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Trace Formulas for a Discrete Schrödinger Operator*

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Dedicated to the memory of Professor M. S. Agranovich

Abstract. The Schrödinger operator with complex decaying potential on a lattice is considered. Trace formulas are derived on the basis of classical results of complex analysis. These formulas are applied to obtain global estimates of all zeros of the Fredholm determinant in terms of the potential.

Key words: trace formula, complex potential, eigenvalues.

1. Introduction. Let us consider a Schrödinger operator $H=\Delta+V$ on $\ell^2(\mathbb{Z}^d)$, $d\geqslant 3$, where Δ is the discrete Laplace operator on the lattice $\mathbb{Z}^d$, which is defined by

$$(\Delta f)(n)=\frac{1}{2}\sum_{|m-n|=1} f_m,\qquad f=(f_n)_{n\in\mathbb{Z}^d}\in\ell^2(\mathbb{Z}^d),\ n=(n_j)_{j=1}^d\in\mathbb{Z}^d.$$

The operator $V=(V_n)_{n\in\mathbb{Z}^d}$, $V_n\in\mathbb{C}$, is a complex potential acting as $(Vf)(n)=V_nf_n$, $n\in\mathbb{Z}^d$, and satisfying the condition

$$V\in\ell^{2/3}(\mathbb{Z}^d). \tag{1}$$

Here $\ell^q(\mathbb{Z}^d)$, $q>0$, is the space of all sequences $f=(f)_{n\in\mathbb{Z}^d}$ such that $\|f\|_q<\infty$, where

$$\|f\|_q=\|f\|_{\ell^q(\mathbb{Z}^d)}=\begin{cases}\sup_{n\in\mathbb{Z}^d}|f_n|, & q=\infty,\\ \big(\sum_{n\in\mathbb{Z}^d}|f_n|^q\big)^{1/q}, & q\in(0,\infty).\end{cases}$$

Note that $\ell^q(\mathbb{Z}^d)$, $q\geqslant 1$, is a Banach space equipped with the norm $\|\cdot\|_q$. It is well known that the spectrum of the Laplace operator is absolutely continuous and satisfies the identity $\sigma(\Delta)=\sigma_{\mathrm{ac}}(\Delta)=[-d,d]$; see, e.g., [10]. Since the perturbation V is of trace class, it follows from Weyl's theorem that the essential spectrum of the Schödinger operator H is

$$\sigma_{\mathrm{ess}}(H)=[-d,d].$$

However, this condition does not exclude the appearance of a singular continuous spectrum on the interval $[-d,d]$. The main goal of this paper is to find new trace formulas for the operator H with complex-valued potential V and use these formulas for estimating the complex eigenvalues in terms of the potential.

Note that some of the results obtained in this paper are new even in the case of real-valued potentials because of the presence of a measure ν (see Theorem 3), which appears in the canonical factorization of the corresponding Fredholm determinant. The nontriviality of such measures is ensured by the condition (1) on the potential V. We believe that it would be interesting to study the connection between properties of the potential V and the measure ν.

Uniform bounds for the eigenvalues of Schrödinger operators on $\mathbb{R}^d$ with complex-valued potentials decaying at infinity have recently attracted the attention of many specialists in the field. Estimates for one eigenvalue were obtained, e.g., in the papers [8], [4], and [19], and estimates for sums of powers of eigenvalues were found in [5], [15], [3], [1], [7], and [20]. The latter results generalize the celebrated Lieb–Thirring inequalities [16] to the non-self-adjoint case. Note that no results on the number of eigenvalues of Schrödinger operators with complex-valued potentials have

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been obtained. We mention the recent paper [6], in which this problem was discussed in detail for odd dimensions.

Most of the results concerning discrete Schrödinger operators were obtained for self-adjoint operators; see, e.g., [22] for the case $\mathbb{Z}^1$. Schrödinger operators with decaying potentials on the lattice $\mathbb{Z}^d$ were considered by Boutet de Monvel with Sahbani [2], Isozaki with Korotyaev [10], Kopylova [13], Rosenblum with Solomjak [18], and Shaban with Vainberg [21] (see also the references therein). In [10] Isozaki and Korotyaev studied the inverse scattering problem for discrete Schrödinger operators with finitely supported potential. In [11] Isozaki and Morioka proved that, in this case, the point spectrum of the operator H does not appear in the interval $(-d, d)$. Scattering on periodic metric graphs was considered by Korotyaev and Saburova [14].

In this paper we use classical results of complex analysis, which lead us to a new class of trace formulas for discrete Schrödinger operators with complex-valued potentials. In particular, we use the canonical factorization of analytic functions in the Hardy space. Such factorizations of Fredholm determinants allow us to obtain trace formulas, which give estimates of complex eigenvalues via the $\ell_p^{2/3}$ norm of the potential.

2. Definitions. Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\} \subset \mathbb{C}$ denote the disc of radius $r > 0$, and let $\mathbb{D} = \mathbb{D}_1$. By $\mathbb{T} = \partial\mathbb{D}$ we denote the boundary of the disc $\mathbb{D}$. It is convenient to introduce a new spectral parameter $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2}\left(z + \frac{1}{z}\right) \in \Lambda := \mathbb{C} \setminus [-d, d], \qquad z \in \mathbb{D}.$$

The function $\lambda(z)$ has the following properties.

- The function $z \to \lambda(z)$ conformally maps the disc $\mathbb{D}$ onto the spectral domain Λ.
- The function $\lambda(z)$ maps $z = 0$ to $\lambda = \infty$ and the boundary $\partial\mathbb{D}$ onto the cut $[-d, d]$.
- The inverse mapping $z(\,\cdot\,)\colon \Lambda \to \mathbb{D}$ is given by $z = d^{-1}(\lambda - \sqrt{\lambda^2 - d^2})$, $\lambda \in \Lambda$, and has the asymptotics $z(\lambda) = (d/2)\lambda^{-1} + O(1)\lambda^{-3}$ as $|\lambda| \to \infty$. This asymptotics defines a branch of $z(\lambda)$.

Let us introduce Hardy spaces $\mathscr{H}_p = \mathscr{H}_p(\mathbb{D})$, $0 < p \leqslant \infty$. Let F be analytic in $\mathbb{D}$. We say that F belongs to the Hardy space $\mathscr{H}_p$ if $\|F\|_{\mathscr{H}_p} < \infty$, where

$$\|F\|_{\mathscr{H}_p} = \begin{cases} \sup_{r\in(0,1)}(\frac{1}{2\pi}\int_{\mathbb{T}} |F(re^{i\vartheta})|^p \, d\vartheta)^{1/p}, & 0 < p < \infty, \\ \sup_{z\in\mathbb{D}} |F(z)|, & p = \infty. \end{cases}$$

3. Main results. Let $\mathscr{B}_1$ denote the space of trace class operators in $\ell^2(\mathbb{Z}^d)$. Since $V \in \mathscr{B}_1$, we can define the Fredholm determinant $D(z)$, $z \in \mathbb{D}$, for the pair Δ, $\Delta + V$ as follows (see [9] for properties of Fredholm determinants):

$$D(z) = \det(I + V(\Delta - \lambda(z))^{-1}), \qquad z \in \mathbb{D}.$$

Note that if $\lambda_0 \in \Lambda$ is an eigenvalue of the operator H, then $z_0 = z(\lambda_0) \in \mathbb{D}$ is a zero of the Fredholm determinant D with the same multiplicity.

Theorem 1. *Let V satisfy condition* (1). *Then the determinant $D(z)$ is analytic in $\mathbb{D}$ and continuous up to the boundary. It has N, $0 \leqslant N \leqslant \infty$, zeros z_j such that*

$$0 < r_0 = |z_1| \leqslant \dots \leqslant |z_j| \leqslant |z_{j+1}| \leqslant \dots, \tag{2}$$

and

$$\|D\|_{\mathscr{H}_\infty} \leqslant e^{C\|V\|_{2/3}}, \tag{3}$$

where the constant C depends only on d. Moreover, the function $\log D(z)$ *defined by the condition* $\log D(0) = 0$ *is analytic in the disc $\mathbb{D}_{r_0}$ whose radius $r_0 > 0$ is defined by* (2), *and its Taylor expansion in $\mathbb{D}_{r_0}$ is*

$$\log D(z) = -c_1 z - c_2 z^2 - c_3 z^3 - c_4 z^4 - \cdots,$$

where

$$c_1 = d_1 a, \quad a = 2/d, \quad c_2 = d_2 a^2, \quad c_3 = d_3 a^3 - c_1, \quad c_4 = d_4 a^4 - c_2, \quad \dots,$$
$$d_1 = \operatorname{Tr} V, \quad d_2 = \operatorname{Tr} V^2, \quad d_3 = \operatorname{Tr}(V^3 + (3d/2)V), \quad \dots, \quad d_n = \operatorname{Tr}(H^n - H_0^n), \quad \dots .$$

The main analytical difficulty in the proof of inequality (3) is the analysis of $A(\lambda) = |V|^{1/2}(\Delta - \lambda)^{-1}|V|^{1/2}$ as an operator function from Λ to $\mathscr{B}_1$. Roughly speaking, it is required to obtain a uniform bound $\sup_{\lambda\in\Lambda}\|A(\lambda)\|_{\mathscr{B}_1} < \infty$. We are not aware of such bounds, but our results impose too strong condition (1) on the potential V.

For the function D, we define a Blaschke product $B(z)$, $z \in \mathbb{D}$, by

$$B = 1 \quad \text{as } N = 0 \quad \text{and} \quad B(z) = \prod_{j=1}^{N} \frac{|z_j|}{z_j} \frac{(z_j - z)}{(1 - \overline{z}_j z)} \quad \text{as } N \geqslant 1. \tag{4}$$

Theorem 2. *Let V satisfy condition* (1), *and let $N \geqslant 2$. Then the zeros z_j of D in the disc $\mathbb{D}$ (see* (2)) *satisfy the inequality*

$$\sum_{j=1}^{N}(1 - |z_j|) < \infty.$$

Moreover, the Blaschke product $B(z)$, $z \in \mathbb{D}$, defined by (4) *is absolutely convergent in the disc $\{|z| < 1\}$, $B \in \mathscr{H}_\infty$, and the function* $\log B$ *has the following Taylor expansion in the disc $\mathbb{D}_{r_0}$:*

$$\log B(z) = B_0 - B_1 z - B_2 z^2 - \cdots \quad \textit{for } z \to 0, \tag{5}$$

where $|B_n| \leqslant 2r_0^{-n}\sum_{j=1}^{N}(1 - |z_j|)$ and

$$B_0 = \log B(0) < 0, \quad B_1 = \sum_{j=1}^{N}\left(\frac{1}{z_j} - \overline{z}_j\right), \quad \dots, \quad B_n = \frac{1}{n}\sum_{j=1}^{N}\left(\frac{1}{z_j^n} - \overline{z}_j^n\right), \quad \dots .$$

Below we introduce the canonical factorization of the determinant D and describe its main properties.

Theorem 3. *Let V satisfy condition* (1). *Then there is a singular measure $\nu \geqslant 0$ on $[-\pi, \pi]$ such that the determinant D has canonical factorization*

$$D(z) = B(z)e^{-K_\nu(z)}e^{K_D(z)}, \quad K_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\nu(t),$$
$$K_D(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\log|D(e^{it})|\, dt, \tag{6}$$

for all $|z| < 1$, where $\log|D(e^{it})| \in L^1(-\pi, \pi)$ and

$$\operatorname{supp}\nu \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}.$$

Remarks. 1. For details on canonical factorization, see, e.g., [12].

2. Note that since $d\nu \geqslant 0$ and $\operatorname{Re}\frac{e^{it}+z}{e^{it}-z} \geqslant 0$ for all $(t, z) \in \mathbb{T} \times \mathbb{D}$, it follows that the function $D_{\mathrm{in}}(z) := B(z)e^{-K_\nu(z)}$ satisfies the inequality $|D_{\mathrm{in}}(z)| < 1$.

Theorem 4 (trace formulas)**.** *Let V satisfy the condition* (1). *Then*

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|D(e^{it})|\, dt \geqslant 0, \tag{7}$$

$$-c_n + B_n = \frac{1}{\pi}\int_{\mathbb{T}} e^{-int}\, d\mu(t), \qquad n = 1, 2, \dots, \tag{8}$$

where $d\mu(t) = \log|D(z)|\,dt - d\nu(t)$, $B_0 = \log B(0) = \log\left(\prod_{j=1}^N |z_j|\right) < 0$, *and the* B_n *are defined in* (5). *In particular*,

$$\sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z}_j\right) = \frac{2}{d}\operatorname{Tr} V + \frac{1}{\pi}\int_{\mathbb{T}} e^{-it}\,d\mu(t), \tag{9}$$

$$\sum_{j=1}^N \left(\frac{1}{z_j^2} - \overline{z}_j^2\right) = \frac{4}{d^2}\operatorname{Tr} V^2 + \frac{1}{\pi}\int_{\mathbb{T}} e^{-2it}\,d\mu(t), \tag{10}$$

and

$$\begin{aligned} \sum_{j=1}^N \operatorname{Im}\lambda_j &= \operatorname{Tr}\operatorname{Im} V - \frac{d}{2\pi}\int_{\mathbb{T}} \sin t\,d\mu(t), \\ \sum_{j=1}^N \operatorname{Re}\sqrt{\lambda_j^2 - d^2} &= \operatorname{Tr}\operatorname{Re} V + \frac{d}{2\pi}\int_{\mathbb{T}} \cos t\,d\mu(t). \end{aligned} \tag{11}$$

There are papers devoted to the spectral shift function for non-self-adjoint trace class perturbations (see, e.g., the recent work [17] and references therein). In [17, pp. 812 and 822] other trace formulas were derived from the factorization of perturbation determinants in $\mathbb{C}_+$. However, the paper [17] does not contain bounds for eigenvalues in terms of the norm of the perturbation, which is the aim of the study of this paper. We believe that formulas (7)–(11) are new.

Let us briefly describe the proof. The main problem is proving the integrability of $\log|D(e^{it})|$, $t \in [-\pi, \pi]$, which follows from (3) and well-known results on Hardy spaces. All functions in (6) have Taylor expansions in the disc. Substituting these expansions into (6) and comparing the coefficients of the same powers of z, we obtain formulas (7) and (8).

Theorem 5. *Let* V *satisfy condition* (1). *Then the following estimates hold*:

$$\sum (1 - |z_j|) \leqslant -B_0 \leqslant C(d)\|V\|_{2/3} - \frac{\nu(\mathbb{T})}{2\pi}, \tag{12}$$

$$\sum_{j=1}^N \operatorname{Im}\lambda_j \leqslant \operatorname{Tr}\operatorname{Im} V + C(d)\|V\|_{2/3}, \qquad \operatorname{Im} V \geqslant 0, \tag{13}$$

$$\sum_{j=1}^N \sqrt{\lambda_j^2 - d^2} \leqslant \operatorname{Tr} V + C(d)\|V\|_{2/3}, \qquad V \geqslant 0. \tag{14}$$

Remarks. 1. Some assertions of Theorems 4 and 5 are new even for real-valued potentials.

2. Let us briefly describe the proof of Theorem 5. The inequality $1 - x \leqslant -\log x$, $x \in (0, 1]$, implies

$$-B_0 = -B(0) = -\sum_j \log|z_j| \geqslant \sum_j (1 - |z_j|).$$

Inequality (3) gives

$$\left|\int_{\mathbb{T}} d\mu(t)\right| \leqslant C(d)\|V\|_{2/3} - \nu(\mathbb{T}). \tag{15}$$

Substituting the last two inequalities into (7), we obtain (12), and substituting (15) into (11), we obtain (13) and (14).

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