Functional Analysis and Its Applications, Vol. 52, *No.* 2, *pp.* 147–150, 2018
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 52, *No.* 2, *pp.* 82–85, 2018

On the Convergence of Solutions of Variational Problems with Implicit Pointwise Constraints in Variable Domains*

A. A. Kovalevsky

Received May 29, 2017

Abstract. Results on the convergence of minimizers and minimum values of integral and more general functionals $J_s\colon W^{1,p}(\Omega_s)\to\mathbb{R}$ on the sets $U_s(h_s)=\{v\in W^{1,p}(\Omega_s): h_s(v)\leqslant 0$ a.e. in $\Omega_s\}$, where $p>1$, $\{\Omega_s\}$ is a sequence of domains contained in a bounded domain Ω of $\mathbb{R}^n$ $(n\geqslant 2)$, and $\{h_s\}$ is a sequence of functions on $\mathbb{R}$, are announced.

Key words: integral functional, variational problem, implicit pointwise constraint, minimizer, minimum value, Γ-convergence, variable domain.

In this note we announce some results on the convergence of minimizers and minimum values of integral and more general functionals $J_s\colon W^{1,p}(\Omega_s)\to\mathbb{R}$ on the sets $U_s(h_s)=\{v\in W^{1,p}(\Omega_s):$ $h_s(v)\leqslant 0$ a.e. in $\Omega_s\}$, where $p>1$, $\{\Omega_s\}$ is a sequence of domains contained in a bounded domain Ω of $\mathbb{R}^n$ $(n\geqslant 2)$, and $\{h_s\}$ is a sequence of functions on $\mathbb{R}$. To prove these results, we need that the spaces $W^{1,p}(\Omega_s)$ be strongly connected with the space $W^{1,p}(\Omega)$ and the functionals under consideration Γ-converge to a functional defined on $W^{1,p}(\Omega)$. Moreover, we impose certain conditions on the relationship of the functions h_s to a function $h\colon\mathbb{R}\to\mathbb{R}$. Actually, these conditions relate the sets $\Phi(h_s)=\{t\in\mathbb{R}: h_s(t)\leqslant 0\}$ to the set $\Phi(h)=\{t\in\mathbb{R}: h(t)\leqslant 0\}$. The convexity of these sets is not required. To our best knowledge, the convergence of minimizers and minimum values of functionals has not been studied before for the specified sets of constraints in such generality nor even in the case where the domains Ω_s coincide with the domain Ω.

It is easy to see that the class of variational problems under consideration includes problems with constraints of the forms $v\geqslant\varphi_s$, $v\leqslant\psi_s$, and $\varphi_s\leqslant v\leqslant\psi_s$ a.e. in Ω_s, where $\{\varphi_s\},\{\psi_s\}\subset\mathbb{R}$. In this connection, we mention that the convergence of solutions of variational problems with constraints of the forms $v\geqslant\varphi$, $v\leqslant\psi$, and $\varphi\leqslant v\leqslant\psi$ a.e. in Ω_s, where $\varphi,\psi\in W^{1,p}(\Omega)$, was studied, for instance, in [1] and [2] for the strongly connected spaces $W^{1,p}(\Omega_s)$ and Γ-convergent functionals defined on these spaces. The notion of strong connectedness of Sobolev spaces goes back to [3], where the condition of strong connectedness of n-dimensional domains was introduced. As for the notion of Γ-convergence of functionals with variable domain of definition, we refer the reader, e.g., to [4]. Concerning the notion of Γ-convergence of functionals with the same domain and related results, see, e.g., [5]. The techniques of Γ-convergence theory was used in [6] and [7] to study the behavior of minimizers and minimum values of variational problems with general varying unilateral and bilateral obstacles in a fixed domain.

Thus, let Ω be a bounded domain of $\mathbb{R}^n$ $(n\geqslant 2)$, and let $p>1$. Suppose given a sequence $\{\Omega_s\}$ of domains of $\mathbb{R}^n$ contained in Ω.

For any $s\in\mathbb{N}$, by q_s we denote the restriction operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\Omega_s)$.

Definition 1. We say that the sequence of spaces $W^{1,p}(\Omega_s)$ is *strongly connected* with the space $W^{1,p}(\Omega)$ if there exists a sequence of linear continuous extension operators $l_s\colon W^{1,p}(\Omega_s)\to W^{1,p}(\Omega)$ such that the sequence of norms $\|l_s\|$ is bounded.

Definition 2. Suppose that $I_s\colon W^{1,p}(\Omega_s)\to\mathbb{R}$ for any $s\in\mathbb{N}$ and $I\colon W^{1,p}(\Omega)\to\mathbb{R}$. We say that the sequence $\{I_s\}$ Γ-*converges* to the functional I if the following conditions are satisfied:

*This work was supported by the Russian Academic Excellence Project (agreement no. 02.A03.21.0006 of August 27, 2013, between the Ministry of Education and Science of the Russian Federation and Ural Federal University).

(i) for any function $v \in W^{1,p}(\Omega)$, there exists a sequence $w_s \in W^{1,p}(\Omega_s)$ such that $\|w_s - q_s v\|_{L^p(\Omega_s)} \to 0$ and $I_s(w_s) \to I(v)$;

(ii) for any function $v \in W^{1,p}(\Omega)$ and any sequence $v_s \in W^{1,p}(\Omega_s)$ such that $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$, we have $\liminf_{s\to\infty} I_s(v_s) \geqslant I(v)$.

Further, suppose given $c_1, c_2 > 0$. Let $\mu_s \in L^1(\Omega_s)$ be such that $\mu_s \geqslant 0$ on Ω_s for any $s \in \mathbb{N}$. We also assume that the sequence of norms $\|\mu_s\|_{L^1(\Omega_s)}$ is bounded.

For each $s \in \mathbb{N}$, let $f_s\colon \Omega_s \times \mathbb{R}^n \to \mathbb{R}$ be a function satisfying the following conditions: for any $\xi \in \mathbb{R}^n$, the function $f_s(\,\cdot\,, \xi)$ is measurable on Ω_s; for a.e. $x \in \Omega_s$, the function $f_s(x, \,\cdot\,)$ is convex on $\mathbb{R}^n$; and, for a.e. $x \in \Omega_s$ and any $\xi \in \mathbb{R}^n$,

$$c_1|\xi|^p - \mu_s(x) \leqslant f_s(x, \xi) \leqslant c_2|\xi|^p + \mu_s(x).$$

In view of the assumptions on the functions f_s and μ_s, for any $s \in \mathbb{N}$ and any $v \in W^{1,p}(\Omega_s)$, the function $f_s(x, \nabla v)$ is integrable on Ω_s.

For each $s \in \mathbb{N}$, we define a functional $F_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ by

$$F_s(v) = \int_{\Omega_s} f_s(x, \nabla v)\, dx, \qquad v \in W^{1,p}(\Omega_s).$$

By the conditions imposed on the functions f_s, for any $s \in \mathbb{N}$, the functional F_s is weakly lower semicontinuous.

Suppose given $c_3 > 0$ and $c_4 \geqslant 0$, and let $G_s\colon W^{1,p}(\Omega_s) \to \mathbb{R}$ be a weakly continuous functional for any $s \in \mathbb{N}$. We assume that, for any $s \in \mathbb{N}$ and any $v \in W^{1,p}(\Omega_s)$, $G_s(v) \geqslant c_3\|v\|^p_{L^p(\Omega_s)} - c_4$. It is clear that, for any $s \in \mathbb{N}$, the functional $F_s + G_s$ is weakly lower semicontinuous and coercive.

Further, for any function $h\colon \mathbb{R} \to \mathbb{R}$, we define

$$\Phi(h) = \{t \in \mathbb{R} : h(t) \leqslant 0\}, \qquad U(h) = \{v \in W^{1,p}(\Omega) : h(v) \leqslant 0 \text{ a.e. in } \Omega\}.$$

Moreover, for any $s \in \mathbb{N}$ and any function $h\colon \mathbb{R} \to \mathbb{R}$, we define

$$U_s(h) = \{v \in W^{1,p}(\Omega_s) : h(v) \leqslant 0 \text{ a.e. in } \Omega_s\}.$$

We note that if $s \in \mathbb{N}$, $h\colon \mathbb{R} \to \mathbb{R}$, and the set $\Phi(h)$ is nonempty and closed, then the set $U_s(h)$ is nonempty and sequentially weakly closed in $W^{1,p}(\Omega_s)$.

Now, in view of the above properties of the functionals $F_s + G_s$ and due to known results on the existence of minimizers of functionals (see, e.g., [8]), we conclude that if $s \in \mathbb{N}$, $h\colon \mathbb{R} \to \mathbb{R}$, and the set $\Phi(h)$ is nonempty and closed, then there exists a function belonging to the set $U_s(h)$ and minimizing the functional $F_s + G_s$ on this set.

We assume that the following conditions are satisfied:

($*_1$) the embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$ is compact;

($*_2$) the sequence of spaces $W^{1,p}(\Omega_s)$ is strongly connected with the space $W^{1,p}(\Omega)$;

($*_3$) there exists a functional $F\colon W^{1,p}(\Omega) \to \mathbb{R}$ such that the sequence $\{F_s\}$ Γ-converges to the functional F;

($*_4$) there exists a functional $G\colon W^{1,p}(\Omega) \to \mathbb{R}$ such that $G_s(v_s) \to G(v)$ for any function $v \in W^{1,p}(\Omega)$ and any sequence $v_s \in W^{1,p}(\Omega_s)$ with the property $\|v_s - q_s v\|_{L^p(\Omega_s)} \to 0$.

Conditions ($*_1$)–($*_4$) are essentially used in the proofs of the theorems stated below. As far as the fulfillment of these conditions is concerned, we refer the reader to the comments and examples given in [2].

Theorem 1. *Assume that, for any sequence of measurable sets $K_s \subset \Omega_s$ such that* $\operatorname{meas} K_s \to 0$, *the integral of the function μ_s over the set K_s tends to zero as $s \to \infty$. For each $s \in \mathbb{N}$, let $h_s\colon \mathbb{R} \to \mathbb{R}$ be a function such that the set $\Phi(h_s)$ is nonempty and closed, and let $h\colon \mathbb{R} \to \mathbb{R}$ be a function such that the set $\Phi(h)$ is nonempty and closed. Assume that the following conditions are satisfied*:

($*'$) *if $t \in \Phi(h)$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $t \in [t_1, t_2] \subset \Phi(h)$;*

($*''$) *if $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, $(t_1, t_2) \subset \Phi(h)$, and $0 < \sigma < (t_2 - t_1)/2$, then there exists an $\bar{s} \in \mathbb{N}$ such that $[t_1 + \sigma, t_2 - \sigma] \subset \Phi(h_s)$ for any $s \in \mathbb{N}$, $s \geqslant \bar{s}$;*

$(*''')$ *if* $t_s \to t$ *in* $\mathbb{R}$, $\{\tilde{s}_j\}$ *is an increasing sequence in* $\mathbb{N}$, *and* $t_{\tilde{s}_j} \in \Phi(h_{\tilde{s}_j})$ *for any* $j \in \mathbb{N}$, *then* $t \in \Phi(h)$.

For each $s \in \mathbb{N}$, *let* u_s *be a function in* $U_s(h_s)$ *minimizing the functional* $F_s + G_s$ *on the set* $U_s(h_s)$. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in U(h)$ *such that the function* u *minimizes the functional* $F + G$ *on the set* $U(h)$, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, *and* $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

In the next theorem, we impose a stronger requirement on the functions μ_s than in Theorem 1. At the same time, the assumptions on the corresponding set $\Phi(h)$ in Theorem 2 are less restrictive than in the previous theorem.

Theorem 2. *Assume that* $\|\mu_s\|_{L^1(\Omega_s)} \to 0$. *For each* $s \in \mathbb{N}$, *let* $h_s \colon \mathbb{R} \to \mathbb{R}$ *be a function such that the set* $\Phi(h_s)$ *is nonempty and closed. Let* $h \colon \mathbb{R} \to \mathbb{R}$ *be a function such that the set* $\Phi(h)$ *is closed and has nonempty interior. Assume that the following condition is satisfied*:

$(*')$ *if* $t \in \Phi(h)$, *then there exists a sequence* $\{t_s\} \subset \mathbb{R}$ *such that* $t_s \to t$ *and* $t_s \in \Phi(h_s)$ *for any* $s \in \mathbb{N}$.

Assume also that conditions $(*'')$ *and* $(*''')$ *of Theorem* 1 *are satisfied. For each* $s \in \mathbb{N}$, *let* u_s *be a function in* $U_s(h_s)$ *minimizing the functional* $F_s + G_s$ *on the set* $U_s(h_s)$. *Then there exist an increasing sequence* $\{s_j\} \subset \mathbb{N}$ *and a function* $u \in U(h)$ *such that the function* u *minimizes the functional* $F + G$ *on the set* $U(h)$, $\|u_{s_j} - q_{s_j} u\|_{L^p(\Omega_{s_j})} \to 0$, *and* $(F_{s_j} + G_{s_j})(u_{s_j}) \to (F + G)(u)$.

A result similar to Theorem 2 also holds in the case where the corresponding set $\Phi(h)$ is nonempty, closed, and has empty interior. Naturally, in this case, condition $(*'')$ of Theorem 1 is not required.

Due to space limitations, here we do not give the proofs of the stated results. We only mention an auxiliary proposition and its analogue that concern the elements of the considered sets of constraints and are essentially used in the proofs of Theorems 1 and 2.

Proposition. *Let* $h \colon \mathbb{R} \to \mathbb{R}$ *be a function such that the set* $\Phi(h)$ *is nonempty and closed. Then* $(\operatorname{ess\,inf}_\Omega v, \operatorname{ess\,sup}_\Omega v) \subset \Phi(h)$ *for any* $v \in U(h)$.

A similar proposition holds for the elements of the sets $U_s(h)$, $s \in \mathbb{N}$, with a given function $h \colon \mathbb{R} \to \mathbb{R}$ such that the set $\Phi(h)$ is nonempty and closed.

We make a number of concluding remarks. It is easy to see that if, for any $s \in \mathbb{N}$, $h_s \colon \mathbb{R} \to \mathbb{R}$ is a function such that the set $\Phi(h_s)$ is nonempty, and if $h \colon \mathbb{R} \to \mathbb{R}$ is a function such that the set $\Phi(h)$ is nonempty, then, first, conditions $(*')$–$(*''')$ of Theorem 1 imply that the sequence $\{\Phi(h_s)\}$ converges to the set $\Phi(h)$ in the sense of Kuratowski [9, Section 29] and, second, condition $(*')$ of Theorem 2, along with condition $(*''')$ of Theorem 1, is satisfied if and only if the sequence $\{\Phi(h_s)\}$ converges to the set $\Phi(h)$ in the sense of Kuratowski. However, examples show that we cannot replace conditions $(*')$–$(*''')$ of Theorem 1 by the requirement that the sequence $\{\Phi(h_s)\}$ converges to the set $\Phi(h)$ in the sense of Kuratowski without violating the conclusion of this theorem. Furthermore, examples show the importance of the conditions of the stated theorems for the validity of the corresponding conclusions.

We also note that, in general, the uniform convergence of functions $h_s \colon \mathbb{R} \to \mathbb{R}$ to a function $h \colon \mathbb{R} \to \mathbb{R}$ on any bounded closed interval of $\mathbb{R}$ does not guarantee that these functions satisfy all the conditions of Theorems 1 and 2. However, if, in addition to this convergence, for instance, the function h is lower semicontinuous on $\mathbb{R}$, the set $\Phi(h)$ has nonempty interior, and $\Phi(h_s) \neq \varnothing$ and $h_s \leqslant h$ in $\Phi(h)$ for any $s \in \mathbb{N}$, then conditions $(*'')$ and $(*''')$ of Theorem 1 and condition $(*')$ of Theorem 2 are satisfied.

Finally, we note that Theorems 1 and 2 cannot be applied to study the convergence of minimizers and minimum values of the functionals $F_s + G_s$ on the sets $U_s(h_s)$, where $h_s \colon \mathbb{R} \to \mathbb{R}$ are periodic functions with periods converging to zero. However, a conclusion similar to the conclusions of the above theorems can be obtained for these minimizers and minimum values with the only but important difference that the set of constraints in the corresponding limit problem is $V = \{v \in$

$W^{1,p}(\Omega) : v = c$ a.e. in Ω, where $c \in \mathbb{R}\}$. It is easy to verify that this set can be written in the form $U(h)$ for no function $h\colon \mathbb{R} \to \mathbb{R}$.

References

[1] A. A. Kovalevskii, in: Current Analysis and its Applications [in Russian], Naukova Dumka, Kiev, 1989, 62–70.

[2] A. A. Kovalevsky, Trudy Inst. Math. Mech. UrO Ross. Akad. Nauk, **22**:1 (2016), 140–152; English transl.: Proc. Steklov Inst. Math., **296, Suppl. 1** (2017), 151–163.

[3] E. Ya. Khruslov, Mat. Sb., **106(148)**:4(8) (1978), 604–621; English transl.: Math. USSR Sb., **35**:2 (1979), 266–282.

[4] A. A. Kovalevskii, Nelinein. Granichnye Zadachi, **4** (1992), 29–39.

[5] V. V. Zhikov, Izv. Akad. Nauk SSSR, Ser. Math., **47**:5 (1983), 961–998; English transl.: Math. USSR Izv., **23**:2 (1984), 243–276.

[6] G. Dal Maso, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Natur. (8), **74**:2 (1983), 55–61.

[7] G. Dal Maso, Ann. Mat. Pura Appl. (4), **129**:1 (1981), 327–366.

[8] M. M. Vainberg, Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations, Wiley, New York, 1973.

[9] K. Kuratowski, Topology, vol. I, Academic Press, New York–London, 1966.

Krasovskii Institute of Mathematics and Mechanics,
Ural Branch of the Russian Academy of Sciences,
Yekaterinburg, Russia
Ural Federal University, Yekaterinburg, Russia
e-mail: alexkvl71@mail.ru

Translated by A. A. Kovalevsky