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On Homogenization for Non-Self-Adjoint Locally Periodic Elliptic Operators*

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Abstract. In this note we consider the homogenization problem for a matrix locally periodic elliptic operator on $\mathbb{R}^d$ of the form $\mathscr{A}^\varepsilon = -\operatorname{div} A(x, x/\varepsilon)\nabla$. The function A is assumed to be Hölder continuous with exponent $s \in [0, 1]$ in the "slow" variable and bounded in the "fast" variable. We construct approximations for $(\mathscr{A}^\varepsilon - \mu)^{-1}$, including one with a corrector, and for $(-\Delta)^{s/2}(\mathscr{A}^\varepsilon - \mu)^{-1}$ in the operator norm on $L_2(\mathbb{R}^d)^n$. For $s \neq 0$, we also give estimates of the rates of approximation.

Key words: homogenization, operator error estimates, locally periodic operators, effective operator, corrector.

1. Introduction. Homogenization theory studies the asymptotic behavior of solutions to differential equations with rapidly oscillating coefficients. The solutions of the classical homogenization problems are known to converge, in a certain sense, to solutions of problems whose coefficients no longer oscillate. It is a matter of considerable interest not only to prove convergence but also to find its rate. "Operator error estimates" make it possible to achieve both of these goals: they yield convergence in the strongest operator topology and, at the same time, provide its rate.

The classical periodic homogenization problems for elliptic operators are rather well studied, and operator error estimates for these problems are now very well known. The next natural step is to extend these estimates to the case of locally periodic operators, when the coefficients depend on both "fast" and "slow" variables. We set $Q = [-1/2, 1/2)^d$. By $C^{0,s}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$ we denote the space of complex-valued functions in $L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ that are Hölder continuous with exponent $s \in [0, 1]$ (uniformly continuous if $s = 0$) in the first variable and periodic with respect to the lattice $\mathbb{Z}^d$ in the second. Let $A = \{A_{kl}\}_{k,l=1}^d$, where $A_{kl} \in C^{0,s}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))^{n\times n}$. Consider the operator $\mathscr{A}^\varepsilon$ from the complex Sobolev space $H^1(\mathbb{R}^d)^n$ to its dual $H^{-1}(\mathbb{R}^d)^n$ given by

$$\mathscr{A}^\varepsilon = D^* A^\varepsilon D = \sum_{k,l=1}^d D_k A_{kl}^\varepsilon D_l, \tag{1}$$

where $D = -i\nabla$ and $A^\varepsilon(x) = A(x, x/\varepsilon)$. Suppose that $\mathscr{A}^\varepsilon$ is coercive uniformly in ε for small ε; in other words, we suppose that there are a $c_A > 0$ and a $C_A \geqslant 0$ such that

$$\operatorname{Re}(A^\varepsilon Du, Du)_{L_2(\mathbb{R}^d)} + C_A\|u\|^2_{L_2(\mathbb{R}^d)} \geqslant c_A\|Du\|^2_{L_2(\mathbb{R}^d)} \tag{2}$$

for all $u \in H^1(\mathbb{R}^d)^n$ and $\varepsilon \in \mathscr{E} = (0, \varepsilon_0]$. It then follows that $\mathscr{A}^\varepsilon$ is m-sectorial, so whenever μ is outside the corresponding sector $\mathscr{S}$, the operator $(\mathscr{A}^\varepsilon - \mu)^{-1}$ is defined, and its norm is uniformly bounded. Our aim is to study the behavior of $(\mathscr{A}^\varepsilon - \mu)^{-1}$ as $\varepsilon \to 0$. In this note we will construct the first two terms of an approximation for $(\mathscr{A}^\varepsilon - \mu)^{-1}$ in the operator norm on $L_2(\mathbb{R}^d)^n$ and the first term of an approximation for $D^s(\mathscr{A}^\varepsilon - \mu)^{-1}$ (here D^s is the differentiation of order s defined via the fractional Laplacian $(-\Delta)^{s/2}$) in the same operator norm. For each approximation, we provide an operator bound for the error.

The results that we discuss here have not been fully known. Operator error estimates for locally periodic elliptic operators were previously studied in [2] and [3], where the leading terms of

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approximations for $(\mathscr{A}^\varepsilon - \mu)^{-1}$ and $D(\mathscr{A}^\varepsilon - \mu)^{-1}$ were obtained. However, in [2] the function A was required to be sufficiently smooth in both variables, and in [3] this function was assumed to be Lipschitz in the first variable (that is, $s = 1$ in our notation). The second term of a uniform approximation for the resolvent was known only in the special case when the fast and slow variables are "separated" (in the sense that $A^\varepsilon(x) = A(x_1, x_2/\varepsilon)$, where $x = (x_1, x_2)$; this means that A is periodic with respect to a lattice of nonfull rank) and the coefficients are Lipschitz in the slow variable [5] (see also [4]). We thus extend these results in two directions: First, we relax the condition on smoothness as much as possible while requiring that A^ε remain measurable. Second, we refine the approximation of the resolvent of $\mathscr{A}^\varepsilon$. Note also that, unlike Borisov [2] and Pastukhova with Tikhomirov [3], we replace the self-adjointness and semiboundedness hypotheses by the assumption that $\mathscr{A}^\varepsilon$ is sectorial (as in [5]). A detailed exposition of the results for $s = 1$ can be found in [6].

2. Main results. In order to construct the approximations, we need, as usual, to introduce a solution of an auxiliary problem on the cell Q. For $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^{d\times n}$, let $N_\xi(x, \cdot)$ be the periodic vector-valued solution of the problem

$$D^* A(x, \cdot)(DN_\xi(x, \cdot) + \xi) = 0, \qquad \int_Q N_\xi(x, y)\, dy = 0, \tag{3}$$

in the cell Q (i.e., $N_\xi(x, \cdot)$ belongs to the periodic Sobolev space $\tilde{H}^1(Q)^n$ and satisfies (3)). Such $N_\xi(x, \cdot)$ exists and is unique by virtue of the uniform coercivity of $\mathscr{A}^\varepsilon$. Moreover, $N_\xi(x, \cdot)$ depends linearly on ξ, so that the map $\xi \mapsto N_\xi$ is an operator of multiplication by a function; we denote this function by N. Clearly, N has the same smoothness in the first variable as A; hence $N \in C^{0,s}(\bar{\mathbb{R}}^d; \tilde{H}^1(Q))$.

The notion of an effective operator is central to homogenization theory. The effective operator $\mathscr{A}^0$ is the map from $H^1(\mathbb{R}^d)^n$ to $H^{-1}(\mathbb{R}^d)^n$ defined by

$$\mathscr{A}^0 = D^* A^0 D, \tag{4}$$

where

$$A^0(x) = \int_Q A(x, y)(I + D_2 N(x, y))\, dy. \tag{5}$$

It follows from the smoothness properties of A and N that $A^0 \in C^{0,s}(\bar{\mathbb{R}}^d)$. Moreover, the operator $\mathscr{A}^0$ turns out to be coercive and satisfy a relation of the form (2) with the same constants as for $\mathscr{A}^\varepsilon$. Thus, we conclude that the effective operator is also m-sectorial (but the sector may differ from $\mathscr{S}$).

Theorem 1. *Let $A \in C^{0,s}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$, and let $\mu \notin \operatorname{spec} \mathscr{A}^0$. If $s = 0$, then $(\mathscr{A}^\varepsilon - \mu)^{-1}$ converges, as $\varepsilon \to 0$, in the operator norm on $L_2(\mathbb{R}^d)^n$ to $(\mathscr{A}^0 - \mu)^{-1}$. On the other hand, if $s \in (0, 1]$, then there is a neighborhood $\mathscr{E}_\mu \subset \mathscr{E}$ of 0 such that, for any $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_2(\mathbb{R}^d)^n$,*

$$\|(\mathscr{A}^\varepsilon - \mu)^{-1} f - (\mathscr{A}^0 - \mu)^{-1} f\|_{L_2(\mathbb{R}^d)} \leqslant C_\mu \varepsilon^s \|f\|_{L_2(\mathbb{R}^d)}.$$

The constant C_μ is explicitly described in terms of s, n, d, μ, c_A, C_A, $\|A\|_{C^{0,s}}$, and the distance from μ to $\operatorname{spec} \mathscr{A}^0$, and the interval $\mathscr{E}_\mu$ depends in addition on ε_0. In particular, $\mathscr{E}_\mu = \mathscr{E}$ provided that $\mu \notin \mathscr{S}$.

The next result concerns an approximation of the resolvent in the fractional Sobolev space $H^s(\mathbb{R}^d)^n$, and hence we will assume that $s \neq 0$. Suppose also that if $s \neq 1$, then the function A belongs to the generalized space $C^{0,s,2}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$ of Hölder continuous functions, i.e., A is uniformly bounded and $[A]_{C^{0,s,2}} = \int_0^\infty t^{-2s-1} \omega(A; t)\, dt$ is finite. Here $\omega(A; t)$ is the modulus of continuity of A as a vector-valued map taking values in $L_\infty(Q)$. The norm of A in this space is $\|A\|_{C^{0,s,2}} = \|A\|_C + [A]_{C^{0,s,2}}$. (Note that the class $C^{0,s,2}$ is embedded in $C^{0,s}$, and it is nothing but the Besov space $B^s_{\infty,2}$.) In this case, we also have $N \in C^{0,s,2}(\bar{\mathbb{R}}^d; \tilde{H}^1(Q))$, and, as a result, $A^0 \in C^{0,s,2}(\bar{\mathbb{R}}^d)$.

The classical corrector of homogenization theory is not quite suitable for our purposes. Instead, we use the operator $\mathscr{K}_\mu^\varepsilon$ from $L_2(\mathbb{R}^d)^n$ to $H^s(\mathbb{R}^d)^n$ defined by

$$\mathscr{K}_\mu^\varepsilon f(x) = \int_Q N(x+\varepsilon z, \varepsilon^{-1}x)\, D(\mathscr{A}^0-\mu)^{-1} f(x+\varepsilon z)\, dz. \tag{6}$$

Under the above assumptions, $\mathscr{K}_\mu^\varepsilon$ is continuous.

Theorem 2. *Suppose that either $s \in (0,1)$ and $A \in C^{0,s,2}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$ or $s=1$ and $A \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$. If $\mu \notin \operatorname{spec} \mathscr{A}^0$, then, for any $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_2(\mathbb{R}^d)^n$,*

$$\|D^s(\mathscr{A}^\varepsilon-\mu)^{-1}f - D^s(\mathscr{A}^0-\mu)^{-1}f - \varepsilon D^s \mathscr{K}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)} \leqslant C_\mu \varepsilon^s \|f\|_{L_2(\mathbb{R}^d)}.$$

The constant C_μ is explicitly described in terms of s, n, d, μ, c_A, C_A, $\|A\|_{C^{0,s,2}}$ (if $s<1$) or $\|A\|_{C^{0,1}}$ (if $s=1$), and the distance from μ to $\operatorname{spec}\mathscr{A}^0$, and the interval $\mathscr{E}_\mu$ is the same as in Theorem 1.

The corrector in this form was first proposed in [3] for the case $s=1$. We note that $\mathscr{K}_\mu^\varepsilon$ involves the rapidly oscillating function $x \mapsto N(x+\varepsilon z, \varepsilon^{-1}x)$, so that the operator norm of $D^s\mathscr{K}_\mu^\varepsilon$ on $L_2(\mathbb{R}^d)^n$ increases without bound as $\varepsilon \to 0$. Nevertheless, thanks to the factor ε, the norm of the term $\varepsilon D^s \mathscr{K}_\mu^\varepsilon f$ is small provided that $s<1$. Thus, Theorem 1 implies the convergence of $D^s(\mathscr{A}^\varepsilon-\mu)^{-1}$. A similar result can be proved for $D^r(\mathscr{A}^\varepsilon-\mu)^{-1}$ with $r \geqslant s$ under even weaker requirements on the coefficients. Below, $\alpha \wedge \beta$ denotes the minimum of α and β.

Theorem 3. *Let $A \in C^{0,s}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$, and let $\mu \notin \operatorname{spec}\mathscr{A}^0$. If $s=0$ and $r \in (0,1)$, then $D^r(\mathscr{A}^\varepsilon-\mu)^{-1}$ converges, as $\varepsilon \to 0$, in the operator norm on $L_2(\mathbb{R}^d)^n$ to $D^r(\mathscr{A}^0-\mu)^{-1}$. On the other hand, if $s \in (0,1)$ and $r \in [s,1)$, then, for any $\varepsilon \in \mathscr{E}_\mu$ and $f \in L_2(\mathbb{R}^d)^n$,*

$$\|D^r(\mathscr{A}^\varepsilon-\mu)^{-1}f - D^r(\mathscr{A}^0-\mu)^{-1}f\|_{L_2(\mathbb{R}^d)} \leqslant C_\mu \varepsilon^{s\wedge(1-r)} \|f\|_{L_2(\mathbb{R}^d)}.$$

The constant C_μ is explicitly described in terms of s, r, n, d, μ, c_A, C_A, $\|A\|_{C^{0,s}}$, and the distance from μ to $\operatorname{spec}\mathscr{A}^0$, and the interval $\mathscr{E}_\mu$ is the same as in Theorem 1.

We emphasize that, under our assumptions, the image of $\mathscr{K}_\mu^\varepsilon$ is contained only in $H^s(\mathbb{R}^d)^n$. Therefore, if we want to better approximate $D^r(\mathscr{A}^\varepsilon-\mu)^{-1}$ for $r>s$, we surely cannot use $\mathscr{K}_\mu^\varepsilon$.

Let us return to the approximation of the resolvent of $\mathscr{A}^\varepsilon$. Recall that Theorem 1 provides the leading term of an approximation. We now construct the second term. It is called a corrector as well, but this corrector is substantially different from $\mathscr{K}_\mu^\varepsilon$ and has much more complicated structure.

Let $(\mathscr{A}^\varepsilon-\mu)^+$ denote the adjoint of $\mathscr{A}^\varepsilon-\mu$. For $(\mathscr{A}^\varepsilon-\mu)^+$, we can define analogues of all objects introduced thus far. We will mark these analogues by the symbol "+" as well. According to elliptic regularity theory, the operator $(\mathscr{A}^0-\mu)^{-1}$ continuously maps $L_2(\mathbb{R}^d)^n$ to $H^{1+s}(\mathbb{R}^d)^n$. If $s \geqslant 1/2$, then the first-order differential operator

$$\mathscr{L} = \int_Q (N^+(\,\cdot\,,y))^* D_1^* A(\,\cdot\,,y)\,(I + D_2 N(\,\cdot\,,y))\, dy$$

is well defined and bounded as a map from $H^{1/2}(\mathbb{R}^d)^n$ to $H^{-1/2}(\mathbb{R}^d)^n$. Therefore, the operator

$$\mathscr{L}_\mu = (\mathscr{A}^0-\mu)^{-1} D^* \mathscr{L} D (\mathscr{A}^0-\mu)^{-1}$$

is continuous on $L_2(\mathbb{R}^d)^n$. Next, let $\varDelta_h A(x,y) = A(x+h,y) - A(x,y)$. We set

$$M_\varepsilon(x) = \varepsilon^{-1} \int_Q (I + D_2 N^+(x, x/\varepsilon+z))^* \varDelta_{\varepsilon z} A(x, x/\varepsilon+z)\,(I + D_2 N(x, x/\varepsilon+z))\, dz$$

and define a bounded operator $\mathscr{M}_\mu^\varepsilon$ on $L_2(\mathbb{R}^d)^n$ as

$$\mathscr{M}_\mu^\varepsilon = (\mathscr{A}^0-\mu)^{-1} D^* M_\varepsilon D (\mathscr{A}^0-\mu)^{-1}.$$

The corrector that we are looking for is then given by

$$\mathscr{C}_\mu^\varepsilon = (\mathscr{K}_\mu^\varepsilon - \mathscr{L}_\mu) - \mathscr{M}_\mu^\varepsilon + ((\mathscr{K}_\mu^\varepsilon)^+ - \mathscr{L}_\mu^+)^*. \tag{7}$$

Theorem 4. *Suppose that either* $s \in [1/2, 1)$ *and* $A \in C^{0,s,2}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$ *or* $s = 1$ *and* $A \in C^{0,1}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$*. If* $\mu \notin \operatorname{spec} \mathscr{A}^0$*, then, for any* $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_2(\mathbb{R}^d)^n$*,*

$$\|(\mathscr{A}^\varepsilon - \mu)^{-1} f - (\mathscr{A}^0 - \mu)^{-1} f - \varepsilon \mathscr{C}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)} \leqslant C_\mu \varepsilon^{2s/(2-s)} \|f\|_{L_2(\mathbb{R}^d)}.$$

The constant C_μ *is explicitly described in terms of* s, n, d, μ, c_A, C_A, $\|A\|_{C^{0,s,2}}$ *(if* $s < 1$*) or* $\|A\|_{C^{0,1}}$ *(if* $s = 1$*), and the distance from* μ *to* $\operatorname{spec} \mathscr{A}^0$*, and the interval* $\mathscr{E}_\mu$ *is the same as in Theorem* 1.

A corrector of this type was first obtained in [1] for the purely periodic case. In [5] (see also [4]) the operator $\mathscr{C}_\mu^\varepsilon$ had a similar form but did not contain the term $\mathscr{M}_\mu^\varepsilon$. However, it is generally impossible to remove $\mathscr{M}_\mu^\varepsilon$ from $\mathscr{C}_\mu^\varepsilon$ while retaining the same order of error; see [6] for an example. This term is therefore a distinguishing feature of nonperiodic problems.

On the other hand, if we replace $\mathscr{C}_\mu^\varepsilon$ by $\mathscr{M}_\mu^\varepsilon$, then the error will be of order $\varepsilon^{1 \wedge 2s/(2-s)}$. It turns out that a similar result holds for any $s \in (0, 1)$ and for a broader class of coefficients.

Theorem 5. *Let* $s \in (0, 1)$*, and let* $A \in C^{0,s}(\bar{\mathbb{R}}^d; \tilde{L}_\infty(Q))$*. If* $\mu \notin \operatorname{spec} \mathscr{A}^0$*, then, for any* $\varepsilon \in \mathscr{E}_\mu$ *and* $f \in L_2(\mathbb{R}^d)^n$*,*

$$\|(\mathscr{A}^\varepsilon - \mu)^{-1} f - (\mathscr{A}^0 - \mu)^{-1} f - \varepsilon \mathscr{M}_\mu^\varepsilon f\|_{L_2(\mathbb{R}^d)} \leqslant C_\mu \varepsilon^{1 \wedge 2s/(2-s)} \|f\|_{L_2(\mathbb{R}^d)}.$$

The constant C_μ *is explicitly described in terms of* s, n, d, μ, c_A, C_A, $\|A\|_{C^{0,s}}$*, and the distance from* μ *to* $\operatorname{spec} \mathscr{A}^0$*, and the interval* $\mathscr{E}_\mu$ *is the same as in Theorem* 1.

3. Sketch of the proof. Our proof is based on results concerning operators with Lipschitz coefficients, and therefore we begin with this case.

For $s = 1$, we develop the method of [5]. The idea of the method is to obtain an operator identity of the form

$$(\mathscr{A}^\varepsilon - \mu)^{-1} - (\mathscr{A}^0 - \mu)^{-1} - \varepsilon \mathscr{K}_\mu^\varepsilon = (\mathscr{A}^\varepsilon - \mu)^{-1} (\ldots) (\mathscr{A}^0 - \mu)^{-1},$$

which is interpreted as a generalized resolvent identity. The key point is that the leading contribution of $(\mathscr{A}^\varepsilon - \mu)^{-1}$ is canceled by that of the sum of $(\mathscr{A}^0 - \mu)^{-1}$ and $\varepsilon \mathscr{K}_\mu^\varepsilon$, so that the remaining terms in brackets on the right-hand side are small. This makes it possible to relatively easily obtain all desired estimates at once. We emphasize that the estimates in Theorems 1, 2, and 4 for $s = 1$ are sharp with respect to order and, generally, cannot be improved.

The case of operators with Hölder continuous coefficients is treated, by way of standard mollifiers, as a "limiting" case of operators with Lipschitz coefficients. Fix a nonnegative function $J \in C_c^\infty(B_1(0))$ such that $\int_{\mathbb{R}^d} J(x)\, dx = 1$. We set $J_\delta(x) = \delta^{-d} J(x/\delta)$ and $A_\delta(x, y) = \int_{\mathbb{R}^d} J_\delta(x - z) A(z, y)\, dz$. According to the results that we have already proved, the resolvent of the regularized operator $\mathscr{A}^\varepsilon(\delta) = D^* A_\delta^\varepsilon D$ can be approximated in terms of the effective operator $\mathscr{A}^0(\delta)$ and the correctors $\mathscr{K}_\mu^\varepsilon(\delta)$ and $\mathscr{C}_\mu^\varepsilon(\delta)$. Further, the resolvents of $\mathscr{A}^\varepsilon(\delta)$ and $\mathscr{A}^0(\delta)$ and the correctors $\mathscr{K}_\mu^\varepsilon(\delta)$ and $\mathscr{C}_\mu^\varepsilon(\delta)$ converge, as $\delta \to 0$, in the corresponding operator norms to the resolvents of $\mathscr{A}^\varepsilon$ and $\mathscr{A}^0$ and the correctors $\mathscr{K}_\mu^\varepsilon$ and $\mathscr{C}_\mu^\varepsilon$, respectively. Moreover, the convergence rates are uniformly bounded by the rate of the convergence of A_δ to A. It remains to choose an optimal subsequence $\delta(\varepsilon)$.

It is noteworthy that, in the case $s = 1$, the estimates involve the Lipschitz seminorm of the coefficients. Since that of A_δ increases without bound as $\delta \to 0$, the orders of the error terms will be worse for $s < 1$. For instance, if $s = 0$, then the above argument shows only that $(\mathscr{A}^\varepsilon - \mu)^{-1}$ converges in norm but says nothing about the rate of convergence.

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