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The Index of a 1-Form on a Real Quotient Singularity*

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Abstract. Let G be a finite Abelian group acting (linearly) on space $\mathbb{R}^n$ and, therefore, on its complexification $\mathbb{C}^n$, and let W be the real part of the quotient $\mathbb{C}^n/G$ (in the general case, $W \neq \mathbb{R}^n/G$). The index of an analytic 1-form on the space W is expressed in terms of the signature of the residue bilinear form on the G-invariant part of the quotient of the space of germs of n-forms on $(\mathbb{R}^n, 0)$ by the subspace of forms divisible by the 1-form under consideration.

Key words: group action, real quotient singularity, 1-form, index, signature formula.

An isolated singular point (zero) of a 1-form on a differentiable manifold has a natural invariant, the index. For a germ $\omega = \sum f_i dx_i$ of a 1-form on $\mathbb{R}^n$ at the origin, the index $\operatorname{ind}(\omega; \mathbb{R}^n, 0)$ is the same as the local degree of the map $F = (f_1, \dots, f_n)\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, or the index of the vector field $\sum f_i \frac{\partial}{\partial x_i}$ at the origin. If a 1-form ω on $(\mathbb{R}^n, 0)$ is analytic and its complexification $\omega_{\mathbb{C}}$ on $(\mathbb{C}^n, 0)$ has an isolated singularity at the origin, one has an algebraic formula for the index $\operatorname{ind}(\omega; \mathbb{R}^n, 0)$ (see [3], [4]). Let $\Omega_{\mathbb{R}^n,0}$ be the space of germs of analytic n-forms on $(\mathbb{R}^n, 0)$. (As a vector space, $\Omega_{\mathbb{R}^n,0}$ is isomorphic to the ring $\mathscr{E}_{\mathbb{R}^n,0}$ of germs of analytic functions on $(\mathbb{R}^n, 0)$. The isomorphism is determined by the choice of a volume form on $(\mathbb{R}^n, 0)$.) Let $\Omega_\omega := \Omega^n_{\mathbb{R}^n,0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n,0}$. This is a finite-dimensional vector space isomorphic (as a vector space) to the algebra $Q_F := \mathscr{E}_{\mathbb{R}^n,0}/\langle f_1, \dots, f_n\rangle$; the isomorphism is determined up to an automorphism of Q_F induced by multiplication by an invertible function. Let $\Omega^{\mathbb{C}}_\omega = \Omega^n_{\mathbb{C}^n,0}/\omega_{\mathbb{C}} \wedge \Omega^{n-1}_{\mathbb{C}^n,0}$ be the complexification of Ω_ω. The space $\Omega^{\mathbb{C}}_\omega$ is endowed with the (canonical) nondegenerate (residue) pairing B_ω defined in the following way: for two elements ζ_i, $i = 1, 2$, represented by n-forms $\varphi_i(\mathbf{x})d\mathbf{x}$, where $d\mathbf{x} := dx_1 \wedge \dots \wedge dx_n$ and $\varphi_i(\mathbf{x}) \in \mathscr{O}_{\mathbb{C}^n,0}$, one has

$$B_\omega(\zeta_1, \zeta_2) = \operatorname{Res}\begin{bmatrix} \varphi_1(\mathbf{x})\varphi_2(\mathbf{x})\, d\mathbf{x} \\ f_1 \cdots f_n \end{bmatrix} := \frac{1}{(2\pi i)^n} \int \frac{\varphi_1(\mathbf{x})\varphi_2(\mathbf{x})}{f_1 \cdots f_n}\, d\mathbf{x},$$

where the integration is over the cycle given by the equations $\|f_k(\mathbf{x})\| = \delta_k$ with sufficiently small positive δ_k. The restriction of the pairing B_ω to the real part Ω_ω gives the (real) residue pairing $B_\omega\colon \Omega_\omega \times \Omega_\omega \to \mathbb{R}$. The Eisenbud–Levine–Khimshiashvili formula ([3], [4]) is that the index $\operatorname{ind}(\omega; \mathbb{R}^n, 0)$ is equal to the signature of the pairing B_ω.

The index of a singular point of a 1-form on a differentiable manifold has a generalization to 1-forms on singular varieties, the so-called radial index [1]. Let $(X, 0) \subset (\mathbb{R}^N, 0)$ be a germ of an analytic or closed semianalytic variety, and let $X = \bigcup_i V_i$ be a Whitney stratification of the germ $(X, 0)$. A 1-form ω on $(X, 0)$ is the restriction of a (continuous) 1-form on $(\mathbb{R}^N, 0)$ (which will be denoted by ω as well). A point $x \in X$ is a singular point of a 1-form ω on X if it is a singular point of the restriction of ω to the stratum V_i containing x. A 1-form ζ defined in a neighborhood of a point $x \in X$ is *radial* at x if, for an arbitrary nontrivial analytic arc $\varphi\colon (\mathbb{R}, 0) \to (X, x)$, the value of the 1-form ζ at the tangent vector $\dot{\varphi}(t)$ is positive for sufficiently small positive t. The (radial) index $\operatorname{ind}(\omega; X, 0)$ is defined for a germ of a 1-form with an isolated singular point at the origin 0. The index $\operatorname{ind}(\omega; X, 0)$ can be defined by the following two properties.

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(1) Assume that $\delta > 0$ is such that the variety X and the 1-form ω are defined in the ball $B_\delta(0)$ of radius δ centered at the origin and ω has no singular points on $X \cap B_\delta(0)$ outside the origin. Let $\widetilde{\omega}$ be a 1-form on $X \cap B_\delta(0)$ (that is, the restriction of a 1-form defined on the ball $B_\delta(0)$) which coincides with ω in a neighborhood of the sphere $S_\delta(0) = \partial B_\delta(0)$ and has only isolated singular points on X. Then

$$\operatorname{ind}(\omega; X, 0) = \sum_{p \in \operatorname{Sing} \widetilde{\omega}} \operatorname{ind}(\widetilde{\omega}; X, p),$$

where the summation is over all singular points p of the 1-form $\widetilde{\omega}$.

(2) Assume that $\widetilde{\omega}$ is a 1-form on $B_\delta(0)$, x_0 is a singular point of $\widetilde{\omega}$ on X in the interior of the ball $B_\delta(0)$, $x_0 \in V_i$ (V_i is the corresponding stratum of the Whitney stratification), and $\dim V_i = k$. There exists a local analytic diffeomorphism $h\colon (\mathbb{R}^N, \mathbb{R}^k, 0) \to (\mathbb{R}^N, V_i, x_0)$. Let $\mathbb{R}^N = \mathbb{R}^k \oplus \mathbb{R}^{N-k}$. Assume that $h^*\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2$, where π_1 and π_2 are the natural projections $\pi_1\colon \mathbb{R}^N \to \mathbb{R}^k$ and $\pi_2\colon \mathbb{R}^N \to \mathbb{R}^{N-k}$, respectively, ω_1 is a germ of a 1-form on $(\mathbb{R}^k, 0)$ with an isolated singular point at the origin, and ω_2 is a radial 1-form on $(\mathbb{R}^{N-k}, 0)$. Then $\operatorname{ind}(\widetilde{\omega}; X, x_0) = \operatorname{ind}(\omega_1; \mathbb{R}^k, 0)$ (the index on the right-hand side is the usual index of a singular point of a 1-form on a smooth manifold).

In particular, if the 1-form ω on $(X, 0)$ is the differential of a real analytic function $f\colon (X,0) \to (\mathbb{R}, 0)$, then $\operatorname{ind}(\omega; X, 0) = -\chi(M_f^-) + 1$, where $M_f^- := \{\mathbf{x} \in X : f(\mathbf{x}) = -\varepsilon\} \cap B_\delta(0)$ with $0 < \varepsilon \ll \delta$ small enough is the "negative" Milnor fiber of the function germ f.

There were attempts to give an analogue of the Eisenbud–Levine–Khimshiashvili formula for an analytic 1-form ω on an isolated singularity $(X, 0)$ of a hypersurface or a complete intersection of dimension n. In particular, analogues of the residue quadratic form on the corresponding spaces, such as $\Omega_\omega = \Omega^n_{X,0}/\omega \wedge \Omega^{n-1}_{X,0}$, have been constructed; see [2]. The main problem was that all these quadratic forms appeared to be, in general, degenerate. This was the main reason for these attempts to fail.

Suppose that a finite group G acts (linearly) on the space $\mathbb{R}^n$ and, thus, on its complexification $\mathbb{C}^n$. The quotient $\mathbb{R}^n/G$ is a semialgebraic variety. Let W be the real part of the quotient $\mathbb{C}^n/G$. In general, $W \neq \mathbb{R}^n/G$. One can show that W is the algebraic closure $\overline{\mathbb{R}^n/G}$ of $\mathbb{R}^n/G$. Another description of the variety W (or, rather, of its preimage $\pi^{-1}(W)$ under the quotient map $\pi\colon \mathbb{C}^n \to \mathbb{C}^n/G$) is as follows. If the order of G is odd, then $W = \mathbb{R}^n/G$ (and $\pi^{-1}(W) = \mathbb{R}^n$). Otherwise, for an element $g \in G$, let $\mathbb{R}^n_{g\pm} := \{\mathbf{x} \in \mathbb{R}^n \mid g\mathbf{x} = \pm\mathbf{x}\}$ (if g is of odd order, $\mathbb{R}^n_{g-} = \{0\}$). One has

$$\pi^{-1}(\overline{\mathbb{R}^n/G}) = \bigcup_{g \in G} (\mathbb{R}^n_{g+} \oplus i\mathbb{R}^n_{g-}).$$

Assume that ω is a real analytic germ of a 1-form on $(W, 0)$ (that is, the restriction of a real analytic germ of a 1-form on an ambient affine space) with an algebraically isolated singular point at the origin. This means that the complexification $\omega_{\mathbb{C}}$ of the 1-form ω (which is a 1-form on $(W_{\mathbb{C}}, 0) = (\mathbb{C}^n/G, 0)$) has an isolated singular point at the origin. Here we give an algebraic formula for the index $\operatorname{ind}(\omega; W, 0)$ as the signature of a certain (nondegenerate) quadratic form.

The lifting $\omega^* = \pi^*\omega$ of the 1-form ω is a G-invariant real analytic 1-form on $(\pi^{-1}(W), 0) \subset (\mathbb{C}^n, 0)$ and, thus, on $(\mathbb{R}^n, 0) \subset (\pi^{-1}(W), 0)$. It has an algebraically isolated singular point on $\mathbb{R}^n$ at the origin. There is the canonical (residue) pairing on the space $\Omega_{\omega^*} = \Omega^n_{\mathbb{R}^n,0}/\omega^* \wedge \Omega^{n-1}_{\mathbb{R}^n,0}$. The group G acts on the module $\Omega^n_{\mathbb{R}^n,0}$ of (analytic) n-forms on $(\mathbb{R}^n, 0)$ and on the quotient Ω_{ω^*}. The residue pairing on Ω_ω is G-invariant. This implies that its restriction to the G-invariant part $\Omega^G_{\omega^*}$ is nondegenerate.

Theorem 1. *Given a finite Abelian group G and a real analytic 1-form ω on the real quotient $(W, 0)$ with an algebraically isolated singular point at the origin, its (radial) index $\operatorname{ind}(\omega; W, 0)$ is equal to the signature $\operatorname{sgn} B^G_{\omega^*}$ of the restriction $B^G_{\omega^*}$ of the residue pairing corresponding to the lifting $\omega^* = \pi^*\omega$ of the 1-form ω to the G-invariant part $\Omega^G_{\omega^*}$.*

The proof of this theorem is based on the following key ingredients. First, it is possible to show that both the index $\operatorname{ind}(\omega; W, 0)$ and the signature $\operatorname{sgn} B^G_{\omega^*}$ satisfy "the law of conservation of number." This means the following. Let $\widetilde{\omega}$ be a small real analytic deformation of the 1-form ω, and let $\operatorname{Sing}\widetilde{\omega} \subset W$ be the set of its singular points in a small neighborhood of the origin. Then one has

$$\operatorname{ind}(\omega; W, 0) = \sum_{p \in \operatorname{Sing}\widetilde{\omega}} \operatorname{ind}(\widetilde{\omega}; W, p), \qquad \operatorname{sgn} B^G_{\omega^*} = \sum_{p \in \operatorname{Sing}\widetilde{\omega}^*} \operatorname{sgn} B^{G_q}_{\widetilde{\omega}^*, q},$$

where q in the second sum is a point in the preimage $\pi^{-1}(p)$ and the signature is computed on the G_q-invariant part of the corresponding space (G_q is the isotropy subgroup of the point q). For the index, this property is a more or less direct consequence of its definition [1]. The proof of this property for the signature is more delicate. Second, any real analytic 1-form ω on $(W, 0)$ can be analytically deformed to a 1-form $\widetilde{\omega}$ such that all singular points of the 1-form $\widetilde{\omega}^*$ (a 1-form on $\mathbb{C}^n$ with real values on $\pi^{-1}(W)$) split from the origin are nondegenerate. This property allows us to verify the statement of Theorem 1 for 1-forms ω such that their liftings ω^* have nondegenerate singular points. (We refer to 1-forms ω of this sort as nondegenerate as well.) For a nondegenerate 1-form ω, the dimension of the space Ω_{ω^*} is equal to 1, and therefore the computation of the signature $\operatorname{sgn} B^G_{\omega^*}$ is not difficult. It is equal to zero if the representation of G on $\mathbb{R}^n$ is not in $\mathrm{SL}(n, \mathbb{R})$ and to $\pm 1 = \operatorname{ind}(\omega^*, \mathbb{R}^n, 0)$ otherwise. However, the computation of the index $\operatorname{ind}(\omega; W, 0)$ leads to a combinatorial problem, which is hard to not only solve but even formulate. It is possible to reduce the problem to those representations of G on $\mathbb{R}^n$ which contain only one-dimensional irreducible representations and do not contain the trivial one. (The implementation of this and the next step requires G to be Abelian.) After that, one constructs a degenerate 1-form on $(W, 0)$ which can be deformed to any nondegenerate 1-form such that its index and signature can be computed (and turn out to coincide). Then the proof of the statement can be completed by using induction on the order of the group G.

A somewhat more general result with a complete proof will be published elsewhere.

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