\delta(x-1)+M\delta(x+1),\qquad L\geqslant 0,\ M\geqslant 0,$$

and consider the behavior as $h\to 0$ of the Kojima–Schur regular semicontinuous linear means

$$U_h(f)=U_h(f;x;\Lambda)=\sum_{n=0}^\infty \lambda_n(h)c_n(f)p_n(x)\qquad(x\in[-1,1])$$

generated by the sequence

$$\Lambda=\{\lambda_n(h)\},\quad \lambda_0(h)=1,\quad \lambda_n(h)=\lambda(x,h)|_{x=n}\quad(n=1,2,\dots),\tag{3}$$

where $\lambda(x,h)$ is a generating function of two variables, $x\in[0,\infty)$, and $h>0$. The sequence (3) is said to be *convex* (*concave*) if

$$\Delta_n^2=\Delta^2\lambda_n(h)\geqslant 0\qquad(\Delta_n^2\leqslant 0),$$
$$\Delta_n^2=\Delta_n-\Delta_{n+1},\quad \Delta_n=\Delta\lambda_n(h)=\lambda_n(h)-\lambda_{n+1}(h),\qquad n=0,1,2,\dots.$$

We say that the sequence (3) is *piecewise convex* if Δ_n^2 changes sign finitely many times.

By $U_*(f;x;\Lambda)$ we denote the maximal function $\sup_{h>0}|U_h(f;x;\Lambda)|$.

Theorem 1. *Let $\mathscr{L}$ be a discrete Sturm–Liouville operator belonging to the class $\mathscr{B}$, and let K be an arbitrary compact set in $(-1,1)$. If the sequence* (3) *is convex* (*concave*) *and its elements satisfy the condition*

$$\lambda_n(h)=O(1/\ln n)\qquad(n\to\infty)\tag{4}$$

for every $h>0$, then the following statements are valid.

1. *If a function $f\in C(K)\cup L_\mu^2(E)$, $E=[-1,1]\setminus K$, is continuous on K, then*

$$\lim_{h\to 0}U_h(f;x;\Lambda)=f(x)\tag{5}$$

uniformly on every compact subset $K_0\subset\operatorname{int}K$.

2. *For the maximal function, the following estimates are valid*:

$$\left(\int_K [U_*(f;x;\Lambda)]^p w(x)\,dx\right)^{1/p} \leqslant C_p\left(\int_K |f(x)|^p w(x)\,dx\right)^{1/p} < \infty \qquad (1 < p < \infty),$$

$$\int_{\{x\in K|U_*(f;x;\Lambda)>\zeta>0\}} w(x)\,dx \leqslant \frac{C}{\zeta^p}\left(\int_K |f(x)|^p w(x)\,dx\right)^{1/p} < \infty \qquad (1 \leqslant p < \infty).$$

3. *If* $f \in L^1_w(K) \cup L^2_\mu(E)$, *then relation* (5) *holds almost everywhere in* K.

Theorem 2. *Suppose that a piecewise convex sequence* (3) *satisfies condition* (4) *and*

$$|\lambda_n(h)| + n|\Delta\lambda_n(h)| \leqslant C \qquad (n = 1, 2, \dots)$$

for all $h > 0$. *Then all statements of Theorem* 1 *hold.*

Corollary 1. *Suppose that a discrete Sturm–Liouville operator* $\mathscr{L}$ *belongs to the class* $\mathscr{B}$ *and* K *is an arbitrary compact set in* $(-1,1)$. *If the sequence*

$$\lambda_n(h) = \exp(-u^\alpha(n)h) \tag{6}$$

is generated by $\lambda(x,h) = \exp(-u^\alpha(x)h)$, *where* $u(x) \in C^2(0,+\infty)$, $u''(x) < 0$, $0 < \alpha \leqslant 1$, *and*

$$\exp(-hu^\alpha(x))\ln x = O(1) \qquad (x \to +\infty) \tag{7}$$

for every $h > 0$, *then, for any function* $f \in L^1_w(K) \cup L^2_\mu(E)$, $E = [-1,1] \setminus K$, *relation* (5) *holds almost everywhere. Moreover, if the function* f *is continuous on* K, *then relation* (5) *is valid on every compact set* $K_0 \subset \operatorname{int} K$.

Corollary 2. *Suppose that a discrete Sturm–Liouville operator* $\mathscr{L}$ *belongs to the class* $\mathscr{B}$, K *is an arbitrary compact set in* $(-1,1)$, *and the sequence* $\lambda_n(h)$ *is defined by formula* (6). *If the function*

$$V(x) = \alpha h u^\alpha(x)\{[u'(x)]^2 - (\alpha-1)[u'(x)]^2 - u(x)u''(x)]\} \qquad (\alpha > 0)$$

has a finite number of zeros on $(0,+\infty)$, *condition* (7) *holds, and there is a constant* $C = C_{u,\alpha} > 0$ *such that*

$$xh\exp(-hu^\alpha(x))u^{\alpha-1}(x)|u'(x)| \leqslant C$$

for all $h > 0$ *and* $x \in (1,+\infty)$, *then the conclusions of Theorem* 2 *are valid for* $\alpha > 1$.

Examples. 1. Let $u(x) = x$, i.e.,

$$\lambda_0(h) = 1, \quad \lambda(x,h) = \exp(-hx^\alpha) \qquad (x > 0). \tag{8}$$

Then, for all $\alpha > 0$, the sequence defined by (8) satisfies all conditions of Corollary 1 and Corollary 2.

2. Suppose fixed a polynomial $\pi_m(x) = a_0x^m + a_1x^{m-1} + \cdots$, $a_0 > 0$, $m = 0,1,2,\dots$. Then, for $U_h(f;x;\Lambda) = \sum_{k=0}^\infty \exp(-h\pi_m(k))c_k(f)p_k(x)$, the statement of Theorem 2 is valid.

3. Generalized heat equations. Let $\{p_n(x)\}_{n=0}^\infty$ be a complete system of polynomials of degree n orthonormal on $[-1,1]$ with respect to the measure μ. These polynomials are eigenfunctions of a differential operator D with respect to the continuous variable x, i.e.,

$$D_x p_n = -\mu_n p_n(x), \quad n = 0,1,2,\dots, \qquad \mu_n \to +\infty, \quad n \to +\infty,$$

and eigenfunctions of a discrete Sturm–Liouville operator $\mathscr{L} \in \mathscr{B}$ with respect to the discrete variable n. The generalized solution (in Bochner's sense [7], [8])) of Dirichlet's problem for the generalized heat equation

$$\frac{\partial u(x,t)}{\partial t} = D_x u(x,t), \quad u(x,0) = f(x),$$

has the form

$$u(x,t) = \sum_{n=0}^{\infty} \exp(-\mu_n t) c_n(f) p_n(x), \quad c_n(f) = \int_{-1}^{1} f(s) p_n(s)\, d\mu(s)$$
$$(n = 0, 1, 2, \dots).$$

We can investigate this series by using the results obtained above.

Remark. For the trigonometric Fourier series, results of this paper were obtained jointly with Nakhman in [2] and [5].

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