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On Unconditional Bases of Reproducing Kernels in Fock-Type Spaces

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Abstract. The existence of unconditional bases of reproducing kernels in the Fock-type spaces $\mathscr{F}_\varphi$ with radial weights φ is studied. It is shown that there exist functions $\varphi(r)$ of arbitrarily slow growth for which $\ln r = o(\varphi(r))$ as $r \to \infty$ and there are no unconditional bases of reproducing kernels in the space $\mathscr{F}_\varphi$. Thus, a criterion for the existence of unconditional bases cannot be given only in terms of the growth of the weight function.

Key words: Hilbert spaces, entire functions, unconditional bases, Riesz bases, reproducing kernels.

Introduction

Given a subharmonic function $\varphi(\lambda)$ in the plane, we set

$$\mathscr{F}_\varphi = \left\{ f \in H(\mathbb{C}) : \|f\|^2 := \int_{\mathbb{C}} |f(\lambda)|^2 e^{-2\varphi(\lambda)}\, dm(\lambda) < \infty \right\},$$

where $H(\mathbb{C})$ is the space of entire functions and dm is the planar Lebesgue measure. Then $\mathscr{F}_\varphi$ is a Hilbert space in which the point functionals $\delta_\lambda\colon f \to f(\lambda)$ are continuous for all $\lambda \in \mathbb{C}$. Since the Hilbert spaces are self-dual, it follows that each of these functionals is generated by an element $K(z,\lambda) \in \mathscr{F}_\varphi$ in the sense that

$$f(\lambda) = \int_{\mathbb{C}} f(z)\overline{K(z,\lambda)} e^{-2\varphi(z)}\, dm(z) \quad \text{for any } f \in \mathscr{F}_\varphi \text{ and } \lambda \in \mathbb{C}.$$

The function $K(z,\lambda)$ is called the *reproducing kernel* of the space $\mathscr{F}_\varphi$ (see [1]); we have $K(z,\lambda) \in \mathscr{F}_\varphi$ for each fixed λ. Obviously,

$$\|\delta_\lambda\|^2 = \|K(\,\cdot\,,\lambda)\|^2 = K(\lambda,\lambda) \quad \text{for any } \lambda \in \mathbb{C}.$$

A basis $\{h_n\}_{n=1}^\infty$ in a Hilbert space is said to be unconditional [2] if there exists a number $P > 1$ such that

$$\frac{1}{P}\sum_{n=1}^N |a_n|^2 \|h_n\|^2 \leqslant \left\| \sum_{n=1}^N a_n h_n \right\|^2 \leqslant P \sum_{n=1}^N |a_n|^2 \|h_n\|^2$$

for any finite set of numbers $a_n \in \mathbb{C}$, $n = 1, \dots, N$.

In [3] the existence of unconditional bases consisting of values $K(z,\lambda_n)$ of the reproducing kernel in a space $\mathscr{F}_\varphi$ was studied (in what follows, instead of "basis consisting of values of the reproducing kernel" we say "basis of reproducing kernels"). This paper considers twice continuously differentiable radial weights $\varphi(z) = \varphi(|z|)$ such that the function

$$\rho(z) = (\Delta\varphi(z))^{-1/2} = \left(\frac{\varphi'(r)}{r} + \varphi''(r)\right)^{-1/2}, \qquad r = |z|,$$

satisfies the conditions

$$0 < \inf_{r>0} \rho(r), \qquad \rho(r) = o(r), \quad r \to \infty,$$
$$\rho(r + \rho(r)) = (1 + o(1))\rho(r), \quad r \to \infty, \qquad \rho(2r) \asymp \rho(r), \quad r > 0.$$

We prove that if $\ln^2 r = o(\varphi(r))$ as $r \to \infty$, then the space $L_2(\varphi)$ has no unconditional bases of reproducing kernels, and if $\varphi(r) = \ln^{\alpha}(r+1)$ for $1 < \alpha \leqslant 2$, then such bases exist.

If was shown in [4] that the space $\mathscr{F}_\varphi$ has unconditional bases of reproducing kernels if $(\varphi(e^t))''$ is a nonincreasing positive function and φ satisfies additional regularity assumptions.

In this paper we show that there exist arbitrarily slowly growing functions $\varphi(r)$ for which $\ln r = o(\varphi(r))$ as $r \to \infty$ and the space $\mathscr{F}_\varphi$ has no unconditional bases of reproducing kernels. Thus, a criterion for the existence of unconditional bases cannot be stated only in terms of the growth of the weight function.

1. Notation, Preliminaries, and Statements of Results

If two nonnegative functions f and g satisfy the condition

$$f(x) \leqslant Cg(x) \quad \text{for any } x \in X,$$

where C is a constant, then we write

$$f(x) \prec g(x), \qquad x \in X.$$

The symbols $\succ$ and $\asymp$ have obvious meaning.

Definition 1. Given a continuous function f on $\overline{B}(z,r)$, we set

$$\|f\|_r = \max_{w \in \overline{B}(z,r)} |f(w)|.$$

Let $d(f,z,r)$ be the distance from f to the space of harmonic functions on $B(z,r)$, i.e.,

$$d(f,z,r) = \inf\{\|f - H\|_r \colon H \text{ is harmonic on } B(z,r)\}.$$

Given a continuous function u on $\mathbb{C}$ and a positive number p, we set

$$\tau(u,z,p) = \sup\{r : d(u,z,r) \leqslant p\}.$$

Lemma 1.1 in [5] asserts that, in the case where u is a continuous subharmonic function, the quantity $\tau = \tau(u,\lambda,p)$ is completely determined by the following condition: If $H(z)$ is the least harmonic majorant of a function u on the disk $B(\lambda,\tau)$, then

$$\max_{z \in \overline{B}(\lambda,\tau)} (H(z) - u(z)) = 2p. \tag{1}$$

For example, it is easy to show that if $u(z) = |z|^2$, then

$$\tau(u,\lambda,p) = \sqrt{p} \quad \text{for any } \lambda \in \mathbb{C} \text{ and } p > 0.$$

Let us state Theorem 5 of [6] as applied to the weight spaces $\mathscr{F}_\varphi$.

Theorem A. *Let $K_\varphi(z,\lambda)$ be the reproducing kernel of the space $\mathscr{F}_\varphi$, and let $K(z) := K_\varphi(z,z)$. Suppose that, for any sufficiently large $p > 0$, there exists a number $\delta = \delta(p) > 0$ and a sequence of disks $B(\zeta_j, R_j)$ (which depends on p) such that the function*

$$\tau(z,p) = \tau(\ln K, z, p)$$

satisfies the conditions

(i) $\inf_{z \in B(\lambda, 2\tau(\lambda))} \tau(z) \geqslant \delta\tau(\lambda)$ *for all* $\lambda \in B(\zeta_j, R_j)$;

(ii) $\max_{z \in \overline{B}(\zeta_j, R_j)} \tau(z) = o(R_j)$ *as* $j \to \infty$.

Then the space $\mathscr{F}_\varphi$ has no unconditional bases of reproducing kernels.

Remark. In the statement of this theorem in [6] the existence of δ was required for all $p > 0$, but the proof used the function $\tau(z) = \tau(\ln K, z, \ln(5P))$, where P is the constant in the definition of an unconditional basis. Therefore, the statement given above is valid, too.

In this paper we prove the following theorem.

Theorem. *Given any positive function $\eta(t)$, $t > 0$, increasing without bound, there exists a radial subharmonic function $\varphi(z)$ such that*

$$\varphi(r) = O(\eta(r)\ln r), \qquad r \to \infty,$$

and the space $\mathscr{F}_\varphi$ has no unconditional bases of reproducing kernels.

Remark. We can always find a positive function $\gamma(t)$, $t > 0$, which increases without bound and satisfies the conditions $\gamma \leqslant \eta$ and

$$\gamma'(t) \leqslant \frac{\gamma(t)}{t\ln t}, \qquad t > 1.$$

We construct a function $\varphi(r) = O(\gamma(r)\ln r)$, $r \to \infty$, which satisfies the condition in the theorem.

2. Proof of the Theorem

1. Scheme of the construction of a radial subharmonic function $\varphi(z)$. Take a differentiable positive function $\mu(t)$, $t > 0$, increasing without bound and such that $\mu(0) = \mu'(0) = 0$. We set

$$u(z) = \int_0^{|z|} \frac{\mu(t)}{t}\,dt.$$

For $r = |z| \neq 0$, we have

$$\Delta u(z) = u''(r) + \frac{u'(r)}{r} = \frac{\mu'(r)}{r} > 0;$$

therefore, $u(z)$ is a twice differentiable positive radial subharmonic function in the plane. Let

$$d\widetilde{\mu}(z) = \frac{\Delta u(z)\,dm(z)}{2\pi} = \frac{\mu'(|z|)\,dm(z)}{2\pi|z|}$$

be the Riesz measure associated with u. Given a measure ν, by $\nu(t)$ we denote the ν-measure of the disk $B(0,t)$. We have

$$\widetilde{\mu}(t) = \mu(t), \qquad t > 0.$$

Take a sequence $T_n > 1$, $n \in \mathbb{N}$, increasing without bound and satisfying the condition $2T_n < T_{n+1}$, $n \in \mathbb{N}$. We denote the restriction of the measure $\widetilde{\mu}$ to the annulus

$$S_n = \{z : T_n < |z| < 2T_n\}$$

by μ_n and set

$$\nu = \sum_n \mu_n.$$

The function

$$\varphi(z) = \int_0^{|z|} \frac{\nu(t)}{t}\,dt$$

is positive, differentiable, and subharmonic in the plane. We impose certain conditions on the function $\mu(t)$ and the sequence T_n in order that the function φ satisfy the assumptions of the theorem.

2. Bounding the function φ from above.

Proposition 1. *If*

$$\gamma_0(t) = \sqrt{\frac{\gamma(t)}{2}}, \quad \mu(t) = \gamma_0(t)\ln^+ t, \quad t > 0, \qquad T_n = \gamma_0^{-1}(n),$$

then

$$\varphi(r) \leqslant \gamma(r)\ln r, \qquad r > 1.$$

Proof. By the definition of the measures μ_n, we have

$$\mu_n(t) = \begin{cases} 0, & t < T_n, \\ \mu(t) - \mu(T_n), & T_n \leqslant t \leqslant 2T_n, \\ \mu(2T_n) - \mu(T_n), & 2T_n < t. \end{cases}$$

Accordingly, if

$$n(r) = \max\{n : T_n \leqslant r\}, \qquad r > 0,$$

i.e., $T_{n(r)} \leqslant r < T_{n(r)+1}$, then

$$\nu(r) = \sum_{n=1}^{n(r)-1} (\mu(2T_n) - \mu(T_n)) + (\mu(r) - \mu(T_{n(r)}), \qquad T_{n(r)} \leqslant r < 2T_{n(r)}, \tag{2}$$

$$\nu(r) = \sum_{n=1}^{n(r)} (\mu(2T_n) - \mu(T_n)), \qquad 2T_{n(r)} \leqslant r < T_{n(r)+1}. \tag{3}$$

Let

$$x_2 = \ln(2T_n), \quad x_1 = \ln T_n.$$

Then, using the assumptions on the function γ, we obtain

$$\mu(2T_n) - \mu(T_n) = \gamma_0(e^{x_2})x_2 - \gamma_0(e^{x_1})x_1 = (\gamma_0(e^x)x)'(x^*)(x_2 - x_1) \leqslant 2\gamma_0(e^{x^*})(x_2 - x_1),$$

where $x^* \leqslant x_2$. If $n \leqslant n(r) - 1$, then $x^* \leqslant x_2 = \ln(2T_n) < \ln T_{n+1} \leqslant \ln T_{n(r)} \leqslant \ln r$. If $n = n(r)$ but $2T_{n(r)} \leqslant r$, then $x^* \leqslant x_2 = \ln(2T_n) \leqslant \ln r$. Thus,

$$\mu(2T_n) - \mu(T_n) \leqslant 2\gamma_0(r) \ln 2 < 2\gamma_0(r).$$

The difference

$$\mu(r) - \mu(T_{n(r)}) < 2\gamma_0(r)$$

with $T_{n(r)} \leqslant r < 2T_{n(r)}$ is estimated in a similar way. These estimates and relations (2)–(3) imply

$$\nu(r) < 2n(r)\gamma_0(r), \qquad r > 0.$$

Let $\beta(t) = \gamma_0^{-1}(t)$. Then $\beta(n(r)) \leqslant r$ or $n(r) \leqslant \gamma_0(r)$, because $T_{n(r)} \leqslant r$. Thus,

$$\nu(r) < 2\gamma_0^2(r) = \gamma(r), \qquad r > 0,$$

and

$$\varphi(r) = \int_1^r \frac{\nu(t)\, dt}{t} \leqslant \gamma(r) \ln r, \qquad r > 1.$$

This completes the proof of Proposition 1. □

Remark. The assumption on the function γ ensures that $2T_n < T_{n+1}$ for $T_n = \beta(n)$.

3. Estimate of the characteristic τ for the functions u and φ.

Proposition 2. *Let $\delta > 0$ be a sufficiently small number, and let*

$$S_n(\delta) := \{z : (1+\delta)T_n \leqslant |z| \leqslant (2-\delta)T_n\}, \qquad n \in \mathbb{N}.$$

For any $p > 0$ and n larger some $n(p, \delta)$, we have

$$\sqrt{\frac{p}{2}}\, \frac{r}{\sqrt{\gamma_0(r)}} \leqslant \tau(\varphi, z, p) \leqslant 6\sqrt{p}\, \frac{r}{\sqrt{\gamma_0(r)}}, \qquad z \in S_n(\delta).$$

Proof. Since the derivative of γ satisfies the condition in the remark to the main theorem, it follows that so does the derivative of γ_0. Therefore, for $r > e$, we have

$$\frac{\gamma_0(r)}{r} \leqslant \mu'(r) = \gamma_0'(r) \ln r + \frac{\gamma_0(r)}{r} \leqslant \frac{\gamma_0(r)}{r} + \frac{\gamma_0(r)}{r} \leqslant 2\frac{\gamma_0(r)}{r}, \qquad r > e.$$

Thus, if $r = |z|$, then

$$\frac{\gamma_0(r)}{r^2} \leqslant \Delta u(z) = \frac{\mu'(r)}{r} \leqslant 2\,\frac{\gamma_0(r)}{r^2}\,. \tag{4}$$

It also follows from the conditions on the function γ that if $r > 4$, then

$$\gamma_0(2r) - \gamma_0(r) = \int_r^{2r} \gamma_0'(t)\,dt \leqslant \gamma_0(2r)\int_r^{2r} \frac{d\ln t}{\ln t} = \gamma_0(2r)\ln\frac{\ln(2r)}{\ln r} \leqslant \frac{\gamma_0(2r)}{2},$$

whence

$$\gamma_0(2r) \leqslant 2\gamma_0(r).$$

Take any point $z_0 \in \mathbb{C}$ and any numbers $\rho, p > 0$. The least harmonic majorant $H(z)$ of the function $v(z) = |z - z_0|^2$ on the disk $B(z_0,\rho)$ equals identically ρ^2, and the measure associated with it equals $(2/\pi)\,dm(z)$. Therefore, if $G(z,w)$ is the Green function of $B(z_0,\rho)$, then

$$\rho^2 = \max_{z\in B(z_0,\rho)} (H(z) - v(z)) = \frac{2}{\pi} \max_{z\in B(z_0,\rho)} \int_{B(z_0,\rho)} G(z,w)\,dm(w).$$

This implies

$$\max_{z\in B(z_0,\rho)} \int_{B(z_0,\rho)} G(z,w)\,dm(w) = \frac{\pi}{2}\,\rho^2. \tag{5}$$

Let $h(z)$ be the least harmonic majorant of u on the disk $B(z_0,\rho)$, where $\rho \leqslant r_0/2$ ($r_0 = |z_0|$). Then the first inequality in (4) and relation (5) give

$$\begin{aligned}\max_{z\in B(z_0,\rho)} (h(z) - u(z)) &= \frac{1}{2\pi} \max_{z\in B(z_0,\rho)} \int_{B(z_0,\rho)} G(z,w)\Delta u(w)\,dm(w)\\ &\geqslant \frac{2\gamma_0(r_0/2)}{9\pi r_0^2} \max_{z\in B(z_0,\rho)} \int_{B(z_0,\rho)} G(z,w)\,dm(w) = \frac{\gamma_0(r_0/2)}{9r_0^2}\,\rho^2 \geqslant \frac{\gamma_0(r_0)\rho^2}{18r_0^2}\,.\end{aligned}$$

If r_0 is such that $\gamma_0(r_0) \geqslant 144p$ and $\rho = 6r_0\sqrt{p/\gamma_0(r_0)}$, then the last estimate implies

$$\max_{z\in B(z_0,\rho)} (h(z) - u(z)) \geqslant 2p.$$

By virtue of relation (1), this means that

$$\tau(u, z_0, p) \leqslant 6\sqrt{p}\,\frac{r_0}{\sqrt{\gamma_0(r_0)}}\,.$$

Similar estimates imply

$$\max_{z\in B(z_0,\rho)} (h(z) - u(z)) \leqslant \frac{4\gamma_0(r_0)}{r_0^2}\,\rho^2,$$

so that if r_0 is such that $\gamma_0(r_0) \leqslant p\sqrt{2}$ and $\rho = (r_0/\sqrt{2})\sqrt{p/\gamma_0(r_0)}$, then

$$\max_{z\in B(z_0,\rho)} (h(z) - u(z)) \leqslant 2p.$$

Thus,

$$\tau(u, z_0, p) \geqslant \sqrt{\frac{p}{2}}\,\frac{r_0}{\sqrt{\gamma_0(r_0)}}\,.$$

We have proved the following inequalities for any positive p and any $r = |z|$ satisfying the condition $r \geqslant \gamma_0^{-1}(144p)$:

$$\sqrt{\frac{p}{2}}\,\frac{r}{\sqrt{\gamma_0(r)}} \leqslant \tau(u, z, p) \leqslant 6\sqrt{p}\,\frac{r}{\sqrt{\gamma_0(r)}}\,. \tag{6}$$

If follows from (6) that if n is so large that

$$\frac{6\sqrt{p}}{\sqrt{\gamma_0(T_n)}} < \frac{\delta}{2},$$

then the disk $B(z,\tau(u,z,p))$ is contained in the annulus $S_n=\{z: T_n<|z|<2T_n\}$ for all $z\in S_n(\delta)$; therefore, by virtue of (1), we have

$$\tau(u,z,p)=\tau(\varphi,z,p).$$

Now Proposition 2 follows from (6). □

4. Estimate of the Bergman function $K(z)$ of the space $\mathscr{F}_\varphi$.

Proposition 3. *The following relation holds*:

$$K(z)\asymp\frac{1}{\tau^2(\varphi,z,1)}\,e^{2\varphi(z)},\qquad z\in S_n(\delta),\ n>n(p,\delta),$$

or, *in view of Proposition* 2,

$$K(z)\asymp\frac{\gamma_0(r)}{r^2}\,e^{2\varphi(z)},\qquad z\in S_n(\delta),\ n>n(p,\delta).$$

Proof. The Bergman function $K(z)=K(z,z)$ equals the squared norm $\|\delta_z\|^2$ of the point functional. The upper bound can be obtained simply by using properties of subharmonic functions. We set $\tau(z)=\tau(\varphi,z,1)$ and let h be the least harmonic majorant of φ on the disk $B(z):=B(z,\tau(z))$, so that

$$0\leqslant h(w)-\varphi(w)\leqslant 1,\qquad w\in B(z).$$

The subharmonicity of the function $e^{-2h}|F|^2$ for $F\in\mathscr{F}_\varphi$ implies

$$e^{-2h(z)}|F(z)|^2\leqslant\frac{1}{\pi\tau^2(z)}\int_{B(z)}e^{-2h(w)}|F(w)|^2\,dm(w).$$

It follows that

$$K(z)\leqslant\frac{e^2}{\pi\tau(z)^2}\,e^{2\varphi(z)}.$$

Proposition 2 gives the upper estimate

$$K(z)\leqslant\frac{2e^2\gamma_0(r)}{\pi r^2}\,e^{2\varphi(z)},\qquad r=|z|,\tag{7}$$

for $z\in S_n(\delta)$ and $n>n(p,\delta)$.

Let us estimate the function $K(z)$ from below. First, note that

$$K(z)=\sum_{n=0}^{\infty}\frac{|z|^{2n}}{a_n^2},$$

where

$$a_n^2=\|z^n\|^2_{F\in\mathscr{F}_\varphi}=2\pi\int_0^\infty t^{2n+1}e^{-2\varphi(t)}\,dt,$$

or

$$a_n^2=2\pi\int_{-\infty}^{\infty}e^{2(n+1)x-2\psi(x)}\,dx;$$

here $\psi(x)=\varphi(e^x)$ is a convex function on $\mathbb{R}$. Let $\widetilde{\psi}$ denote the Young complement of ψ, and let

$$\widetilde{\psi}(y)=\sup_{x\in\mathbb{R}}(xy-\psi(x)).$$

By x_n we denote the point at which the supremum is attained for $y=n+1$, i.e.,

$$\psi'(x_n)=n+1.\tag{8}$$

Performing straightforward calculations and taking into account (4), we see that if $t=e^x\in\widetilde{S}_m:=(T_m;2T_m)$ for some m, then

$$\psi''(x)=\nu'(t)t=\mu'(t)t\asymp\gamma_0(t).\tag{9}$$

We denote the interval $\{t:|r-t|\leqslant\tau(z)\}$ by $I(r)$, where $r=|z|$, and take any positive $\delta<1/4$. Let $r=|z|\in\widetilde{S}_k(2\delta):=[(1+2\delta)T_k;(2-2\delta)T_k]$. Then $I(r)\subset\widetilde{S}_k(\delta)$.

According to Theorem 2(a) in [7], for any $p > 0$, we have

$$a_n^2 = 2\pi \int_{-\infty}^{\infty} e^{2(n+1)x-2\psi(x)}\,dx \asymp \frac{1}{\rho_2(\widetilde{\psi}, n+1, p)}\, e^{2\widetilde{\psi}(n+1)}, \qquad n \in \mathbb{N}.$$

Here $\rho_2 = \rho_2(\widetilde{\psi}, y, p)$ is defined by [7, formula (1)], i.e.,

$$\rho_2 = \sup\left\{t > 0 : \int_{y-t}^{y+t} |\widetilde{\psi}'(\tau) - \widetilde{\psi}'(y)|\,d\tau \leqslant p\right\}.$$

Thus,

$$K(z) \succ \frac{1}{r^2} \sum_{r_n \in I(r)} e^{2((n+1)\ln r - \widetilde{\psi}(n+1))} \rho_2(\widetilde{\psi}, n+1, p). \tag{10}$$

Let us prove that

$$\rho_2(\widetilde{\psi}, n+1, 4) \succ \sqrt{\gamma_0(r_n)}, \qquad r_n = e^{x_n}, \tag{11}$$

for sufficiently large n. First, we show that if $|t-(n+1)| \leqslant \sqrt{\gamma_0(r_n)}$, then $e^{\widetilde{\psi}'(t)} \in \widetilde{S}_k$. Indeed, take $t = \psi'(\ln T)$, where $T = T_k$ or $T = 2T_k$. We have

$$|t-(n+1)| = |\psi'(\ln T) - \psi'(\ln r_n)| = \psi''(t_1)|\ln(T/r_n)|,$$

where t_1 is between $\ln T$ and $\ln r_n$. Therefore, $e^{t_1} \in \widetilde{S}_k$ and

$$|t-(n+1)| \succ \gamma_0(e^{t_1}) \succ \gamma_0(r_n).$$

We set $\rho = \sqrt{\gamma_0(r_n)}$. If $|t-(n+1)| \leqslant \rho$, then it follows from the above considerations that $e^{\widetilde{\psi}'(t)} \in \widetilde{S}_k$; hence

$$\begin{aligned}\int_{y-\rho}^{y+\rho} |\widetilde{\psi}'(t) - \widetilde{\psi}'(n+1)|\,dt &\leqslant \sup_{|t-(n+1)|\leqslant\rho} \widetilde{2}\psi''(t)\rho^2 = 2\gamma_0(r_n) \sup_{|t-(n+1)|\leqslant\rho} \widetilde{\psi}''(t) \\ &= 2\gamma_0(r_n) \sup_{|t-(n+1)|\leqslant\rho} \frac{1}{\psi''(\widetilde{\psi}'(t))} \leqslant 2\gamma_0(r_n) \sup_{|t-(n+1)|\leqslant\rho} \frac{1}{\gamma_0(e^{\widetilde{\psi}'(t)})} \\ &\leqslant \frac{2\gamma_0(2T_k)}{\gamma_0(T_k)} < \frac{2\gamma_0(T_{k+1})}{\gamma_0(T_k)} = \frac{2(k+1)}{k} \leqslant 4.\end{aligned}$$

This implies estimate (11).

From (10) and (11) we obtain

$$K(z) \succ \frac{1}{r^2} \sum_{r_n \in I(r)} e^{2((n+1)\ln r - \widetilde{\psi}(n+1))} \sqrt{\gamma_0(r_n)}\,. \tag{12}$$

Let y_r denote the point at which the supremum $\sup_{y\in\mathbb{R}}(y \ln r - \widetilde{\psi}(y))$ is attained. Then

$$\psi(\ln r) = y_r \ln r - \widetilde{\psi}(y_r), \qquad \widetilde{\psi}'(y_r) = \ln r.$$

It follows from properties of Young complementary functions that if $y(t)$ is the point at which $\sup_{y\in\mathbb{R}}(yt - \widetilde{\psi}(y)) = \psi(t)$ is attained, then $\widetilde{\psi}'(y(t)) = t$ and

$$ty - \widetilde{\psi}(y) - \psi(t) = -\widetilde{\psi}(y) + \widetilde{\psi}(y(t)) + \widetilde{\psi}'(y(t))(y - y(t)).$$

Considering the Taylor expansion of $\widetilde{\psi}(y)$ centered at $y(t)$, we obtain

$$ty - \widetilde{\psi}(y) - \psi(t) = -\frac{\widetilde{\psi}''(y_1)}{2}(y - y(t))^2,$$

where y_1 is a point between y and $y(t)$. Assuming that $r_n \in I(r)$ and $t = \ln r$, we set $y = n+1$ and $y(t) = y_r$ and use the known formulas

$$\psi'(\widetilde{\psi}'(x)) \equiv x, \qquad \psi''(\widetilde{\psi}'(x))\widetilde{\psi}''(x) \equiv 1.$$

The relations

$$n+1=\psi'(\ln r_n),\quad \ln r_n=\widetilde{\psi}'(n+1),\quad \widetilde{\psi}'(y_r)=\ln r,\quad \psi'(\ln r)=y_r$$

imply

$$(n+1)\ln r-\widetilde{\psi}(n+1)-\psi(\ln r)=-\frac{1}{2\psi''(\widetilde{\psi}'(y_1))}(\psi'(\ln r_n)-\psi'(\ln r))^2.$$

Let $r_1 = e^{\widetilde{\psi}'(y_1)}$. Since y_1 is between $n+1$ and y_r, it follows that r_1 is between the points $e^{\widetilde{\psi}'(n+1)}=r_n$ and $e^{\widetilde{\psi}'(y_r)}=r$. In particular, $r_1\in\widetilde{S}_k$, whence $\psi''(\ln r_1)\asymp\gamma_0(r_1)\leqslant\gamma_0(r)/2$. Thus,

$$(n+1)\ln r-\widetilde{\psi}(n+1)-\psi(\ln r)\geqslant-\frac{1}{\gamma_0(r)}(\psi'(\ln r_n)-\psi'(\ln r))^2. \tag{13}$$

By virtue of the mean value theorem, we have

$$|\psi'(\ln r_n)-\psi'(\ln r)|=\left|\frac{\psi''(\ln t)}{t}\right||r_n-r|,$$

where t is a point between r_n and r. Therefore,

$$|\psi'(\ln r_n)-\psi'(\ln r)|\prec\left|\frac{\gamma_0(t)}{t}\right|\tau(z)\asymp\sqrt{\gamma_0(r)}. \tag{14}$$

Substituting this estimate into (13), we obtain

$$(n+1)\ln r-\widetilde{\psi}(n+1)-\psi(\ln r)\succ-1.$$

We return to estimate (12). If $N(r)$ is the number of positive integers n for which $r_n\in I(r)$, then

$$K(z)\succ\frac{N(r)\sqrt{\gamma_0(r)}}{r^2}\,e^{2\psi(\ln r)}.$$

If $r_n\in I(r)$, then (14) implies

$$|(n+1)-y_r|=|\psi'(\ln r_n)-\psi'(\ln r)|\prec\sqrt{\gamma_0(r)},$$

whence

$$N(r)\asymp\sqrt{\gamma_0(r)}.$$

Thus,

$$K(z)\succ\frac{\gamma_0(r)}{r^2}e^{2\varphi(r)}.$$

This completes the proof of Proposition 3. □

5. Estimate of the characteristic τ for the function $\ln K(z)$. We set

$$K_0(z)=\frac{\gamma_0(r)}{r^2}e^{2\varphi(r)},\qquad r=|z|.$$

It is easy to derive from the condition on the function γ in the remark to the main theorem that

$$\lim_{r\to\infty}\ln\frac{\gamma_0(3r/2)}{\gamma_0(r/2)}=0.$$

Proposition 4. *For sufficiently large n,*

$$\sqrt{\frac{p}{8}}\,\frac{r}{\sqrt{\gamma_0(r)}}\leqslant\tau(\ln K_0,z,p)\leqslant 6\sqrt{p}\,\frac{r}{\sqrt{\gamma_0(r)}},\qquad z\in S_n(\delta),\ r=|z|.$$

Proof. Take any $p>0$ and any positive integer N so large that

$$\ln\frac{\gamma_0(3r/2)}{\gamma_0(r/2)}<\frac{p}{2},\qquad \tau(z):=\tau(\varphi,z,p)<\frac{r}{2} \tag{15}$$

for all $r \in \widetilde{S}_n$ with $n \geqslant N$. We set $\tau_0(z) := \tau(\ln K_0, z, p)$. Take $r \in \widetilde{S}_n(\delta)$, $n \geqslant N$, and positive $t = \min(\tau_0(z), r/2)$. Let $2h$ be the harmonic function on $B(z,t)$ nearest to $\ln K_0$ on this disk in the sense of the distance in Definition 1. Then

$$\begin{aligned}\sup_{w\in B(z,t)} &|2\varphi(w) + \ln\gamma_0(r) - 2\ln|w| - 2h(w)| \\ &\leqslant \sup_{w\in B(z,t)} |2\varphi(w) + \ln\gamma_0(|w|) - 2\ln|w| - 2h(w)| + \sup_{w\in B(z,t)} |\ln\gamma_0(r) - \ln\gamma_0(|w|)| \\ &\leqslant \sup_{w\in B(z,t)} |\ln K_0(w) - 2h(w)| + \ln\frac{\gamma_0(3r/2)}{\gamma_0(r/2)}\,.\end{aligned}$$

By the choice of N and t, we have

$$\sup_{w\in B(z,t)} |2\varphi(w) + \ln\gamma_0(r) - 2\ln|w| - 2h(w)| < 2p;$$

hence the harmonic function $h_1(w) = h(w) - \frac{1}{2}\ln\gamma_0(r) + \ln|w|$ on $B(z,t)$ obeys the estimate

$$\sup_{w\in B(z,t)} |\varphi(w) - h_1(w)| < p.$$

This means that $\tau(z) = \tau(\varphi, z, p) \geqslant t$. If $t = \tau_0(z)$, then $\tau(z) \geqslant \tau_0(z)$. If $t = r/2$, then $\tau(z) \geqslant r/2$, which contradicts the choice of N. By virtue of Proposition 2, we have

$$\tau(\ln K_0, r, p) \leqslant 6\sqrt{p}\,\frac{r}{\sqrt{\gamma_0(r)}}\,.$$

To obtain a lower bound, we set $t = \tau(\varphi, z, p/4)$ and use h to denote the harmonic function on $B(z,t)$ nearest to φ on this disk in the sense of the distance in Definition 1. Then

$$\begin{aligned}\sup_{w\in B(z,t)} &|\ln K_0(w) - 2h(w) + 2\ln|w| - \ln\gamma_0(r)| \\ &\leqslant \sup_{w\in B(z,t)} |2\varphi(w) - 2h(w)| + \sup_{w\in B(z,t)} |\ln\gamma_0(|w|) - \ln\gamma_0(r)| < p,\end{aligned}$$

because, according to (15), $t = \tau(\varphi, z, p/4) \leqslant \tau(\varphi, z, p) < r/2$. This means that, for the harmonic function $h_1(w) = 2h(w) - 2\ln|w| + \ln\gamma_0(r)$, we have

$$\sup_{w\in B(z,t)} |\ln K_0(w) - h_1(w)| < p.$$

Therefore,

$$\tau(\ln K_0, z, p) \geqslant t = \tau\left(\varphi, z, \frac{p}{4}\right) \geqslant \sqrt{\frac{p}{8}}\,\frac{r}{\sqrt{\gamma_0(r)}}\,.$$

This proves Proposition 4. □

Proposition 3 implies

$$e^{-C}K_0(z) \leqslant K(z) \leqslant e^{C}K_0(z), \qquad z \in \bigcup_n S_n(\delta),$$

for some $C > 0$. Using these inequalities, in the same way as above, we obtain

$$\begin{aligned}\tau(\ln K, z, p + C) &\geqslant \tau(\ln K_0, z, p), \qquad z \in S_n(2\delta),\\ \tau(\ln K_0, z, p + C) &\geqslant \tau(\ln K, z, p), \qquad z \in S_n(2\delta),\end{aligned}$$

for all $p > 0$ and sufficiently large n. Therefore, if $p \geqslant 2C$, then

$$\tau(\ln K_0, z, p/2) \leqslant \tau(\ln K_0, z, p - C) \leqslant \tau(\ln K, z, p) \leqslant \tau(\ln K_0, z, 3p/2),$$

and Proposition 4 implies

$$\frac{\sqrt{p}}{4}\,\frac{r}{\sqrt{\gamma_0(r)}} \leqslant \tau(\ln K, z, p) \leqslant 6\sqrt{\frac{3p}{2}}\,\frac{r}{\sqrt{\gamma_0(r)}}, \qquad z \in S_n(2\delta),\ r = |z|,$$

for sufficiently large n. It remains to apply Theorem A for the disks $B(\zeta_j, R_j)$, where

$$|\zeta_j| = 3T_j/2, \quad R_j = (1/4 - \delta)T_j$$

for $\delta < 1/4$.

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