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Probabilistic Approximation of the Evolution Operator*

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Abstract. A method for approximation of the operator e^{-itH}, where $H=-\frac{1}{2}\frac{d^2}{dx^2}+V(x)$, in the strong operator topology is proposed. The approximating operators have the form of expectations of functionals of a certain random point field.

Key words: evolution equation, limit theorem, Feynman–Kac formula.

1. Introduction

Consider the operator

$$H=-\frac{1}{2}\,\frac{d^2}{dx^2}+V(x) \tag{1}$$

with domain $W_2^2(\mathbb{R})$. The potential V is assumed to be real and bounded, which implies the self-adjointness of the operator H. We introduce the free Hamiltonian

$$H_0=-\frac{1}{2}\,\frac{d^2}{dx^2}\,. \tag{2}$$

The family of operators e^{-itH} forms a group of unitary operators on $L_2(\mathbb{R})$. The operator e^{-itH} takes each function $\varphi\in W_2^2(\mathbb{R})$ to the solution $u(t,x)$ of the Cauchy problem for the Schrödinger equation

$$i\,\frac{\partial u}{\partial t}=Hu \tag{3}$$

with $u(0,x)=\varphi(x)$.

As is known (see [1], [5], and [13]), for the heat equation $\partial u/\partial t=-\sigma^2Hu$ (here σ^2 is any positive parameter), the solution of the Cauchy problem with initial function $u(0,x)=\varphi(x)$ admits a probabilistic representation in the form of the expectation of a functional of a Wiener process (the Feynman–Kac formula); namely,

$$u(t,x)=e^{-t\sigma^2H}\varphi(x)=\mathbf{E}[\varphi(x+\sigma w(t))e^{-\sigma^2\int_0^t V(x+\sigma w(\tau))\,d\tau}], \tag{4}$$

where $w(t)$ is a standard Wiener process.

This relation means that the evolution of the initial function φ under the action of the heat operator can be modeled by statistical methods; for this purpose, it is sufficient to have means for generating trajectories of a Wiener process.

In this paper, an ideologically similar approach is developed for the operator e^{-itH}. We construct an approximation of e^{-itH} in the strong operator topology by the expectations of functionals of a certain random point field. As in the case of the heat operator, this approach makes it possible to model the evolution of a wave function by statistical methods, generating realizations of a random point field. Note also that the squared modulus of a wave function always equals a probability density function. The evolution of a wave function generates the corresponding evolution of the probability density. In the literature, such an evolution of a probability density is also called a

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quantum random walk (see, e.g., [10]). The approach which we propose makes it possible, in particular, to model a quantum random walk by classical probabilistic-statistical methods. We emphasize that, in this paper, we do not pretend to study the spectrum of the operator or the asymptotic behavior of solutions.

The main idea of our approach is as follows. We rewrite Eq. (3) in the form

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + U_0(x)u, \tag{5}$$

where $\sigma = e^{i\pi/4}$ and $U_0(x) = -iV(x)$.

Let us consider (5) as the heat equation but with a complex coefficient multiplying $\partial^2/\partial x^2$.

If σ were a real number, then, by virtue of the Feynman–Kac formula (4), the solution of the Cauchy problem $u(0,x) = \varphi(x)$ for Eq. (5) would admit the probabilistic representation

$$u(t,x) = \mathbf{E}[\varphi(x+\sigma w(t))e^{\int_0^t U_0(x+\sigma w(\tau))\,d\tau}], \tag{6}$$

where $w(t)$ is a standard Wiener process. The expectation in (6) can be expressed in terms of an integral with respect to the Wiener measure P_W on the space $C_0[0,\infty)$ of continuous functions $w(\,\cdot\,)$ satisfying the condition $w(0) = 0$ as

$$\int_{C_0[0,\infty)} [\varphi(x+\sigma w(t))e^{\int_0^t U_0(x+\sigma w(\tau))\,d\tau}]\mathrm{P}_W(dw(\,\cdot\,)). \tag{7}$$

The main purpose of this paper is to attach meaning to the expression on the right-hand side of (6) *for* $\sigma = e^{i\pi/4}$, $\varphi \in L_2(\mathbb{R})$, *and* $V \in L_\infty(\mathbb{R})$. In this case, the expectation in (6) is no longer a Lebesgue integral with respect to a probability measure (and, therefore, cannot be expressed in terms of the integral (7) with respect to the Wiener measure), because φ and U_0 are functions of a real variable, and we cannot substitute a complex variable into them without additional assumptions. Thus, we can aim only at constructing a regularization of this integral.

Importantly, such a regularization cannot be constructed by simply approximating the functions φ and U_0 by entire analytic functions. Indeed, even if these functions can be extended over the whole complex plane to entire functions (see [4]), there immediately arise insurmountable difficulties related to the presence of the expectation in (6), because the function U_0 is bounded on the real axis and, hence, grows at least exponentially in the complex plane. *To obviate the difficulties caused by the rapid growth of entire analytic functions, in this paper, instead of Wiener process we use the family* $\xi_\varepsilon^{(1)}(t)$ *of jump centered Lévy processes with a Lévy measure of a special form concentrated on the positive half-axis.* This family of Lévy processes weakly converges (in the Skorokhod space) to a Wiener process as $\varepsilon \to 0$, and its trajectories are always bounded from below; finally, for each fixed ε, the process $\xi_\varepsilon^{(1)}(t)$ generates an operator semigroup whose generator is easy to write out. Apparently, such an approximation of a Wiener process first appeared in [2]; subsequently, this approach has been effectively used to construct probabilistic representations of solutions to the Cauchy problem for evolution equations with generators not satisfying the maximum principle, such as the differentiation operator of order higher than 2 or the Riemann–Liouville operator (see [11] and [12]).

Let us explain the above considerations by the substantially simpler example in which the potential V vanishes. In this case, for $\varphi \in L_2(\mathbb{R})$ and real σ, representation (6) can be rewritten as

$$u(t,x) = \mathbf{E}\varphi(x+\sigma w(t)) = \frac{1}{2\pi}\mathbf{E}\int_{-\infty}^{\infty} e^{-ipx}e^{-i\sigma p w(t)}\widehat{\varphi}(p)\,dp. \tag{8}$$

The integral (8) with $\sigma = e^{i\pi/4}$ generally diverges for any function $\varphi \in L_2(\mathbb{R})$. One of the possible ways to regularize this divergent integral is to approximate the function $\widehat{\varphi}$ by $\widehat{\varphi}\cdot\mathbf{1}_{[-M,M]}$ with large M. This corresponds to the approximation in $L_2(\mathbb{R})$ of the function φ itself by entire functions of exponential type. This regularization method was used in [7]; from the point of view of operator theory, this method is most natural: in fact, the approximation described above means that a differential operator is approximated by bounded operators with the use of cuts of its symbol in the Fourier image. Nevertheless, *from the probabilistic point of view, this method turns out to be very inconvenient*. Its drawback is that, in this case, for any fixed M, (8) gives the expectation of an exponentially growing function of $w(t)$, so that the obtained formula is of little use for practical calculations.

In [6] the authors proposed another method for regularizing the divergent integral (8), which evaluated the expectation of only bounded functionals of processes. The initial function φ was approximated by the sum of two functions, one of which had bounded analytic continuation to any subset of the form $\operatorname{Im} z \geqslant -M$ in the complex plane and the second, to a subset of the form $\operatorname{Im} z \leqslant M$ with any $M > 0$. Simultaneously, an asymmetric approximation of the Wiener process $w(t)$ by centered Lévy processes $\xi_\varepsilon^{(1)}(t)$ with Lévy measure concentrated on $\mathbb{R}_+$ was used. The trajectories of these processes are always bounded from below by some constant (depending on ε). Then, the solution was approximated by a quantity similar to (8) (see (10) and Theorem 1 in Section 2), but the argument of the first of the two functions was $x + \sigma\xi_\varepsilon^{(1)}(t)$ and that of the second was $x - \sigma\xi_\varepsilon^{(1)}(t)$. The quantity under the sign of expectation turned out to be bounded: the structure of the formulas is such that the analytic function can increase only in a part of the complex plane never reached by the process $x + \sigma\xi_\varepsilon^{(1)}(t)$ (or $x - \sigma\xi_\varepsilon^{(1)}(t)$).

In Sections 3 and 4 of this paper we use the above "asymmetric" approximation of a Wiener process to construct a regularization of integral (6) for any bounded potential V. Unlike in the case of Wiener processes, as the probability space for Lévy processes it is convenient to take the space of point configurations with Poisson measure rather than the space of trajectories (because the processes used for approximation have pure jump trajectories).

The main results of this paper are Theorems 5 and 6. Note that the methods which we use are easy to generalize to the case of any dimension (i.e., to the case of the operator $e^{-it\Delta/2}$). It is also easy to see that the requirement that the potential be real can be replaced by the weaker condition $\operatorname{Im} V \leqslant 0$; the only difference is that, in the latter case, the corresponding operator exponential ceases to be a unitary operator. In fact, the only essential condition is the boundedness of the potential. Note also that the construction of the regularization of integral (6) is the same for potentials tending to zero at infinity and those not tending to zero (e.g., periodic), although these operators may have quite different spectral properties.

In this paper we do not touch the question of representing solutions of the Schrödinger equation in terms of integrals with respect to the Feynman measure, because such a representation is not probabilistic: the Feynman measure, in contrast to the Wiener measure, is only a finitely additive complex-valued (and hence not probabilistic) set function. A detailed exposition of the theory of integration with respect to the Feynman measure can be found in the book [15], which also contains an extensive survey of the literature on the theory of Feynman integral.

All random variables and random processes considered in this paper are assumed to be defined on a certain base probability space $(\Omega, \mathscr{F}, \mathbf{P})$; the symbol $\mathbf{E}$ is used to denote mathematical expectation (Lebesgue integral) with respect to the measure $\mathbf{P}$.

By $W_2^k(\mathbb{R})$ we denote the Sobolev space of functions defined on $\mathbb{R}$ and having square integrable generalized derivatives up to order k. We endow the space $W_2^k(\mathbb{R})$ with the following norm (which is equivalent to the standard one):

$$\|\psi\|^2_{W_2^k(\mathbb{R})} = \int_{\mathbb{R}} (1 + |p|^{2k})|\widehat{\psi}(p)|^2\,dp,$$

where $\widehat{\psi}$ denotes the direct Fourier transform of the function ψ, which is defined in this paper as

$$\widehat{\psi}(p) = \int_{\mathbb{R}} e^{ipx}\psi(x)\,dx.$$

2. The Case of the Absence of a Potential

In this section we largely describe ideas and approaches of [6].

Suppose given a function $\varphi \in L_2(\mathbb{R})$. We represent it in the form of a sum as

$$\varphi = \varphi_+ + \varphi_- = P_+\varphi + P_-\varphi,$$

where P_+ and P_- are the Riesz projections acting on the function φ (provided that $\widehat{\varphi} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$) so that

$$\varphi_+(x) = \frac{1}{2\pi}\int_{-\infty}^{0} e^{-ipx}\widehat{\varphi}(p)\,dp, \qquad \varphi_-(x) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-ipx}\widehat{\varphi}(p)\,dp.$$

Note that the functions φ_+ and φ_- can be analytically continued to the upper and lower half-planes, respectively.

Let $\nu(dt, dx)$ be the Poisson random measure on $[0,\infty) \times [0,\infty)$ with intensity measure

$$\mathbf{E}\nu(dt, dx) = \frac{dt\,dx}{x^3}\,.$$

For each $\varepsilon > 0$, we define the random process

$$\xi_\varepsilon(t) = \int_0^t \int_\varepsilon^{e\varepsilon} x\,\nu(ds, dx) \geqslant 0.$$

Note that this is a homogeneous random process with independent increments (see [14]). Consider the centered process

$$\xi_\varepsilon^{(1)}(t) = \xi_\varepsilon(t) - \mathbf{E}\xi_\varepsilon(t). \tag{9}$$

.

For $p \in \mathbb{R}$, we have

$$\mathbf{E}e^{ip\xi_\varepsilon^{(1)}(t)} = \exp\left(t\int_\varepsilon^{e\varepsilon}(e^{ipx} - 1 - ipx)\frac{dx}{x^3}\right).$$

Next, following [6], we consider the semigroup of the operators P_ε^t on $L_2(\mathbb{R})$ defined by

$$P_\varepsilon^t\varphi(x) = \mathbf{E}[\varphi_+ * h_\varepsilon^t(x + \sigma\xi_\varepsilon^{(1)}(t)) + \varphi_- * h_\varepsilon^t(x - \sigma\xi_\varepsilon^{(1)}(t))] \tag{10}$$

for $\varphi \in L_2(\mathbb{R})$, where $\sigma = e^{i\pi/4}$ and the function h_ε^t is determined by its Fourier transform

$$\widehat{h}_\varepsilon^t(p) = \widehat{h}_\varepsilon^t(|p|) = \exp(-\tfrac{1}{6}\sigma|p|^3 t\varepsilon(e-1)). \tag{11}$$

The proofs of the following assertions can be found in [6].

Theorem 1. 1. *For any $t \geqslant 0$ and $\varepsilon > 0$, P_ε^t is a pseudodifferential operator with symbol*

$$e^{-itp^2/2}H(t,\varepsilon,p),$$

where

$$H(t,\varepsilon,p) = \exp\left(t\int_\varepsilon^{e\varepsilon}\left(e^{i|p|\sigma x} - 1 - i|p|\sigma x - \frac{1}{2}(i|p|\sigma x)^2 - \frac{1}{6}(i|p|\sigma x)^3\right)\frac{dx}{x^3}\right). \tag{12}$$

2. *For any $t \geqslant 0$, $\varepsilon > 0$, and $p \in \mathbb{R}$,*

$$|H(t,\varepsilon,p)| \leqslant 1. \tag{13}$$

Note that, by virtue of (12), the generator A_ε of the semigroup of P_ε^t is a pseudodifferential operator with symbol $\widehat{g}_\varepsilon(p)$, where

$$\widehat{g}_\varepsilon(p) = -\frac{ip^2}{2} + \int_\varepsilon^{e\varepsilon}\left(e^{i|p|\sigma x} - 1 - i|p|\sigma x - \frac{1}{2}(i|p|\sigma x)^2 - \frac{1}{6}(i|p|\sigma x)^3\right)\frac{dx}{x^3}\,. \tag{14}$$

Theorem 2. *There exists a constant $C > 0$ such that*

$$\|P_\varepsilon^t\varphi - e^{-itH_0}\varphi\|_{L_2} \leqslant C\,t\varepsilon^2\|\varphi\|_{W_2^4}$$

for any function $\varphi \in W_2^4(\mathbb{R})$ and any $t \geqslant 0$.

Corollary. *For any* $t \geqslant 0$ *and* $\varphi \in L_2(\mathbb{R})$,

$$\|P_\varepsilon^t \varphi - e^{-itH_0}\varphi\|_{L_2} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

In the following sections, we generalize Theorem 2 (and its corollary) to the case of a Hamiltonian of the general form (1). For this purpose, we need to introduce some more notation.

Given $\varepsilon > 0$ and $t > 0$, we define an operator $\mathrm{R}_\varepsilon^t \colon L_2(\mathbb{R}) \to L_2(\mathbb{R})$ by setting

$$\begin{aligned}\mathrm{R}_\varepsilon^t \varphi(y) &= \varphi_+ * h_\varepsilon^t(y) + \varphi_- * h_\varepsilon^t(-y)\\ &= \frac{1}{2\pi}\int_{-\infty}^0 e^{-ipy}\widehat{h}_\varepsilon^t(|p|)\widehat{\varphi}(p)\,dp + \frac{1}{2\pi}\int_0^\infty e^{ipy}\widehat{h}_\varepsilon^t(|p|)\widehat{\varphi}(p)\,dp \qquad (15)\end{aligned}$$

for $\varphi \in L_2(\mathbb{R})$, where the function h_ε^t is defined by (11).

Throughout the paper, T_a denotes the shift operator: $\mathrm{T}_a\varphi(x) = \varphi(a + x)$. In this notation, formula (10) defining the semigroup of operators P_ε^t can be rewritten as

$$P_\varepsilon^t \varphi(x) = \mathbf{E}F_\varepsilon^t(x, \sigma\xi_\varepsilon^{(1)}(t)), \qquad (16)$$

where the function F_ε^t is defined by

$$F_\varepsilon^t(x, y) = \mathrm{R}_{\varepsilon,y}^t \mathrm{T}_x \varphi(y) = \varphi_+ * h_\varepsilon^t(x + y) + \varphi_- * h_\varepsilon^t(x - y). \qquad (17)$$

In this formula, the subscript y in the notation $\mathrm{R}_{\varepsilon,y}^t$ of an operator means that this operator acts on the variable y.

Let us show that the quantity under the sign of expectation in (16) is bounded. First, note that, given any function $\varphi \in L_2(\mathbb{R})$, the function $F_\varepsilon^t(x, y)$ with fixed ε, t, and x can be continued to an entire analytic function of y.

Since the process $\xi_\varepsilon(t)$ is nonnegative, it follows that

$$\xi_\varepsilon^{(1)}(t) = \xi_\varepsilon(t) - \mathbf{E}\xi_\varepsilon(t) \geqslant -\mathbf{E}\xi_\varepsilon(t) = -t\varepsilon^{-1}\frac{e-1}{e}. \qquad (18)$$

We set $A = (e - 1)/(\sqrt{2}e)$ and $B = (e - 1)/(6\sqrt{2})$. Relation (18) implies the inequality (recall that $\sigma = e^{i\pi/4}$)

$$\operatorname{Im}(\sigma\xi_\varepsilon^{(1)}(t)) \geqslant -tA\varepsilon^{-1}.$$

In what follows, we need z-uniform estimates of the uniform norm and the L_2-norm of the function $F_\varepsilon^t(\,\cdot\,, z)$ on $\{z : \operatorname{Im} z \geqslant -tA\varepsilon^{-1}\}$. The following assertions are valid.

Lemma 1. *For any* $t > 0$,

$$\sup_{x\in\mathbb{R}} \sup_{\operatorname{Im} z \geqslant -tA\varepsilon^{-1}} |F_\varepsilon^t(x, z)| \leqslant t^{-1/6} G(\varepsilon, t)\|\varphi\|_2,$$

where

$$G^2(\varepsilon, t) = \int_0^\infty e^{2(t^{2/3}A\varepsilon^{-1}v - Bv^3\varepsilon)}\,dv.$$

Proof. By virtue of (15) and (11), for any z satisfying the condition $\operatorname{Im} z \geqslant -tA\varepsilon^{-1}$, we have

$$\begin{aligned}|F_\varepsilon^t(x, z)| &\leqslant \frac{1}{2\pi}\int_{-\infty}^\infty e^{At|p|\varepsilon^{-1}} e^{-Bt|p|^3\varepsilon}|\widehat{\varphi}(p)|\,dp\\ &\leqslant \frac{1}{2\pi}\biggl(\int_{-\infty}^\infty e^{2At|p|\varepsilon^{-1}} e^{-2Bt|p|^3\varepsilon}\,dp\biggr)^{1/2}\|\widehat{\varphi}(p)\|_2\\ &= \frac{1}{\sqrt{\pi}}\biggl(\int_0^\infty e^{2Atp\varepsilon^{-1}} e^{-2Btp^3\varepsilon}\,dp\biggr)^{1/2}\|\varphi\|_2 = t^{-1/6}G(\varepsilon, t)\|\varphi\|_2. \qquad \square\end{aligned}$$

Lemma 1 immediately implies that the quantity under the sign of expectation in (16) is bounded.

Lemma 2. *There exists a constant $D > 0$ such that, for any $t > 0$,*

$$\sup_{\operatorname{Im} z \geqslant -tA\varepsilon^{-1}} \|F_\varepsilon^t(\,\cdot\,, z)\|_2 \leqslant e^{tD\varepsilon^{-2}} \|\varphi\|_2.$$

Proof. Relations (15) and (17) imply

$$\begin{aligned} F_\varepsilon^t(x,z) &= \frac{1}{2\pi} \int_{-\infty}^0 e^{-ip(x+z)} \widehat{h}_\varepsilon^t(|p|)\widehat{\varphi}(p)\,dp + \frac{1}{2\pi} \int_0^\infty e^{-ip(x-z)} \widehat{h}_\varepsilon^t(|p|)\widehat{\varphi}(p)\,dp \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-ipx} e^{i|p|z} \widehat{h}_\varepsilon^t(|p|)\widehat{\varphi}(p)\,dp. \end{aligned}$$

Thus, for any z satisfying the condition $\operatorname{Im} z \geqslant -tA\varepsilon^{-1}$, we have

$$\begin{aligned} \|F_\varepsilon^t(\,\cdot\,, z)\|_2 &\leqslant \max_{p>0} |e^{ipz} \widehat{h}_\varepsilon^t(p)| \cdot \|\varphi\|_2 \leqslant \max_{p>0} (e^{-tp^3 B\varepsilon} e^{tpA\varepsilon^{-1}}) \cdot \|\varphi\|_2 \\ &= e^{tD\varepsilon^{-2}} \|\varphi\|_2, \quad \text{where } D = \frac{2(e-1)}{3e\sqrt{e}}. \end{aligned}$$

□

3. Construction of an Evolution Family of Operator Functionals

In the preceding two sections we showed how to construct a probabilistic approximation of the evolution operator in the case of the absence of a potential. As an approximation of the operator e^{-itH_0} we used the operator $P_\varepsilon^t = e^{tA_\varepsilon}$, where A_ε is the pseudodifferential operator with symbol (14). The operator P_ε^t admits the probabilistic representation (10) in the form of the expectation of a functional of a stochastic process.

The idea of the further considerations is to construct an analogue of the Feynman–Kac formula for the operator $A_\varepsilon + U_0$. To do this, first of all, we must construct the functional of a process trajectory under the sign of expectation. This functional is constructed in the form of an integral with respect to the Poisson random measure with Lebesgue intensity on the positive half-axis. We begin with constructing such an integral representation for the functional used in the Feynman–Kac formula (6), and then we show how to "adapt" it to the operator $A_\varepsilon + U_0$.

Let U_0 be a bounded measurable function, and let $\gamma(\,\cdot\,)$ be a locally bounded measurable function on $[0, \infty)$. Given $\gamma(\,\cdot\,)$ and $0 \leqslant s \leqslant t$, we define an operator $\Phi_{st}(\gamma)$ acting on the function φ as

$$[\Phi_{st}(\gamma)\varphi](x) = e^{\int_s^t U_0(x+\gamma(\tau)-\gamma(s))\,d\tau}\, \varphi(x + \gamma(t) - \gamma(s)). \tag{19}$$

We make two observations. First, the operator $\Phi_{st}(\gamma)$ is determined by the restriction of the function γ to the interval $[s, t]$ rather than by the whole function γ, and secondly, the operators $\Phi_{st}(\gamma)$ form an evolution family in the sense that, for any u and $s \leqslant u \leqslant t$, we have

$$\Phi_{st}(\gamma) = \Phi_{su}(\gamma)\Phi_{ut}(\gamma); \tag{20}$$

in this formula, it is assumed that first we apply the operator $\Phi_{ut}(\gamma)$ and then the operator $\Phi_{su}(\gamma)$.

Now, we obtain another expression for $\Phi_{st}(\gamma)$. Denoting the function U_0+1 by U and expanding the exponential in (19) in a Taylor series, we obtain

$$\begin{aligned} [\Phi_{st}(\gamma)\varphi](x) &= e^{-(t-s)} e^{\int_s^t U(x+\gamma(\tau)-\gamma(s))\,d\tau}\, \varphi(x+\gamma(t)-\gamma(s)) \\ &= e^{-(t-s)} \sum_{k=0}^\infty \frac{1}{k!} \biggl(\int_s^t U(x+\gamma(\tau)-\gamma(s))\,d\tau \biggr)^k \varphi(x+\gamma(t)-\gamma(s)) \\ &= e^{-(t-s)} \sum_{k=0}^\infty \int_{s<t_1<\cdots<t_k<t} U(x+\gamma(t_1)-\gamma(s)) \cdots U(x+\gamma(t_k)-\gamma(s)) \\ &\qquad\qquad \times \varphi(x+\gamma(t)-\gamma(s))dt_1 \cdots dt_k. \end{aligned} \tag{21}$$

This expression can be written as an integral with respect to a Poisson random field. Namely, let $\mathscr{X} = \mathscr{X}(\mathbb{R}_+)$ be a space of configurations on $\mathbb{R}_+$. Each point X in the space $\mathscr{X}$ is a strictly increasing sequence $\{t_1, t_2, \dots\}$ of positive numbers, and this sequence is locally finite, i.e., each bounded interval contains only finitely many terms of the sequence. Next, let $\mathbf{P}_0$ be the Poisson measure on $\mathscr{X}$ whose intensity measure is the Lebesgue measure (see, e.g., [9]).

In this notation, formula (21) takes the form

$$[\Phi_{st}(\gamma)\varphi](x) = \int_{\mathscr{X}} \mathbf{P}_0(dX)\Bigg(\prod_{\tau \in X \cap (s,t)} U(x+\gamma(\tau)-\gamma(s))\Bigg)\varphi(x+\gamma(t)-\gamma(s)). \tag{22}$$

Note that the space $\mathscr{X}$ with probability measure $\mathbf{P}_0$ is in no way related to our initial probability space $(\Omega, \mathscr{F}, \mathbf{P})$ and serves only the purpose of simplifying long expressions.

Let us transform the integrand in (22) for fixed $X = \{t_1, t_2, \dots\} \in \mathscr{X}$. First, we introduce a family of operators $N_k(y_1, \dots, y_{k+1})$ parameterized by $k = 0, 1, 2, \dots$ and $y_1, \dots, y_{k+1}$, $y_j \in \mathbb{R}$. We define them as

$$[N_k(y_1, \dots, y_{k+1})\varphi](x) = U(x+y_1)U(x+y_1+y_2)\cdots U(x+y_1+\dots+y_k)\varphi(x+y_1+\dots+y_{k+1}). \tag{23}$$

We refer to the family of operators N_k as the base family.

Using the base family of operators, we define a two-parameter family of operators $H_{s,t}(\gamma, X)$, which already depend on a function γ and a configuration $X \in \mathscr{X}$. We define the operators $H_{s,t}(\gamma, X)$ only for those s and t which do not belong to X. This is enough for our purposes.

Let $X = \{t_1, t_2, \dots\}$. For $j \in \mathbb{N}$, we use the notation

$$m(t_{j-1}, t_j) = \gamma(t_j) - \gamma(t_{j-1}).$$

Let $l = \operatorname{card}(X \cap (0, s))$, and let $k = \operatorname{card}(X \cap (s, t))$. For $s, t \notin X$, we set

$$\begin{aligned}[H_{s,t}(\gamma, X)\varphi](x) &= [N_k(m(s, t_{l+1}), m(t_{l+1}, t_{l+2}), \dots, m(t_{l+k-1}, t_{l+k}), m(t_{l+k}, t))\varphi](x) \\ &= U(x+m(s, t_{l+1}))U(x+m(s, t_{l+2}))\cdots U(x+m(s, t_{l+k}))\varphi(x+m(s,t)).\end{aligned}$$

A direct calculation shows that, for fixed X and γ, the operators $H_{s,t}(\gamma, X)$ form an evolution family, i.e., for any fixed $s \leqslant u \leqslant t$ such that $s, u, t \notin X$, we have

$$H_{s,t}(\gamma, X) = H_{s,u}(\gamma, X)H_{u,t}(\gamma, X).$$

Next, using (22), we can easily show that the operator $\Phi_{st}(\gamma)$ has the integral representation

$$\Phi_{st}(\gamma) = \int_{\mathscr{X}} \mathbf{P}_0(dX)H_{s,t}(\gamma, X) = \int_{\mathscr{X} \cap [s,t]} \mathbf{P}_0(dX)H_{s,t}(\gamma, X). \tag{24}$$

Indeed, as is known (see, e.g., [9, Chap. 2, Sec. 2.4]), the Poisson measure has the property that the conditional measure given that a fixed interval $[s, t]$ contains precisely k points of a configuration X (the number k can take values $0, 1, 2, \dots$) does not depend on $X \cap (\mathbb{R} \setminus [s, t])$ and coincides with the uniform distribution on the simplex $\{\tau = (\tau_1, \dots, \tau_k) : s < \tau_1 < \tau_2 < \dots < \tau_k < t\}$ up to the multiplier $e^{-(t-s)}$.

Note also that the evolution property (20) of the family $\Phi_{st}(\gamma)$ follows from (24) and the independence of the values of the Poisson measure on disjoint intervals. Moreover, the condition $s, t \notin X$ imposed in the definition of the operators $H_{s,t}(\gamma, X)$ is not restrictive, because it holds for $\mathbf{P}_0$-almost all X at fixed s and t.

For our purposes, we need yet another evolution family, which is constructed by analogy with the preceding one but has another base family of operators. First, we introduce notation. As above, by T_a we denote the operator of shift by a; the expression $\mathrm{T}_a^{y_1}$ means that this operator acts on the variable y_1: $\mathrm{T}_a^{y_1}\varphi(y_1, y_2) = \varphi(y_1 + a, y_2)$. In this notation, formula (23) can be rewritten as

$$[N_k(y_1, \dots, y_{k+1})\varphi](x) = \mathrm{T}_x^{y_1}U(y_1)\mathrm{T}_{y_1}^{y_2}U(y_2)\dots\mathrm{T}_{y_{k-1}}^{y_k}U(y_k)\mathrm{T}_{y_k}^{y_{k+1}}\varphi(y_{k+1}). \tag{25}$$

The new base family of operators is constructed from the chain of operators (25) by "interweaving" the operators R_ε^t into this chain. Namely, we define a new family of operators

$N_k^R(\tau_1, \dots, \tau_{k+1}, y_1, \dots, y_{k+1})$, which additionally depend on nonnegative parameters $\tau_1, \dots, \tau_{k+1}$, by setting

$$[N_k^R(\tau_1, \dots, \tau_{k+1}, y_1, \dots, y_{k+1})\varphi](x) \\ = \mathrm{R}_{\varepsilon,y_1}^{\tau_1}\mathrm{T}_x^{y_1}U(y_1)\mathrm{R}_{\varepsilon,y_2}^{\tau_2}\mathrm{T}_{y_1}^{y_2}U(y_2)\cdots\mathrm{T}_{y_{k-1}}^{y_k}U(y_k)\mathrm{R}_{\varepsilon,y_{k+1}}^{\tau_{k+1}}\mathrm{T}_{y_k}^{y_{k+1}}\varphi(y_{k+1}). \quad (26)$$

In this formula, the second subscript in the notation of the operator R indicates the variable on which this operator acts. The objects constructed above depend also on ε, but we do not show this dependence in the notation.

As above, using the base family of operators, we define a two-parameter family of operators $H_{s,t}^R(\gamma, X)$ depending now on a function γ and a configuration $X \in \mathscr{X}$. We again assume that $s, t \notin X$. Let $l = \mathrm{card}(X \cap (0, s))$, and let $k = \mathrm{card}(X \cap (s, t))$. We set

$$[H_{s,t}^R(\gamma, X)\varphi](x) = [N_k^R(t_{l+1} - s, t_{l+2} - t_{l+1}, \dots, t_{l+k} - t_{l+k-1}, t - t_{l+k}, m(s, t_{l+1}), \\ m(t_{l+1}, t_{l+2}), \dots, m(t_{l+k-1}, t_{l+k}), m(t_{l+k}, t))\varphi](x). \quad (27)$$

Lemma 3. *For fixed X and γ, the operators $H_{s,t}^R(\gamma, X)$ form an evolution family, i.e., for any $s \leqslant u \leqslant t$ such that $s, u, t \notin X$,*

$$H_{s,t}^R(\gamma, X) = H_{s,u}^R(\gamma, X)H_{u,t}^R(\gamma, X).$$

Proof. We prove only the assertion

$$H_{0,t}^R(\gamma, X) = H_{0,s}^R(\gamma, X)H_{s,t}^R(\gamma, X)$$

under the assumption $X \cap (0, t) = \{t_1, t_2\}$, where $0 < t_1 < s < t_2 < t$; this means that, in (27), we have $l = \mathrm{card}(X \cap (0, s)) = 1$ and $k = \mathrm{card}(X \cap (s, t)) = 1$. The general case is treated in a similar way.

First, we prove an auxiliary assertion, namely, the relation

$$N_1^R(s_1, s_2, y_1, y_2)N_1^R(s_3, s_4, y_3, y_4) = N_2^R(s_1, s_2 + s_3, s_4, y_1, y_2 + y_3, y_4) \quad (28)$$

for any $s_1, s_2, s_3, s_4 > 0$ and any y_1, y_2, y_3, and y_4.

Let $\varphi \in L_2(\mathbb{R})$. We set

$$\psi(x) = [N_1^R(s_3, s_4, y_3, y_4)\varphi](x) = \mathrm{R}_{\varepsilon,y_3}^{s_3}\mathrm{T}_x^{y_3}U(y_3)\mathrm{R}_{\varepsilon,y_4}^{s_4}\mathrm{T}_{y_3}^{y_4}\varphi(y_4).$$

Then

$$[N_1^R(s_1, s_2, y_1, y_2)N_1^R(s_3, s_4, y_3, y_4)\varphi](x) \\ = [N_1^R(s_1, s_2, y_1, y_2)\psi](x) = \mathrm{R}_{\varepsilon,y_1}^{s_1}\mathrm{T}_x^{y_1}U(y_1)\mathrm{R}_{\varepsilon,y_2}^{s_2}\mathrm{T}_{y_1}^{y_2}\mathrm{R}_{\varepsilon,y_3}^{s_3}\mathrm{T}_{y_2}^{y_3}U(y_3)\mathrm{R}_{\varepsilon,y_4}^{s_4}\mathrm{T}_{y_3}^{y_4}\varphi(y_4). \quad (29)$$

The middle part of this expression is easy to transform. Namely, by virtue of (15), we have

$$\begin{aligned}\mathrm{R}_{\varepsilon,y_2}^{s_2}\mathrm{T}_{y_1}^{y_2}\mathrm{R}_{\varepsilon,y_3}^{s_3}\mathrm{T}_{y_2}^{y_3}g(y_3) &= \mathrm{R}_{\varepsilon,y_2}^{s_2}\mathrm{T}_{y_1}^{y_2}(g_+ * h_\varepsilon^{s_3}(y_2 + y_3) + g_- * h_\varepsilon^{s_3}(y_2 - y_3)) \\ &= g_+ * h_\varepsilon^{s_2+s_3}(y_1 + y_2 + y_3) + g_- * h_\varepsilon^{s_2+s_3}(y_1 - y_2 - y_3) \\ &= \mathrm{R}_{\varepsilon,z}^{s_2+s_3}\mathrm{T}_{y_1}^{z}g(z)|_{z=y_2+y_3}\end{aligned}$$

for any $g \in L_2(\mathbb{R})$. Substituting the last expression into the right-hand side of (29), we obtain (28).

Next, using (28), we obtain

$$\begin{aligned}H_{0,s}^R(\gamma, X)H_{s,t}^R(\gamma, X) &= N_1^R(t_1, s - t_1, m(0, t_1), m(t_1, s))N_1^R(t_2 - s, t - t_2, m(s, t_2), m(t_2, t)) \\ &= N_2^R(t_1, t_2 - t_1, t - t_2, m(0, t_1), m(t_1, t_2), m(t_2, t)) = H_{0,t}^R(\gamma, X). \qquad \square\end{aligned}$$

Now, we define a new family of operators $\Phi_{st}^R(\gamma)$ depending only on γ (and on ε) by setting

$$\Phi_{st}^R(\gamma) = \int_{\mathscr{X}} \mathbf{P}_0(dX)H_{s,t}^R(\gamma, X) = \int_{\mathscr{X}\cap[s,t]} \mathbf{P}_0(dX)H_{s,t}^R(\gamma, X). \quad (30)$$

The independence of the values of a Poisson measure on disjoint intervals and Lemma 3 imply that the family of operators $\Phi^R_{st}(\gamma)$ has the evolution property

$$\Phi^R_{st}(\gamma) = \Phi^R_{su}(\gamma)\Phi^R_{ut}(\gamma) \quad \text{for any } s \leqslant u \leqslant t. \tag{31}$$

Note also that, for any constant C, we have $\Phi^R_{st}(\gamma+C) = \Phi^R_{st}(\gamma)$. Moreover, using the invariance of a Poisson measure with respect to shifts, we can easily show that

$$\Phi^R_{s+a,t+a}(\gamma) = \Phi^R_{st}(\mathrm{T}_a\gamma) \tag{32}$$

for each $a > 0$.

4. The Case of a Hamiltonian of General Form

In this section we obtain results similar to those of Section 2 but for the case of an arbitrary real bounded potential V. As above, we introduce the notation

$$U_0(x) = -iV(x), \qquad U(x) = U_0(x) + 1.$$

Now, for each $t \geqslant 0$, we define an operator Q^t_ε by setting

$$Q^t_\varepsilon\varphi(x) = \mathbf{E}[\Phi^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,))\varphi](x) \tag{33}$$

for $\varphi \in L_2(\mathbb{R})$, where $\sigma = e^{i\pi/4}$ and $\xi^{(1)}_\varepsilon(\,\cdot\,)$ is the random process defined by (9). The operator $\Phi^R_{0t}(\gamma)$, which depends on the trajectory γ as a parameter, is defined by (30) at $s = 0$, but in (33) we use the trajectory of the process $\sigma\xi^{(1)}_\varepsilon$ instead of γ.

First, we check that the operator Q^t_ε is well defined, because in the initial definition the argument of $\Phi^R_{0t}(\gamma)$ was assumed to be a real process γ. The following assertion is valid.

Theorem 3. 1. *There exists a constant $C = C(\varepsilon, V) > 0$ such that*

$$\sup_{x\in\mathbb{R}} |[\Phi^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,))\varphi](x)| \leqslant C\|\varphi\|_2$$

with probability 1.

2. *For any $x \in \mathbb{R}$, the function $[H^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,), X)\varphi](x)$ regarded as a function on $\Omega \times \mathscr{X}$ is integrable with respect to the measure $\mathbf{P} \times \mathbf{P}_0$ and*

$$\begin{aligned}\mathbf{E}[\Phi^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,))\varphi](x) &= \int_{\mathscr{X}} \mathbf{P}_0(dX)\mathbf{E}[H^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,), X)\varphi](x)\\ &= \sum_{k=0}^{\infty}\int_{\mathscr{X}_k} \mathbf{P}_0(dX)\mathbf{E}[H^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,), X)\varphi](x),\end{aligned} \tag{34}$$

where $\mathscr{X}_k = \{X \in \mathscr{X} : \operatorname{card}(X \cap (s,t) = k\}$.

3. *For any $k = 0, 1, \dots,$*

$$\begin{aligned}&\int_{\mathscr{X}_k} \mathbf{P}_0(dX)\mathbf{E}[H^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,), X)\varphi](x)\\ &\qquad = e^{-t}\int_{0<t_1<\dots<t_k<t} dt_1\cdots dt_k P^{t_1}_\varepsilon(UP^{t_2-t_1}_\varepsilon(\cdots P^{t_k-t_{k-1}}_\varepsilon(UP^{t-t_k}_\varepsilon\varphi)\cdots))(x).\end{aligned} \tag{35}$$

Proof. Using (26) and Lemmas 1 and 2, we obtain the following inequality for all $k \geqslant 1$ (here $t_0 = 0$ and $t_{k+1} = t$):

$$\sup_{x\in\mathbb{R}}\ \sup_{\operatorname{Im} z_j \geqslant -(t_j-t_{j-1})A\varepsilon} |[N^R_k(t_1, t_2-t_1\dots, t-t_k, z_1,\dots,z_{k+1})\varphi](x)| \leqslant \|U\|^k_\infty G(\varepsilon,t_1)t_1^{-1/6}e^{Dt\varepsilon^{-2}}\|\varphi\|_2.$$

Thus, for $X \in \mathscr{X}_k$, we obtain

$$|[H^R_{0t}(\sigma\xi^{(1)}_\varepsilon(\,\cdot\,), X)\varphi](x)| \leqslant \|U\|^k_\infty t_1^{-1/6}G(\varepsilon,t)\,e^{kDt\varepsilon^{-2}}\|\varphi\|_2.$$

Integrating the last inequality with respect to the measure $\mathbf{P}_0$, we obtain assertions 1 and 2 of the theorem. Assertion 3 follows from the independence of the increments of the process $\xi^{(1)}_\varepsilon$. $\square$

Theorem 4. *The operators Q_ε^t form a one-parameter semigroup (i.e., $Q_\varepsilon^{t+s} = Q_\varepsilon^t Q_\varepsilon^s$) with generator $A_\varepsilon + U_0$ (the symbol of the pseudodifferential operator A_ε is defined by* (14)*).*

Proof. The semigroup property of the operators Q_ε^t follows from relations (31), (32), and (33) and the fact that the process $\xi_\varepsilon^{(1)}$ has independent homogeneous increments.

To calculate the generator of the semigroup, we use formulas (34) and (35). Clearly, in (34) it suffices to retain the two terms corresponding to $k = 0$ and $k = 1$. For $k = 0$, we have

$$e^{-t} P_\varepsilon^t \varphi(x) = \varphi(x)(1-t) + tA_\varepsilon \varphi(x) + o(t), \qquad t \to 0,$$

and for $k = 1$, we have

$$e^{-t} \int_0^t dt_1 P_\varepsilon^{t_1} U P_\varepsilon^{t-t_1} \varphi(x) = tU(x)\varphi(x) + o(t), \qquad t \to 0.$$

The last two relations imply that the generator of the semigroup Q_ε^t is $A_\varepsilon + U - 1 = A_\varepsilon + U_0$. □

Now, let us show that, for small ε, the operator Q_ε^t approximates e^{-itH}. First, we additionally assume that the potential V has four bounded derivatives. We set

$$L = \max(\|V\|_\infty, \|V^{(1)}\|_\infty, \|V^{(2)}\|_\infty, \|V^{(3)}\|_\infty, \|V^{(4)}\|_\infty).$$

Theorem 5. *If $V \in C^{(4)}$, then there exists a constant $C > 0$ such that*

$$\|Q_\varepsilon^t \varphi - e^{-itH}\varphi\|_2 \leqslant Ct(1+t^4)\varepsilon^2 \|\varphi\|_{W_2^4}$$

for any function $\varphi \in W_2^4(\mathbb{R})$ and any $t \geqslant 0$.

Proof. By virtue of Theorem 4 and relation (2), we have

$$Q_\varepsilon^t = e^{-it(iA_\varepsilon + V)}, \qquad e^{-itH} = e^{-it(H_0+V)}.$$

In view of Theorems 1 and 2, it is natural to regard the operator iA_ε as a small perturbation of H_0. We need a series of auxiliary assertions.

Lemma 4. *For any $\varepsilon > 0$,*

$$\|Q_\varepsilon^t\|_{L_2 \to L_2} \leqslant 1.$$

Proof. The generator D_ε of the semigroup Q_ε^t has the form $D_\varepsilon = A_\varepsilon - iV$, where $-A_\varepsilon$ is an m-accretive operator (see, e.g., [8, Chap. V, Sec. 3, Subsec. 10; Chap. IX, Sec. 1, Subsec. 2]) and the operator V is bounded and self-adjoint. Thus, $-D_\varepsilon$ is m-accretive, which proves the lemma. □

Lemma 5. *There exists a positive constant C such that*

$$\|e^{-itH}\|_{W_2^4 \to W_2^4} \leqslant C(1 + L^4 t^4).$$

Proof. The proof of this lemma repeats that of Lemma 2 in [7] with almost no changes. □

To prove the theorem, we use the identity (see [8, Chap. IX, Sec. 2, Subsec. 1])

$$Q_\varepsilon^t \varphi = e^{-iHt}\varphi + \int_0^t (Q_\varepsilon^{t-\tau} B e^{iH\tau})\varphi \, d\tau, \tag{36}$$

where $B = A\varepsilon - iH_0$ and $\varphi \in W_2^4$.

By virtue of Lemmas 4 and 5 and identity (36), to prove the theorem, it suffices to estimate $\|B\|_{W_2^4 \to L_2}$. First, note that, by virtue of (36), B is the pseudodifferential operator with symbol $\widehat{b}_\varepsilon(p) = ip^2/2 + \widehat{g}_\varepsilon(p)$. For $\varphi \in W_2^4(\mathbb{R})$, we have

$$\|B\varphi\|_{L_2}^2 = \frac{1}{2\pi} \int |\widehat{\varphi}(p)|^2 |\widehat{b}_\varepsilon(p)|^2 \, dp.$$

Note that, for $|p|\varepsilon \leqslant 1$, we have $|\widehat{b}_\varepsilon(p)|^2 \leqslant C|p|^4\varepsilon^2$, and for $|p|\varepsilon > 1$, we have $|\widehat{b}_\varepsilon(p)|^2 \leqslant C|p|^3\varepsilon$. Thus,

$$\|B\varphi\|_{L_2}^2 \leqslant C\varepsilon^4 \int_{|p|\varepsilon \leqslant 1} |\widehat{\varphi}(p)|^2 |p|^8 dp + C\varepsilon^2 \int_{|p|\varepsilon > 1} |\widehat{\varphi}(p)|^2 |p|^6 dp \leqslant C\varepsilon^4 \|\varphi\|_{W_2^4}^2.$$

The last inequality directly implies the estimate

$$\|B\|_{W_2^4\to L_2} \leqslant C\varepsilon^2. \qquad \square$$

Theorem 6. *Let V be any bounded real potential. Then, for any function $\varphi\in L_2(\mathbb{R})$,*

$$\lim_{\varepsilon\to 0}\|Q_\varepsilon^t\varphi - e^{-itH}\varphi\|_2 = 0.$$

Proof. First, suppose that $V\in C^{(4)}(\mathbb{R})$. Then we have $\|e^{-itH}\|_{L_2\to L_2} = 1$ and $\|Q_\varepsilon^t\|_{L_2\to L_2}\leqslant 1$, and the Sobolev class $W_2^4(\mathbb{R})$ is dense in the space $L_2(\mathbb{R})$. Therefore, the assertion of the theorem follows directly from Theorem 5 and the Banach–Steinhaus theorem (see, e.g., [3, II.1.18]).

In the general case, we choose and fix a sequence $\{V_n\}$ of functions in the class $C^{(4)}(\mathbb{R})$ which converge to V almost everywhere with respect to the Lebesgue measure and are uniformly bounded. Let H_n denote the Hamiltonian $-\frac{1}{2}\frac{d^2}{dx^2}+V_n(x)$.

Note that, for each $\varphi\in L_2(\mathbb{R})$, we have

$$\|(H_n-H)\varphi\|_2 = \|(V_n-V)\varphi\|_2\to 0, \qquad n\to\infty. \tag{37}$$

It follows from (37) that, for all $\varphi\in L_2(\mathbb{R})$, we have

$$e^{-itH_n}\varphi - e^{-itH}\varphi = -i\int_0^t e^{-i(t-\tau)H_n}(H_n-H)e^{-i\tau H}\varphi\,d\tau.$$

This relation and (37) immediately imply

$$\|e^{-itH_n}\varphi - e^{-itH}\varphi\|_2\to 0, \qquad n\to\infty,$$

which proves the theorem. $\square$

References

[1] A. D. Wentzell, Course in the Theory of Stochastic Processes, McGraw-Hill, New York, 1981.

[2] A. M. Vershik and N. V. Smorodina, "Nonsingular transformations of symmetric Lévy processes," Zap. Nauchn. Sem. POMI, **408** (2012), 102–114; English transl.: J. Math. Sci., **119**:2 (2014), 123–129.

[3] N. Dunford and J. T. Schwartz, Linear Operators, vol. 1: General Theory, Intersci., New York, 1958.

[4] H. Doss, "Sur une résolution stochastique de l'equation de Schrödinger à coefficients analytiques," Commun. Math. Phys., **73**:3 (1980), 247–264.

[5] J. Glimm and A. Jaffe, Quantum Physics. A Functional Integral Point of View, Springer-Verlag, New York–Heidelberg–Berlin, 1987.

[6] I. A. Ibragimov, N. V. Smorodina, and M. M. Faddeev, "On a limit theorem related to a probabilistic representation of the solution to the Cauchy problem for the Schrödinger equation," Zap. Nauchn. Sem. POMI, **454** (2016), 158–176; English transl.: J. Math. Sci., **229**:6 (2018), 702–713.

[7] M. M. Faddeev, I. A. Ibragimov, and N. V. Smorodina, "A stochastic interpretation of the Cauchy problem solution for the equation $\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+V(x)u$ with complex σ," Markov Processes Related Fields, **22**:2 (2016), 203–226.

[8] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1976.

[9] J. F. C. Kingman, Poisson Processes, Clarendon Press, Oxford, 1992.

[10] D. Reitzner, D. Nagaj, and V. Bužek, "Quantum walks," Acta Physica Slovaca, **61**:6 (2011), 603–725.

[11] M. V. Platonova, "Probabilistic representation of the solution to the Cauchy problem for the evolution equation with differentiation operator of high order," J. Math. Sci., **229**:6 (2018), 744–755.

[12] M. V. Platonova, "Probabilistic representation of the solution to the Cauchy problem for the evolution equation with Riemann–Liouville operator," Teor. Veroyantn. Primen., **61**:3 (2016), 417–438; English transl.: Theory Probab. Appl., **61**:3 (2017), 387–407.

[13] M. C. Reed and B. Simon, Methods of Modern Mathematical Physics, vol. 2, Academic, New York, 1975.
[14] A. V. Skorokhod, Random Processes with Independent Increments [in Russian], Nauka, Moscow, 1986.
[15] O. G. Smolyanov and E. T. Shavgulidze [in Russian], Path Integrals, 2nd ed., rev. suppl., LENAND, Moscow, 2015.

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