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# Lagrangian Subspaces, Delta-Matroids, and Four-Term Relations*

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Abstract. Finite-order invariants (Vassiliev invariants) of knots are expressed in terms of weight systems, that is, functions on chord diagrams (embedded graphs with a single vertex) satisfying the four-term relations. Weight systems have graph analogues, the so-called 4-invariants of graphs, i.e., functions on graphs that satisfy the four-term relations for graphs. Each 4-invariant determines a weight system.

The notion of a weight system is naturally generalized to the case of embedded graphs with an arbitrary number of vertices. Such embedded graphs correspond to links; to each component of a link there corresponds a vertex of an embedded graph. Recently, two approaches have been suggested to extend the notion of 4-invariants of graphs to the case of combinatorial structures corresponding to embedded graphs with an arbitrary number of vertices. The first approach is due to V. Kleptsyn and E. Smirnov, who considered functions on Lagrangian subspaces in a $2n$-dimensional space over $\mathbb{F}_2$ endowed with a standard symplectic form and introduced four-term relations for them. The second approach, due to V. Zhukov and S. Lando, gives four-term relations for functions on binary delta-matroids.

In this paper, these two approaches are proved to be equivalent.

Key words: Vassiliev invariants, weight system, 4-invariants, chord diagrams, symplectic spaces, Lagrangian subspaces, binary delta-matroids, Hopf algebra, embedded graphs.

Finite-order invariants (Vassiliev invariants) of knots are expressed in terms of weight systems, that is, functions on chord diagrams satisfying four-term relations. The vector space over $\mathbb{C}$ spanned by chord diagrams considered modulo four-term relations is endowed with a Hopf algebra structure. The notion of a weight system is naturally extended from functions on chord diagrams (which can be interpreted as embedded graphs with a single vertex) to functions on arbitrary embedded graphs.

In [1] to each embedded graph a Lagrangian subspace in a symplectic space over the field $\mathbb{F}_2$ was associated. Kleptsyn and Smirnov rediscovered this construction in [7]. They introduced four-term relations in the vector space spanned by Lagrangian subspaces and showed that linear functionals satisfying these four-term relations produce weight systems. They also constructed a Hopf algebra of Lagrangian subspaces and its quotient by the four-term relations.

Meanwhile, in [10] Lando and Zhukov constructed a Hopf algebra of binary delta-matroids, introduced four-term relations for them, and constructed the quotient Hopf algebra by the four-term relations. The correspondence between delta-matroids and embedded graphs allows one to associate a weight system to a linear functional on the quotient Hopf algebra. The main result of the present paper is the proof of the equivalence of these two approaches; in particular, we establish an isomorphism between the Hopf algebra of Lagrangian subspaces and the Hopf algebra of binary delta-matroids. This isomorphism is given by a mapping $\nu_E$, which establishes (according to Theorem 2.1) a one-to-one correspondence between the set of Lagrangian subspaces in $V_E$, which is the vector space spanned by the elements of a finite set $E$ and their duals, and binary delta-matroids on the set $E$.

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## 1. Necessary Information About Delta-Matroids

A *set system* $(E; \Phi)$ is a pair consisting of a finite set $E$ and a set $\Phi \subset 2^E$ of subsets of $E$. The set $E$ is called the *ground set*, and the elements of $\Phi$ are called the *feasible subsets* of this system.

Two set systems $(E_1; \Phi)$ and $(E_2; \Phi_2)$ are said to be *isomorphic* if there exists a one-to-one correspondence $E_1 \to E_2$, which identifies $\Phi_1 \subset 2^{E_1}$ with $\Phi_2 \subset 2^{E_2}$. Below we will not distinguish between isomorphic set systems.

A set system $(E; \Phi)$ is said to be *proper* if the set $\Phi$ is nonempty. In our paper we consider only proper set systems, unless otherwise is stated explicitly. We denote the symmetric difference operation by $\Delta$: $A \Delta B = (A \setminus B) \sqcup (B \setminus A)$. A *delta-matroid* is a set system $(E; \Phi)$ that satisfies the following *symmetric exchange axiom* (SEA): *for any two feasible subsets $\phi_1, \phi_2 \in \Phi$ and any element $e \in \phi_1 \Delta \phi_2$, there exists an element $e' \in \phi_1 \Delta \phi_2$ such that $\phi_1 \Delta \{e, e'\} \in \Phi$.*

Let $G$ be an (abstract) simple graph. We will consider more general objects, namely, framed graphs, that is, graphs each of whose vertices is labeled by an element 0 or 1 of the field $\mathbb{F}_2$. To each framed graph $G$ with vertex set $V(G)$ there corresponds its adjacency matrix $A(G)$ (of size $|V(G)| \times |V(G)|$), in which the entry at the intersection of row $v$ and column $v'$ ($v \neq v'$) is the element 1 of the field $\mathbb{F}_2$ if the vertices $v$ and $v'$ are neighboring (that is, connected by an edge) and the element 0 otherwise. In turn, the diagonal elements are equal to the labels of the corresponding vertices.

A framed graph $G$ is said to be *nondegenerate* if its adjacency matrix $A(G)$, considered as a matrix over the field $\mathbb{F}_2$, is nonsingular, i.e., its determinant equals 1. We define a set system $(V(G); \Phi(G))$, $\Phi(G) \subset 2^{V(G)}$, in the following way:

$$V(G) \text{ is the vertex set of } G,$$
$$\Phi(G) = \{U \subset V(G) \mid G_U \text{ is nondegenerate}\},$$

where $G_U$ denotes the subgraph in $G$ induced by the vertex set $U$.

**Theorem 1.1** [2]**.** *The set system $(V(G); \Phi(G))$ is a delta-matroid.*

We refer to this delta-matroid as the *nondegeneracy delta-matroid* of the graph $G$.

The nondegeneracy delta-matroids of framed graphs are examples of binary delta-matroids. To introduce the notion of a binary delta-matroid, we need the operation of twisting. For a set system $D = (E; \Phi)$ and a subset $E' \subset E$, we define the twist $D * E'$ of the set system $D$ by the subset $E'$ as

$$D * E' = (E; \Phi \Delta E') = (E; \{\phi \Delta E' \mid \phi \in \Phi\}).$$

Obviously, the twisting of set systems by a given subset is an involution: $D * E' * E' = D$.

**Theorem 1.2** [4]**.** *The twist of the nondegeneracy delta-matroid of a framed graph by any subset is a delta-matroid.*

**Definition 1.1** [4]**.** A *delta-matroid* is said to be *binary* if it results from the twisting of the nondegeneracy delta-matroid of a framed graph by some (possibly empty) subset.

We denote the set of binary delta-matroids with ground set $E$ by $\mathscr{B}_E$.

## 2. Binary Delta-Matroids and Lagrangian Subspaces (a Set-Theoretic Bijection)

In this section we establish a one-to-one correspondence between the set of binary delta-matroids (on a finite set $E$) and the set of Lagrangian subspaces in the symplectic space $V_E$ over $\mathbb{F}_2$ associated with the set $E$.

Let $E$ be a finite set, and let $E^\vee$ be its copy. We denote the element of $E^\vee$ corresponding to an element $e$ in $E$ by $e^\vee$. Let ${}^\vee \colon E \sqcup E^\vee \to E \sqcup E^\vee$ be the bijection of $E \sqcup E^\vee$ which interchanges $e$ and $e^\vee$ for all $e \in E$. For $Y \subset E \sqcup E^\vee$, by $Y^\vee$ we denote the image of $Y$ under the mapping ${}^\vee$.

A *symplectic structure* on a vector space is a nondegenerate skew symmetric form on it. Symplectic structures exist only on even-dimensional spaces. Let $V_E$ denote the $2|E|$-dimensional space over $\mathbb{F}_2$ spanned by the elements of the set $E \sqcup E^\vee$. We introduce a symplectic structure $(\,\cdot\,,\,\cdot\,)$ on

$V_E$ by the rule $(e, e^\vee) = (e^\vee, e) = 1$ (the pairings of the remaining pairs of basis vectors are set to zero).

A subspace $L$ of a symplectic space is said to be *isotropic* if the restriction of the symplectic form to $L$ is zero, i.e., $(u, v) = 0$ for all $u$ and $v$ in $L$. The dimension of an isotropic subspace of a symplectic space cannot exceed half the dimension of the symplectic space itself. An isotropic subspace whose dimension is half the dimension of the symplectic space is called a *Lagrangian subspace*. Let $\mathscr{L}_E$ denote the set of Lagrangian subspaces in $V_E$.

**Definition 2.1** (of a mapping $\nu_E$)***.** Let $L$ be an arbitrary Lagrangian subspace in $V_E$, and let $\nu_E(L)$ denote the set system $\nu_E(L) = (E; \Psi_L)$, where a subset $Y \subset E$ belongs to $\Psi_L$ if and only if $L \cap \langle Y^\vee \sqcup (E \backslash Y)\rangle = 0$; here the angle brackets denote the vector subspace in $V_E$ spanned by the elements in brackets and 0 is the zero vector of the space $V_E$.

**Example 2.2.** Let $E$ be a 2-element set, $E = \{1, 2\}$; then $L = \langle 1^\vee + 2 + 2^\vee, 1 + 2\rangle$ is a Lagrangian subspace in $V_E$. It consists of four elements, namely, $0$, $1^\vee + 2 + 2^\vee$, $1 + 2$, and $1 + 1^\vee + 2^\vee$. In this case, $\nu_E(L) = (E; \{\{1\}, \{2\}, \{1, 2\}\})$. (In [10] this set system was denoted by $s_{25}$.) Indeed, we have

$$\begin{aligned}
&\text{for } Y = \varnothing, && \langle Y^\vee \sqcup (E \setminus Y)\rangle = \langle 1, 2\rangle, && L \cap \langle 1, 2\rangle \ni 1 + 2,\\
&\text{for } Y = \{1\}, && \langle Y^\vee \sqcup (E \setminus Y)\rangle = \langle 1^\vee, 2\rangle, && L \cap \langle 1^\vee, 2\rangle = 0,\\
&\text{for } Y = \{2\}, && \langle Y^\vee \sqcup (E \setminus Y)\rangle = \langle 1, 2^\vee\rangle, && L \cap \langle 1, 2^\vee\rangle = 0,\\
&\text{for } Y = \{1, 2\}, && \langle Y^\vee \sqcup (E \setminus Y)\rangle = \langle 1^\vee, 2^\vee\rangle, && L \cap \langle 1^\vee, 2^\vee\rangle = 0.
\end{aligned}$$

**Theorem 2.1.** *The mapping $\nu_E$ is a bijection between the set $\mathscr{L}_E$ of Lagrangian subspaces and the set $\mathscr{B}_E$ of binary delta-matroids on $E$.*

We split the proof of this theorem into several lemmas.

**Definition 2.3.** We say that a Lagrangian subspace $L$ in $V_E$ is *graphic* if, for each $e \in E$, there exists an element $v_e \in L$ such that $(v_e, e) = 1$ and $(v_e, e') = 0$ for all $e' \in E$, $e' \neq e$.

From dimension considerations, the collection $\{v_e\}$, $e \in E$, of such elements forms a basis in the space $L$.

**Example 2.4.** The Lagrangian subspace $L$ in Example 2.2 is not graphic. Indeed, for the element $e = 1 \in E$, there are two elements $v_e$ such that $(e, v_e) = 1$ (namely, $1^\vee + 2 + 2^\vee$ and $1 + 1^\vee + 2^\vee$), but for any such element $v_e$, the equality $(2, v_e) = 1$ holds as well.

The subspace $\langle 1^\vee, 2^\vee\rangle$ is an example of a graphic Lagrangian subspace in $V_{\langle 1^\vee, 2^\vee\rangle}$. (For $e = 1$, we can take $v_e = 1^\vee$, and for $e = 2$, $v_e = 2^\vee$.)

**Lemma 2.5.** *The mapping $\nu_E$ determines a bijection between the graphic Lagrangian subspaces of $V_E$ and the nondegeneracy delta-matroids of framed graphs on the vertex set $E$.*

**Proof.** Let $L \subset V_E$ be a graphic Lagrangian subspace; to this subspace we assign a symmetric $|E| \times |E^\vee|$-matrix $A(L)$ over $\mathbb{F}_2$ in which the element at the intersection of row $e$ and column $e'^\vee$ is $(v_e, e'^\vee)$ for any $e \in E$ and $e'^\vee \in E^\vee$. (The symmetry of this matrix follows from the isotropy of $L$: indeed, for any different $e$ and $e'$, the relations $(v_e, e) = (v_{e'}, e') = 1$ (for $e \neq e'^\vee$), $(v_e, e') = (v_{e'}, e) = 0$, and $(v_e, v_{e'}) = 0$ imply $(v_e, e'^\vee) = (v_{e'}, e^\vee)$.) One can obtain an arbitrary symmetric matrix in this way. Conversely, from any symmetric matrix one can uniquely reconstruct a Lagrangian subspace. Indeed, $L$ is the Lagrangian subspace in $V_E$ spanned by the vectors $v_e = e^\vee + \sum_{e' \in E} A(L)_{e, e'^\vee} e'$.

On the other hand, to each framed graph $G$ with vertex set $E$ its adjacency matrix $A(G)$ over $\mathbb{F}_2$ is associated. Putting $A(L) = A(G)$, we obtain a one-to-one correspondence between the two sets. Let us prove that under this correspondence the set system $\nu_E(L)$ assigned to the Lagrangian subspace $L$ is matched to the nondegeneracy delta-matroid of the graph $G$. Indeed, the subset $Y \subset E$ is feasible, i.e., $Y \in \Phi_L$, if and only if the submatrix $A|_Y$ is nonsingular over $\mathbb{F}_2$, or, equivalently, the subspace $L \cap \langle Y^\vee \sqcup (E \backslash Y)\rangle$ contains only the zero vector.

*A similar mapping was considered in [12].

Let us prove the last statement, that the subspace $L\cap\langle Y^\vee\sqcup(E\setminus Y)\rangle$ contains a nonzero vector if and only if there exists a nonzero linear combination $\sum_{e\in E}\lambda_e v_e$ (here $v_e=e^\vee+\sum_{e'\in E}A(L)_{e,e'^\vee}e'$) in $L$ belonging to $\langle Y^\vee\sqcup(E\setminus Y)\rangle$. This means that there exist $\lambda_e\in\mathbb{F}_2$, $e\in E$, not all equal to 0 and such that $\sum_{e\in E}\lambda_e v_e^*=0$, where

$$v_e^*=\begin{cases}e^\vee+\sum_{e'\in Y}A(L)_{e,e'^\vee}e' & \text{if } e\in E\setminus Y,\\ \sum_{e'\in Y}A(L)_{e,e'^\vee}e' & \text{if } e\in Y\end{cases}$$

(here the $v_e^*$ are the restrictions of $v_e$ to $Y\sqcup(E^\vee\setminus Y)$). This statement is equivalent to the singularity of the matrix $\begin{pmatrix}0 & A|_Y\\ E & *\end{pmatrix}$ (here 0 is the zero matrix of appropriate size) and hence of the matrix $A|_Y$. We have arrived at a contradiction. □

Given any $L\in\mathscr{L}_E$ and $e\in E$, let $L*e$ denote the Lagrangian subspace obtained from $L$ by the linear transformation of the space $V_E$ interchanging $e$ and $e^\vee$ and acting trivially on the other vectors of the basis. The operation $*e$ is called the *local duality* at $e\in E$ for Lagrangian subspaces.

**Lemma 2.6.** *For any $L\in\mathscr{L}_E$ and $e\in E$, $\nu_E(L)*e=\nu_E(L*e)$. In other words, the local duality of Lagrangian subspaces descends to twisting of delta-matroids under the mapping $\nu_E$.*

**Proof.** Let $Y\subset E$ be an arbitrary subset. Note that

$$(L*e)\cap\langle Y^\vee\sqcup(E\setminus Y)\rangle=L\cap\langle(Y^\vee\Delta\{e^\vee\})\sqcup(E\setminus(Y\Delta\{e\}))\rangle.$$

It follows that $Y$ is a feasible subset for $\nu_E(L*e)$ if and only if $L\cap\langle(Y^\vee\Delta\{e^\vee\})\sqcup(E\setminus(Y\Delta\{e\}))\rangle=0$. Thus, $Y\Delta e$ is feasible for $\nu_E(L)$, or, equivalently, $Y$ is feasible for $\nu_E(L)*e$. □

Clearly, the operations $*e$ and $*e'$ at (not necessarily distinct) elements $e,e'\in E$ commute with each other; therefore, the operation $*E'$ is well defined for an arbitrary subset $E'\subset E$.

**Lemma 2.7.** *For any Lagrangian subspace $L\in\mathscr{L}_E$, there exists a subset $E'\subset E$ such that the Lagrangian subspace $L*E'$ is graphic.*

**Proof.** We begin with choosing a "good" basis of $L$. We proceed as follows.

Choose a vector $e_1$ in the standard basis $E\sqcup E^\vee$ of $V_E$ so that there exists a vector $v_1\in L$ such that $(e_1,v_1)=1$. (We pick $v_1$ for the first element of the "good basis.") Then we pick a vector $e_2$ in the standard basis in $V_E$ so that there exists a vector $v_2\in L$ for which $(e_2,v_2)=1$ and add the vector $v_2'=v_2-(e_1,v_2)v_1$ to the "good basis." We proceed in this way until we obtain a basis in $L$ (like in the Gram–Schmidt process). Then to $L$ we apply local duality over the set of those $e_1,\dots,e_{|E|}$ that belong to $E^\vee$. We obtain a subspace $L_1$. (The corresponding matrix $A(L_1)$ is symmetric, because $L_1$ is a Lagrangian space.) □

Lemmas 2.5, 2.6, and 2.7 imply the following assertion.

**Corollary 2.8.** *The mapping $\nu_E$ takes every Lagrangian subspace in $V_E$ to a binary delta-matroid over the set $E$.*

Now we can complete the proof of Theorem 2.1.

Let us prove that $\nu_E\colon\mathscr{L}_E\to\mathscr{B}_E$ is an injection. Suppose the contrary. Then there exist distinct Lagrangian subspaces $L_1,L_2\in\mathscr{L}_E$ such that $\nu_E(L_1)=\nu_E(L_2)$. Let $E'\subset E$ be the set corresponding to $L_1$ in Lemma 2.7. Then

$$\nu_E(L_1*E')=\nu_E(L_1)*E'=\nu_E(L_2)*E'=\nu_E(L_2*E')$$

by Lemma 2.6. But it was shown in Lemma 2.5 that the equality $\nu_E(L_1*E')=\nu_E(L_2*E')$ implies $L_1*E'=L_2*E'$. Therefore, $L_1*E'*E'=L_2*E'*E'$, i.e., $L_1=L_2$, which is a contradiction.

Now let us prove that $\nu_E\colon L_E\to B_E$ is a surjection. Indeed, for every binary delta-matroid $B\in\mathscr{B}_E$, there exists a subset $E'\subset E$ such that $B*E'$ is a graphic delta-matroid. There exists a Lagrangian subspace $L\in\mathscr{L}_E$ such that $\nu_E(L)=B*E'$. Hence $\nu_E(L)*E'=B$ and, by Lemma 2.6, $\nu_E(L)*E'=\nu_E(L*E')$, i.e., $\nu_E(L*E')=B$.

This completes the proof of Theorem 2.1.

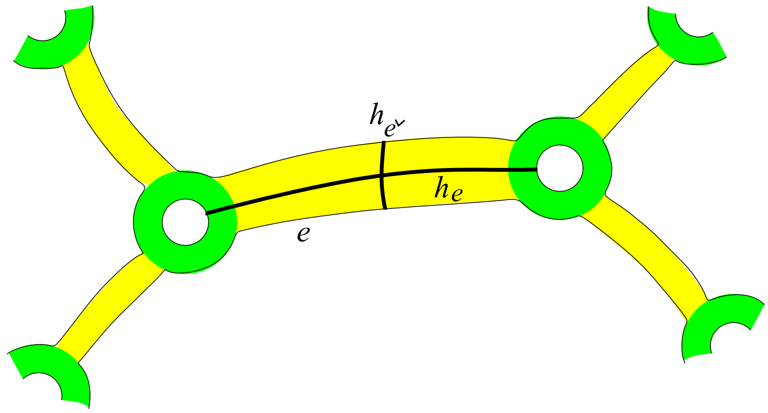


**Fig. 1.** A ribbon graph without disks around the centers of vertices and the elements $h_e$ and $h_{e^\vee}$ of the first relative homology group $H_1(F_\Gamma, \partial F_\Gamma)$ assigned to the edge $e$

### 3. Lagrangian Subspaces and Binary Delta-Matroids of Embedded Graphs

Let $\mathscr{G}_E$ the set of connected ribbon graphs with edges labeled by the elements of $E$.

In [1] a mapping from $\mathscr{G}_E$ to $\mathscr{L}_E$ was constructed. It has the following form. Let $\Gamma$ be a connected ribbon graph with edge set $E$ regarded as a 2-surface formed by closed topological disk-vertices to which closed topological disk-edges are attached so that

- edges and vertices intersect in disjoint line segments;
- each such segment lies in the closure of precisely one edge and one vertex;
- each edge contains two such segments.

Given a ribbon graph $\Gamma$, let $F_\Gamma$ denote the two-dimensional surface with boundary obtained by removing small open disks from the centers of the disk-vertices.

To each $e \in E$ we assign an element $h_e$ of the relative homology group $H_1(F_\Gamma, \partial F_\Gamma)$. This element is represented by a segment along the edge $e$ connecting the boundaries of the disks that are removed from the vertices incident to $e$. In turn, to each element $e^\vee \in E^\vee$ we assign the element $h_{e^\vee}$ in $H_1(F_\Gamma, \partial F_\Gamma)$ represented by a segment across the edge $e$ connecting the opposite sides of this edge (see Fig. 1).

To each continuous cycle $\gamma\colon S^1 \to F_\Gamma$ we associate the vector $\sum_{e\in E}((\gamma, h_e)h_e + (\gamma, h_{e^\vee})h_{e^\vee})$ in $V_E$. (The parentheses $(\,\cdot\,,\,\cdot\,)$ in this formula denote the intersection form of the absolute and relative first homology groups for the given surface with boundary $F_\Gamma$.)

As shown in [1] and [7], the subspace of $V_E$ formed by the vectors corresponding to all cycles $\gamma$ is Lagrangian. We denote this subspace by $\pi_E(\Gamma)$.

On the other hand, Bouchet [4] associated each ribbon graph with a set system whose ground set is the set of edges of the graph: a subset of edges is feasible if the restriction of the given graph to this subset is a *quasi-tree*, that is, a ribbon graph with connected boundary. Bouchet showed that the set system assigned to a ribbon graph in such a way is a delta-matroid. We denote this delta-matroid by $\rho_E(\Gamma)$.

**Theorem 3.1.** *The mapping $\nu_E$ is compatible with the mappings $\pi_E$ and $\rho_E$ in the sense that $\rho_E(\Gamma) = \nu_E(\pi_E(\Gamma))$ for an arbitrary $\Gamma \in \mathscr{G}_E$.*

**Proof.** Let first $\Gamma$ be a ribbon graph with a single vertex, i.e., a (framed) chord diagram. Then the statement is true, since both mappings are compatible with the mapping which takes a chord diagram $\Gamma$ to the adjacency matrix of its intersection graph. Conversely, each of the mappings is compatible with the twist operation on the corresponding ribbon graphs $\rho_E(\Gamma * e) = (\pi_E(\Gamma)) * e$.

Given an arbitrary ribbon graph $\Gamma$, take a set $E' \subset E$ such that $\Gamma * E'$ has a single vertex; then $(\rho_E(\Gamma)) * E' = \nu_E(\pi_E(\Gamma) * E') = \nu_E(\pi_E(\Gamma)) * E'$, i.e., $(\rho_E(\Gamma)) * E' = \nu_E(\pi_E(\Gamma)) * E'$, and hence $\rho_E(\Gamma) = \nu_E(\pi_E(\Gamma))$ is as required. □

## 4. Hopf Algebra Isomorphism

Let $n = |E|$, and let $\mathscr{L}_n$ denote the set of isomorphism classes of Lagrangian subspaces $\mathscr{L}_E \subset V_E$ with respect to the bijections of $n$-element sets.

Let $\mathscr{B}_n$ denote the set of isomorphism classes of binary delta-matroids on $n$ elements.

In [7] Kleptsyn and Smirnov introduced the structure of a graded commutative and cocommutative Hopf algebra on the infinitely dimensional vector space

$$\mathbb{C}\mathscr{L} = \mathbb{C}\mathscr{L}_0 \oplus \mathbb{C}\mathscr{L}_1 \oplus \cdots ,$$

where $\mathbb{C}\mathscr{L}_n$ is the vector space over $\mathbb{C}$ freely spanned by the set $\mathscr{L}_n$. Multiplication in this Hopf algebra is given by the operation of direct sum of Lagrangian subspaces in the direct sum of symplectic spaces, which is extended to $\mathbb{C}\mathscr{L}$ by linearity. The comultiplication $\mathbb{C}\mathscr{L} \to \mathbb{C}\mathscr{L} \otimes \mathbb{C}\mathscr{L}$ takes a Lagrangian subspace $L \subset V_E$ to a sum of tensor products of Lagrangian subspaces:

$$L \mapsto \sum_{I \subset E} L_I \otimes L_{E \setminus I},$$

where, given a subset $I$ of $E$, the Lagrangian subspace $L_I \subset V_I$ is the symplectic reduction of the Lagrangian subspace $L$ (see [7]). This multiplication can be naturally transferred to the vector space $\mathbb{C}\mathscr{L}$ spanned by the Lagrangian subspaces considered up to renumbering the elements of finite sets.

Meanwhile, in [10], a graded Hopf algebra

$$\mathbb{C}\mathscr{B} = \mathbb{C}\mathscr{B}_0 \oplus \mathbb{C}\mathscr{B}_1 \oplus \cdots$$

of binary delta-matroids was constructed; here the subspace $\mathbb{C}\mathscr{B}_n$ is freely spanned over $\mathbb{C}$ by the set $\mathscr{B}_n$. Multiplication in this Hopf algebra is given by the direct sum of set systems extended to $\mathbb{C}\mathscr{B}$ by linearity. The coproduct of a given set system $(E; \Psi)$ is the sum

$$\mu(E; \Psi) = \sum_{E' \subset E} \Psi|_{E'} \otimes \Psi|_{E \setminus E'},$$

where the set $\Psi|_{E'}$ consists of those elements of $\Psi$ that are contained in $E'$.

The mapping $\nu_E$ (see Definition 2.1) is equivariant with respect to the bijections of finite sets both on the set of Lagrangian subspaces and on the set of binary delta-matroids. Hence the set of such mappings defines a graded linear mapping

$$\nu\colon \mathbb{C}\mathscr{L} \to \mathbb{C}\mathscr{B}, \qquad \nu_n\colon \mathbb{C}\mathscr{L}_n \to \mathbb{C}\mathscr{B}_n, \qquad n = 0, 1, 2, \dots .$$

This linear mapping turns out to be an isomorphism.

**Theorem 4.1.** *The mapping $\nu\colon \mathbb{C}\mathscr{L} \to \mathbb{C}\mathscr{B}$ is a graded isomorphism of Hopf algebras.*

**Proof.** The mapping $\nu$ transfers multiplication and comultiplication in the Hopf algebra of Lagrangian subspaces to multiplication and comultiplication, respectively, in the algebra of binary delta-matroids. This follows directly from the definitions.

## 5. Four-Term Relations and Weight Systems

In [14] Vassiliev introduced the four-term relations for functions on chord diagrams. He proved that any graph invariant of order at most $n$ determines a function on chord diagrams that satisfies these relations. Such a function is called a *weight system*. Every four-term relation corresponds to a chord diagram and to a pair of chords with neighboring endpoints in this diagram. The remaining three diagrams in this relation can be built from the initial one by applying one of the two (mutually commuting) Vassiliev moves and their compositions. In [9] Vassiliev moves were extended to framed diagrams, which are chord diagrams associated to ribbon graphs with possibly twisted ribbons, and the corresponding four-term relations were described.

In [7] Kleptsyn and Smirnov extended Vassiliev moves to Lagrangian subspaces. As above, let $E$ be a finite set, and let $V_E$ be the vector space over $\mathbb{F}_2$ spanned by the elements of the set $E \sqcup E^\vee$. Given two distinct elements $e, e' \in E$, the *first Vassiliev move* assigned to the pair $e$, $e'$ is the linear mapping $V_E \to V_E$ preserving all basis vectors except for the vectors $e^\vee$ and $e'^\vee$, on which it acts as follows:

$$e^\vee \mapsto e^\vee + e'; \qquad e'^\vee \mapsto e'^\vee + e.$$

Note that the first Vassiliev move is symmetric with respect to the transposition of the elements $e$ and $e'$.

The *second Vassiliev move* for the pair $e, e'$ is the linear mapping $V_E \to V_E$ obtained from the first move by conjugation with respect to the twist along $e' \in E$; see Section 2. The description of the second move, in contrast to that of the first one, depends on the order of elements in the pair $e$, $e'$. The action of each Vassiliev move on the set of Lagrangian subspaces is induced by its action on $V_E$.

In [10] the first and second Vassiliev moves for binary delta-matroids $\mathscr{B}_E$ were defined. The second Vassiliev move was defined by using the recently introduced (see [13]) concept of handle sliding for delta-matroids. It was also shown in [10] (see Proposition 4.10) that the action of the first and second Vassiliev moves on the space $V_E$ as defined by Kleptsyn and Smirnov generates that defined by the author and Lando for binary delta-matroids. Taking into account Theorem 2.1, we obtain the following statement.

**Theorem 5.1.** *The graded Hopf algebra isomorphism $\nu\colon \mathbb{C}\mathscr{L} \to \mathbb{C}\mathscr{B}$ descends to a graded isomorphism $\nu\colon \mathscr{F}\mathbb{C}\mathscr{L} \to \mathscr{F}\mathbb{C}\mathscr{B}$ of the quotients of the Hopf algebras $\mathbb{C}\mathscr{L}$ and $\mathbb{C}\mathscr{B}$ by the corresponding four-term relations.*

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